2 ☐ LIMITS AND DERIVATIVES

2.1 The Tangent and Velocity Problems

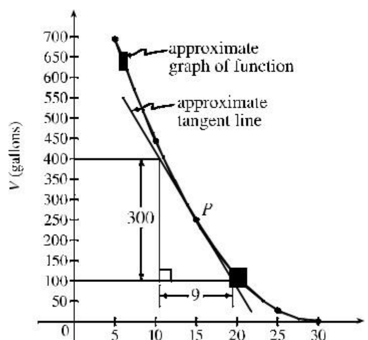
(a) Using (15 250), we construct the following table:

		slope =
5	(5 694)	$\frac{694-250}{5-15} = -\frac{444}{10} = -444$
10	(10 444)	$\frac{444-250}{10-15} = -\frac{194}{5} = -388$
20	(20 111)	$\frac{111-250}{20-15} = -\frac{139}{5} = -278$
25	(25 28)	$\frac{28-250}{25-15} = -\frac{222}{10} = -222$
30	(30 0)	$\frac{0-250}{30-15} = -\frac{250}{15} = -166$

Using the values of that correspond to the points closest to (= 10 and = 20), we have

$$\frac{-38\ 8 + (-27\ 8)}{2} = -33\ 3$$

(c) From the graph, we can estimate the slope of the tangent line at to be $\frac{-300}{9} = -33\ 3$



(a) Slope = $\frac{2948-2530}{42-36} = \frac{418}{6} \approx 6967$

(b) Slope = $\frac{2948-2661}{42-38} = \frac{287}{4} = 7175$

(c) Slope = $\frac{2948-2806}{42-40} = \frac{142}{2} = 71$

(d) Slope = $\frac{3080-2948}{44-42} = \frac{132}{2} = 66$

From the data, we see that the patient's heart rate is decreasing from 71 to 66 heartbeats minute after 42 minutes. After being stable for a while, the patient's heart rate is dropping.

3. (a) $= 1 -$, $(2 -1)$

		$(1(1-))$	
(i)	1 5	(1 5 −2)	2
(ii)	1 9	(1 9 −1 111 111)	1 111 111
(iii)	1 99	(1 99 −1 010 101)	1 010 101
(iv)	1 999	(1 999 −1 001 001)	1 001 001
(v)	2 5	(2 5 −0 666 667)	0 666 667
(vi)	2 1	(2 1 −0 909 091)	0 909 091
(vii)	2 01	(2 01 −0 990 099)	0 990 099
(viii)	2 001	(2 001 −0 999 001)	0 999 001

The slope appears to be 1.

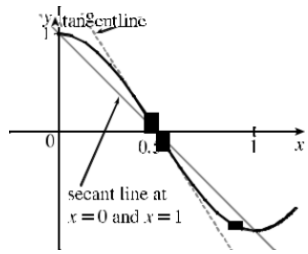
Using $= 1$, an equation of the tangent line to the curve at $(2 -1)$ is $-(-1) = 1(-2)$, or $= -3$.

4. (a) = cos , (0 5 0)

(i)	0	(0 1)	−2
(ii)	0 4	(0 4 0 309017)	−3 090170
(iii)	0 49	(0 49 0 031411)	−3 141076
(iv)	0 499	(0 499 0 003142)	−3 141587
(v)	1	(1 −1)	−2
(vi)	0 6	(0 6 −0 309017)	−3 090170
(vii)	0 51	(0 51 −0 031411)	−3 141076
(viii)	0 501	(0 501 −0 003142)	−3 141587

The slope appears to be − .

− 0 = − (− 0 5) or = − + $\frac{1}{2}$.



5. (a) = ( ) = 40 − 16 ^{2}. At = 2, = 40(2) − 16(2)^2 = 16. The average velocity between times 2 and 2 + is

ave = (2+) − (2) / (2+) − 2 = 40(2 +) − 16(2 +)^2 − 16 / = 24 −16 ^{2} / = 24 16 , if 6 = .0

(i) [2 2 5]: = 0 5, ave = −32 ft s (ii) [2 2 1]: = 0 1, ave = −25 6 ft s

(iii) [2 2 05]: = 0 05, ave = −24 8 ft s (iv) [2 2 01]: = 0 01, ave = −24 16 ft s

The instantaneous velocity when = 2 (approaches 0) is −24 ft s.

(a) = ( ) = 10 − 1 86 ^{2}. At = 1, = 10(1) − 1 86(1)^2 = 8 14. The average velocity between times 1 and 1 + is

ave = (1+) − (1) / (1+) − 1 = 10(1+)−186(1+)^2 −814 / = 628 −186 ^{2} / =628 − 1 86 , if 6 = .0

(i) [1 2]: = 1, ave = 4 42 m s (ii) [1 1 5]: = 0 5, ave = 5 35 m s

(iii) [1 1 1]: = 0 1, ave = 6 094 m s (iv) [1 1 01]: = 0 01, ave = 6 2614 m s

[1 1 001]: = 0 001, ave = 6 27814 m s

The instantaneous velocity when = 1 (approaches 0) is 6 28 m s.

(a) (i) On the interval [2 4] , ave = (4) − (2) / 4−2 = 79 2 − 20 6 / 2 = 29 3 ft s.

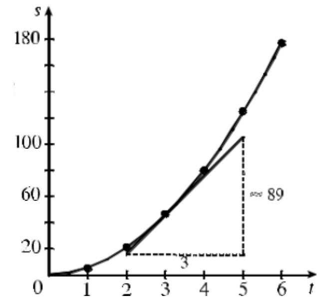
(ii) On the interval [3 4], ave = (4) − (3) / 4 − 3 = 792−465 / 1 = 32 7 ft s.

(iii) On the interval [4 5], ave = (5) − (4) / 5 − 4 = 1248−792 / 1 = 45 6 ft s.

(iv) On the interval [4 6], ave = (6) − (4) / 6 − 4 = 1767−792 / 2 = 48 75 ft s.

(b) Using the points (2 16) and (5 105) from the approximate tangent line, the instantaneous velocity at = 3 is about

$$\frac{105-16}{5-2} = \frac{89}{3} \approx 29\ 7 \text{ ft s.}$$



8. (a) (i) = ( ) = 2 sin + 3 cos . On the interval [1 2], ave = $\frac{(2) - (1)}{2-1} = \frac{3-(-3)}{1} = 6$ cm s.

On the interval [1 1 1], ave = $\frac{(1\ 1) - (1)}{11 - 101} \approx \frac{-3\ 471 - (-3)}{} = -4\ 71$ cm s.

On the interval [1 1 01], ave = $\frac{(1\ 01) - (1)}{101 - 1001} \approx \frac{-3\ 0613 - (-3)}{} = -6\ 13$ cm s.

On the interval [1 1 001], ave = $\frac{(1\ 001) - (1)}{1001 - 10001} \approx \frac{-3\ 00627 - (-3)}{} = -6\ 27$ cm s.

The instantaneous velocity of the particle when = 1 appears to be about −6 3 cm s.

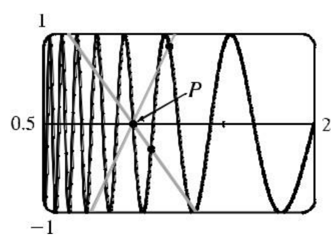
(a) For the curve = sin(10) and the point (1 0):

2	(2 0)	0
1 5	(1 5 0 8660)	1 7321
1 4	(1 4 −0 4339)	−1 0847
1 3	(1 3 −0 8230)	−2 7433
1 2	(1 2 0 8660)	4 3301
1 1	(1 1 −0 2817)	−2 8173

0 5	(05 0)	0
0 6	(0 6 0 8660)	−2 1651
0 7	(0 7 0 7818)	−2 6061
0 8	(08 1)	−5
0 9	(0 9 −0 3420)	3 4202

As approaches 1, the slopes do not appear to be approaching any particular value.

(b)



We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at that we need to take -values much closer to 1 in order to get accurate estimates of its slope.

If we choose = 1 001, then the point is (1 001 −0 0314) and ≈ −31 3794. If = 0 999, then is (0 999 0 0314) and = −31 4422. The average of these slopes is −31 4108. So we estimate that the slope of the tangent line at is about −31 4.

NOT FOR SALE

2.2 The Limit of a Function

As approaches 2, ( ) approaches 5. [Or, the values of ( ) can be made as close to 5 as we like by taking sufficiently close to 2 (but $\neq 2$).] Yes, the graph could have a hole at $(2\ 5)$ and be defined such that $(2) = 3$.

As approaches 1 from the left, ( ) approaches 3; and as approaches 1 from the right, ( ) approaches 7. No, the limit does not exist because the left- and right-hand limits are different.

3. (a) $\lim_{\to -3}$ ( ) $= \infty$ means that the values of ( ) can be made arbitrarily large (as large as we please) by taking sufficiently close to -3 (but not equal to -3).

(b) $\lim_{\to 4^+}$ ( ) $= -\infty$ means that the values of ( ) can be made arbitrarily large negative by taking sufficiently close to 4 through values larger than 4.

(a) As approaches 2 from the left, the values of ( ) approach 3, so $\lim_{\to 2^-}$ ( ) $= 3$.

(b) As approaches 2 from the right, the values of ( ) approach 1, so $\lim_{\to 2^+}$ ( ) $= 1$.

(c) $\lim_{\to 2}$ ( ) does not exist since the left-hand limit does not equal the right-hand limit.

(d) When $= 2$, $= 3$, so $(2) = 3$.

(e) As approaches 4, the values of ( ) approach 4, so $\lim_{\to 4}$ ( ) $= 4$.

(f) There is no value of ( ) when $= 4$, so (4) does not exist.

5. (a) As approaches 1, the values of ( ) approach 2, so $\lim_{\to 1}$ ( ) $= 2$.

(b) As approaches 3 from the left, the values of ( ) approach 1, so $\lim_{\to 3^-}$ ( ) $= 1$.

(c) As approaches 3 from the right, the values of ( ) approach 4, so $\lim_{\to 3^+}$ ( ) $= 4$.

(d) $\lim_{\to 3}$ ( ) does not exist since the left-hand limit does not equal the right-hand limit.

(e) When $= 3$, $= 3$, so $(3) = 3$.

6. (a) ( ) approaches 4 as approaches -3 from the left, so $\lim_{\to -3^-}$ ( ) $= 4$.

(b) ( ) approaches 4 as approaches -3 from the right, so $\lim_{\to -3^+}$ ( ) $= 4$.

(c) $\lim_{\to -3}$ ( ) $= 4$ because the limits in part (a) and part (b) are equal.

(d) (-3) is not defined, so it doesn't exist.

(e) ( ) approaches 1 as approaches 0 from the left, so $\lim_{\to 0^-}$ ( ) $= 1$.

(f) ( ) approaches -1 as approaches 0 from the right, so $\to 0+$ ( ) $= -1$.

(g) $\lim_{\to 0}$ ( ) does not exist because the limits in part (e) and part (f) are not equal.

(h) $(0) = 1$ since the point $(0\ 1)$ is on the graph of .

(i) Since $\lim_{\to 2^-}$ ( ) $= 2$ and $\lim_{\to 2^+}$ ( ) $= 2$, we have $\lim_{\to 2}$ ( ) $= 2$.

(j) (2) is not defined, so it doesn't exist.

NOT FOR SALE

(k) ( ) approaches 3 as approaches 5 from the right, so $\lim_{\to 5^+}$ ( ) = 3.

(l) ( ) does not approach any one number as approaches 5 from the left, so $\lim_{\to 5^-}$ ( ) does not exist.

7. (a) $\lim_{\to 0^-}$ ( ) = 1 (b) $\lim_{\to 0^+}$ ( ) = 2

(c) $\lim_{\to 0}$ ( ) does not exist because the limits in part (a) and part (b) are not equal.

(d) $\lim_{\to 2^-}$ ( ) = 2 (e) $\lim_{\to 2^+}$ ( ) = 0

(f) $\lim_{\to 2}$ ( ) does not exist because the limits in part (d) and part (e) are not equal.

(g) (2) = 1 (h) $\lim_{\to 4}$ ( ) = 3

8. (a) $\lim_{\to -3}$ ( ) = ∞ (b) $\lim_{\to 2}$ ( ) does not exist. (c) $\lim_{\to 2^-}$ ( ) = $-\infty$

(d) $\lim_{\to 2^+}$ ( ) = ∞ (e) $\lim_{\to -1}$ ( ) = $-\infty$

(f) The equations of the vertical asymptotes are = −3, = −1 and = 2.

9. (a) $\lim_{\to -7}$ ( ) = $-\infty$ (b) $\lim_{\to -3}$ ( ) = ∞ (c) $\lim_{\to 0}$ ( ) = ∞

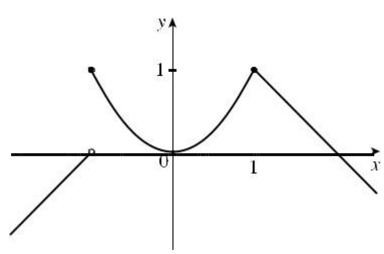
(d) $\lim_{\to 6^-}$ ( ) = $-\infty$ (e) $\lim_{\to 6^+}$ ( ) = ∞

(f) The equations of the vertical asymptotes are = −7, = −3, = 0, and = 6.

10. $\lim_{\to 12^-}$ ( ) = 150 mg and $\lim_{\to 12^+}$ ( ) = 300 mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at = 12 h. The left-hand limit represents the amount of the drug just before the fourth injection. The right-hand limit represents the amount of the drug just after the fourth injection.

11. From the graph of

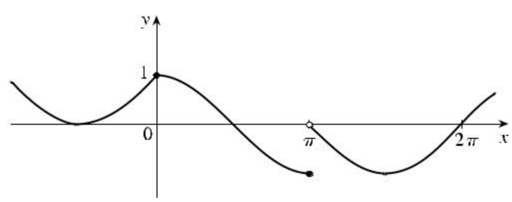
( ) = 1 + if −1, 2 if −1 1, 2 − if ≥ 1



we see that $\lim_{\to}$ ( ) exists for all except = −1. Notice that the right and left limits are different at = −1.

12. From the graph of

( ) = 1 + sin if 0, cos if 0 ≤ ,

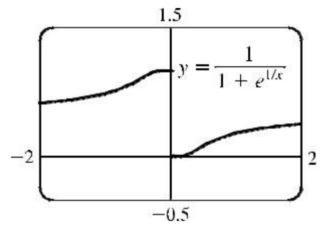


we see that $\lim_{\to}$ ( ) exists for all except = . Notice that the right and left limits are different at = .

(a) $\lim_{\to 0^-} (\ ) = 1$

$\lim_{\to 0^+} (\ ) = 0$

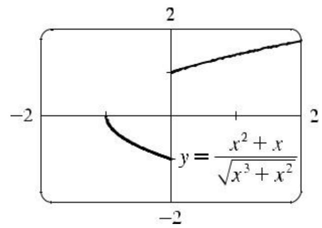
(c) $\lim_{\to 0} (\ )$ does not exist because the limits in part (a) and part (b) are not equal.



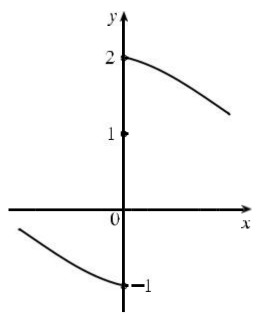
(a) $\lim_{\to 0^-} (\ ) = -1$

$\lim_{\to 0^+} (\ ) = 1$

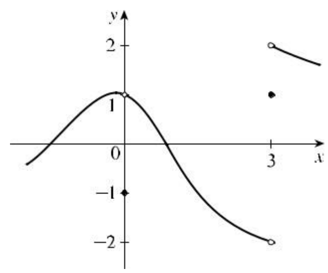
(c) $\lim_{\to 0} (\ )$ does not exist because the limits in part (a) and part (b) are not equal.



15. $\lim_{\to 0^-} (\ ) = -1$, $\lim_{\to 0^+} (\ ) = 2$ (0) = 1

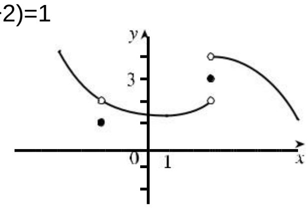


$\lim_{\to 0} (\ ) = 1$, $\lim_{\to 3^-} (\ ) = -2$, $\lim_{\to 3^+} (\ ) = 2$, (0) = −1, (3) = 1

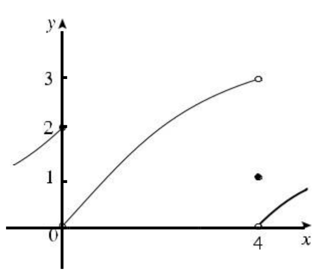


17. $\lim_{\to 3^+} (\ ) = 4$, $\lim_{\to 3^-} (\ ) = 2$, $\lim_{\to -2} (\ ) = 2$,

(3) = 3, (−2) = 1



18. $\lim_{\to 0^-} (\ ) = 2$, $\lim_{\to 0^+} (\ ) = 0$, $\lim_{\to 4^-} (\ ) = 3$, $\lim_{\to 4^+} (\ ) = 0$, (0) = 2, (4) = 1



For $(\ ) = \dfrac{^2 - 3}{^2 - 9}$:

	( )
3 1	0 508 197
3 05	0 504 132
3 01	0 500 832
3 001	0 500 083
3 0001	0 500 008

	( )
2 9	0 491 525
2 95	0 495 798
2 99	0 499 165
2 999	0 499 917
2 9999	0 499 992

It appears that $\lim_{\to 3} \dfrac{^2 - 3}{^2 - 9} = \dfrac{1}{2}$.

NOT FOR SALE

For $(\ ) = \dfrac{2 - 3}{2 - 9}$:

	( )
−2 5	−5
−2 9	−29
−2 99	−299
−2 999	−2999
−2 9999	−29,999

	( )
−3 5	7
−3 1	31
−3 01	301
−3 001	3001
−3 0001	30,001

It appears that $\lim\limits_{\to -3^+} (\ ) = -\infty$ and that

$\lim\limits_{\to -3^-} (\ ) = \infty$, so $\lim\limits_{\to -3} \dfrac{2 - 3}{2 - 9}$ does not exist.

21. For $(\ ) = \dfrac{5 - 1}{}$:

	( )
0 5	22 364 988
0 1	6 487 213
0 01	5 127 110
0 001	5 012 521
0 0001	5 001 250

	( )
−0 5	1 835 830
−0 1	3 934 693
−0 01	4 877 058
−0 001	4 987 521
−0 0001	4 998 750

It appears that $\lim\limits_{\to 0} \dfrac{5 - 1}{} = 5$.

22. For $(\ ) = \dfrac{(2+\ )^5 - 32}{}$:

	( )
0 5	131 312 500
0 1	88 410 100
0 01	80 804 010
0 001	80 080 040
0 0001	80 008 000

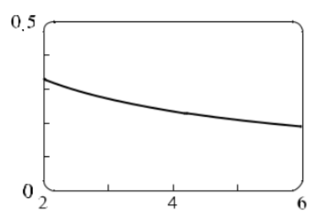
	( )
−0 5	48 812 500
−0 1	72 390 100
−0 01	79 203 990
−0 001	79 920 040
−0 0001	79 992 000

It appears that $\lim\limits_{\to 0} \dfrac{(2+\ )^5 - 32}{} = 80$.

For $(\ ) = \dfrac{\ln - \ln 4}{- 4}$:

	( )
3 9	0 253 178
3 99	0 250 313
3 999	0 250 031
3 9999	0 250 003

	( )
4 1	0 246 926
4 01	0 249 688
4 001	0 249 969
4 0001	0 249 997

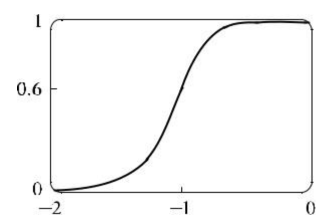


It appears that $\lim\limits_{\to 4} (\ ) = 0\ 25$. The graph confirms that result.

For $(\ ) = \dfrac{1 + 9}{1 + 15}$:

	( )
−1 1	0 427 397
−1 01	0 582 008
−1 001	0 598 200
−1 0001	0 599 820

	( )
−0 9	0 771 405
−0 99	0 617 992
−0 999	0 601 800
−0 9999	0 600 180



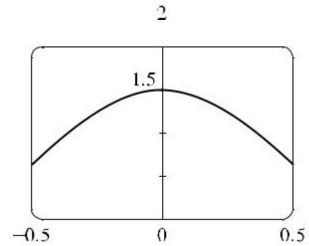
It appears that $\lim\limits_{\to -1} (\ ) = 0\ 6$. The graph confirms that result.

For ( ) = $\frac{\sin 3}{\tan 2}$:

	( )
±0 1	1 457 847
±0 01	1 499 575
±0 001	1 499 996
±0 0001	1 500 000

It appears that $\lim_{\to 0} \frac{\sin 3}{\tan 2} = 15.$

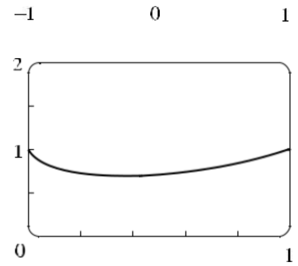
The graph confirms that result.



For ( ) = $\frac{5 - 1}{}$:

	( )		( )

0 1	1 746 189	−0 1	1 486 601
0 01	1 622 459	−0 01	1 596 556
0 001	1 610 734	−0 001	1 608 143
0 0001	1 609 567	−0 0001	1 609 308



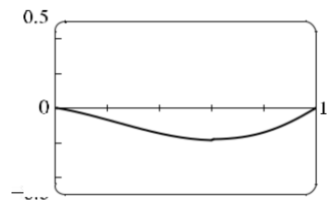
It appears that $\lim_{\to 0}$ ( ) ≈ 1 6094. The graph confirms that result.

For ( ) = :

	( )
0 1	0 794 328
0 01	0 954 993
0 001	0 993 116
0 0001	0 999 079

It appears that $\lim_{\to 0^+}$ ( ) = 1.

The graph confirms that result.



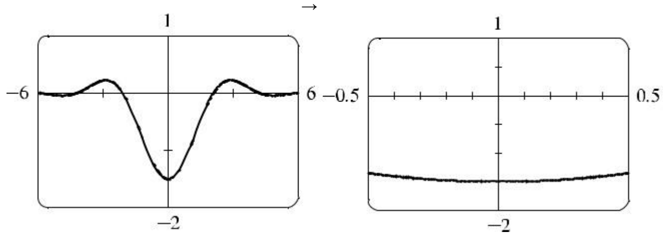
For ( ) = ^{2} ln :

	( )
0 1	−0 023 026
0 01	−0 000 461
0 001	−0 000 007
0 0001	−0 000 000

It appears that $\lim_{\to 0^+}$ ( ) = 0.

The graph confirms that result.

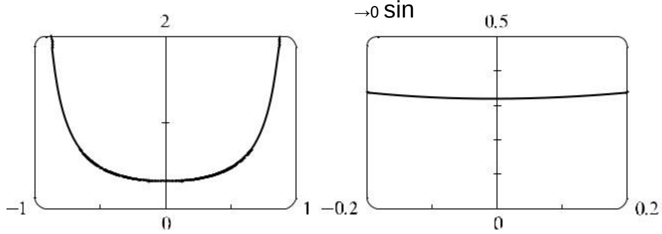
29. (a) From the graphs, it seems that $\lim_{\to 0} \frac{\cos 2 - \cos}{2} = -1\ 5$.



(b)

	( )
±0 1	−1 493 759
±0 01	−1 499 938
±0 001	−1 499 999
±0 0001	−1 500 000

30. (a) From the graphs, it seems that $\lim_{\to 0} \frac{\sin}{\sin} = 0.32$.



(b)

	( )
±0.1	0.323 068
±0.01	0.318 357
±0.001	0.318 310
±0.0001	0.318 310

Later we will be able to show that the exact value is $\frac{1}{}$.

31. $\lim_{\to 5^+} \frac{+1}{-5} = \infty$ since the numerator is positive and the denominator approaches 0 from the positive side as $\to 5^+$.

32. $\lim_{\to 5^-} \frac{+1}{-5} = -\infty$ since the numerator is positive and the denominator approaches 0 from the negative side as $\to 5^-$.

33. $\lim_{\to 1} \frac{2-}{(-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $\to 1$.

34. $\lim \frac{\sqrt{}}{} = -\infty$ since the numerator is positive and the denominator approaches 0 from the negative side as $\to 3^-$.

35. Let $= {}^2 - 9$. Then as $\to 3^+$, $\to 0^+$, and $\lim_{\to 3^+} \ln({}^2 - 9) = \lim_{\to 0^+} \ln = -\infty$ by (5).

$\lim_{\to 0^+} \ln(\sin) = -\infty$ since $\sin \to 0^+$ as $\to 0^+$.

37. $\lim_{\to} \frac{1}{} \sec = \infty$ since $\frac{1}{}$ is positive and $\sec \to \infty$ as $\to (\ 2)^+$.

$\lim \cot = \lim \frac{\cos}{} = 0$

38. $\lim_{\to^-} \cot = \lim_{\to^-} \frac{\cos}{\sin} = -\infty$ since the numerator is negative and the denominator approaches 0 through positive values as $\to^-$.

39. $\lim_{\to 2^-} \csc = \lim_{\to 2^-} \frac{1}{\sin} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $\to 2^-$.

40. $\lim_{\to 2^-} \frac{{}^2 - 2}{{}^2 - 4 + 4} = \lim_{\to 2^-} \frac{(-2)}{(-2)^2} = \lim_{\to 2^-} \frac{}{-2} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $\to 2^-$.

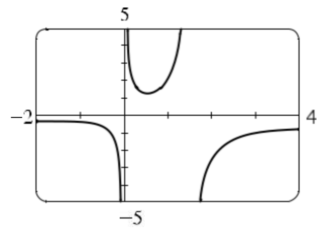
41. $\lim_{\to 2^+} \frac{{}^2 - 2 - 8}{{}^2 - 5 + 6} = \lim_{\to 2^+} \frac{(-4)(+2)}{(-3)(-2)} = \infty$ since the numerator is negative and the denominator approaches 0 through negative values as $\to 2^+$.

42. $\lim\limits_{\to 0^+} \left(\frac{1}{} - \ln \right) = \infty$ since $\frac{1}{} \to \infty$ and $\ln \to -\infty$ as $\to 0^+$.

$\lim\limits_{\to 0} (\ln {}^2 - {}^{-2}) = -\infty$ since $\ln {}^2 \to -\infty$ and ${}^{-2} \to \infty$ as $\to 0$.

44. (a) The denominator of $= \dfrac{^2+1}{3-2^2} = \dfrac{^2+1}{(3-2)}$ is equal to zero when $=0$ and $=\frac{3}{2}$ (and the numerator is not), so $=0$ and $=1\ 5$ are vertical asymptotes of the function.

(b)



45. (a) $(\ ) = \dfrac{1}{^3-1}$.

From these calculations, it seems that $\lim\limits_{\to 1^-} (\ ) = -\infty$ and $\lim\limits_{\to 1^+} (\ ) = \infty$.

	( )
0 5	−1 14
0 9	−3 69
0 99	−33 7
0 999	−333 7
0 9999	−3333 7
0 99999	−33,333 7

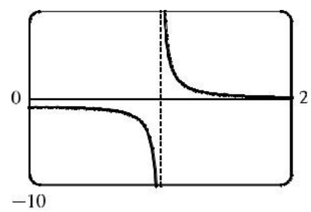
	( )
1 5	0 42
1 1	3 02
1 01	33 0
1 001	333 0
1 0001	3333 0
1 00001	33,333 3

(b) If is slightly smaller than 1, then $^3-1$ will be a negative number close to 0, and the reciprocal of $^3-1$, that is, $(\ )$, will be a negative number with large absolute value. So $\lim\limits_{\to 1^-} (\ ) = -\infty$.

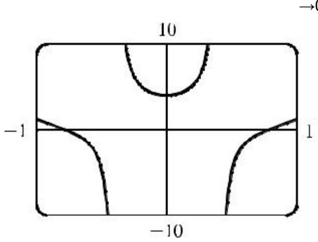
If is slightly larger than 1, then $^3-1$ will be a small positive number, and its reciprocal, $(\ )$, will be a large positive number. So $\lim\limits_{\to 1^+} (\ ) = \infty$.

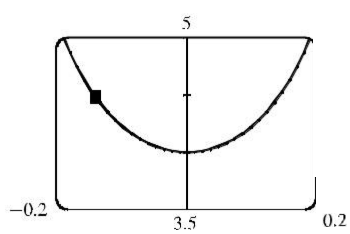
(c) It appears from the graph of that

$\lim\limits_{\to 1^-} (\ ) = -\infty$ and $\lim\limits_{\to 1^+} (\ ) = \infty$.



(a) From the graphs, it seems that $\lim\limits_{\to 0} \dfrac{\tan 4}{} = 4$.





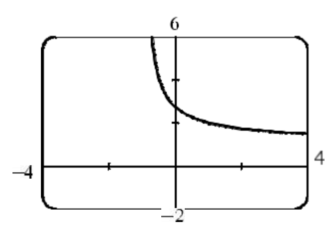
(b)

	( )
±0 1	4 227 932
±0 01	4 002 135
±0 001	4 000 021
±0 0001	4 000 000

47. (a) Let $(\ ) = (1+\ )^1$.

( )	
−0 001	2 71964
−0 0001	2 71842
−0 00001	2 71830
−0 000001	2 71828
0 000001	2 71828
0 00001	2 71827
0 0001	2 71815
0 001	2 71692

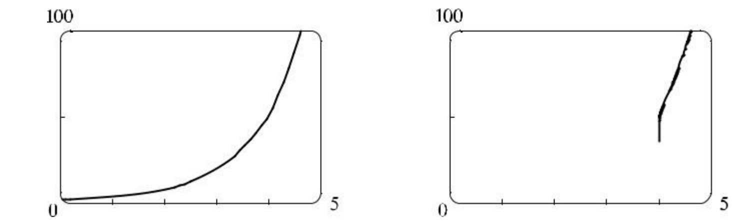
(b)



It appears that $\lim\limits_{\to 0} (1+\ )^1 \approx 2\ 71828$, which is approximately .

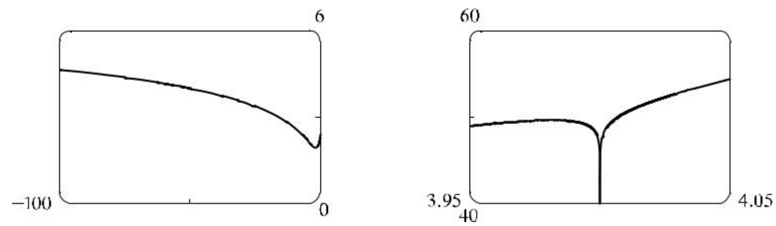
In Section 3.6 we will see that the value of the limit is exactly .

48. (a)

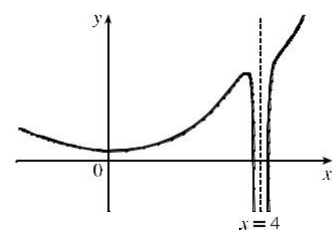


No, because the calculator-produced graph of ( ) = + ln | − 4| looks like an exponential function, but the graph of has an infinite discontinuity at = 4. A second graph, obtained by increasing the `numpoints` option in Maple, begins to reveal the discontinuity at = 4.

There isn't a single graph that shows all the features of . Several graphs are needed since looks like ln | − 4| for large negative values of and like for 5, but yet has the infinite discontiuity at = 4.



A hand-drawn graph, though distorted, might be better at revealing the main features of this function.



49. For ( ) = ^{2} − (2 1000):

(a)

	( )
1	0 998 000
0 8	0 638 259
0 6	0 358 484
0 4	0 158 680
0 2	0 038 851
0 1	0 008 928
0 05	0 001 465

It appears that $\lim\limits_{\to 0}$ ( ) = 0.

(b)

( )	
0 04	0 000 572
0 02	−0 000 614
0 01	−0 000 907
0 005	−0 000 978
0 003	−0 000 993
0 001	−0 001 000

It appears that $\lim\limits_{\to 0}$ ( ) = −0 001.

For ( ) = $\dfrac{\tan \; - \;}{\;^3}$:

(a)

( )	
1 0	0 55740773
0 5	0 37041992
0 1	0 33467209
0 05	0 333 667 00
0 01	0 333 346 67
0 005	0 333 336 67

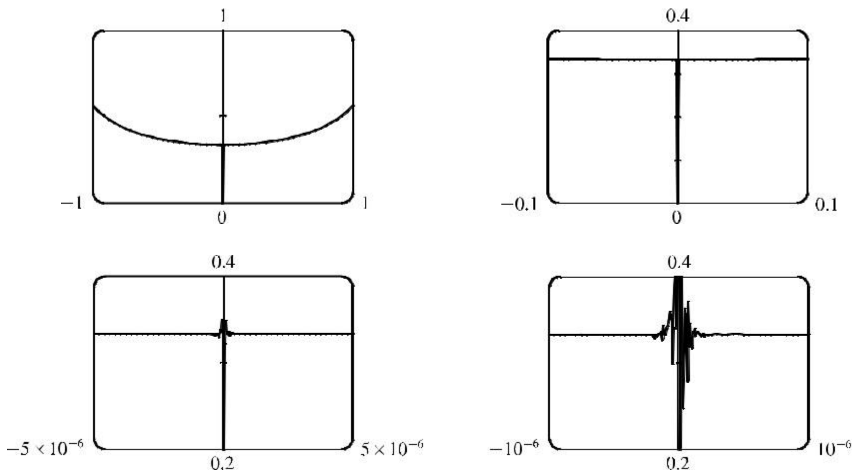
(b) It seems that $\lim\limits_{\to 0}$ ( ) = $\frac{1}{3}$.

(c)

	( )
0 001	0 333 333 50
0 0005	0 333 333 44
0 0001	0 333 330 00
0 00005	0 333 336 00
0 00001	0 333 000 00
0 000001	0 000 000 00

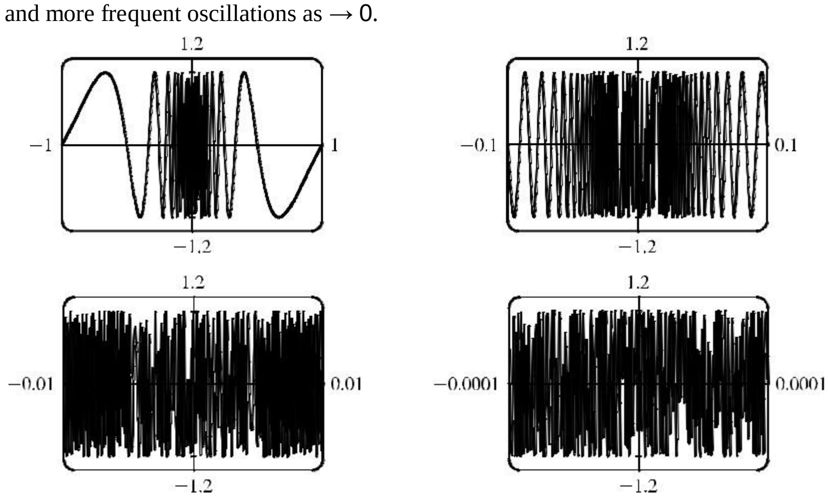
Here the values will vary from one calculator to another. Every calculator will eventually give *false values*.

(d) As in part (c), when we take a small enough viewing rectangle we get incorrect output.



No matter how many times we zoom in toward the origin, the graphs of ( ) = sin( ) appear to consist of almost-vertical lines.

This indicates more and more frequent oscillations as → 0.



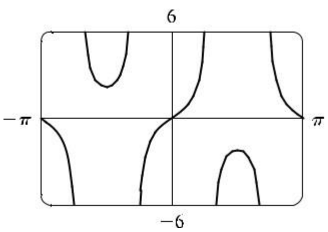
(a) For any positive integer , if = 1/ , then ( ) = tan 1/ = tan( ) = 0. (Remember that the tangent function has period .)

(b) For any nonnegative number , if $= \frac{4}{(4 \ +1)}$, then

$$( ) = \tan \frac{1}{} = \tan \frac{(4 \ + 1)}{4} = \tan \frac{4}{4} + = \tan + = \tan \frac{}{4} = 1$$

(c) From part (a), $( ) = 0$ infinitely often as $\to 0$. From part (b), $( ) = 1$ infinitely often as $\to 0$. Thus, $\lim_{\to 0} \tan \frac{1}{}$

does not exist since $( )$ does not get close to a fixed number as $\to 0$.

53. 

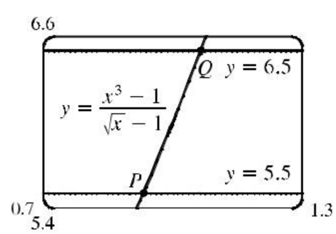
There appear to be vertical asymptotes of the curve $= \tan(2 \sin \ )$ at $\approx \pm 0\ 90$ and $\approx \pm 2\ 24$. To find the exact equations of these asymptotes, we note that the graph of the tangent function has vertical asymptotes at $= \frac{}{2} +$. Thus, we must have $2 \sin \ = \frac{}{2} +$, or equivalently, $\sin \ = \frac{}{4} + \frac{}{2}$. Since $-1 \le \sin \ \le 1$, we must have $\sin \ = \pm \frac{}{4}$ and so $= \pm \sin^{-1} \frac{}{4}$ (corresponding to $\approx \pm 0\ 90$). Just as 150° is the reference angle for 30°, $- \sin^{-1} \frac{}{4}$ is the reference angle for $\sin^{-1} \frac{}{4}$. So $= \pm \left(- \sin^{-1} \frac{}{4} \right)$ are also equations of vertical asymptotes (corresponding to $\approx \pm 2\ 24$).

54. $\lim_{\to^-} \ = \lim_{\to^-} \frac{0}{\sqrt{1 - \ ^2 \ ^2}}$. As $\to^-$, $\sqrt{1 - \ ^2 \ ^2} \to 0^+$, and $\to \infty$.

55. (a) Let $= \frac{^3 - 1}{\sqrt{\ } - 1}$.

From the table and the graph, we guess that the limit of as approaches 1 is 6.

0 99	5 925 31
0 999	5 992 50
0 9999	5 999 25
1 01	6 075 31
1 001	6 007 50
1 0001	6 000 75



(b) We need to have $5\ 5 < \frac{^3 - 1}{\sqrt{\ } - 1} < 6\ 5$. From the graph we obtain the approximate points of intersection $(0\ 9314\ 5\ 5)$ and $(1\ 0649\ 6\ 5)$. Now $1 - 0\ 9314 = 0\ 0686$ and $1\ 0649 - 1 = 0\ 0649$, so by requiring that be within $0\ 0649$ of 1, we ensure that is within $0\ 5$ of 6.

2.3 Calculating Limits Using the Limit Laws

1. (a) $\lim_{\to 2} [( ) + 5 ( )] = \lim_{\to 2} ( ) + \lim_{\to 2} [5 ( )]$ [Limit Law 1]

$= \lim_{\to} ( ) + 5 \lim_{\to} ( )$ [Limit Law 3]

$= 4 + 5(-2) = -6$

(b) $\lim_{\to 2} [( )]^3 = \left[\lim_{\to 2} ( ) \right]^3$ [Limit Law 6]

(c) $\lim_{\to 2} \sqrt{()} = \sqrt{\lim_{\to 2} ()}$ [Limit Law 11]

$= \sqrt{4} = 2$

$\lim_{\to 2} \frac{3()}{()} = \frac{\lim_{\to 2} [3()]}{\lim_{\to 2} ()}$ [Limit Law 5]

$= \frac{3 \lim_{\to 2} ()}{\lim_{\to 2} ()}$ [Limit Law 3]

$= \frac{3(4)}{-2} = -6$

$\lim_{\to 2} \frac{()}{()}$, does not exist because the limit of the denominator is 0, we can't use Limit Law 5. The given limit, because the denominator approaches 0 while the numerator approaches a nonzero number.

$\lim_{\to 2} \frac{()()}{()} = \frac{\lim_{\to 2} [()()]}{\lim_{\to 2} ()}$ [Limit Law 5]

$= \frac{\lim_{\to 2} () \cdot \lim_{\to 2} ()}{\lim_{\to 2} ()}$ [Limit Law 4]

$= \frac{-2 \cdot 0}{4} = 0$

2. (a) $\lim_{\to 2} [() + ()] = \lim_{\to 2} () + \lim_{\to 2} ()$ [Limit Law 1]

$-1 + 2$

1

$\lim_{\to 0} ()$ exists, but $\lim_{\to 0} ()$ does not exist, so we cannot apply Limit Law 2 to $\lim_{\to 0} [() - ()]$. The limit does not exist.

(c) $\lim_{\to -1} [()()] = \lim_{\to -1} () \cdot \lim_{\to -1} ()$ [Limit Law 4]

$1 \cdot 2$

2

$\lim_{\to 3} () = 1$, but $\lim_{\to 3} () = 0$, so we cannot apply Limit Law 5 to $\lim_{\to 3} \frac{()}{()}$. The limit does not exist.

Note: $\lim_{\to 3^-} \frac{()}{()} = \infty$ since $() \to 0^+$ as $\to 3^-$ and $\lim_{\to 3^+} \frac{()}{()} = -\infty$ since $() \to 0^-$ as $\to 3^+$.

Therefore, the limit does not exist, even as an infinite limit.

(e) $\lim_{\to} {}^2() = \lim_{\to} {}^2 \lim_{\to} ()$ [Limit Law 4]

$= 2^2 \cdot (-1)$

$= -4$

$(-1) + \lim_{\to -1} ()$ is undefined since (-1) is not defined.

3. $\lim_{\to 3} (5^3 - 3^2 + -6) = \lim_{\to 3} (5^3) - \lim_{\to 3} (3^2) + \lim_{\to 3} - \lim_{\to 3} 6$ [Limit Laws 2 and 1]

$= 5 \lim_{\to 3} {}^3 - 3 \lim_{\to 3} {}^2 + \lim_{\to 3} - \lim_{\to 3} 6$ [3]

$= 5(3^3) - 3(3^2) + 3 - 6$ [9, 8, and 7]

$= 105$

4. $\lim_{\to -1}(^4 - 3)(^2 + 5 + 3) = \lim_{\to -1}(^4 - 3) \lim_{\to -1}(^2 + 5 + 3)$ [Limit Law 4]

$\lim_{\to -1} {}^4 - \lim_{\to -1} 3 \lim_{\to -1} {}^2 + \lim_{\to -1} 5 + \lim_{\to -1} 3$ [2, 1]

$\lim_{\to -1} {}^4 - 3 \lim_{\to -1} \lim_{\to -1} {}^2 + 5 \lim_{\to -1} + \lim_{\to -1} 3$ [3]

$(1+3)(1-5+3)$ [9, 8, and 7]

$4(-1) = -4$

5. $\lim_{\to -2} \frac{^4 - 2}{2^2 - 3 + 2} = \frac{\lim_{\to -2} (^4 - 2)}{\lim_{\to -2} (2^2 - 3 + 2)}$ [Limit Law 5]

$= \frac{\lim_{\to -2} {}^4 - \lim_{\to -2} 2}{2 \lim_{\to -2} {}^2 - 3 \lim_{\to -2} + \lim_{\to -2} 2}$ [1, 2, and 3]

$= \frac{16 - 2}{2(4) - 3(-2) + 2}$ [9, 7, and 8]

$= \frac{14}{16} = \frac{7}{8}$

6. $\lim \sqrt{^4 \ldots} = \sqrt{\lim (^4 + 3 + 6)}$ [11]

$= \sqrt{\lim {}^4 + 3 \lim + \lim 6}$ [1, 2, and 3]

$= \sqrt{(-2)^4 + 3(-2) + 6}$ [9, 8, and 7]

$= \sqrt{16 - 6 + 6} = \sqrt{16} = 4$

7. $\lim_{\to 8} (1 + \sqrt[3]{\ }) (2 - 6^2 + ^3) = \lim_{\to 8} (1 + \sqrt[3]{\ }) \cdot \lim_{\to 8} (2 - 6^2 + ^3)$ [Limit Law 4]

$= \lim_{\to 8} 1 + \lim_{\to 8} \sqrt[3]{\ } \cdot \lim_{\to 8} 2 - 6 \lim_{\to 8} {}^2 + \lim_{\to 8} {}^3$ [1, 2, and 3]

$= 1 + \sqrt[3]{8} \cdot 2 - 6 \cdot 8^2 + 8^3$ [7, 10, 9]

$= (3)(130) = 390$

8. $\lim_{\to 2} \left(\frac{^2 - 2}{^3 - 3 + 5}\right)^2 = \left(\lim_{\to 2} \frac{^2 - 2}{^3 - 3 + 5}\right)^2$ [Limit Law 6]

$= \left(\frac{\lim_{\to 2}(^2 - 2)}{\lim_{\to 2}(^3 - 3 + 5)}\right)^2$ [5]

$= \left(\frac{\lim_{\to 2} {}^2 - \lim_{\to 2} 2}{\lim_{\to 2} {}^3 - 3 \lim_{\to 2} + \lim_{\to 2} 5}\right)^2$ [1, 2, and 3]

$= \dfrac{4-2}{8-3(2)+5}$ [9, 7, and 8]

$2^2=4$

$\overline{7}$ 49

lim 2 2 + 1 3 = lim 2 2+1 [Limit Law 11]

−2→2

→2 3 − 2

lim (2 2 + 1)

→2 [5]

= lim (3 2)

2 lim 2 + lim 1

= →2 →2 [1, 2, and 3]

lim lim 2

3 2 2

→ →

2(2)2 + 1 9 3 [9, 8, and 7]

= 3(2) − 2 = 4 =2

(a) The left-hand side of the equation is not defined for = 2, but the right-hand side is.

Since the equation holds for all 6= ,2it follows that both sides of the equation approach the same limit as → 2, just as in Example 3. Remember that in finding lim → ( ), we never consider = .

lim 2 − 6 + 5 = lim (− 5)(− 1) = lim (− 1) = 5 − 1 = 4

− 5− 5→5→5→5

12. lim 2 + 3 = lim (+3) = lim = −3 = 3

→−3 2− −12 →−3 (−4)(+3) →−3 − 4 −3−4 7

lim 2 − 5 + 6 does not exist since − 5 → 0, but 2 − 5 + 6 → 6 as → 5.

− 5→5

14. →4 2 + 3 (+3) −∞ and

lim 2− −12 = lim →4(−4)(+3) = lim →4 The last limit does not exist since →4− −4 lim =

lim

→4+ −4 =∞.

15. lim 2 − 9 = lim (+3)(−3) = lim − 3 = −3−3 = −6 = 6

→−3 22+7 +3 →−3 (2 +1)(+3) →−3 2 + 1 2(3)+1 5 5

− −

lim 2 2 + 3 + 1 = lim (2 + 1)(+ 1) = lim 2 + 1 = 2(−1) + 1 = −1 = 1 2−2 −3 →−1 (

−3)(+1) →−1 −3 −1−3 −4 4→−1

lim (−5 +)2 − 25 = lim (25 − 10 + 2) − 25 = lim −10 + 2 = lim (−10 +) = lim (−10 +) = −10

→0→0→0→0→0→0

lim (2+)3 − 8 = lim 8+12 +6 2+ 3 − 8 = lim 12+6 2+3

18. →0 →0 →0

= lim 2 =12+0+0=12

By the formula for the sum of cubes, we have

lim + 2 = lim + 2 = lim 1 = 1 = 1 .

→−2 3 + 8 →−2 (+2)(2−2 +4) →−2 2−2 +4 4 + 4 + 4 12

NOT FOR SALE

We use the difference of squares in the numerator and the difference of cubes in the denominator.

lim $\frac{4 - 1}{3 - 1}$ = lim (^{2}−1)(^{2}+1) = lim (−1)(+1)(^{2}+1) = lim (+1)(^{2}+1) = $\frac{2(2)}{3}$ = $\frac{4}{3}$

→1 →1 (−1)(^{2}+ +1) →1 (−1)(^{2}+ +1) →1 ^{2}+ +1

√ √ √ √ 9 + 2 3^2 9

lim 9 + 3 lim 9 + 3 9+ +3 = lim (9+)

lim

21. →0 − = →0 − √9+ +3 →0 √9 + + 3 →0 √9+ +3

1

→0 1 1

= lim lim 3+3 6 = √ = 2 − 3^2

lim √4 + 1 3 lim √4 + 1 3 √ 4 +1+3 = lim 4 + 1

22. →2 2 −

4

4

= lim + 1 − 9 = lim (− 2)

= 2 √ (2) 4 +1+3 (3 2) 4 +1+3

lim

→ 4 +1+3 9 + 3

1 1 1 1

− 3 − 3 3 1

23. lim − 3 = lim − 3 3 = lim 3 − = lim −1 = −9

→3 →3 →3 3(−3) →3 3

(3 +)$^{-1}$ − 3^{-1} 1 − 1

24. lim = lim 3 + 3 = lim 3−(3+) = lim −

→0 →0 →0 (3+)3 →0 (3+)3

$= \lim \frac{1}{\;} \cdot \frac{1}{\;} \quad 1 \quad . \quad 1$

$\to 0$

25. $\lim \frac{\sqrt{1+} - \sqrt{1-}}{\;} = \lim \frac{\sqrt{1+} - \sqrt{1-}}{\;} \cdot \frac{\sqrt{1+} + \sqrt{1-}}{\sqrt{1+} + \sqrt{1-}} = \lim \frac{\sqrt{1+}^{\,2} - \sqrt{1-}^{\,2}}{\;}$

$= \lim \frac{(1+) - (1-)}{\;(\sqrt{1+} + \sqrt{1-})} = \lim \frac{2}{\;(\sqrt{1+} + \sqrt{1})} = \lim \frac{2}{\sqrt{1+} + \sqrt{1}}$

$= \frac{2}{\sqrt{1} + \sqrt{1}} = \frac{2}{2} = 1$

26. $\lim_{\to 0} \left(\frac{1}{\;} - \frac{1}{^2 +}\right) = \lim_{\to 0} \frac{1}{\;} \cdot \frac{1}{(+1)} = \lim_{\to 0} \frac{+1-1}{(+1)} = \lim_{\to 0} \frac{1}{+1} = \frac{1}{0+1} = 1$

27. $\lim_{\to 16} \frac{4 - \sqrt{\;}}{16 - ^2} = \lim_{\to 16} \frac{(4 - \sqrt{\;})(4 + \sqrt{\;})}{(16 - ^2)(4 + \sqrt{\;})} = \lim_{\to 16} \frac{16 -}{(16 -)(4 + \sqrt{\;})}$

$= \lim_{\to 16} \frac{1}{(4 + \sqrt{\;})} = \frac{1}{16(4 + \sqrt{16})} = \frac{1}{16(8)} = \frac{1}{128}$

28. $\lim_{\to 2} \frac{^2 - 4 + 4}{^4 - 3^2 - 4} = \lim_{\to 2} \frac{(-2)^2}{(^2 - 4)(^2 + 1)} = \lim_{\to 2} \frac{(-2)^2}{(+2)(-2)(^2 + 1)}$

$= \lim_{\to 2} \frac{-2}{(+2)(^2 + 1)} = \frac{0}{4 \cdot 5} = 0$

NOT FOR SALE

$\to 0$ $\sqrt{}+11+$ $\sqrt{1+}$ $\to 0$ $\sqrt{1+}$ $1+$ $\sqrt{1+}$

29. lim $\frac{1}{}$ $\frac{1}{}$ = lim $1-\frac{\sqrt{1+}}{}$ = lim = lim $-$

$\to 0$ 1 $\sqrt{1+}$ 1 1

= lim = = = =

30. $\lim\limits_{\to -4} \frac{\sqrt{^2+9}-5}{+4} = \lim\limits_{\to -4} \frac{\sqrt{^2+9}-5}{} \cdot \frac{\sqrt{^2+9}+5}{\sqrt{^2+9}+5} = \lim\limits_{\to -4} \frac{(^2+9)-25}{(+4)\sqrt{^2+9}+5}$

$= \lim\limits_{\to -4} \frac{^2-16}{(+4)\sqrt{^2+9}+5} = \lim\limits_{\to -4} \frac{(+4)(-4)}{(+4)\sqrt{^2+9}+5}$

$= \lim\limits_{\to -4} \frac{-4}{\sqrt{^2+9}+5} = \frac{-4-4}{\sqrt{16+9}+5} = \frac{-8}{5+5} = -\frac{4}{5}$

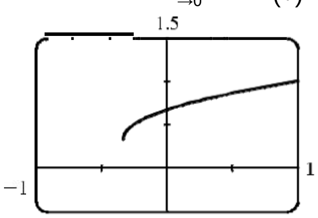
$\lim\limits_{\to 0} \frac{(+)^3-^3}{} = \lim\limits_{\to 0} \frac{(^3+3^2+3^2+^3)-^3}{} = \lim\limits_{\to 0} \frac{3^2+3^2+^3}{}$

$= \lim\limits_{\to 0} \frac{(3^2+3+^2)}{} = \lim\limits_{\to 0} (3^2+3+^2) = 3^2$

32. $\lim\limits_{\to 0} \frac{\frac{1}{(+)^2}-\frac{1}{^2}}{} = \lim\limits_{\to 0} \frac{\frac{^2-(+)^2}{(+)^2\,^2}}{} = \lim\limits_{\to 0} \frac{^2-(^2+2+^2)}{^2(+)^2} = \lim\limits_{\to 0} \frac{-(2+)}{^2(+)^2}$

$= \lim\limits_{\to 0} \frac{-(2+)}{^2(+)^2} = \frac{-2}{^2\cdot ^2} = -\frac{2}{^3}$

33. (a)



$\lim\limits_{\to 0} \frac{}{\sqrt{1+3}-1} \approx \frac{2}{3}$

(b)

	( )
−0 001	0 666 166 3
−0 000 1	0 666 616 7
−0 000 01	0 666 661 7
−0 000 001	0 666 666 2
0 000 001	0 666 667 2
0 000 01	0 666 671 7
0 0001	0 666 716 7
0 001	0 667 166 3

The limit appears to be $\frac{2}{3}$.

(c) $\lim\limits_{\to 0} \left(\frac{}{\sqrt{1+3}-1} \cdot \frac{\sqrt{1+3}+1}{\sqrt{1+3}+1} \right) = \lim\limits_{\to 0} \frac{(\sqrt{1+3}+1)}{(1+3)-1} = \lim\limits_{\to 0} \frac{(\sqrt{1+3}+1)}{3}$

$= \frac{1}{3} \lim\limits_{\to 0} \left(\sqrt{1+3}+1 \right)$ [Limit Law 3]

$= \frac{1}{3} \left[\sqrt{\lim\limits_{\to 0}(1+3)} + \lim\limits_{\to 0} 1 \right]$ [1 and 11]

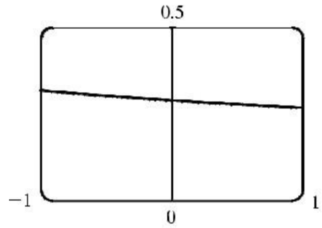
$= \frac{1}{3} \left(\sqrt{\lim\limits_{\to 0} 1 + 3 \lim\limits_{\to 0}} + 1 \right)$ [1, 3, and 7]

$= \frac{1}{}$

$$= \frac{1}{3}\sqrt{1+3\cdot 0}+1$$ [7 and 8]

$$= \frac{1}{3}(1+1) = \frac{2}{3}$$

34. (a)



$\lim_{\to 0} \frac{\sqrt{3+}-\sqrt{3}}{} \approx 0\ 29$

(b)

	( )
−0 001	0 288 699 2
−0 000 1	0 288 677 5
−0 000 01	0 288 675 4
−0 000 001	0 288 675 2
0 000 001	0 288 675 1
0 000 01	0 288 674 9
0 0001	0 288 672 7
0 001	0 288 651 1

The limit appears to be approximately 0 2887.

(c) $\lim_{\to 0} \frac{\sqrt{3+}-\sqrt{3}}{} \cdot \frac{\sqrt{3+}+\sqrt{3}}{\sqrt{3+}+\sqrt{3}} = \lim_{\to 0} \frac{(3+)-3}{\sqrt{3+}+\sqrt{3}} = \lim_{\to 0} \frac{1}{\sqrt{3+}+\sqrt{3}}$

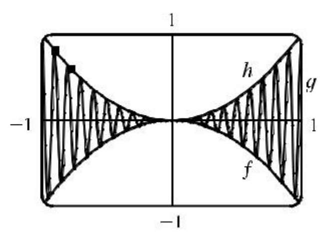
$= \frac{\lim_{\to 0} 1}{\lim_{\to 0}\sqrt{3+} + \lim_{\to 0}\sqrt{3}}$ [Limit Laws 5 and 1]

$= \frac{1}{\sqrt{\lim_{\to 0}(3+)} + \sqrt{3}}$ [7 and 11]

$= \frac{1}{\sqrt{3+0}+\sqrt{3}}$ [1, 7, and 8]

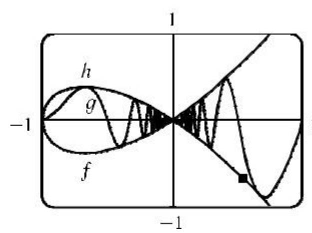
$= \frac{1}{2\sqrt{3}}$

35. Let $( ) = -{}^2$, $( ) = {}^2 \cos 20$ and $( ) = {}^2$. Then



$-1 \le \cos 20 \le 1 \Rightarrow -{}^2 \le {}^2\cos 20 \le {}^2 \Rightarrow ( ) \le ( ) \le ( )$.
So since $\lim_{\to 0} ( ) = \lim_{\to 0} ( ) = 0$, by the Squeeze Theorem we have $\lim_{\to 0} ( ) = 0$.

36. Let $( ) = -\sqrt{{}^3 + {}^2}$, $( ) = \sqrt{{}^3 + {}^2}\sin( )$, and $( ) = \sqrt{{}^3 + {}^2}$. Then



$-1 \le \sin( ) \le 1 \Rightarrow -\sqrt{{}^3+{}^2} \le \sqrt{{}^3+{}^2}\sin( ) \le \sqrt{{}^3+{}^2} \Rightarrow$

$( ) \le ( ) \le ( )$. So since $\lim_{\to} ( ) = \lim_{\to} ( ) = 0$, by the Squeeze Theorem we have $\lim ( ) = 0$.

37. $\lim_{\to 4}(4 - 9) = 4(4) - 9 = 7$ and $\lim_{\to 4}({}^2 - 4 + 7) = 4^2 - 4(4)+7=7$. Since $4 - 9 \le ( ) \le {}^2 - 4 + 7$ for ≥ 0, $\lim_{\to 4} ( ) = 7$ by the Squeeze Theorem.

We have $\lim_{\to 1}(2) = 2(1) = 2$ and $\lim_{\to 1}({}^4 - {}^2 + 2) = 1^4 - 1^2 + 2 = 2$. Since $2 \le ( ) \le {}^4 - {}^2 + 2$ for all ,
$\lim_{\to 1} ( ) = 2$ by the Squeeze Theorem.

$-1 \le \cos(2) \le 1 \Rightarrow -{}^4 \le {}^4\cos(2) \le {}^4$. Since $\lim_{\to 0}(-{}^4) = 0$ and $\lim_{\to 0}{}^4 = 0$, we have $\lim_{\to 0}{}^4\cos(2) = 0$ by the Squeeze Theorem.

39.

NOT FOR SALE

40. $-\le$ 1 sin( ) 1 $\le$ $\Rightarrow$ -1 $\le$ sin() $\le$ 1 $\Rightarrow$ $\sqrt{}$ $-$ / $\le$ $\sqrt{}$ $-$ sin() $\le$ $\sqrt{}$ $-$. Since $\lim_{\to 0^+}$ $(\sqrt{}-/)=0$ and

$\lim_{\to 0^+}(\sqrt{}\ ) = 0$, we have $\lim_{\to 0^+}\sqrt{}$ $-$ sin() $= 0$ by the Squeeze Theorem.

41. $|-3| =$ -3 if $-3 \ge 0$ (3) if 3 0 $=$ 3 if ≥ 3 $3 -$ if3

Thus, $\lim_{\to 3^+}(2 + |-3|) = \lim_{\to 3^+}(2 + -3) = \lim(3 \quad 3) = 3(3) \quad 3 = 6$ and

$\lim_{\to 3^-}(2 + |-3|) = \lim_{\to 3^-}(2 + 3 -) = \lim_{\to 3^-}(+ 3) = 3 + 3 = 6$. Since the left and right limits are equal,

$\lim_{\to}(2 + 3) = 6$

42. $|+6| =$ $+6$ if $+6 \ge 0$ $= +6$ if ≥ -6 $-$ $-$ $-$

We'll look at the one-sided limits.

$$\lim_{\to -6^+}\frac{2+12}{|+6|} = \lim_{\to -6^+}\frac{2(+6)}{+6} = 2 \quad \text{and} \quad \lim_{\to -6^-}\frac{2+12}{|+6|} = \lim_{\to -6^-}\frac{2(+6)}{-(+6)} = -2$$

The left and right limits are different, so $\lim_{\to -6}\frac{2+12}{|+6|}$ does not exist.

$|2^3 - 2| = |^2(2-1)| = |^2| \cdot |2-1| = {}^2|2-1|$

$|2-1| =$ $2 - 1$ if $2-1 \ge 0$ $2 - 1$ if ≥ 0.5 $-(2 - 1)$ if $2-1 < 0$ $= -(2 - 1)$ if < 0.5

So $|2^3 - 2| = {}^2[-(2-1)]$ for < 0.5.

Thus, $\lim_{\to 0.5^-}\frac{2-1}{|2^3-2|} = \lim_{\to 0.5^-}\frac{2-1}{^2[-(2-1)]} = \lim_{\to 0.5^-}\frac{-1}{2} = \frac{-1}{(0.5)^2} = \frac{-1}{0.25} = -4$.

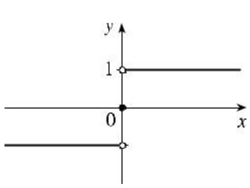
Since $|\ | = -$ for < 0, we have $\lim_{\to -2}\frac{2-|\ |}{2+} = \lim_{\to -2}\frac{2-(-)}{2+} = \lim_{\to -2}\frac{2+}{2+} = \lim_{\to -2} 1 = 1$.

$= \quad 0 \quad \lim \frac{1}{} \frac{1}{} \quad \lim \frac{1}{} \frac{1}{} = \lim \frac{2}{}$

45. Since $|\ | = -$ for < 0, we have $\lim_{\to 0^-}\frac{-}{|\ |} = \lim_{\to 0^-}\frac{-}{-}$ $\to 0^-$, which does not exist since the denominator approaches 0 and the numerator does not.

Since $|\ | =$ for > 0, we have $\lim_{\to 0^+}\left(\frac{1}{} - \frac{1}{|\ |}\right) = \lim_{\to 0^+}\left(\frac{1}{} - \frac{1}{}\right) = \lim_{\to 0^+} 0 = 0$.

47. (a)



(b) (i) Since sgn $= 1$ for > 0, $\lim_{\to 0^+}$ sgn $= \lim_{\to 0^+} 1 = 1$.

(ii) Since $= -1$ for < 0, $\lim_{\to 0^-}$ sgn $= \lim_{\to 0^-} -1 = -1$.

sgn = 1 0 lim sgn lim 1 = 1

(iii) Since lim sgn 6= lim sgn, lim sgn does not exist.
→0− →0+ →0

Since |sgn | = 1 for 6= 0, lim |sgn | = lim 1 = 1.
→0 →0

48. (a) $( ) = \operatorname{sgn}(\sin\ ) = \begin{cases} -1 & \text{if } \sin 0 \\ 0 & \text{if } \sin = 0 \\ 1 & \text{if } \sin 0 \end{cases}$

(i) $\lim_{\to 0^+} ( ) = \lim_{\to 0^+} \operatorname{sgn}(\sin\ ) = 1$ since $\sin$ is positive for small positive values of .

$\lim ( ) = \lim \operatorname{sgn}(\sin\ ) = -1$ since $\sin$ is negative for small negative values of .

(iii) $\lim_{\to 0} ( )$ does not exist since $\lim_{\to 0^+} ( ) \neq \lim_{\to 0^-} ( )$.

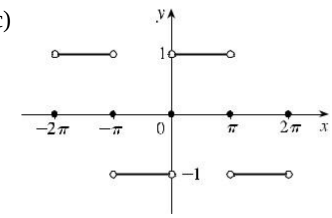
(iv) $\lim_{\to +} ( ) = \lim_{\to +} \operatorname{sgn}(\sin\ ) = -1$ since $\sin$ is negative for values of slightly greater than .

(v) $\lim_{\to -} ( ) = \lim_{\to -} \operatorname{sgn}(\sin\ ) = 1$ since $\sin$ is positive for values of slightly less than .

(vi) $\lim ( )$ does not exist since $\lim ( ) \neq \lim ( )$.

The sine function changes sign at every integer multiple of , so the signum function equals 1 on one side and -1 on the other side of , an integer. Thus, $\lim_{\to} ( )$ does not exist for $=$, an integer.

(c)



49. (a) (i) $\lim_{\to 2^+} ( ) = \lim_{\to 2^+} \dfrac{^2 + -6}{| -2|} = \lim_{\to 2^+} \dfrac{(+3)(-2)}{| -2|}$

$= \lim_{\to 2^+} \dfrac{(+3)(-2)}{-2}$ [since $-2 \; 0$ if $\to 2^+$]

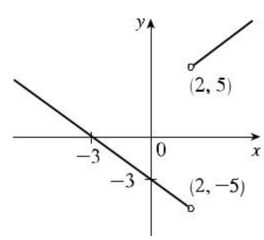
$= \lim_{\to 2^+} (+ 3) = 5$

(ii) The solution is similar to the solution in part (i), but now $| -2| = 2 -$ since $-2 \; 0$ if $\to 2^-$.

Thus, $\lim_{\to 2^-} ( ) = \lim_{\to 2^-} -(+ 3) = -5$.

(b) Since the right-hand and left-hand limits of at $= 2$ are not equal, $\lim_{\to 2} ( )$ does not exist.

(c)

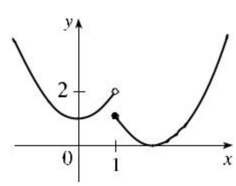


50. (a) $( ) = \begin{cases} ^2 + 1 & \text{if } 1 \\ (-2)^2 & \text{if } \geq 1 \end{cases}$

$\lim_{\to 1^-} ( ) = \lim_{\to 1^-} (^2 + 1) = 1^2 + 1 = 2$, $\lim_{\to 1^+} ( ) = \lim_{\to 1^+} (- 2)^2 = (-1)^2 = 1$

(b) Since the right-hand and left-hand limits of at $= 1$ are not equal, $\lim_{\to 1} ( )$ does not exist.

(c)



51. For the $\to 2$ to exist, the one-sided limits at must be equal. $\to 2^-$ $\to 2^-$ $-\,2$ $-$ and

lim ( ) = 2 lim ( ) = lim 4 $\frac{1}{\,}$ = 4 1 = 3

. Now .

lim ( ) = lim $\sqrt{\;}+\;=\sqrt{\;}2+$ $3=\sqrt{\;}2+$ $9=2+$ $=7$

(a) (i) $\lim\limits_{\to 1^-}(\;)=\lim\limits_{\to 1^-}=1$

$\lim\limits_{\to 1^+}(\;)=\lim\limits_{\to 1^+}(2-{}^2)=2-1^2=1$. Since $\lim\limits_{\to 1^-}(\;)=1$ and $\lim\limits_{\to 1^+}(\;)=1$, we have $\lim\limits_{\to 1}(\;)=1$.

Note that the fact (1) = 3 does not affect the value of the limit.

When = 1, ( ) = 3, so (1) = 3.

$\lim\limits_{\to 2^-}(\;)=\lim\limits_{\to 2^-}(2-{}^2)=2-2^2=2-4=-2$

$\lim\limits_{\to 2^+}(\;)=\lim\limits_{\to 2^+}(\;-3)=2-3=-1$

(vi) $\lim\limits_{\to 2}(\;)$ does not exist since $\lim\limits_{\to 2^-}(\;)\ \ne\ \lim\limits_{\to 2^+}(\;)$.

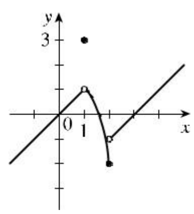
(b)

( ) = $_2$ if 1

2 $-$ if 1 $\le$ 2

3 if $=1$

-3 if 2



53. (a) (i) $[[\;]]=-2$ for $-2\le\;<-1$, so $\lim\limits_{\to -2^+}[[\;]]=\lim\limits_{\to -2^+}(-2)=-2$

(ii) $[[\;]]=-3$ for $-3\le\;<-2$, so $\lim\limits_{\to -2^-}[[\;]]=\lim\limits_{\to -2^-}(-3)=-3$.

The right and left limits are different, so $\lim\limits_{\to -2}[[\;]]$ does not exist.

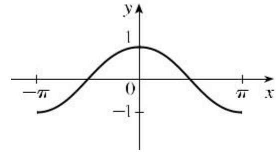
(iii) $[[\;]]=-3$ for $-3\le\;<-2$, so $\lim\limits_{\to -2.4}[[\;]]=\lim\limits_{\to -2.4}(-3)=-3$.

(b) (i) $[[\;]]=-1$ for $-1\le\;<$, so $\lim\limits_{\to^-}[[\;]]=\lim\limits_{\to^-}(-1)=-1$.

(ii) $[[\;]]=$ for $\le\;<+1$, so $\lim\limits_{\to^+}[[\;]]=\lim\limits_{\to^+}=$.

(c) $\lim\limits_{\to}[[\;]]$ exists $\Leftrightarrow$ is not an integer.

54. (a) See the graph of = cos .

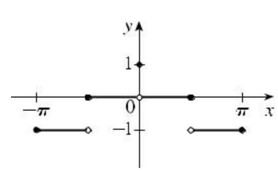


Since $-1\le\cos\;<0$ on $[-\;-\;2)$, we have $=(\;)=[[\cos\;]]=-1$ on $[-\;-\;2)$.

Since $0\le\cos\;<1$ on $[-\;2\;0)\cup(0\;2]$, we have $(\;)=0$ on $[-\;2\;0)\cup(0\;2]$.

Since $-1\le\cos\;<0$ on $(\;2\;]$, we have $(\;)=-1$ on $(\;2\;]$.

Note that (0) = 1.



(i) lim ( ) = 0 and lim ( ) = 0, so lim ( ) = 0.
→0- →0+ →0

(ii) As →(2) - , ( ) → 0, so lim ( )=0.
→

(iii) As →(2) + , ( ) → −1, so lim ( ) = −1.
→

(iv) Since the answers in parts (ii) and (iii) are not equal, lim ( ) does not exist.

(c) lim ( ) exists for all in the open interval ( ) except → 2 − 2 and − = 2 .
→

The graph of ( ) = [[ ]] + [[−]] is the same as the graph of ( ) = −1 with holes at each integer, since ( ) = 0 for any integer . Thus, lim ( ) = −1 and lim ( ) = −1, so lim ( ) = −1. However,
→2- →2+ →2

(2) = [[2]] + [[−2]] = 2 + (−2) = 0, so lim ( ) 6= (2).
→2

lim 2 =√

56. →− 0 1 − 2 0 1 − 1 = 0. As the velocity approaches the speed of light, the length approaches 0.

A left-hand limit is necessary since is not defined for .

Since ( ) is a polynomial, ( ) = 0 + 1 + 2 2 + · · · +. Thus, by the Limit Laws,

lim ( ) = lim 0 + 1 + 2 2 + · · · + = 0 + 1 lim + 2 lim 2 + · · · + lim
→ → → → →

0+1 +2 2 +···+= ()

Thus, for any polynomial , lim ( ) = ( ).
→

Let ( ) = ( ) where ( ) and ( ) are any polynomials, and suppose that ( ) 6= .0Then

( )

lim ( ) = lim ( ) = lim ( ) = ( ) [Exercise 57] = ( ).
→ → ( ) lim ( ) [Limit Law 5]
→ →

59. lim [( ) 8] = lim ( )−8 (1) = lim ( )−8 lim(1)=10 0=0.
→1 − →1 − 1 · − →1 − 1 →1 − .

Thus, 1 lim ( ) = lim [( ) 8]+8 = lim [( ) 8] + lim 8 = 0 + 8 = 8
→ → → →

Note: The value of lim ( )−8 does not affect the answer since it's multiplied by 0. What's important is that
→1 − 1

lim ( )−8 exists.
→1 − 1

60. (a) lim ( ) = lim ( ) 2 = lim ( ) lim 2 = 5 0 = 0
→0 →0 2 · →0 2 · →0 .

(b) lim ( ) = lim ( ) = lim ( ) lim = 5 0 = 0
→0 →0 2 · →0 2 →0 .

Observe that $0 \le (\ ) \le {}^2$ for all , and $\lim_{\to 0} 0 = 0 = \lim_{\to 0} {}^2$. So, by the Squeeze Theorem, $\lim_{\to 0} (\ ) = 0$.

62. Let $(\,) = [[\,]]$ and $(\,) = -[[\,]]$. Then $\lim_{\to 3} (\,)$ and $\lim_{\to 3} (\,)$ do not exist [Example 10] but

$$\lim_{\to 3} [(\,) + (\,)] = \lim_{\to 3} ([[\,]] - [[\,]]) = \lim_{\to 3} 0 = 0.$$

Let $(\,) = (\,)$ and $(\,) = 1 - (\,)$, where is the Heaviside function defined in Exercise 1.3.59.

Thus, either or is 0 for any value of . Then $\lim_{\to 0} (\,)$ and $\lim_{\to 0} (\,)$ do not exist, but $\lim_{\to 0} [(\,)(\,)] = \lim_{\to 0} 0 = 0$.

$$\lim \frac{\sqrt{6} - 2}{\ } = \frac{\sqrt{6}}{2} \cdot \frac{\sqrt{6} + 2}{\ } \cdot \frac{\sqrt{3} + 1}{\ }$$

$$= \lim_{\to 2} \frac{\sqrt{6} - 2}{\sqrt{3}} \cdot \frac{\sqrt{3} + 1}{\sqrt{6} + 2} = \lim \frac{6 - 4}{3 - 1} \cdot \frac{\sqrt{3} + 1}{\sqrt{6} + 2}$$

$$= \lim_{\to 2} \frac{(2 \;)\sqrt{3} + 1}{(2 \;)\sqrt{6} + 2} = \lim_{\to 2} \frac{\sqrt{3} + 1}{\sqrt{6} + 2} = \frac{1}{2}$$

Since the denominator approaches 0 as $\to -2$, the limit will exist only if the numerator also approaches 0 as $\to -2$. In order for this to happen, we need $\lim_{\to -2} (3\,^2 + + + 3) = 0$ $\Leftrightarrow$ $3(-2)^2 + (-2) + + 3 = 0$ $\Leftrightarrow$ $12 - 2 + + 3 = 0$ $\Leftrightarrow$ $= 15$. With $= 15$, the limit becomes

$$\lim_{\to -2} \frac{3\,^2 + 15 + 18}{^2 + - 2} = \lim_{\to -2} \frac{3(+2)(+3)}{(-1)(+2)} = \lim_{\to -2} \frac{3(+3)}{-1} = \frac{3(-2+3)}{-2-1} = \frac{3}{-3} = -1.$$

Solution 1: First, we find the coordinates of and as functions of . Then we can find the equation of the line determined by these two points, and thus find the -intercept (the point), and take the limit as $\to 0$. The coordinates of are $(0\;)$.

The point is the point of intersection of the two circles $^2 + {}^2 = {}^2$ and $(-1)^2 + {}^2 = 1$. Eliminating from these equations, we get $^2 - {}^2 = 1 - (-1)^2$ $\Leftrightarrow$ $^2 = 1 + 2 - 1$ $\Leftrightarrow$ $= \frac{1}{2}\,^2$. Substituting back into the equation of the shrinking circle to find the -coordinate, we get $\left(\frac{1}{2}\,^2\right)^2 + {}^2 = {}^2$ $\Leftrightarrow$ $^2 = {}^2\left(1 - \frac{1}{4}\,^2\right)$ $\Leftrightarrow$ $= \sqrt{1 - \frac{1}{4}\,^2}$ (the positive -value). So the coordinates of are $\left(\frac{1}{2}\,^2 \;\; \sqrt{1 - \frac{1}{4}\,^2}\right)$. The equation of the line joining and is thus

$$- = \frac{\sqrt{1 - \frac{1}{4}\,^2} - }{\frac{1}{2}\,^2 - 0}(- 0).$$

We set $= 0$ in order to find the -intercept, and get

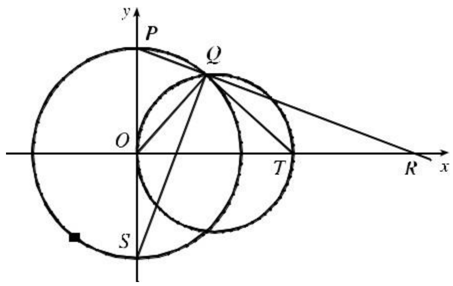
$$= \frac{\frac{1}{2}\,^2}{\sqrt{1 - \frac{1}{4}\,^2} - 1} = \frac{\frac{1}{2}\,^2\left(\sqrt{1 - \frac{1}{4}\,^2} + 1\right)}{1 - \frac{1}{4}\,^2 - 1} = 2\left(\sqrt{1 - \frac{1}{4}\,^2} + 1\right)$$

Now we take the limit as → 0

$$\lim = \lim 2 \quad 1 \quad ^2+1 = \lim 2 \quad \sqrt{1} + 1 \quad = 4$$

So the limiting position of is the point (4 0).

Solution 2: We add a few lines to the diagram, as shown. Note that ∠ = 90° (subtended by diameter). So ∠ = 90° = ∠ (subtended by diameter). It follows that ∠ = ∠ . Also ∠ = 90° − ∠ = ∠ . Since 4 is isosceles, so is 4 , implying that = . As the circle 2 shrinks, the point plainly approaches the origin, so the point must approach a point twice as far from the origin as , that is, the point (4 0), as above.



2.4 The Precise Definition of a Limit

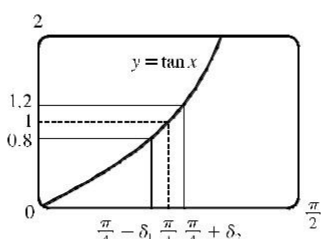
1. If | ( ) − 1| 0 2, then −0 2( ) − 1 0 2 ⇒ 0 8( ) 1 2. From the graph, we see that the last inequality is true if 0 71 1, so we can choose = min {1 − 0 7 1 1 − 1} = min {0 3 0 1} = 0 1 (or any smaller positive number).

2. If | ( ) − 2| 0 5, then −0 5( ) − 2 0 5 ⇒ 1 5( ) 2 5. From the graph, we see that the last inequality is true if 2 6 3 8, so we can take = min {3 − 2 6 3 8 − 3} = min {0 4 0 8} = 0 4 (or any smaller positive number). Note that 6= .3

The leftmost question mark is the solution of √ = 1 6 and the rightmost, √ = 2 4. So the values are 1 6^2 = 2 56 and 2 4^2 = 5 76. On the left side, we need | − 4| |2 56 − 4| = 1 44. On the right side, we need | − 4| |5 76 − 4| = 1 76. To satisfy both conditions, we need the more restrictive condition to hold — namely, | − 4| 1 44. Thus, we can choose = 1 44, or any smaller positive number.

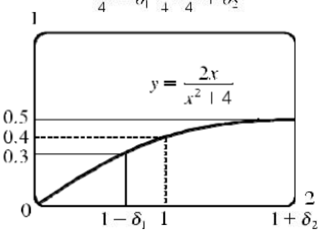
4. The leftmost question mark is the positive solution of = 2, that is, = √2, and the rightmost question mark is the positive solution of = 2, that is, = ≈ 0 292 (rounding down to be safe). On the right side, we need | − | 1 2/3 1 ≈ 0 224. The more restrictive of these two conditions must apply, so we choose = 0 224 (or any smaller positive number).

5.

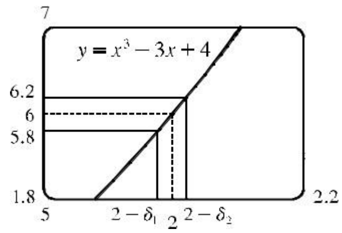


From the graph, we find that = tan = 0 8 when ≈ 0 675, so 4 − 1 ≈ 0 675 ⇒ 1 ≈ 4 − 0 675 ≈ 0 1106. Also, = tan = 1 2 when ≈ 0 876, so 4 + 2 ≈ 0 876 ⇒ 2 = 0 876 − 4 ≈ 0 0906. Thus, we choose = 0 0906 (or any smaller positive number) since this is the smaller of 1 and 2.

6.

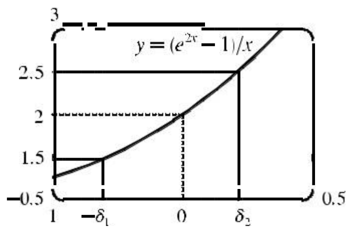


From the graph, we find that = 2 (2 + 4) = 0 3 when = 3 2, so 1 − 1 = 3 2 ⇒ 1 = 3 1. Also, = 2 (2 + 4) = 0 5 when = 2, so 1 + 2 = 2 ⇒ 2 = 1. Thus, we choose = 3 1 (or any smaller positive number) since this is the smaller of 1 and 2.

7. 

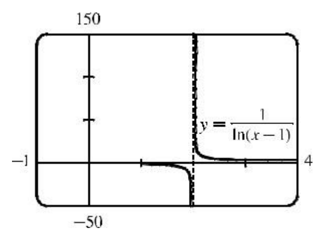
From the graph with $= 0\ 2$, we find that $= {}^3 - 3 + 4 = 5\ 8$ when $\approx 1\ 9774$, so $2 - {}_1 \approx 1\ 9774 \Rightarrow {}_1 \approx 0\ 0226$. Also, $= {}^3 - 3 + 4 = 6\ 2$ when $\approx 2\ 022$, so $2 + {}_2 \approx 2\ 0219 \Rightarrow {}_2 \approx 0\ 0219$. Thus, we choose $= 0\ 0219$ (or any smaller positive number) since this is the smaller of ${}_1$ and ${}_2$.

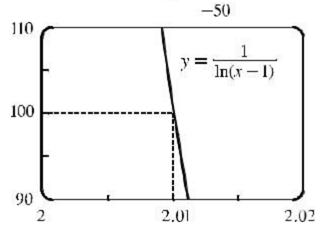
For $= 0\ 1$, we get ${}_1 \approx 0\ 0112$ and ${}_2 \approx 0\ 0110$, so we choose $= 0\ 011$ (or any smaller positive number).

8. 

From the graph with $= 0\ 5$, we find that $= ({}^2 - 1) = 1\ 5$ when $\approx -0\ 303$, so ${}_1 \approx 0\ 303$. Also, $= ({}^2 - 1) = 2\ 5$ when $\approx 0\ 215$, so ${}_2 \approx 0\ 215$. Thus, we choose $= 0\ 215$ (or any smaller positive number) since this is the smaller of ${}_1$ and ${}_2$.

For $= 0\ 1$, we get ${}_1 \approx 0\ 052$ and ${}_2 \approx 0\ 048$, so we choose $= 0\ 048$ (or any smaller positive number).



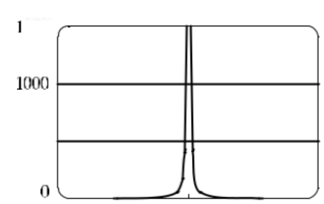
9. (a) 

The first graph of $= \dfrac{1}{\ln(- 1)}$ shows a vertical asymptote at $= 2$. The second graph shows that $= 100$ when $\approx 2\ 01$ (more accurately, $2\ 01005$). Thus, we choose $= 0\ 01$ (or any smaller positive number).

From part (a), we see that as gets closer to 2 from the right, increases without bound. In symbols,

$$\lim_{\to 2^+} \frac{1}{\ln(- 1)} = \infty.$$

10. We graph $= \csc^2$ and $= 500$. The graphs intersect at $\approx 3\ 186$, so we choose $= 3\ 186 - \approx 0\ 044$. Thus, if $0 \quad | - | \quad 0\ 044$, then $\csc^2 \quad 500$. Similarly, for $= 1000$, we get $= 3\ 173 - \approx 0\ 031$.



11. (a) $=$ ${}_2$ and $= 1000\ \text{cm}_2 \Rightarrow {}_2 = 1000 \Rightarrow {}_2 = \frac{}{1000} \Rightarrow = \frac{}{1000}$ $\frac{\pi}{2}$ (0) $\approx 17\ 8412$ cm. $\frac{3\pi}{2}$

(b) $| - 1000| \le 5 \Rightarrow -5 \le {}^2 - 1000 \le 5 \Rightarrow 1000 - 5 \le {}^2 \le 1000 + 5 \Rightarrow$ $\frac{}{995}$ $\frac{}{1005}$ $\le \le$

8. 0 04406 and 0 04455. So

⇒ 17 7966

1
0
0
0

9
9
5

1
0
0
5

.

1
0
0
0

≤

≤

1
7

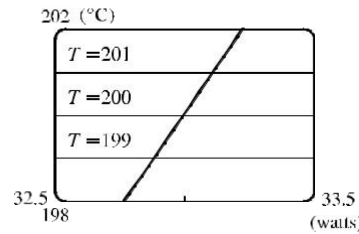
8
8
5

if the machinist gets the radius within 0.0445 cm of 17.8412, the area will be within 5 cm^2 of 1000.

is the radius, ( ) is the area, is the target radius given in part (a), is the target area (1000), is the tolerance in the is the area (5), and tolerance in the radius given in part (b).

12. (a) = 0 1 ^{2} + 2 155 + 20 and = 200 ⇒

0 1 ^{2} + 2 155 + 20 = 200 ⇒ [by the quadratic formula or from the graph] ≈ 33 0 watts (0)



(b) From the graph, 199 ≤ ≤ 201 ⇒ 32 89 33 11.

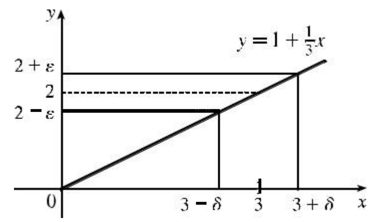
is the input power, ( ) is the temperature, is the target input power given in part (a), is the target temperature (200), is the tolerance in the temperature (1), and is the tolerance in the power input in watts indicated in part (b) (0 11 watts).

13. (a) |4 −8|=4| −2| 01 ⇔|−2| $\frac{0\,1}{4}$, so = $\frac{0\,1}{4}$ = 0 025.

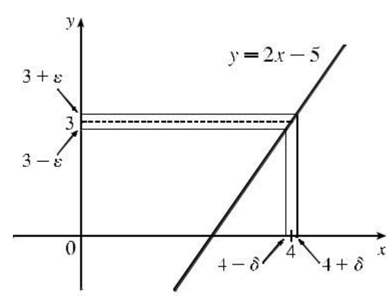
(b) |4 −8|=4| −2| 001 ⇔|−2| $\frac{0\,01}{4}$, so = $\frac{0\,01}{4}$ = 0 0025.

14. |(5 − 7) − 3| = |5 − 10| = |5(− 2)| = 5 | − 2|. We must have | ( ) − | , so 5 | − 2|⇔ | − 2|5. Thus, choose = 5. For = 0 1, = 0 02; for = 0 05, = 0 01; for = 0 01, = 0 002.

15. Given0, we need0 such that if 0 | − 3| , then (1 + $\frac{1}{3}$) − 2. But (1+ $\frac{1}{3}$)−2⇔ $\frac{1}{3}$ − 1 ⇔ $\frac{1}{3}$ 3 3 3. So if we choose = 3 , then 0 | − | 3 ⇒ |(1 + $\frac{1}{3}$) − 2. Thus, lim$_{\to 3}$ (1 + $\frac{1}{3}$) = 2 by the definition of a limit.



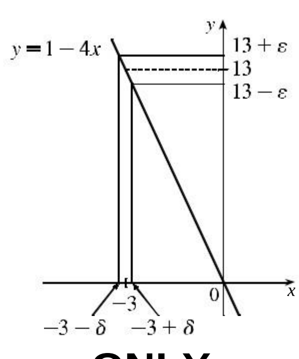
16. Given0, we need0 such that if 0 | − 4| , then |(2 − 5) − 3| . But |(2 − 5) − 3|⇔ |2 − 8| ⇔ |2|| −4| ⇔ | − 4|2. So if we choose = 2, then 0 | −4| ⇒ |(2 − 5) − 3| . Thus, lim$_{\to}$ (2 − 5) = 3 by the definition of a limit.



17. Given0, we need0 such that if 0 | − (−3)| , then |(1 − 4) − 13| . But |(1 − 4) − 13|⇔

|−4 − 12|⇔ |−4| | + 3| ⇔ | − (−3)| 4. So if we choose = 4, then 0 | − (−3)| ⇒ |(1 − 4) − 13| .

Thus, $\lim_{\to -3}$ (1 − 4) = 13 by the definition of a limit.



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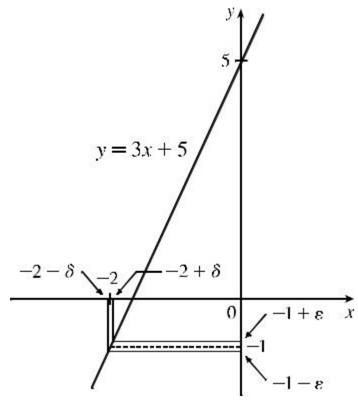
18. Given 0, we need 0 such that if 0 | − (−2)| , then |(3 + 5) − (−1)| .

But |(3 + 5) − (−1)| ⇔

|3 + 6| ⇔ |3| | + 2| ⇔ | + 2| 3. So if we choose = 3, then 0 | + 2| ⇒ |(3

+ 5) − (−1)| . Thus, lim (3 + 5) = −1 by the definition of a limit.

→−2



19. Given0, we need0 such that if 0 | − 1|, then $\frac{+4}{3}$ − 2. But $\frac{2+4}{3}$ − 2 ⇔

4 4 ⇔ 4 3 |−1|⇔|−1| 3 4 . So if we choose = 3 4 , then 0 | − 1| ⇒

2 + 4 lim $\frac{2+4}{}$ = 2 by the definition of a limit.

20. Given4 8 5 0, we need 4 5 0 such that if 0 | − 10 | 10, then 3 4 (5) . But 3 4 (5) ⇔

− 4 ⇔ − | − | 4 ⇔ |− | . So if we choose = 4 5, then 0 | − | 10 ⇒ . Thus, lim by the definition of a limit.

→10

21. Given0, we need0 such that if 0 4, then 2 −2 −8 6

− 4) (+ 2

0 |−4|⇒|−4| ⇒|+2−6| ⇒ $\frac{(\ \ 4)(\ +2)}{\ \ 4}$ − 6 [6=4]⇒

2 2 8 − 2 2 8

22. Given0, we need0 such that if 0+ 1 5, then $\frac{9-4\ ^2}{3+2}$ − 6 ⇔

(3+2)(3 2)

+ 1 52. So choose = 2. Then 0 + 1 5 +152 2 + 1 5

| | | | ⇒ | | ⇒ |−|| | ⇒

|− 2 3 − | ⇒ | − 3 2 6 − | By the definition of a limit, lim (3+2) ⇒ 3 3+2 − 6 − 2) 6 9 4 2

− = 6.

[= 15]

9

−

4
2

6

.

⇒

3+2 −

→−1 5

3 + 2

Given0, we need0 such that if 0 | − | , then | − | . So = will work.

Given 0, we need 0 such that if 0 | − | , then | − | . But | − | = 0, so this will be true no matter what we pick.

25. 0 0 such that if 0 0, then 2 0 2 √ =. Take = √ =.

Given , we need 2 0 | − | . Thus, lim 2 = 0 by the definition of a limit. Then 0 0 3 − ⇔ 3 ⇔ | | √ 3 √ 3

26. | − | ⇒ − →0

0 0 such that if 0 0, then 0 − = −

Given , we need 3 0 3 | = − | . Thus, lim = 0 by the definition of a limit. Then 0 0 3 − ⇔ | | ⇔ | | . Take .

|− | ⇒ − →0

27. Given 0, we need 0 such that if 0 | − 0| , then | |−0 . But | | = | |. So this is true if we pick = .

Thus, lim = 0 by the definition of a limit.
→

28. Given 0, we need 0 such that if 0 (6), then √8 6 + 0. But √8 6 + 0

8 6 + 8 − − 8 − =8 − (6) ⇔ (6). So if we choose , then 0

⇔ ⇔ − − √ by the definition of a right-hand limit. − − ⇒

√8 lim 8 6 + = 0

29. Given 2 , we need such that if | − | , then − − ⇔ 2 −

(2) . So take = √ −. Then 0 2 2 √ − (2) . Thus,

− 2 | − | ⇔|−| ⇔ −
→2
lim 4+ 5 = 1 by the definition of a limit.

30. Given 0, we need 0 such that if 0 2 (2+2 7) 1 (2+2 7) 1

2+ 2 8 + 4 | − | , then − − . But − − ⇔. Thus our goal is to make 2 small enough so that its product with + 4

is less than . Suppose we first require that 2 1. Then 1 2 1 13 5+47 |− | − − ⇒ ⇒ ⇒

| + 4| 7, and this gives us 7 | − 2| ⇒ | − 2| 7. Choose = min {1 7}. Then if 0 | − 2| , we

have 27 and + 4 7, so (2 + 2 7) 1 = (+4)(2) = + 4 2 7(7) = , as

desired. Thus, | − | lim (2 + 2 | | − 7) = 1 by the definition of a limit. − − | − | | ||−|
→

31. Given 0, we need 0 such that if 0 | − − |, then (2) 2 2 1 3 or upon simplifying we need

4 whenever 0 | | + 2. Notice that if + 2 1, then 1 | | + 2 1 ⇒ − 5 2 3 −− ⇒

2 5. So take = min { 5 1 }. Then 0 | − | | + 2 | ⇒ |− | 2 5 and | | + 25, so

− 1 − 3 = |(+ 2)(− 2)| = | + 2| | − 2| (5)(5) = . Thus, by the definition of a limit, →−2 lim (2 1)=3 .

Given 2 , we need such that if 2 | − | , 2 − −

32. 0 0 0 2 then 3 8. Now 3 8 = (2) ^{2}+2 +4 .

If 2 1, that is, 13, then + 2 + 4 3 + 2(3) +4 = 19 and so

|− |

1

3 −8 =| −2| 2 +2 +4 | − |. So if we take 9 , then 3 | − | ⇒

3^2 19 2 = min 1 — 0 2

− |− | 19 →2

33. 0 = min 2 If 0 3, then 3 2 2 3 2

Given , we let 8 • | − | 2 | − | ⇒ − − ⇒ 2

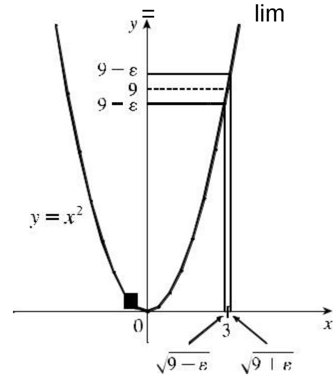
⇒ | | . Also | √ 8 . Thus, →3 = 9.

4+38 + 3 8 3 = lim

34. From the figure, our choices for are 1 = 3 − √ 9 − and

2 = 9 + − 3. The *largest* possible choice for is the minimum √

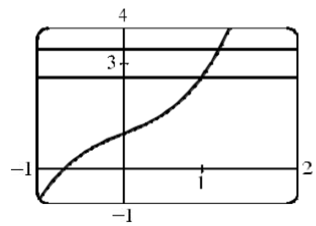
value of { 1 2}; that is, = min{ 1 2} = 2 = 9 +− 3.



35. (a) The points of intersection in the graph are (1 2 6) and (2 3 4)

with 1 ≈ 0 891 and 2 ≈ 1 093. Thus, we can take to be the

smaller of 1 − 1 and 2 − 1. So = 2 − 1 ≈ 0 093.



(b) Solving 3 + + 1 = 3 + gives us two nonreal complex roots and one real root, which is √

( ) = 216 + 108 + 12 336+324 +81 2 6 216 + 108 + 12 √336 + 324 + 81 2 2 3 1 3 12 . Thus, = ( ) − 1.

(c) If = 0 4, then ( ) ≈ 1 093 272 342 and = ( ) − 1 ≈ 0 093, which agrees with our answer in part (a).

36. 1. Guessing a value for Let be given. We have to find a number 1 1 such that whenever

| − | 1 − 1 2 2 2 − 2 | 2 1 ⇒

02. But = = − | . We find a positive constant such that

| 2 2 | − | 2 and we can make | − | 2 by taking | − | 2 = . We restrict to lie in the interval | |

| − 2| 1 ⇒ 13 so 1 1 1 3 ⇒ 1 6 1 2 1 2 ⇒ 1 |2| 1 2. So = 1 2 is suitable. Thus, we should

choose = min {1 2 }.

2. Showing that works Given 0 we let = min {1 2 }. If 0 | − 2| , then | − 2| 1 ⇒ 13 ⇒ | 2 2 | − | 2 2 | 2 | →2 2

(as in part 1). Also 2 2 , so = − 2| 2 = . This shows that lim (1) =

1. Guessing a value for Given 0, we must find 0 such that | √ − √ | whenever 0 | − |. But

√ − √ = | − | √ (from the hint). Now if we can find a positive constant such that √ + √ then

| − | + √

$|\ -|\quad |-|$, and we take . We can find this number by restricting to lie in some interval $|-|$

centered at . If $|-|$ $\frac{1}{2}$, then $-\frac{1}{2}$ $-\frac{1}{2}$ $\Rightarrow \frac{1}{2}$ $\frac{3}{2}$ $\Rightarrow$ + $\frac{1}{2}$ +, and so

$=\frac{1}{2}+\sqrt{}$ is a suitable choice for the constant. So $|-|$ $\frac{1}{2}+\sqrt{}$. This suggests that we let

$=\min\ \frac{1}{2}\ \frac{1}{2}+\sqrt{}$.

2. Showing that works Given 0, we let $=\min\ \frac{1}{2}\ \frac{1}{2}+\sqrt{}$. If 0 $|-|$, then

$|-|\ \frac{1}{2}\Rightarrow\sqrt{}+\sqrt{}\ \frac{1}{2}+\sqrt{}$ (as in part 1). Also $|-|\ \frac{1}{2}+\sqrt{}$, so

$|\ -\ |$ $2+\sqrt{}$ = . Therefore, lim =

$2+\sqrt{}$

38. Suppose that 0 . Given $\frac{}{2}$, there exists such that $||\Rightarrow|\ -|$ $\Leftrightarrow$

$-\frac{1}{2}$ $()+\frac{1}{2}$. For 0, $()=1$, so 1+ $\frac{1}{2}\Rightarrow\frac{1}{2}$. For -0, $()=0$,

1 $\frac{1}{}$ $\frac{1}{}$ lim ()

lim () = $=1$ 0 0 () 1

39. Suppose that 0 . Given $\frac{}{2}$, there exists such that $||\Rightarrow|\ -|$. Take any rational

number with 0 $||$. Then $()=0$, so $|0-|$ $\frac{1}{2}$, so $\leq||\ \frac{1}{2}$. Now take any irrational number with

0 $()=1$ 1 $\frac{1}{}$ lim ()

exist.

40. First suppose that lim $()=$. Then, given 0 there exists 0 so that 0 $|-|$ $\Rightarrow|()-|$.

Then $-$ $\Rightarrow 0$ $|$ $-|$ so $|\ ()-$ $|$ lim . Thus, $\to-$ ()

lim ()=

Now suppose lim ()= = lim (). Let 0 be given. Since lim () = , there exists 1

$-\ _1$ $\Rightarrow|()-|$. Since $\to+$ lim () = , there exists $_2$ 0 so that $+\ _2\Rightarrow$

$|()-\ |$. Let be the smaller of $_1$ and $_2$. Then 0 $|-|$ $\Rightarrow -$ or $+\ _2$ so

$()$ lim lim () = $\Leftrightarrow$ lim () = = lim ().

41. $\frac{1}{(+3)^4}$ 10,000 $\Leftrightarrow(+3)^4\ \frac{1}{10{,}000}\Leftrightarrow|+3|\ \frac{1}{\sqrt[4]{10{,}000}}\Leftrightarrow|-(-3)|\ \frac{1}{10}$

42. Given 0, we need 0 such that 0 $|+3|$ $\Rightarrow 1(+3)^4$. Now $\frac{1}{(+3)^4}$ $\Leftrightarrow$

$\frac{1}{(+3)^4} \Leftrightarrow |+3| \frac{1}{\sqrt[4]{}}$. So take $= \frac{1}{\sqrt[4]{}}$. Then $0 \quad | + 3| = \frac{1}{\sqrt[4]{}} \Rightarrow \frac{1}{(+3)^4}$, so

$\lim_{\to -3} \frac{1}{(+3)^4} = \infty$.

Given 0 we need 0 so that ln whenever 0; that is, = ln whenever 0. This suggests that we take . If , then = 0 ln ln = . By the definition of a limit, $\to 0^+$ lim ln = −∞.

44. (a) Let be given. Since lim ( ) = ∞, there exists 1 such that | − | 1 ⇒ ( ) ( ) +1 − 1. Since lim ( ) = → 0 0 0 0, there exists 2 such that | − | 2 ⇒ | − | ⇒ − . Let be the smaller of 1 and 2. Then 0 | − | ⇒ ( ) + ( ) (+ 1 −) + (− 1) = . Thus, lim [( ) + ( )] = ∞.

(b) Let 0 be given. Since lim ( ) = 0, there exists 1 0 such that 0 | − | 1 ⇒ | ( ) − | 2 ⇒ ( ) 2. Since lim ( ) = ∞, there exists 2 0 such that 0 2 ⇒ ( ) 2. Let = min { 1 2}. Then 0 | − | ⇒ ( ) ( ) 2 = , so lim ( ) ( ) = ∞.

(c) Let 0 be given. Since lim ( ) = 0, there exists 1 0 such that 0 | − | 1 ⇒ | ( ) − | − 2 ⇒ ( ) 2. Since lim ( ) = ∞, there exists 2 0 such that 0 | 2 ⇒ ( ) 2 . (Note that 0 and 0 ⇒ 2 0.) Let = min { 1 2}. Then 0 | − | ⇒ ( ) 2 ⇒ ( ) ( ) 2 = , so lim ( ) ( ) = −∞.

2.5 Continuity

From Definition 1, $\lim_{\to 4}$ ( ) = (4).

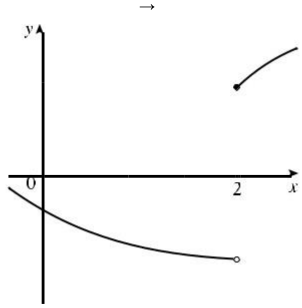
The graph of has no hole, jump, or vertical asymptote.

(a) is discontinuous at −4 since (−4) is not defined and at −2, 2, and 4 since the limit does not exist (the left and right limits are not the same).

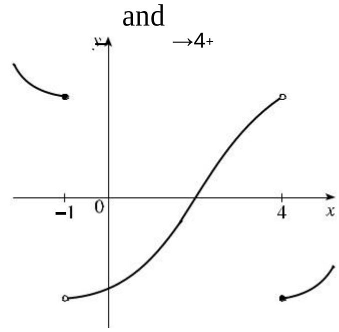
(b) is continuous from the left at −2 since $\lim_{\to -2^-}$ ( ) = (−2). is continuous from the right at 2 and 4 since $\lim_{\to 2^+}$ ( ) = (2) and $\lim_{\to 4^+}$ ( ) = (4). It is continuous from neither side at − since − is undefined.

From the graph of , we see that is continuous on the intervals [−3 −2), (−2 −1), (−1 0], (0 1), and (1 3].

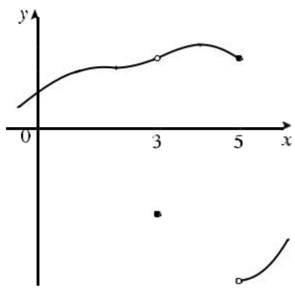
5. The graph of = ( ) must have a discontinuity at = 2 and must show that $\lim_{\to 2^+}$ ( ) = (2).



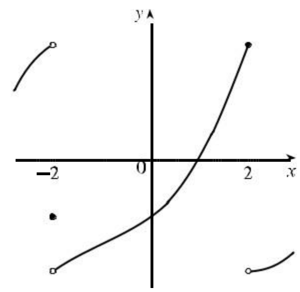
6. The graph of = ( ) must have discontinuities at = −1 and = 4. It must show that $\lim_{\to -1^-}$ ( ) = (−1) and $\lim_{\to 4^+}$ ( ) = (4).



7. The graph of $=$ ( ) must have a removable discontinuity (a hole) at $= 3$ and a jump discontinuity at $= 5$.

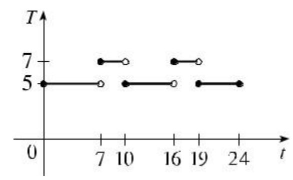


8. The graph of $=$ ( ) must have a discontinuity at $= -2$ with $\lim_{\to -2^-}$ ( ) $= (-2)$ and $\lim_{\to -2^+}$ ( ) $6= (-2)$. It must also show that $\lim_{\to 2^-}$ ( ) $= (2)$ and $\lim_{\to 2^+}$ ( ) $6= (2)$.



9. (a) The toll is \$7 between 7:00 AM and 10:00 AM and between 4:00 PM and 7:00 PM.

(b) The function has jump discontinuities at $= 7, 10, 16$, and 19. Their significance to someone who uses the road is that, because of the sudden jumps in the toll, they may want to avoid the higher rates between $= 7$ and $= 10$ and between $= 16$ and $= 19$ if feasible.



(a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.

Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values — at a cliff, for example.

Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.

Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

$\lim_{\to -1}$ ( ) $= \lim_{\to -1} + 2\,{}^3\,{}^4 = \lim_{\to -1} + 2\lim_{\to -1}{}^3\,{}^4 = -1 + 2(-1)^3\,{}^4 = (-3)^4 = 81 =$ (-1).

By the definition of continuity, is continuous at $= -1$.

12. $\lim_{\to 2}$ ( ) $= \lim_{\to 2} \dfrac{{}^2 + 5}{2 + 1} = \dfrac{\lim_{\to 2}({}^2 + 5\,)}{\lim_{\to 2}(2 + 1)} = \dfrac{\lim_{\to 2}{}^2 + 5\lim_{\to 2}}{2\lim_{\to 2} + \lim_{\to 2} 1} = \dfrac{2^2 + 5(2)}{2(2) + 1} = \dfrac{14}{5} =$ (2).

By the definition of continuity, is continuous at $= 2$.

13. = 2

lim ( ) = lim 2 $\sqrt{3^2 + 1}$ = 2 lim $\sqrt{}$ 2 lim(3 ^{2} + 1) = 2 3 lim 2 + lim 1

= 2 $\sqrt{3(1)^2 + 1}$ = 2 $\sqrt{4}$ = 4 = (1)

By the definition of continuity, is continuous at = 1.

14. →2 →2 − →2 − →2

lim ( ) = lim 3 ^{4} 5 + $\sqrt[3]{^2 + 4}$ = 3 lim ^{4} 5 lim + $_3$ lim (^{2} + 4)

= 3(2)^4 − 5(2) + $\sqrt[3]{2^2 + 4}$ = 48 − 10 + 2 = 40 = (2)

By the definition of continuity, is continuous at = 2.

For 4, we have

lim ( ) = lim (+ $\sqrt{ - 4}$) = lim + lim $\sqrt{ - 4}$ [Limit Law 1]

→ → → →

= + lim lim 4 [8, 11, and 2]

= + $\sqrt{ - 4}$ [8 and 7]

( )

So is continuous at = for every in (4 ∞). Also, $\lim_{\to 4^+}$ ( ) = 4 = (4), so is continuous from the right at 4.

Thus, is continuous on [4 ∞).

For −2, we have

lim ( ) = lim $\dfrac{-1}{3+6}$ = $\dfrac{\lim (- 1)}{\lim (3 + 6)}$ [Limit Law 5]

→ →3+6 → →

lim lim 1

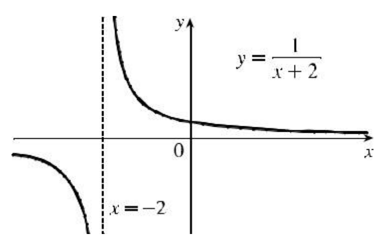
= $\dfrac{- }{3 \lim + \lim 6}$ [2 1 and 3]

→ →

= $\dfrac{-1}{3 + 6}$ [8 and 7]

Thus, is continuous at = for every in (−∞ −2); that is, is continuous on (−∞ −2).

17. ( ) = $\dfrac{1}{+2}$ is discontinuous at = −2 because (−2) is undefined.



18. ( ) = $\dfrac{1}{+2}$ if 6=−2

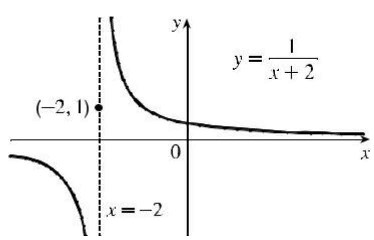
1 if = 2

Here (−2) = 1, but →−2

−∞ and →−2+ ∞,

lim $^-$ ( ) = lim ( ) =

so $\lim_{\to -2}$ ( ) does not exist and is discontinuous at −2.



NOT FOR SALE

19. ()= 2 + 3 if ≤ −1
if 1

− −

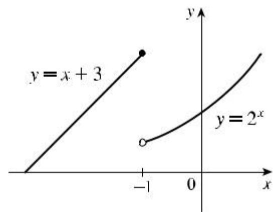
→−1⁻ →−1⁻ and

lim ( ) = lim (+3)= 1+3=2

lim ( ) = lim 2 =2^{-1}= $\frac{1}{2}$. Since the left-hand and the
→−1$^+$ →−1$^+$

right-hand limits of at −1 are not equal, lim ( ) does not exist, and
→−1

is discontinuous at −1.



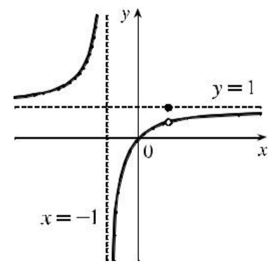
20. ( )= $\frac{2\,-}{2\,-1}$ if 6 = 1

1 if = 1

lim ( ) = lim $\frac{2\ -}{2\ -1}$ = lim $\frac{(-1)}{(+1)(-1)}$ = lim $\frac{}{+1}$ = $\frac{1}{2}$,

→1 →1 →1 →1

but (1) = 1, so is discontinous at 1

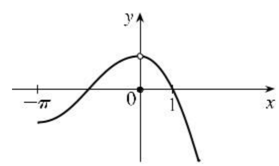


cos if 0

()= 0 if = 0

1 − 2 if 0

lim ( ) = 1, but (0) = 0 6= ,1so is discontinuous at 0.
→0

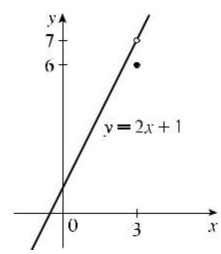


22. ( )= $\frac{2\,^2\ -5\ -3}{-3}$ if 6= 3

6 if = 3

lim ( ) = lim $\frac{2\,^2-5\,-3}{-3}$ = lim $\frac{(2\,+1)(\,-3)}{-3}$ = lim (2 + 1) = 7,
→3 →3 →3 →3

but (3) = 6, so is discontinuous at 3.



( ) = $\frac{2\,-\,-2}{-2}$ = $\frac{(\,-2)(\,+1)}{-2}$ = + 1 for 6= .2Since lim ( ) = 2 + 1 = 3, define (2) = 3. Then is
→2

continuous at 2.

( ) = $\frac{3\,-8}{2-4}$ = $\frac{(\,-2)(\,2+2\,+4)}{(\,-2)(\,+2)}$ = $\frac{2\,+2\,+4}{+2}$ for 6= .2Since lim ( ) = $\frac{4+4+4}{2+2}$ = 3, define (2) = 3.
→2

Then is continuous at 2.

( ) = $\frac{2\,2\,-\,-1}{2\,+1}$ is a rational function, so it is continuous on its domain, (−∞ ∞), by Theorem 5(b).

26. ( )= $\frac{2\,+1}{2\,2-\,-1}$ = $\frac{2\,+1}{(2\,+1)(\,-1)}$ is a rational function, so it is continuous on its domain,

−∞ −$\frac{1}{2}$ ∪ −$\frac{1}{2}$ 1 ∪ (1 ∞), by Theorem 5(b).

27. $-\ ^3 \ ^{2=0} \Rightarrow\ ^3 \ ^{=2} \Rightarrow \sqrt{3}\ = 2$, so $(\ ) = \dfrac{\sqrt{\ ^3 - 2}}{3 - 2}$ has domain $-\infty\ \sqrt{3}\ ^2 \cup \sqrt{3}\ ^2\ \infty$. Now $\ ^3 - \ ^2$ is continuous everywhere by Theorem 5(a) and $\sqrt{\ ^3 - 2}$ is continuous everywhere by Theorems 5(a), 7, and 9. Thus, is continuous on its domain by part 5 of Theorem 4.

28. The domain of $(\ ) = \dfrac{\sin}{2 + \cos}$ is $(-\infty\ \infty)$ since the denominator is never 0 [$\cos\ \geq -1 \Rightarrow 2 + \cos\ \geq 1$]. By Theorems 7 and 9, sin and cos are continuous on R. By part 1 of Theorem 4, $2 + \cos$ is continuous on R and by part 5 of Theorem 4, is continuous on R.

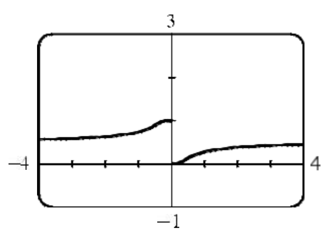
By Theorem 5(a), the polynomial $1 + 2$ is continuous on R. By Theorem 7, the inverse trigonometric function arcsin is continuous on its domain, $[-1\ 1]$. By Theorem 9, $(\ ) = \arcsin(1 + 2\ )$ is continuous on its domain, which is $\{\ |-1\leq 1+2\ \leq 1\}=\{\ |-2\leq 2\ \leq 0\}=\{\ |-1\leq\ \leq 0\}=[-1\ 0]$.

By Theorem 7, the trigonometric function tan is continuous on its domain, $|\ 6= 2 +$. By Theorems 5(a), 7, and 9, the composite function $\sqrt{4 - \ ^2}$ is continuous on its domain $[-2\ \ 2]$. By part 5 of Theorem 4, $(\ ) = \dfrac{\tan}{\sqrt{4 - \ ^2}}$ is continuous on its domain, $(-2\ -\ 2) \cup (-\ 2\ \ 2) \cup (\ 2\ 2)$.

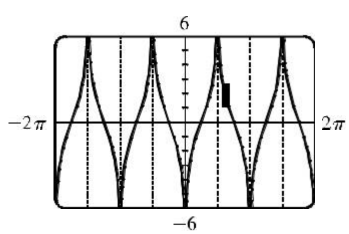
31. $(\ ) = \sqrt{1 + \dfrac{1}{}} = \sqrt{\dfrac{+1}{}}$ is defined when $\dfrac{+1}{} \geq 0 \Rightarrow +1 \geq 0$ and 0 or $+1 \leq 0$ and $0 \Rightarrow 0$ or ≤ -1, so has domain $(-\infty\ -1] \cup (0\ \infty)$. is the composite of a root function and a rational function, so it is continuous at every number in its domain by Theorems 7 and 9.

By Theorems 7 and 9, the composite function $-\ ^2$ is continuous on R. By part 1 of Theorem 4, $1 + -\ ^2$ is continuous on R. By Theorem 7, the inverse trigonometric function $\tan^{-1}$ is continuous on its domain, R. By Theorem 9, the composite function $(\ ) = \tan^{-1}\ 1 + \ ^{-\ 2}$ is continuous on its domain, R.

33. The function $= \dfrac{1}{1 + \ ^1}$ is discontinuous at $= 0$ because the left- and right-hand limits at $= 0$ are different.



34. The function $= \tan^2$ is discontinuous at $= 2 +$, where is any integer. The function $= \ln \tan^2$ is also discontinuous where $\tan^2$ is 0, that is, at $=$. So $= \ln\tan^2$ is discontinuous at $= \frac{}{2}$, any integer.



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35. Because is continuous on R and $\sqrt{20 - \ ^2}$ is continuous on its domain, $-\sqrt{20} \le \ \le \sqrt{20}$, the product

$(\ ) = \ \sqrt{20 - \ ^2}$ is continuous on $-\sqrt{20} \le \ \le \sqrt{20}$. The number 2 is in that domain, so is continuous at 2, and $\lim_{\to 2} (\ ) = (2) = 2\sqrt{16} = 8$.

Because is continuous on R, sin is continuous on R, and + sin is continuous on R, the composite function

$(\ ) = \sin(\ + \sin\ )$ is continuous on R, so $\lim_{\to} (\ ) = (\ ) = \sin(\ + \sin\ ) = \sin\ = 0$.

37. The function $(\ ) = \ln \dfrac{5 - \ ^2}{1 +}$ is continuous throughout its domain because it is the composite of a logarithm function and a rational function. For the domain of , we must have $\dfrac{5 - \ ^2}{1 +}$ 0, so the numerator and denominator must have the same sign, that is, the domain is $(-\infty\ -\sqrt{5}\,] \cup (-1\ \ \sqrt{5}\,]$. The number 1 is in that domain, so is continuous at 1, and $\lim_{\to 1} (\ ) = (1) = \ln \dfrac{5 - 1}{1 + 1} = \ln 2$.

38. The function $(\ ) = 3^{\sqrt{\ ^2 - 2\ - 4}}$ is continuous throughout its domain because it is the composite of an exponential function, a root function, and a polynomial. Its domain is

$$|\ ^2 - 2\ - 4 \ge 0 = |\ ^2 - 2\ + 1 \ge 5 = |(\ - 1)^2 \ge 5$$

$$= |\ - 1| \ge \sqrt{5} \qquad = (-\infty\ 1 - \sqrt{5}\,] \cup [1 + \sqrt{5}\ \ \infty)$$

$\sqrt{16}$ 8 4 2

$\to 4$

39. $(\ ) = \begin{cases} 1 - \ ^2 & \text{if} \quad \le 1 \\ \ln & \text{if} \quad 1 \end{cases}$

By Theorem 5, since $(\ )$ equals the polynomial $1 - \ ^2$ on $(-\infty\ 1]$, is continuous on $(-\infty\ 1]$.

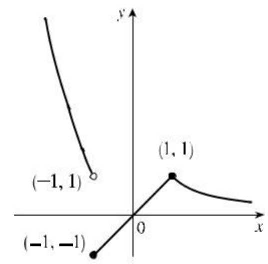
By Theorem 7, since $(\ )$ equals the logarithm function ln on $(1\ \infty)$, is continuous on $(1\ \infty)$.

$\lim (\ ) = \lim (1 - \ ^2) = 1 - 1^2 = 0$ and $\lim (\ ) = \lim \ln\ = \ln 1 = 0$. Thus, $\lim_{\to 1} (\ )$ exists and equals 0. Also, $(1) = 1 - 1^2 = 0$. Thus, is continuous at $= 1$. We conclude that is continuous on $(-\infty\ \infty)$.

40. $(\ ) = \begin{cases} \sin & \text{if} \quad 4 \\ \cos & \text{if} \ge 4 \end{cases}$

By Theorem 7, the trigonometric functions are continuous. Since $(\ ) = \sin$ on $(-\infty\ \ 4)$ and $(\ ) = \cos$ on $(\ 4\ \ \infty)$, is continuous on $(-\infty\ \ 4) \cup (\ 4\ \ \infty)$. $\lim_{\to(\ 4)^-} (\ ) = \lim_{\to(\ 4)^-} \sin\ = \sin \ 4 = 1\ \sqrt{2}$ since the sine function is continuous at 4 Similarly, $\lim_{\to(\ 4)^+} (\ ) = \lim_{\to(\ 4)^+} \cos\ = 1\ \sqrt{2}$ by continuity of the cosine function at 4. Thus, $\lim_{\to(\ 4)} (\ )$ exists and equals $1\ \sqrt{2}$, which agrees with the value $(\ 4)$. Therefore, is continuous at 4, so is continuous on $(-\infty\ \infty)$.

41. ( )= 2 if −1; if −1 1; 1if ≥ 1



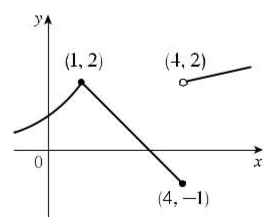
is continuous on (−∞ −1), (−1 1), and (1 ∞), where it is a polynomial, a polynomial, and a rational function, respectively.

Now $\lim_{\to -1^-}$ ( ) = $\lim_{\to -1^-}$ 2 = 1 and $\lim_{\to -1^+}$ ( ) = $\lim_{\to -1^+}$ = − 1,

so is discontinuous at −1. Since (−1) = −1, is continuous from the right at − 1. Also, $\lim_{\to 1^-}$ ( ) = $\lim_{\to 1^-}$ = 1 and

$\lim_{\to}$ ( ) = $\lim_{\to}$ 1 = 1 = (1), so is continuous at 1.

42. ( )= 2 if ≤ 1; 3 − if 1 ≤ 4; if 4

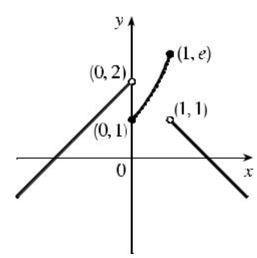


is continuous on (−∞ 1), (1 4), and (4 ∞), where it is an exponential, a polynomial, and a root function, respectively.

Now $\lim_{\to 1^-}$ ( ) = $\lim_{\to 1^-}$ 2 = 2 and $\lim_{\to 1^+}$ ( ) = $\lim_{\to 1^+}$ (3 −) = 2. Since (1)=2 we have continuity at 1. Also,

$\lim_{\to 4^-}$ ( ) = $\lim_{\to 4^-}$ (3 −) = − 1= (4) and $\lim_{\to 4^+}$ ( ) = $\lim_{\to 4^+}$ √ = 2, so is discontinuous at 4, but it is continuous from the left at 4.

43. ( )= + 2 if 0; if 0 ≤ ≤ 1; 2 − if 1



is continuous on (−∞ 0) and (1 ∞) since on each of these intervals it is a polynomial; it is continuous on (0 1) since it is an exponential.

Now $\lim_{\to 0^-}$ ( ) = $\lim_{\to 0^-}$ (+ 2) = 2 and $\lim_{\to 0^+}$ ( ) = $\lim_{\to 0^+}$ = 1, so is discontinuous at 0. Since (0) = 1, is continuous from the right at 0. Also $\lim_{\to 1^-}$ ( ) = $\lim_{\to 1^-}$ = and $\lim_{\to 1^+}$ ( ) = $\lim_{\to 1^+}$ (2 −) = 1, so is discontinuous at 1. Since (1) = , is continuous from the left at 1.

44. By Theorem 5, each piece of is continuous on its domain. We need to check for continuity at = . $\lim_{\to^-}$ ( ) = $\lim_{\to^-}$ 3 = 2 and $\lim_{\to^+}$ ( ) = $\lim_{\to^+}$ 2 = 2, so $\lim_{\to}$ ( ) = 2. Since ( ) = 2, is continuous at . Therefore, is a continuous function of .

45. ( ) = 2 + 2 if 2; 3 − if ≥ 2

is continuous on −∞ and ∞ . Now $\lim_{\to 2^-}$ + 2 = 4 + 4 and

$\lim_{\to 2^+} () = \lim_{\to 2^+} 3 - = 8 - 2$. So is continuous $\Leftrightarrow 4 + 4 = 8 - 2 \Leftrightarrow 6 = 4 \Leftrightarrow = \frac{2}{3}$. Thus, for to be continuous on $(-\infty, \infty)$, $= \frac{2}{3}$.

46. $() = \begin{cases} \frac{2 - 4}{- 2} & \text{if } < 2 \\ 2 - + 3 & \text{if } 2 \le < 3 \\ 2 - + & \text{if } \ge 3 \end{cases}$

At $= 2$: $\lim_{\to 2^-} () = \lim_{\to 2^-} \frac{2 - 4}{- 2} = \lim_{\to 2^-} \frac{(+2)(-2)}{- 2} = \lim_{\to 2^-} (+ 2) = 2 + 2 = 4$

$\lim_{\to 2^+} () = \lim_{\to 2^+} (2 - + 3) = 4 - 2 + 3$

We must have $4 - 2 + 3 = 4$, or $4 - 2 = 1$ **(1)**.

At $= 3$: $\lim_{\to 3^-} () = \lim_{\to 3^-} (2 - + 3) = 9 - 3 + 3$

$\lim_{\to 3^+} () = \lim_{\to 3^+} (2 - +) = 6 - +$

We must have $9 - 3 + 3 = 6 - +$, or $10 - 4 = 3$ **(2)**.

Now solve the system of equations by adding -2 times equation **(1)** to equation **(2)**.

$$\begin{aligned} -8 + 4 &= -2 \\ 10 - 4 &= 3 \\ \hline 2 &= 1 \end{aligned}$$

So $= \frac{1}{2}$. Substituting $\frac{1}{2}$ for in **(1)** gives us $-2 = -1$, so $= \frac{1}{2}$ as well. Thus, for to be continuous on $(-\infty, \infty)$, $= = \frac{1}{2}$.

47. If and are continuous and $(2) = 6$, then $\lim_{\to 2} [3 () + () ()] = 36 \Rightarrow$

$3 \lim_{\to 2} () + \lim_{\to 2} () \cdot \lim_{\to 2} () = 36 \Rightarrow 3 (2) + (2) \cdot 6 = 36 \Rightarrow 9 (2) = 36 \Rightarrow (2) = 4$.

48. (a) $() = \frac{1}{}$ and $() = \frac{1}{2}$, so $(\circ)() = (()) = (1/2) = 1/(1/2) = 2$.

The domain of $\circ$ is the set of numbers in the domain of (all nonzero reals) such that $()$ is in the domain of (also all nonzero reals). Thus, the domain is $\{ \mid 6= 0 \text{ and } \frac{1}{2} 6= 0\} = \{ \mid 6= 0\}$ or $(-\infty, 0) \cup (0, \infty)$. Since $\circ$ is the composite of two rational functions, it is continuous throughout its domain; that is, everywhere except $= 0$.

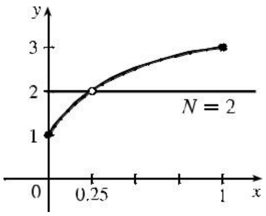
49. (a) $() = \frac{4 - 1}{- 1} = \frac{(2 + 1)(2 - 1)}{- 1} = \frac{(2 + 1)(+ 1)(- 1)}{- 1} = (2 + 1)(+ 1)$ [or $3 + 2 + + 1$]

for $6= 1$. The discontinuity is removable and $() = 3 + 2 + + 1$ agrees with for $6= 1$ and is continuous on $\mathbb{R}$.

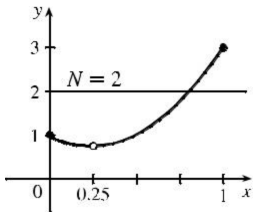
(b) $() = \frac{3 - 2 - 2}{- 2} = \frac{(2 - - 2)}{- 2} = \frac{(- 2)(+ 1)}{- 2} = (+ 1)$ [or $2 +$] for $6= 2$. The discontinuity is removable and $() = 2 +$ agrees with for $6= 2$ and is continuous on $\mathbb{R}$.

(c) $\lim_{\to^-} () = \lim_{\to^-} [[\sin]] = \lim_{\to^-} 0 = 0$ and $\lim_{\to^+} () = \lim_{\to^+} [[\sin]] = \lim_{\to^+} (-1) = -1$, so $\lim_{\to} ()$ does not exist. The discontinuity at = is a jump discontinuity.

50.



does not satisfy the conclusion of the Intermediate Value Theorem.



does satisfy the conclusion of the Intermediate Value Theorem.

( ) = ^{2} + 10 sin is continuous on the interval [31 32], (31) ≈ 957, and (32) ≈ 1030. Since 957 1000 1030, there is a number c in (31 32) such that ( ) = 1000 by the Intermediate Value Theorem. *Note:* There is also a number c in (−32 −31) such that ( ) = 1000

Suppose that (3) 6. By the Intermediate Value Theorem applied to the continuous function on the closed interval [2 3], the fact that (2) = 8 6 and (3) 6 implies that there is a number in (2 3) such that ( ) = 6. This contradicts the fact that the only solutions of the equation ( ) = 6 are = 1 and = 4. Hence, our supposition that (3) 6 was incorrect. It follows that (3) ≥ 6. But (3) 6= 6because the only solutions of ( ) = 6 are = 1 and = 4. Therefore, (3) 6.

( ) = ^{4} + − 3 is continuous on the interval [1 2] (1) = −1, and (2) = 15. Since −1 0 15, there is a number in (1 2) such that ( ) = 0 by the Intermediate Value Theorem. Thus, there is a root of the equation ^{4} + − 3 = 0 in the interval (1 2)

The equation ln = − √ is equivalent to the equation ln − + √ = 0. ( ) = ln − + √ is continuous on the interval [2 3], (2) = ln 2 − 2 + √2 ≈ 0 107, and (3) = ln 3 − 3 + √3 ≈ −0 169. Since (2) 0 (3), there is a number in (2 3) such that ( ) = 0 by the Intermediate Value Theorem. Thus, there is a root of the equation ln − + √ = 0, or ln = − √ , in the interval (2 3).

The equation = 3 − 2 is equivalent to the equation + 2 − 3 = 0. ( ) = + 2 − 3 is continuous on the interval [0 1], (0) = −2, and (1) = − 1 ≈ 1 72. Since −2 0 − 1, there is a number in (0 1) such that ( ) = 0 by the Intermediate Value Theorem. Thus, there is a root of the equation + 2 − 3 = 0, or = 3 − 2 , in the interval (0 1).

The equation sin = ^{2} − is equivalent to the equation sin − ^{2} + = 0. ( ) = sin − ^{2} + is continuous on the interval [1 2] (1) = sin 1 ≈ 0 84, and (2) = sin 2 − 2 ≈ −1 09. Since sin 1 0 sin 2 − 2, there is a number in (1 2) such that ( ) = 0 by the Intermediate Value Theorem. Thus, there is a root of the equation sin − ^{2} + = 0, or sin = ^{2} − , in the interval (1 2).

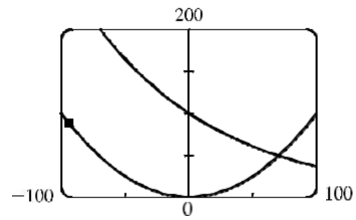
(a) ( ) = cos − ^{3} is continuous on the interval [0 1], (0) = 1 0, and (1) = cos 1 − 1 ≈ −0 46 0. Since 1 0 −0 46, there is a number in (0 1) such that ( ) = 0 by the Intermediate Value Theorem. Thus, there is a root of the equation cos − ^{3} = 0, or cos = ^{3}, in the interval (0 1).

(0 86) ≈ 0 016 0 and (0 87) ≈ −0 014 0, so there is a root between 0 86 and 0 87, that is, in the interval (0 86 0 87).

(a) ( ) = ln − 3 + 2 is continuous on the interval [1 2], (1) = −1 0, and (2) = ln 2 + 1 ≈ 1 7 0. Since −1 0 1 7, there is a number in (1 2) such that ( ) = 0 by the Intermediate Value Theorem. Thus, there is a root of the equation ln − 3 + 2 = 0, or ln = 3 − 2 , in the interval (1 2).

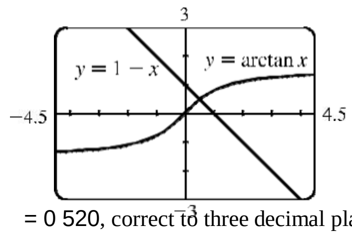
(1 34) ≈ −0 03 0 and (1 35) ≈ 0 0001 0, so there is a root between 1 34 and 1 35 that is, in the interval (1 34 1 35).

(a) Let ( ) = 100 − 100 − 0 01 2 Then (0) = 100 0 and (100) = 100 −1 − 100 ≈ −63 2 0. So by the Intermediate Value Theorem, there is a number in (0 100) such that ( ) = 0. This implies that 100 − 100 = 0 01 2.



(b) Using the intersect feature of the graphing device, we find that the root of the equation is = 70 347, correct to three decimal places.

(a) Let ( ) = arctan + − 1. Then (0) = −1 0 and (1) = 4 0. So by the Intermediate Value Theorem, there is a number in (0 1) such that ( ) = 0. This implies that arctan = 1 − .



(b) Using the intersect feature of the graphing device, we find that the root of the equation is = 0 520, correct to three decimal places.

Let ( ) = sin 3. Then is continuous on [1 2] since is the composite of the sine function and the cubing function, both of which are continuous on R. The zeros of the sine are at , so we note that 0 1 32 2 8 3 , and that the pertinent cube roots are related by 1 3 3 [call this value] 2. [By observation, we might notice that = √3 and = √3 2 are zeros of .]

Now (1) = sin 1 0, ( ) = sin 32 = −1 0, and (2) = sin 8 0. Applying the Intermediate Value Theorem on [1] and then on [2], we see there are numbers and in (1) and (2) such that ( ) = ( ) = 0. Thus, has at least two -intercepts in (1 2).

Let ( ) = 2 − 3 + 1 . Then is continuous on (0 2] since is a rational function whose domain is (0 ∞). By inspection, we see that 1 4 = 17 16 0 (1)= − 1 0 , and (2) = 3 2 0. Appling the Intermediate Value Theorem on 1 4 1 and then on [1 2], we see there are numbers and in 1 4 1 and (1 2) such that ( ) = ( ) = 0. Thus, has at least two -intercepts in (0 2).

(⇒) If is continuous at , then by Theorem 8 with ( ) = + , we have

$\lim_{\to 0} (+) = \lim_{\to 0} (+) = ()$.

($\Leftarrow$) Let 0. Since $\lim_{\to}^{0} (+) = ()$, there exists 0 such that $0 \; | \; |$ $\Rightarrow$

$|(+) - ()|$. So if $0 \; | - |$, then $|() - ()| = |(+ (-)) - ()|$.

Thus, $\lim_{\to} () = ()$ and so is continuous at .

$\lim_{\to 0} \sin(+) = \lim_{\to 0} (\sin \cos + \cos \sin) = \lim_{\to 0} (\sin \cos) + \lim_{\to 0} (\cos \sin)$

$\to 0 \quad \sin\to 0 \quad \to 0 \quad \to 0$

$= \lim \quad \lim \cos + \lim \cos \quad \lim \sin = (\sin)(1) + (\cos)(0) = \sin$

As in the previous exercise, we must show that $\lim_{\to 0} \cos(+) = \cos$ to prove that the cosine function is continuous.

$\lim_{0} \cos(+) = \lim_{0} (\cos \cos \sin \sin) = \lim (\cos \cos)_{0} - \lim_{0} (\sin \sin)$

$\to 0 \quad \cos\to 0 \quad - \quad \to 0 \quad \to 0 \quad -$

$= \lim \quad \lim \cos \quad \lim \sin \quad \lim \sin = (\cos)(1) \quad (\sin)(0) = \cos$

(a) Since is continuous at , $\lim_{\to} () = ()$. Thus, using the Constant Multiple Law of Limits, we have

$\lim_{\to} ( )() = \lim_{\to} () = \lim_{\to} () = () = ( )()$. Therefore, is continuous at .

(b) Since and are continuous at , $\lim_{\to} () = ()$ and $\lim_{\to} () = ()$. Since $() \; 6 = 0$, we can use the Quotient Law

of Limits: $\lim_{\to} \_ () = \lim_{\to} \frac{()}{()} = \frac{\lim_{\to} ()}{\lim_{\to} ()} = \frac{()}{()} = \_ ()$. Thus, $\_$ is continuous at .

67. $() = \begin{cases} 0 & \text{if is rational} \\ 1 & \text{if is irrational} \end{cases}$ is continuous nowhere. For, given any number and any 0, the interval $(- +)$

contains both infinitely many rational and infinitely many irrational numbers. Since $() = 0$ or 1, there are infinitely many numbers with $0 \; | - |$ and $|() - ()| = 1$. Thus, $\lim_{\to} () \; 6 = ()$. [In fact, $\lim_{\to} ()$ does not even exist.]

68. $() = \begin{cases} 0 & \text{if is rational} \\ & \text{if is irrational} \end{cases}$ is continuous at 0. To see why, note that $-| | \leq () \leq | |$, so by the Squeeze Theorem

$\lim_{\to 0} () = 0 = (0)$. But is continuous nowhere else. For if $6 = 0$ and 0, the interval $(- +)$ contains both infinitely many rational and infinitely many irrational numbers. Since $() = 0$ or , there are infinitely many numbers with $| - |$ and $|() - ()| \; | | 2$. Thus, $\lim_{\to} () \; 6 = ()$.

69. If there is such a number, it satisfies the equation $^3 + 1 = \Leftrightarrow {}^3 - + 1 = 0$. Let the left-hand side of this equation be called $()$. Now $(-2) = -5 \; 0$, and $(-1) = 1 \; 0$. Note also that $()$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number between -2 and -1 such that $() = 0$, so that $= {}^3 + 1$.

70. $\frac{}{^3 + 2\,^2 - 1} + \frac{}{^3 + - 2} = 0 \Rightarrow ({}^3 + - 2) + ({}^3 + 2\,^2 - 1) = 0$. Let $()$ denote the left side of the last equation. Since is continuous on $[-1\ 1]$, $(-1) = -40$, and $(1) = 2 \; 0$, there exists a in $(-1\ 1)$ such that

( ) = 0 by the Intermediate Value Theorem. Note that the only root of either denominator that is in (−1 1) is $(-1+\sqrt{5})\,2=$, but ( ) = (3 $\sqrt{5}-9)$ 2 6= .0Thus, is not a root of either denominator, so ( ) = 0 ⇒ = is a root of the given equation.

( ) = $^4 \sin(1\ )$ is continuous on (−∞ 0) ∪ (0 ∞) since it is the product of a polynomial and a composite of a trigonometric function and a rational function. Now since −1 ≤ sin(1) ≤ 1, we have $-\ ^4 \le\ ^4 \sin(1\ ) \le\ ^4$. Because $\lim_{\to 0}(-\ ^4) = 0$ and $\lim_{\to 0}\ ^4 = 0$, the Squeeze Theorem gives us $\lim_{\to 0}(\ ^4 \sin(1\ )) = 0$, which equals (0). Thus, is continuous at 0 and, hence, on (−∞ ∞).

72. (a) $\lim_{\to 0^+}$ ( ) = 0 and $\lim_{\to 0^-}$ ( ) = 0, so $\lim_{\to 0}$ ( ) = 0, which is (0), and hence is continuous at = if = 0. For 0, $\lim_{\to}$ ( ) = $\lim_{\to}$ = = ( ). For0, $\lim_{\to}$ ( ) = $\lim_{\to}$ (−) = − = ( ). Thus, is continuous at = ; that is, continuous everywhere.

(b) Assume that is continuous on the interval . Then for ∈ , $\lim_{\to}$|()|= $\lim_{\to}$ ()= | () | by Theorem 8. (If is an endpoint of , use the appropriate one-sided limit.) So | | is continuous on .

(c) No, the converse is false. For example, the function ( ) = $\begin{cases} 1 & \text{if } \ge 0 \\ -1 & \text{if } < 0 \end{cases}$ is not continuous at = 0, but | ( )| = 1 is continuous on R.

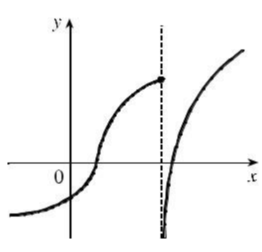
Define ( ) to be the monk's distance from the monastery, as a function of time (in hours), on the first day, and define ( ) to be his distance from the monastery, as a function of time, on the second day. Let be the distance from the monastery to the top of the mountain. From the given information we know that (0) = 0, (12) = , (0) = and (12) = 0. Now consider the function − , which is clearly continuous. We calculate that (−)(0) = − and (−)(12) = . So by the Intermediate Value Theorem, there must be some time $_0$ between 0 and 12 such that (−)($_0$) = 0 ⇔ ($_0$) = ($_0$). So at time $_0$ after 7:00 AM, the monk will be at the same place on both days.

2.6 Limits at Infinity; Horizontal Asymptotes

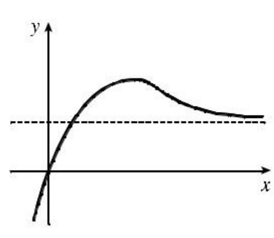
(a) As becomes large, the values of ( ) approach 5.

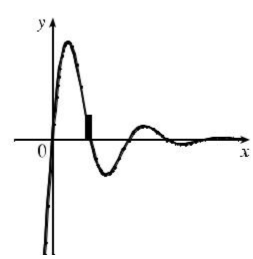
As becomes large negative, the values of ( ) approach 3.

2. (a) The graph of a function can intersect a vertical asymptote in the sense that it can meet but not cross it.

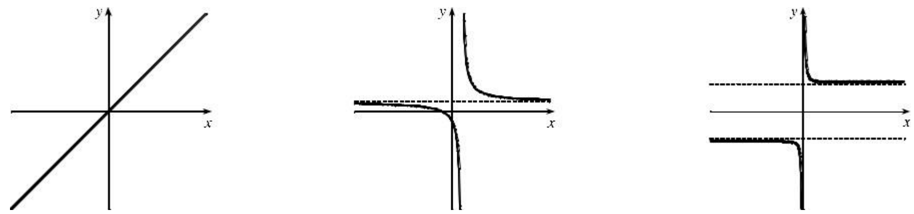


The graph of a function can intersect a horizontal asymptote. It can even intersect its horizontal asymptote an infinite number of times.





(b) The graph of a function can have 0, 1, or 2 horizontal asymptotes. Representative examples are shown.



No horizontal asymptote One horizontal asymptote Two horizontal asymptotes

3. (a) lim ( ) = −2 →∞ (b) lim ( ) = 2 →−∞ (c) lim 1 ( ) = ∞ →

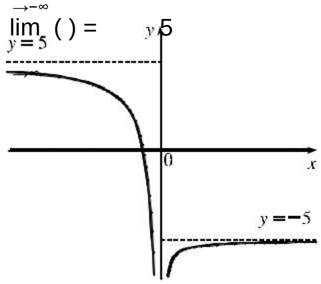
(d) 3 −∞ (e) Vertical: , ; horizontal: = − , = 2

4. (a) lim ( ) = 2 →∞ (b) lim ( ) = −1 →−∞ (c) lim 0 ( ) = −∞ →

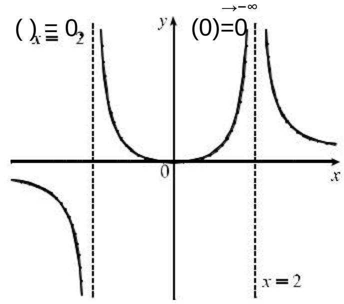
(d) →2− lim ( ) = −∞ (e) →2+ lim ( ) = ∞ (f) Vertical: , ; = 0 = 2

horizontal: = −1, = 2

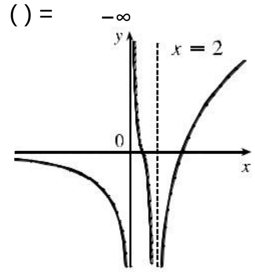
5. →0 lim ( ) = −∞, lim ( ) = 5, →−∞ lim ( ) = −5 →∞



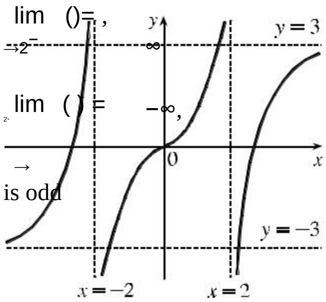
6. →2 lim ( ) = ∞, →−2+ lim ( ) = ∞, lim ( ) = −∞, →−2− lim ( ) = 0, lim ( ) = 0 →−∞ →∞ (0) = 0



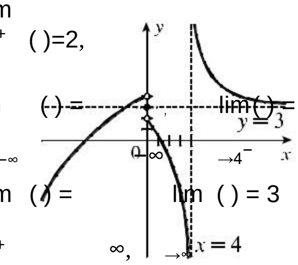
7. →2 lim ( ) = −∞, lim ( ) = ∞, →∞ lim ( ) = 0, lim ( ) = , →−∞ lim ( ) = −∞ → −



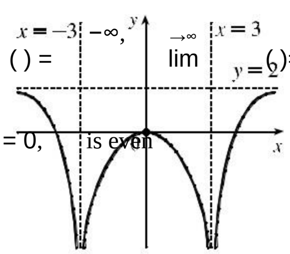
8. →∞ lim ( ) = 3, lim ( ) = ∞, →2− lim ( ) = −∞, →2+ is odd



9. (0) = 3, →0− lim ( ) = 4, lim →0+ ( ) = 2, lim ( ) = −∞, →−∞ lim ( ) = −∞, →4− lim ( ) = ∞, →4+ lim ( ) = 3 →∞



10. →3 lim ( ) = −∞, →∞ lim ( ) = 2, (0) = 0, is even



If ( ) = ^{2} 2 , then a calculator gives (0) = 0, (1) = 0 5, (2) = 1, (3) = 1 125, (4) = 1, (5) = 0 78125, (6) = 0 5625, (7) = 0 3828125, (8) = 0 25, (9) = 0 158203125, (10) = 0 09765625, (20) ≈ 0 00038147, (50) ≈ 2 2204 × 10^{-12}, (100) ≈ 7 8886 × 10^{-27}. It appears that $\lim_{\to\infty}$ ^{2} 2 = 0.

(a) From a graph of ( ) = (1 − 2) in a window of [0 10,000] by [0 0 2], we estimate that $\lim_{\to\infty}$ ( ) = 0 14 (to two decimal places.)

(b) From the table, we estimate that $\lim_{\to\infty}$ ( ) = 0 1353 (to four decimal places.)

	( )
10,000	0 135 308
100,000	0 135 333
1,000,000	0 135 335

13. $\lim_{\to\infty} \dfrac{2^2 - 7}{5^2 + -3} = \lim_{\to\infty} \dfrac{(2^2 - 7)/^2}{(5^2 + -3)/^2}$ [Divide both the numerator and denominator by ^{2} (the highest power of that appears in the denominator)]

$= \dfrac{\lim_{\to\infty} (2 - 7/^2)}{\lim_{\to\infty}}$ [Limit Law 5]

$= \dfrac{\lim_{\to\infty} 2 - \lim_{\to\infty} (7/^2)}{\lim_{\to\infty} \quad \lim_{\to\infty} \quad \lim_{\to\infty}}$ [Limit Laws 1 and 2]

$= \dfrac{2 - 7 \lim_{\to\infty} (1/^2)}{\lim_{\to\infty} \quad \lim_{\to\infty}}$ [Limit Laws 7 and 3]

$= \dfrac{2 - 7(0)}{5 + 0 + 3(0)}$ [Theorem 2.6.5]

$= \dfrac{2}{5}$.

14. $\lim_{\to\infty} \sqrt{\dfrac{9^3 + 8 - 4}{3 - 5 + ^3}} = \sqrt{\lim_{\to\infty} \dfrac{9^3 + 8 - 4}{3 - 5 + ^3}}$ [Limit Law 11]

$= \sqrt{\lim_{\to\infty} \dfrac{9 + 8/^2 - 4/^3}{3/^3 - 5/^2 + 1}}$ [Divide by ^{3}]

$= \sqrt{\dfrac{\lim_{\to\infty} (9 + 8/^2 - 4/^3)}{\lim_{\to\infty} (3/^3 - 5/^2 + 1)}}$ [Limit Law 5]

$= \sqrt{\dfrac{\lim_{\to\infty} 9 + \lim_{\to\infty} (8/^2) - \lim_{\to\infty} (4/^3)}{\lim_{\to\infty} (3/^3) - \lim_{\to\infty} (5/^2) + \lim_{\to\infty} 1}}$ [Limit Laws 1 and 2]

$= \sqrt{\dfrac{9 + 8 \lim_{\to\infty} (1/^2) - 4 \lim_{\to\infty} (1/^3)}{3 \lim_{\to\infty} (1/^3) - 5 \lim_{\to\infty} (1/^2) + 1}}$ [Limit Laws 7 and 3]

$= \sqrt{\dfrac{9 + 8(0) - 4(0)}{3(0) - 5(0) + 1}}$ [Theorem 2.6.5]

$= \sqrt{\dfrac{9}{1}} = \sqrt{9} = 3$

15. $\lim\limits_{\to\infty} \frac{3 - 2}{2 + 1} = \lim\limits_{\to\infty} \frac{(3 - 2)}{(2 + 1)} = \lim\limits_{\to\infty} \frac{3 - 2}{2 + 1} = \frac{\lim\limits_{\to\infty} 3 - 2 \lim\limits_{\to\infty} 1}{\lim 2 + \lim 1} = \frac{3 - 2(0)}{2 + 0} = \frac{3}{2}$

16. $\lim\limits_{\to\infty} \frac{1 - 2}{3 - +1} = \lim\limits_{\to\infty} \frac{(1 - 2)\,3}{(3 - +1)\,3} = \lim\limits_{\to\infty} \frac{1\,3 - 1}{1 - 1\,2 + 1\,3}$

$= \frac{\lim\limits_{\to\infty} 1\,3 - \lim\limits_{\to\infty} 1}{\lim\limits_{\to\infty} 1 - \lim\limits_{\to\infty} 1\,2 + \lim\limits_{\to\infty} 1\,3} = \frac{0 - 0}{1 - 0 + 0} = 0$

17. $\lim\limits_{\to-\infty} \frac{2 - 2}{2 + 1} = \lim\limits_{\to-\infty} \frac{(2 - 2)\,2}{(2 + 1)\,2} = \lim\limits_{\to-\infty} \frac{1 - 2\,2}{1 + 1\,2} = \frac{\lim\limits_{\to-\infty} 1 - 2 \lim\limits_{\to-\infty} 1\,2}{\lim\limits_{\to-\infty} 1 + \lim 1\,2} = \frac{0 - 2(0)}{1 + 0} = 0$

18. $\lim\limits_{\to-\infty} \frac{4\,3 + 6\,2 - 2}{2\,3 - 4 + 5} = \lim\limits_{\to-\infty} \frac{(4\,3 + 6\,2 - 2)\,3}{(2\,3 - 4 + 5)\,3} = \lim\limits_{\to-\infty} \frac{4 + 6 - 2\,3}{2 - 4\,2 + 5\,3} = \frac{4 + 0 - 0}{2 - 0 + 0} = 2$

19. $\lim\limits_{\to\infty} \frac{\sqrt{\;} + 2}{2 - 2} = \lim\limits_{\to\infty} \frac{(\sqrt{\;} + 2)\,2}{(2 - 2)\,2} = \lim\limits_{\to\infty} \frac{1\,32 + 1}{2 - 1} = \frac{0 + 1}{0 - 1} = -1$

20. $\lim\limits_{\to\infty} \frac{\sqrt{\;} - 3\,2}{2 + 3 - 5} = \lim\limits_{\to\infty} \frac{(\sqrt{\;} - 3\,2)\,32}{(2\,3\,2 + 3 - 5)\,32} = \lim\limits_{\to\infty} \frac{1\,12 - 1}{2 + 3\,12 - 5\,32} = \frac{0 - 1}{2 + 0 - 0} = -\frac{1}{2}$

21. $\lim\limits_{\to\infty} \frac{(2\,2 + 1)\,2}{(-1)\,2(\,2 +)} = \lim\limits_{\to\infty} \frac{(2\,2 + 1)\,2\,4}{[(-1)\,2(\,2 +)]\,4} = \lim\limits_{\to\infty} \frac{[(2\,2 + 1)\,2]\,2}{[(-2 + 1)\,2][(\,2 +)\,2]}$

$= \lim\limits_{\to\infty} \frac{(2 + 1\,2)\,2}{(1 - 2 + 1\,2)(1 + 1)} = \frac{(2 + 0)\,2}{(1 - 0 + 0)(1 + 0)} = 4$

22. $\lim\limits_{\to\infty} \frac{2}{\sqrt{4} + 1} = \lim\limits_{\to\infty} \frac{2\,2}{\sqrt{4 + 1}\,2} = \lim\limits_{\to\infty} \frac{1}{\sqrt{(4 + 1)\,4}}$ [since $2 = \sqrt{4}$ for > 0]

$= \lim\limits_{\to\infty} \frac{1}{\sqrt{1 + 1\,4}} = \frac{1}{\sqrt{1 + 0}} = 1$

23. $\lim\limits_{\to\infty} \frac{\sqrt{1 + 4\,6}}{2 - 3} = \lim\limits_{\to\infty} \frac{\sqrt{1 + 4\,6}\,3}{(2 - 3)\,3} = \frac{\lim\limits_{\to\infty} \sqrt{(1 + 4\,6)\,6}}{\lim\limits_{\to\infty} (2\,3 - 1)}$ [since $3 = \sqrt{6}$ for > 0]

$= \frac{\lim\limits_{\to\infty} \sqrt{1\,6 + 4}}{\lim\limits_{\to\infty} (2\,3) - \lim\limits_{\to\infty} 1} = \frac{\sqrt{\lim\limits_{\to\infty} (1\,6) + \lim\limits_{\to\infty} 4}}{\lim\limits_{\to\infty} (2\,3) - \lim\limits_{\to\infty} 1}$

$= \frac{\sqrt{0 + 4}}{-1} = \frac{2}{-1} = -2$

24. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{1+4x^6}}{2-x^3} = \lim\limits_{x\to-\infty} \dfrac{\dfrac{\sqrt{1+4x^6}}{x^3}}{\dfrac{2-x^3}{x^3}} = \dfrac{\lim\limits_{x\to-\infty} -\sqrt{\dfrac{1+4x^6}{x^6}}}{\lim\limits_{x\to-\infty} \left(\dfrac{2}{x^3} - 1\right)}$ [since $x^3 = -\sqrt{x^6}$ for $x<0$]

$$= \frac{\lim\limits_{x\to-\infty} -\sqrt{\dfrac{1}{x^6}+4}}{\lim\limits_{x\to-\infty} \left(\dfrac{2}{x^3}-1\right)} = \frac{-\sqrt{\lim\limits_{x\to-\infty} \left(\dfrac{1}{x^6}\right) + \lim\limits_{x\to-\infty} 4}}{-1}$$

$$= \frac{-\sqrt{0+4}}{-1} = \frac{-2}{-1} = 2$$

NOT FOR SALE

25. lim $\sqrt{\;\;^2 + 3}$ = lim $\sqrt{\;\;^2}$ — lim $(\;+3\;^2)^2$ [since $\sqrt{2}$ = for 0]

$\to\infty$ 4 1 $\to\infty$ (4 1) lim (4 1)

$$= \frac{\lim_{\to\infty} \sqrt{1 + 3}}{\lim_{\to\infty} 4 \;\; \lim_{\to\infty}(1\;)} = \frac{\sqrt{\lim_{\to\infty}(1\;) + \lim_{\to\infty} 3}}{4 - 0} = \frac{\sqrt{0+3}}{4} = \frac{\sqrt{3}}{4}$$

26. $\lim_{\to\infty} \frac{+3\;^2}{4\;-1} = \lim_{\to\infty} \frac{(+3\;^2)}{(4\;-1)} = \lim_{\to\infty} \frac{1+3}{4-1}$ $= \infty$ since $1 + 3 \to \infty$ and $4 - 1 \to 4$ as $\to \infty$.

27. $\lim_{\to\infty} \sqrt{9\;^2 + \;} - 3 = \lim_{\to\infty} \frac{\sqrt{9\;^2+\;} - 3}{1} \cdot \frac{\sqrt{9\;^2+\;} + 3}{\sqrt{9\;^2+\;}+3} = \lim_{\to\infty} \frac{(\sqrt{9\;^2+\;})^2 - (3\;)^2}{\sqrt{9\;^2+\;} + 3}$

$$= \lim_{\to\infty} \frac{9\;^2 + \; - 9\;^2}{\sqrt{9\;^2+\;}+3} = \lim_{\to\infty} \frac{\;}{\sqrt{9\;^2+\;}+3} \cdot \frac{1/}{1/}$$

$$= \lim_{\to\infty} \frac{1}{\sqrt{9\;^2/\;^2 + /\;^2} + 3/} = \lim_{\to\infty} \frac{1}{\sqrt{9 + 1/} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{3+3} = \frac{1}{6}$$

28. $\lim_{\to-\infty} \sqrt{4\;^2 + 3\; + 2} = \lim_{\to-\infty} \sqrt{4\;^2+3\;} + 2 \cdot \frac{\sqrt{4\;^2+3\;} - 2}{\sqrt{4\;^2+3\;} - 2}$

$$= \lim_{\to-\infty} \frac{4\;^2 + 3\; - (2\;)^2}{\sqrt{4\;^2+3\;} - 2} = \lim_{\to-\infty} \frac{3}{\sqrt{4\;^2+3\;} - 2}$$

$$= \lim_{\to-\infty} \frac{3}{\sqrt{4\;^2+3\;}-2} = \lim_{\to-\infty} \frac{3}{-\sqrt{4 + 3/} - 2} \quad [\text{since } \sqrt{\;^2} = - \text{ for } < 0]$$

$$= \frac{3}{-\sqrt{4+0} - 2} = -\frac{3}{4}$$

29. $\lim_{\to\infty} \left(\sqrt{\;^2 + \;} - \sqrt{\;^2 + \;}\right) = \lim_{\to\infty} \frac{\sqrt{\;^2+\;} - \sqrt{\;^2+\;}}{1} \cdot \frac{\sqrt{\;^2+\;} + \sqrt{\;^2+\;}}{\sqrt{\;^2+\;}+\sqrt{\;^2+\;}}$

$$= \lim_{\to\infty} \frac{(\;^2+\;) - (\;^2+\;)}{\sqrt{\;^2+\;} + \sqrt{\;^2+\;}} = \lim_{\to\infty} \frac{[(\;-\;)]/}{\left(\sqrt{\;^2+\;} + \sqrt{\;^2+\;}\right)/\sqrt{\;^2}}$$

$$= \lim_{\to\infty} \frac{-}{\sqrt{1+/} + \sqrt{1+/}} = \frac{-}{\sqrt{1+0} + \sqrt{1+0}} = \frac{-}{2}$$

For > 0, $\sqrt{\;^2+1} > \sqrt{\;^2} = $. So as $\to \infty$, we have $\sqrt{\;^2+1} \to \infty$, that is, $\lim_{\to\infty} \sqrt{\;^2+1} = \infty$.

31. $\lim_{\to\infty} \frac{\;^4 - 3\;^2 + }{\;^3 - +2} = \lim_{\to\infty} \frac{(\;^4 - 3\;^2 + \;)/\;^3}{(\;^3 - +2)/\;^3}$ $\left[\begin{array}{l}\text{divide by the highest power}\\ \text{of in the denominator}\end{array}\right]$ $= \lim_{\to\infty} \frac{-3/ + 1/\;^2}{1 - 1/\;^2 + 2/\;^3} = \infty$

since the numerator increases without bound and the denominator approaches 1 as $\to \infty$.

32. $\lim\limits_{\to\infty} (^{-} + 2\cos 3\,)$ does not exist. $\lim\limits_{\to\infty} {}^{-} = 0$, but $\lim\limits_{\to\infty} (2\cos 3\,)$ does not exist because the values of $2\cos 3$ oscillate between the values of -2 and 2 infinitely often, so the given limit does not exist.

33. $\lim\limits_{\to -\infty} (\ ^2 + 2\ ^7) = \lim\limits_{\to -\infty}\ ^7 \left(\frac{1}{\ ^5} + 2\right)$ [factor out the largest power of] $= -\infty$ because $^7 \to -\infty$ and $1\ ^5 + 2 \to 2$ as $\to -\infty$.

Or: $\lim\limits_{\to -\infty} \ ^2 + 2\ ^7 = \lim\limits_{\to -\infty}\ ^2\ 1 + 2\ ^5 = -\infty$.

34. $\lim\limits_{\to -\infty} \frac{1 + \ ^6}{\ ^4 + 1} = \lim\limits_{\to -\infty} \frac{(1 + \ ^6)\ \ ^4}{(\ ^4 + 1)\ \ ^4}$ divide by the highest power of in the denominator $= \lim\limits_{\to -\infty} \frac{1\ ^4 + \ ^2}{1 + 1\ ^4} = \infty$

since the numerator increases without bound and the denominator approaches 1 as $\to -\infty$.

35. Let $=$. As $\to \infty$, $\to \infty$. $\lim\limits_{\to\infty} \arctan(\ ) = \lim\limits_{\to\infty} \arctan\ = \frac{}{2}$ by (3).

36. Divide numerator and denominator by ^{3}: $\lim\limits_{\to\infty} \frac{\ ^3 - \ ^{-3}}{\ ^3 + \ ^{-3}} = \lim\limits_{\to\infty} \frac{1 - \ ^{-6}}{1 + \ ^{-6}} = \frac{1 - 0}{1 + 0} = 1$

37. $\lim\limits_{\to\infty} \frac{1 - }{} = \lim\limits_{\to\infty} \frac{(1 - \ )}{} = \lim\limits_{\to\infty} \frac{1}{} - 1 = 0 - 1 = \frac{1}{}$

38. Since $0 \le \sin^2 \le 1$, we have $0 \le \frac{\sin^2}{\ ^2 + 1} \le \frac{1}{\ ^2 + 1}$. We know that $\lim\limits_{\to\infty} 0 = 0$ and $\lim\limits_{\to\infty} \frac{1}{\ ^2 + 1} = 0$, so by the Squeeze Theorem, $\lim\limits_{\to\infty} \frac{\sin^2}{\ ^2 + 1} = 0$.

Since $-1 \le \cos \le 1$ and $^{-2} 0$, we have $-\ ^{-2} \le \ ^{-2} \cos \le \ ^{-2}$. We know that $\lim\limits_{\to\infty} (-\ ^{-2}) = 0$ and $\lim\limits_{\to\infty}\ ^{-2} = 0$, so by the Squeeze Theorem, $\lim\limits_{\to\infty} (\ ^{-2} \cos\ ) = 0$.

40. Let $= \ln$. As $\to 0^+$, $\to -\infty$. $\lim\limits_{\to 0^+} \tan^{-1}(\ln\ ) = \lim\limits_{\to -\infty} \tan^{-1} = -\frac{}{2}$ by (4).

41. $\lim\limits_{\to\infty} [\ln(1 + \ ^2) - \ln(1 + \ )] = \lim\limits_{\to\infty} \ln \frac{1 + \ ^2}{1 + } = \ln \lim\limits_{\to\infty} \frac{1 + \ ^2}{1 + } = \ln \lim\limits_{\to\infty} \frac{\frac{1}{} + }{\frac{1}{} + 1} = \infty$, since the limit in parentheses is ∞.

42. $\lim\limits_{\to\infty} [\ln(2 + \ ) - \ln(1 + \ )] = \lim\limits_{\to\infty} \ln \frac{2 + }{1 + } = \lim\limits_{\to\infty} \ln \frac{\frac{2}{} + 1}{\frac{1}{} + 1} = \ln \frac{1}{1} = \ln 1 = 0$

43. (a) (i) $\lim\limits_{\to 0^+} (\ ) = \lim\limits_{\to 0^+} \frac{}{\ln} = 0$ since $\to 0^+$ and $\ln \to -\infty$ as $\to 0^+$.

(ii) $\lim\limits_{\to 1^-} (\ ) = \lim\limits_{\to 1^-} \frac{}{\ln} = -\infty$ since $\to 1$ and $\ln \to 0^-$ as $\to 1^-$.

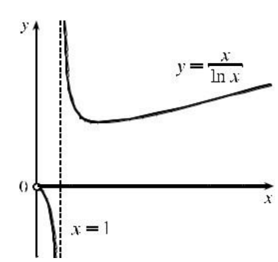
(iii) $\lim\limits_{\to 1^+} (\ ) = \lim\limits_{\to 1^+} \frac{}{\ln} = \infty$ since $\to 1$ and $\ln \to 0^+$ as $\to 1^+$.

(b)

	$(\ )$
10,000	1085 7
100,000	8685 9
1,000,000	72,382 4

It appears that $\lim\limits_{\to\infty} (\ ) = \infty$.

(c)



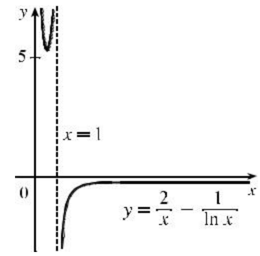
44. (a) $\lim\limits_{\to\infty} (\ ) = \lim\limits_{\to\infty} \left(\frac{2}{} - \frac{1}{\ln} \right) = 0$

since $\frac{2}{} \to 0$ and $\frac{1}{\ln} \to 0$ as $\to \infty$.

(b) $\lim\limits_{\to 0^+} (\ ) = \lim\limits_{\to 0^+} \left(\frac{2}{} - \frac{1}{\ln} \right) = \infty$

since $\frac{2}{} \to \infty$ and $\frac{1}{\ln} \to 0$ as $\to 0^+$.

(e)



(c) $\lim\limits_{\to 1^-} ( ) = \lim\limits_{\to 1^-} \frac{2}{} \frac{1}{} = \infty$ since $\frac{2}{} \to 2$ and $\frac{1}{} \to -\infty$ as $\to 1^-$.

(d) $\lim\limits_{\to 1^+} ( ) = \lim\limits_{\to 1^+} \frac{2}{} \frac{1}{} = -\infty$ since $\frac{2}{} \to 2$ and $\frac{1}{\ln} \to \infty$ as $\to 1^+$.

45. (a)

From the graph of $( ) = \sqrt{2 + + 1} +$, we

(b)

	$( )$
−10,000	−0 499 962 5
−100,000	−0 499 996 2
−1,000,000	−0 499 999 6

From the table, we estimate the limit to be −0 5.

estimate the value of $\lim_{\to -\infty} ()$ to be $-0\ 5$.

−

−

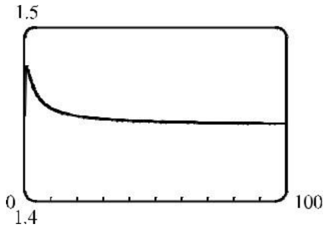
(c) $\lim\limits_{\to-\infty} \left(\sqrt{\ ^2+\ +1}+\ \right) = \lim\limits_{\to-\infty} \left(\sqrt{\ ^2+\ +1}+\ \right)\dfrac{\sqrt{\ ^2+\ +1}-\ }{\sqrt{\ ^2+\ +1}-\ } = \lim\limits_{\to-\infty} \dfrac{\left(\ ^2+\ +1\right)-\ ^2}{\sqrt{\ ^2+\ +1}-\ }$

$= \lim\limits_{\to-\infty} \dfrac{(\ +1)(1\ )}{\left(\sqrt{\ ^2+\ +1}-\ \right)(1\ )} = \lim\limits_{\to-\infty} \dfrac{1+(1\ )}{-\sqrt{1+(1\ )+(1\ ^2)}-1}$

$= \dfrac{1+0}{-\sqrt{1+0+0}-1} = \dfrac{1}{-2} = -\dfrac{1}{2}$

Note that for 0, we have $\sqrt{\ ^2} = |\ | = -\ $, so when we divide the radical by , with 0, we get $\dfrac{1}{\ }\sqrt{\ ^2+\ +1} = -\dfrac{1}{\sqrt{\ ^2}}\sqrt{\ ^2+\ +1} = -\sqrt{1+(1\ )+(1\ ^2)}$.

46. (a)



From the graph of

$(\ ) = \sqrt{3\ ^2+8\ +6} - \sqrt{3\ ^2+3\ +1}$, we estimate

(to one decimal place) the value of $\lim\limits_{\to\infty} (\ )$ to be $1\ 4$.

(b)

	( )
10,000	1 443 39
100,000	1 443 38
1,000,000	1 443 38

From the table, we estimate (to four decimal places) the limit to be $1\ 4434$.

(c) $\lim_{\to\infty} (\,) = \lim_{\to\infty} \left(\sqrt{3^2+8+6} - \sqrt{3^2+3+1}\right) \cdot \dfrac{\sqrt{3^2+8+6}+\sqrt{3^2+3+1}}{\sqrt{3^2+8+6}+\sqrt{3^2+3+1}}$

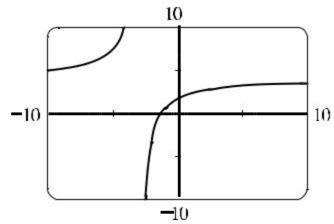
$= \lim \dfrac{(3^2+8+6)-(3^2+3+1)}{\sqrt{3^2+8+6}+\sqrt{3^2+3+1}} = \lim \dfrac{(5+5)(1/\ )}{\left(\sqrt{3^2+8+6}+\sqrt{3^2+3+1}\right)(1/\ )}$

$= \lim_{\to\infty} \dfrac{5+5/}{\sqrt{3+8/+6/^2}+\sqrt{3+3/+1/^2}} = \dfrac{5}{\sqrt{3}+\sqrt{3}} = \dfrac{5}{2\sqrt{3}} = \dfrac{5\sqrt{3}}{6} \approx 1.443376$

47. $\lim_{\to\pm\infty} \dfrac{5+4}{+3} = \lim_{\to\pm\infty} \dfrac{(5+4)/}{(+3)/} = \lim_{\to\pm\infty} \dfrac{5+4/}{1+3/} = \dfrac{0+4}{1+0} = 4$, so

$= 4$ is a horizontal asymptote. $= (\,) = \dfrac{5+4}{+3}$, so $\lim_{\to-3^+} (\,) = -\infty$

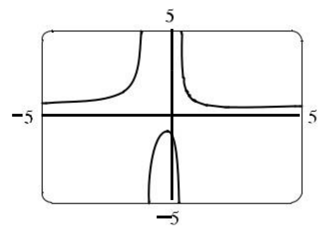
since $5+4 \to -7$ and $+3 \to 0^+$ as $\to -3^+$. Thus, $= -3$ is a vertical asymptote. The graph confirms our work.



48. $\lim_{\to\pm\infty} \dfrac{2^2+1}{3^2+2-1} = \lim_{\to\pm\infty} \dfrac{(2^2+1)/^2}{(3^2+2-1)/^2}$

$= \lim_{\to\pm\infty} \dfrac{2+1/^2}{3+2/-1/^2} = \dfrac{2}{3}$

so $= \frac{2}{3}$ is a horizontal asymptote. $= (\,) = \dfrac{2^2+1}{3^2+2-1} = \dfrac{2^2+1}{(3-1)(+1)}$.



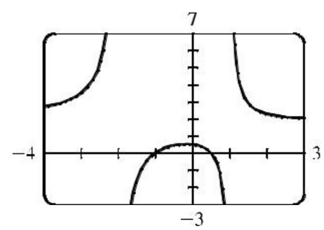
The denominator is zero when $= \frac{1}{3}$ and -1, but the numerator is nonzero, so $= \frac{1}{3}$ and $= -1$ are vertical asymptotes. The graph confirms our work.

49. $\lim_{\to\pm\infty} \dfrac{2^2+-1}{^2+-2} = \lim_{\to\pm\infty} \dfrac{\frac{2^2+-1}{^2}}{\frac{^2+-2}{^2}} = \lim_{\to\pm\infty} \dfrac{2+\frac{1}{}-\frac{1}{^2}}{1+\frac{1}{}-\frac{2}{^2}} = \dfrac{\lim_{\to\pm\infty}\left(2+\frac{1}{}-\frac{1}{^2}\right)}{\lim_{\to\pm\infty}\left(1+\frac{1}{}-\frac{2}{^2}\right)}$

$= \dfrac{\lim_{\to\pm\infty} 2 + \lim_{\to\pm\infty} \frac{1}{} - \lim_{\to\pm\infty} \frac{1}{^2}}{\lim_{\to\pm\infty} 1 + \lim_{\to\pm\infty} \frac{1}{} - 2\lim_{\to\pm\infty} \frac{1}{^2}} = \dfrac{2+0-0}{1+0-2(0)} = 2$, so $= 2$ is a horizontal asymptote.

$= (\,) = \dfrac{2^2+-1}{^2+-2} = \dfrac{(2-1)(+1)}{(+2)(-1)}$, so $\lim_{\to-2^-} (\,) = \infty$, $\lim_{\to-2^+} (\,) = -\infty$, $\lim_{\to1^-} (\,) = -\infty$, and $\lim_{\to1^+} (\,) = \infty$. Thus, $= -2$

and $= 1$ are vertical asymptotes. The graph confirms our work.



$\dfrac{1+^4}{} \quad \dfrac{1}{} \quad \lim \dfrac{1}{}+1 \quad \lim \dfrac{1}{} + \lim 1$

50. $\lim_{x\to\pm\infty} \frac{1+4}{2-4} = \lim_{x\to\pm\infty} \frac{+1}{2-4} = \lim_{x\to\pm\infty} \frac{1}{1} \ldots$

$\frac{0+1}{0-1} = -1$, so $y = -1$ is a horizontal asymptote.

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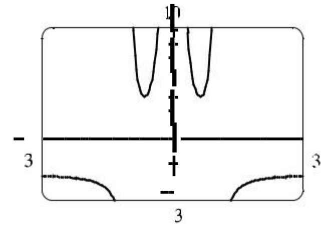
$= () = \dfrac{1+ \quad ^4}{-} = \dfrac{1+ \quad ^4}{(1- \;)} = \dfrac{1+ \quad ^4}{(1+\;)(1-\;)}$. The denominator is zero when $=0, -1$, and 1, but the numerator is nonzero, so $=0$, $=-1$, and $=1$ are vertical asymptotes. Notice that as $\to 0$, the numerator and denominator are both positive, so $\lim_{\to 0} (\;) = \infty$. The graph confirms our work.

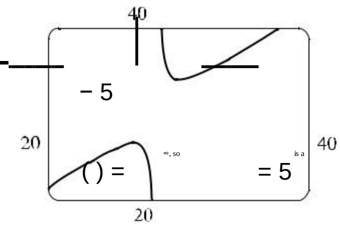


51. $= () = \dfrac{^3 -}{^2 - 6 + 5} = \dfrac{(^2 - 1)}{(-1)(-5)} = \dfrac{(+1)(-1)}{(-1)(-5)} = \dfrac{(+1)}{-5} = (\;)$ for $= .1$

6

The graph of is the same as the graph of with the exception of a hole in the graph of at $=1$. By long division, $(\;) = \dfrac{^2 +}{-5} = +6+ \dfrac{30}{-5}$.

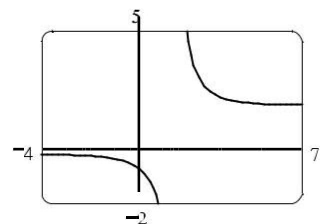
As $\to \pm\infty$, $(\;) \to \pm\infty$, so there is no horizontal asymptote. The denominator of is zero when $=5$. $\lim (\;) = -\infty$ and $\lim (\;) = \infty$, so $=5$ is a vertical asymptote. The graph confirms our work.



52. $\lim_{\to\infty} \dfrac{2}{-5} = \lim_{\to\infty} \dfrac{2}{-5} \cdot \dfrac{1}{1} = \lim_{\to\infty} \dfrac{2}{1-(5\;)} = \dfrac{2}{1-0} = 2$, so $=2$ is a horizontal asymptote.

$\lim_{\to -\infty} \dfrac{2}{-5} = \dfrac{2(0)}{0-5} = 0$, so $=0$ is a horizontal asymptote. The denominator is zero (and the numerator isn't) when $-5=0 \Rightarrow \;=5 \Rightarrow \;=\ln 5$.

$\lim_{\to(\ln 5)^+} \dfrac{2}{-5} = \infty$ since the numerator approaches 10 and the denominator approaches 0 through positive values as $\to (\ln 5)^+$. Similarly, $\lim_{\to(\ln 5)^-} \dfrac{2}{-5} = -\infty$. Thus, $=\ln 5$ is a vertical asymptote. The graph confirms our work.



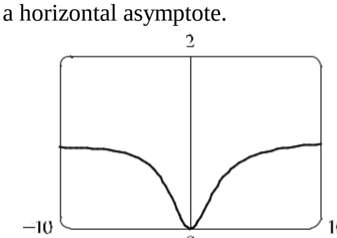
53. From the graph, it appears $=1$ is a horizontal asymptote.

$$\lim_{\to\pm\infty} \frac{3^3 + 500^2}{^3 + 500^2 + 100 + 2000} = \lim_{\to\pm\infty} \frac{\frac{3^3+500^2}{^3}}{\frac{^3+500^2+100+2000}{^3}}$$

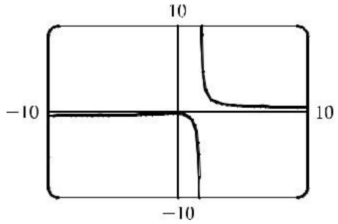
$$= \lim_{\to\pm\infty} \frac{3+(500\;)}{1+(500\;)+(100\;^2)+(2000\;^3)}$$

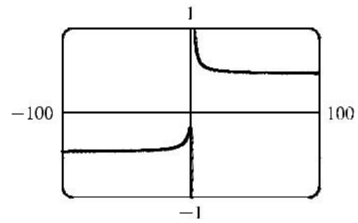
$$= \frac{3+0}{1+0+0+0} = 3, \text{ so } =3 \text{ is a horizontal asymptote.}$$

The discrepancy can be explained by the choice of the viewing window. Try $[-100{,}000\;100{,}000]$ by $[-1\;4]$ to get a graph that lends credibility to our calculation that $=3$ is a horizontal asymptote.



54. (a)





From the graph, it appears at first that there is only one horizontal asymptote, at ≈ 0 and a vertical asymptote at ≈ 1.7. However, if we graph the function with a wider and shorter viewing rectangle, we see that in fact there seem to be two horizontal asymptotes: one at ≈ 0.5 and one at ≈ -0.5. So we estimate that

$$\lim_{\to\infty} \frac{\sqrt{2\,^2+1}}{3\,-5} \approx 0.5 \quad \text{and} \quad \lim_{\to-\infty} \frac{\sqrt{2\,^2+1}}{3\,-5} \approx -0.5$$

(b) $(1000) \approx 0.4722$ and $(10{,}000) \approx 0.4715$, so we estimate that $\lim_{\to\infty} \frac{\sqrt{2\,^2+1}}{3\,-5} \approx 0.47$.

$(-1000) \approx -0.4706$ and $(-10{,}000) \approx -0.4713$, so we estimate that $\lim_{\to-\infty} \frac{\sqrt{2\,^2+1}}{3\,-5} \approx -0.47$.

(c) $\lim_{\to\infty} \frac{\sqrt{2\,^2+1}}{3\,-5} = \lim_{\to\infty} \frac{\sqrt{2+1/\,^2}}{3-5/} \quad [\text{since } \sqrt{\,^2} = \text{ for } 0] \quad = \frac{\sqrt{2}}{3} \approx 0.471404$.

For 0, we have $\sqrt{\,^2} = |\,| = -$, so when we divide the numerator by , with 0, we get $\frac{1}{}\sqrt{2\,^2+1} = -\frac{1}{\sqrt{\,^2}}\sqrt{2\,^2+1} = -\sqrt{2+1/\,^2}$. Therefore,

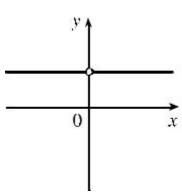
$$\lim_{\to-\infty} \frac{\sqrt{2\,^2+1}}{3\,-5} = \lim_{\to-\infty} \frac{-\sqrt{2+1/\,^2}}{3-5/} = -\frac{\sqrt{2}}{3} \approx -0.471404.$$

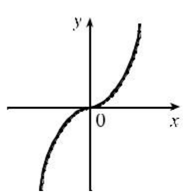
Divide the numerator and the denominator by the highest power of in $()$.

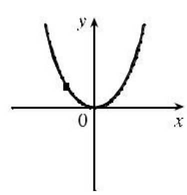
If deg deg , then the numerator $\to 0$ but the denominator doesn't. So $\lim_{\to\infty} [\,()\,/\,()] = 0$.

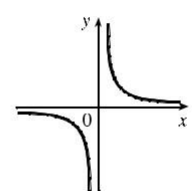
(b) If deg deg , then the numerator $\to \pm\infty$ but the denominator doesn't, so $\lim_{\to\infty} [\,()\,/\,()] = \pm\infty$ (depending on the ratio of the leading coefficients of and).

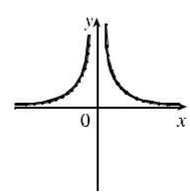
56.











(i) $= 0$ (ii) 0 (odd) (iii) 0 (even) (iv) 0 (odd) (v) 0 (even)

From these sketches we see that

(a) $\lim_{\to} = \begin{cases} 1 & \text{if } = 0 \\ 0 & \text{if } 0 \end{cases}$

(b) $\lim_{\to} = \begin{cases} 1 & \text{if } = 0 \\ 0 & \text{if } 0 \\ -\infty & \text{if } 0, \text{ odd} \end{cases}$

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(c) $\lim_{\to\infty} =$ 1 if $= 0$; ∞ if 0

(d) $\lim_{\to-\infty} =$ 1 if $= 0$; ∞ if 0, odd; $-\infty$ if 0, even; if 0

Let's look for a rational function.

(1) $\lim_{\to\pm\infty} ( ) = 0$ $\Rightarrow$ degree of numerator degree of denominator

(2) $\lim_{\to 0} ( ) = -\infty$ $\Rightarrow$ there is a factor of ^{2} in the denominator (not just , since that would produce a sign change at $= 0$), and the function is negative near $= 0$.

(3) $\lim_{\to 3^-} ( ) = \infty$ and $\lim_{\to 3^+} ( ) = -\infty$ $\Rightarrow$ vertical asymptote at $= 3$; there is a factor of $(- 3)$ in the denominator.

(4) $(2) = 0$ $\Rightarrow$ 2 is an -intercept; there is at least one factor of $(- 2)$ in the numerator.

Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits gives us

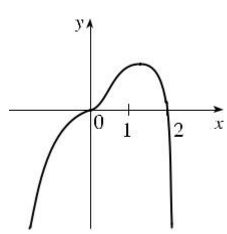
$( ) = \dfrac{2 - }{^2(- 3)}$ as one possibility.

Since the function has vertical asymptotes $= 1$ and $= 3$, the denominator of the rational function we are looking for must have factors $(- 1)$ and $(- 3)$. Because the horizontal asymptote is $= 1$, the degree of the numerator must equal the degree of the denominator, and the ratio of the leading coefficients must be 1. One possibility is $( ) = \dfrac{^2}{(- 1)(- 3)}$.

(a) We must first find the function . Since has a vertical asymptote $= 4$ and -intercept $= 1$, $- 4$ is a factor of the denominator and $- 1$ is a factor of the numerator. There is a removable discontinuity at $= -1$, so $- (-1) = + 1$ is a factor of both the numerator and denominator. Thus, now looks like this: $( ) = \dfrac{(- 1)(+ 1)}{(- 4)(+ 1)}$, where is still to be determined. Then $\lim_{\to -1} ( ) = \lim_{\to -1} \dfrac{(- 1)(+ 1)}{(- 4)(+ 1)} = \lim_{\to -1} \dfrac{(- 1)}{ - 4} = \dfrac{(-1 - 1)}{(-1 - 4)} = \dfrac{2}{5}$, so $\dfrac{2}{5} = 2$, and $= 5$. Thus $( ) = \dfrac{5(- 1)(+ 1)}{(- 4)(+ 1)}$ is a ratio of quadratic functions satisfying all the given conditions and $(0) = \dfrac{5(-1)(1)}{(-4)(1)} = \dfrac{5}{4}$.

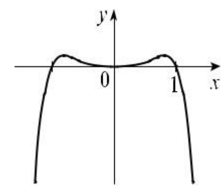
(b) $\lim_{\to\infty} ( ) = 5 \lim_{\to\infty} \dfrac{^2 - 1}{^2 - 3 - 4} = 5 \lim_{\to\infty} \dfrac{(^2/^2) - (1/^2)}{(^2/^2) - (3/^2) - (4/^2)} = 5\,\dfrac{1 - 0}{1 - 0 - 0} = 5(1) = 5$

60. $= ( ) = 2^3 - ^4 = ^3(2 -)$. The -intercept is $(0) = 0$. The -intercepts are 0 and 2. There are sign changes at 0 and 2 (odd exponents on and $2 - $). As $\to \infty$, $( ) \to -\infty$ because $^3 \to \infty$ and $2 - \to -\infty$. As

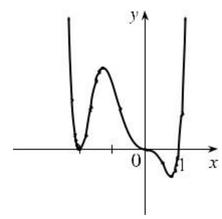


$\rightarrow -\infty$, $(\ ) \rightarrow -\infty$ because $^3 \rightarrow -\infty$ and $2 - \rightarrow \infty$. Note that the graph of near $= 0$ flattens out (looks like $=$ $_3$).

61. = ( ) = ^{4} − ^{6} = ^{4}(1 − ^{2}) = ^{4}(1 +)(1 −). The -intercept is (0) = 0. The -intercepts are 0, −1, and 1 [found by solving ( ) = 0 for]. Since $_4$ 0 for 6= ,0 doesn't change sign at = 0. The function does change sign at = −1 and = 1. As → ±∞, ( ) = ^{4}(1 − ^{2}) approaches −∞ because ^{4} → ∞ and (1 − ^{2}) → −∞.



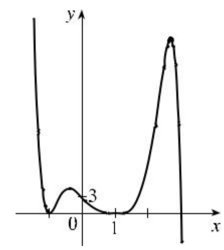
62. = ( ) = ^{3}(+ 2)^2(− 1). The -intercept is (0) = 0. The -intercepts are 0, −2, and 1. There are sign changes at 0 and 1 (odd exponents on and − 1). There is no sign change at −2. Also, ( ) → ∞ as → ∞ because all three factors are large. And ( ) → ∞ as → −∞ because ^{3} → −∞, (+ 2)^2 → ∞, and (− 1) → −∞. Note that the graph of at = 0 flattens out (looks like = − $_3$).



63. = ( ) = (3 −)(1 +)^2(1 −)^4. The -intercept is (0) = 3(1)^2(1)^4 = 3. The -intercepts are 3, −1, and 1. There is a sign change at 3, but not at −1 and 1. When is large positive, 3 − is negative and the other factors are positive, so lim ( ) = −∞. When is large negative, 3 − is positive, so

→∞

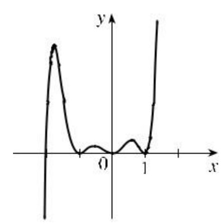
lim ( ) = ∞.
→−∞



64. = ( ) = ^{2}(^{2} − 1)^2(+ 2) = ^{2}(+ 1)^2(− 1)^2(+ 2). The -intercept is (0) = 0. The -intercepts are 0, −1, 1 and −2. There is a sign change at −2, but not at 0, −1, and 1. When is large positive, all the factors are positive, so lim ( ) = ∞. When is large negative, only + 2 is negative, so

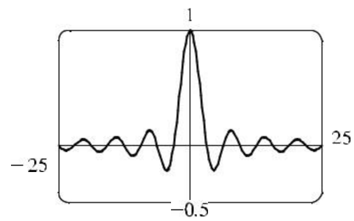
→∞

lim ( ) = −∞.
→−∞



(a) Since −1 ≤ sin ≤ 1 for all − 1 ≤ sin ≤ 1 for 0. As → ∞, −1 → 0 and 1 → 0, so by the Squeeze Theorem, (sin) → 0. Thus, lim sin = 0.
→∞

(b) From part (a), the horizontal asymptote is = 0. The function = (sin) crosses the horizontal asymptote whenever sin = 0; that is, at = for every integer . Thus, the graph crosses the asymptote *an infinite number of times.*

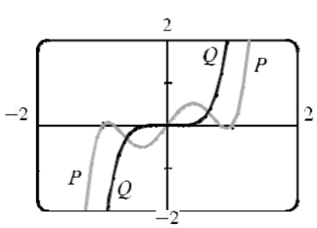


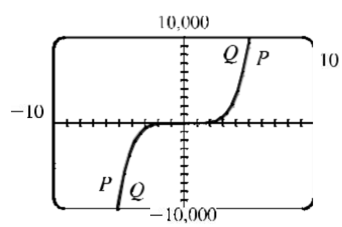
66. (a) In both viewing rectangles,

lim ( ) = lim ( ) = ∞ and
→∞ →∞

lim ( ) = lim ( ) = −∞.
→−∞ →−∞

In the larger viewing rectangle, and become less distinguishable.





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(b) $\lim_{\to\infty} \frac{(\;)}{(\;)} = \lim_{\to\infty} \frac{3^5 - 5^3 + 2}{3^5} = \lim_{\to\infty} \left(1 - \frac{5}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{4}\right) = 1 - \frac{5}{3}(0) + \frac{2}{3}(0) = 1 \Rightarrow$

and have the same end behavior.

67. $\lim_{\to\infty} \frac{5\sqrt{\;}}{\sqrt{\;} - 1} \cdot \frac{1/\sqrt{\;}}{1/\sqrt{\;}} = \lim_{\to\infty} \frac{5}{1 - (1/\sqrt{\;})} = \frac{5}{1-0} = 5$ and

$\lim_{\to\infty} \frac{10 - 21}{2} \cdot \frac{1}{1} = \lim_{\to\infty} \frac{10 - (21\;)}{2} = \frac{10 - 0}{2} = 5$. Since $\frac{10 - 21}{2} < (\;) < \frac{5\sqrt{\;}}{\sqrt{\;} - 1}$,

we have $\lim_{\to\infty} (\;) = 5$ by the Squeeze Theorem.

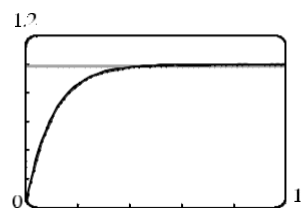
(a) After minutes, 25 liters of brine with 30 g of salt per liter has been pumped into the tank, so it contains

$(5000 + 25\;)$ liters of water and $25 \cdot 30 = 750$ grams of salt. Therefore, the salt concentration at time will be

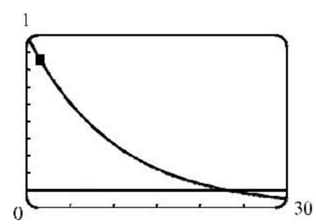
$(\;) = \frac{750}{5000 + 25} = \frac{30}{200 + } \frac{\text{g}}{\text{L}}$.

(b) $\lim_{\to\infty} (\;) = \lim_{\to\infty} \frac{30}{200 + } = \lim_{\to\infty} \frac{30}{200 + } = \frac{30}{0 + 1} = 30$. So the salt concentration approaches that of the brine being pumped into the tank.

(a) $\lim_{\to\infty} (\;) = \lim_{\to\infty} {}^*\left(1 - {}^{-}\right) = {}^*(1 - 0) = {}^*$



(b) We graph $(\;) = 1 - {}^{-9.8}$ and $(\;) = 0.99\,{}^*$, or in this case, $(\;) = 0.99$. Using an intersect feature or zooming in on the point of intersection, we find that ≈ 0.47 s.

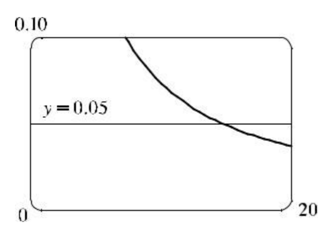


70. (a) $= {}^{-\;/10}$ and $= 0.1$ intersect at $_1 \approx 23.03$.

If $> _1$, then ${}^{-\;/10} < 0.1$.

(b) ${}^{-\;/10} < 0.1 \Rightarrow -\;/10 < \ln 0.1 \Rightarrow$

$> -10 \ln \frac{1}{10} = -10 \ln 10^{-1} = 10 \ln 10 \approx 23.03$

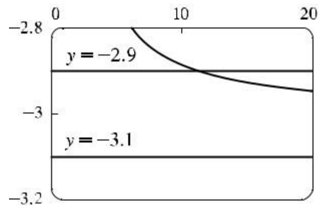


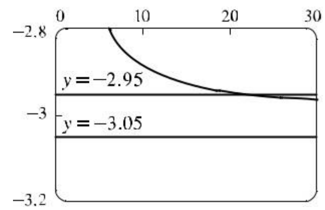
71. Let $(\;) = \frac{3^2 + 1}{2^2 + \;+ 1}$ and $(\;) = |(\;) - 1.5|$. Note that $\lim_{\to\infty} (\;) = \frac{3}{2}$ and $\lim_{\to\infty} (\;) = 0$. We are interested in finding the -value at which $(\;) < 0.05$. From the graph, we find that ≈ 14.804, so we choose $= 15$ (or any larger number).

72. We want to find a value of such that $\Rightarrow \left|\frac{1 - 3}{\sqrt{^2 + 1}} - (-3)\right| <$, or equivalently,

$-3 - < \frac{1 - 3}{\sqrt{^2 + 1}} < -3 +$. When $= 0.1$, we graph $= (\;) = \frac{1 - 3}{\sqrt{^2 + 1}}$, $= -3.1$, and $= -2.9$. From the graph,

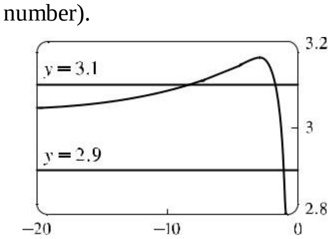
we find that ( ) = −2 9 at about = 11 283, so we choose = 12 (or any larger number). Similarly for = 0 05, we find that ( ) = −2 95 at about = 21 379, so we choose = 22 (or any larger number).

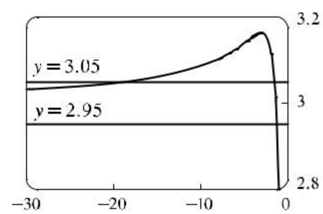




73. We want a value of such that 1 3 ... 1 3 ... 1 3

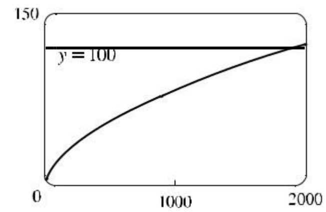
we graph = ( ) = $\frac{1-3}{\sqrt{2+1}}$, = 3 1, and = 2 9. From the graph, we find that ( ) = 3 1 at about = 8 092, so we choose = −9 (or any lesser number). Similarly for = 0 05, we find that ( ) = 3 05 at about = −18 338, so we choose = −19 (or any lesser number).





74. We want to find a value of such that ⇒ $\frac{\sqrt{\ }}{\ln}$ 100.

We graph = ( ) = $\frac{\sqrt{\ }}{\ln}$ and = 100. From the graph, we find that ( ) = 100 at about = 1382 773, so we choose = 1383 (or any larger number).



75. (a) 1 ^{2} 0 0001 ⇔ ^{2} 1 0 0001 = 10000 ⇔ 100 (0)

(b) If 0 is given, then 1 ^{2} ⇔ ^{2} 1 ⇔ 1 $\sqrt{\ }$. Let = 1 $\sqrt{\ }$.

Then ⇒ $\frac{1}{\ }$ $\frac{1}{\ }$ 0 = $\frac{1}{\ }$, so $\lim\limits_{\to\infty} \frac{1}{\ } = 0$

76. (a) 1 $\sqrt{\ }$ 0 0001 ⇔ 0 0001 = 10^4 ⇔ 10^8

(b) If 0 is given, then 1 $\sqrt{\ }$ ⇔ $\sqrt{\ }$ 1 ⇔ 1 ^{2}. Let = 1 ^{2}.

Then ⇒ $\frac{1}{\sqrt{\ }}$ $\frac{1}{\sqrt{\ }}$ 0 = $\frac{1}{\sqrt{\ }}$, so $\lim \frac{1}{\sqrt{\ }} = 0$.

77. For 0, |1 − 0| = −1 . If 0 is given, then −1 ⇔ −1 .
Take = −1 . Then ⇒ −1 ⇒ |(1) − 0| = −1, so $\lim\limits_{\to-\infty} (1\ ) = 0$.

78. Given 0, we need 0 such that ⇒ ^{3}. Now ^{3} ⇔ $\sqrt[3]{\ }$, so take = $\sqrt[3]{\ }$. Then = $\sqrt[3]{\ }$ ⇒ ^{3}, so $\lim\limits_{\to\infty} {}^3 = \infty$.

Given 0, we need 0 such that = max(1 ln). ⇒ . Now ⇔ ln , so take

(This ensures that 0.) Then = max(1 ln) ⇒ max( ) ≥ ,

so $\lim_{\to\infty}$ = ∞.

80. Definition Let be a function defined on some interval $(-\infty$). Then $\lim_{\to-\infty}$ ( ) = −∞ means that for every negative number there is a corresponding negative number such that ( ) whenever . Now we use the definition to prove that $\lim_{\to-\infty}$ 1 + ^{3} = −∞. Given a negative number , we need a negative number such that ⇒ 1 + ^{3}. Now 1 + ^{3} ⇔ ^{3} −1 ⇔ $\sqrt[3]{\;\;-1}$. Thus, we take = $\sqrt[3]{\;\;-1}$ and find that

⇒ 1 + ^{3}. This proves that $\lim_{\to-\infty}$ 1 + ^{3} = −∞.

(a) Suppose that $\lim_{\to\infty}$ ( ) = . Then for every 0 there is a corresponding positive number such that | ( ) − | whenever . If = 1 , then ⇔ 0 1 1 ⇔ 0 1 . Thus, for every 0 there is a corresponding 0 (namely 1) such that | (1) − | whenever 0 . This proves that

$\lim_{\to 0^+}$ (1) = = $\lim_{\to\infty}$ ( ).

Now suppose that $\lim_{\to-\infty}$ ( ) = . Then for every 0 there is a corresponding negative number such that | ( ) − | whenever . If = 1 , then ⇔ 1 1 0 ⇔ 1 0. Thus, for every 0 there is a corresponding 0 (namely −1) such that | (1) − | whenever − 0. This proves that

$\lim_{\to 0^-}$ (1) = = $\lim_{\to-\infty}$ ( ).

(b) $\lim_{\to}$ sin $\frac{1}{}$ = $\lim_{\to}$ sin $\frac{1}{}$ [let =]

= $\lim_{\to\infty}$ $\frac{1}{}$ sin [part (a) with = 1]

= $\lim_{\to\infty}$ $\frac{\sin}{}$ [let =]

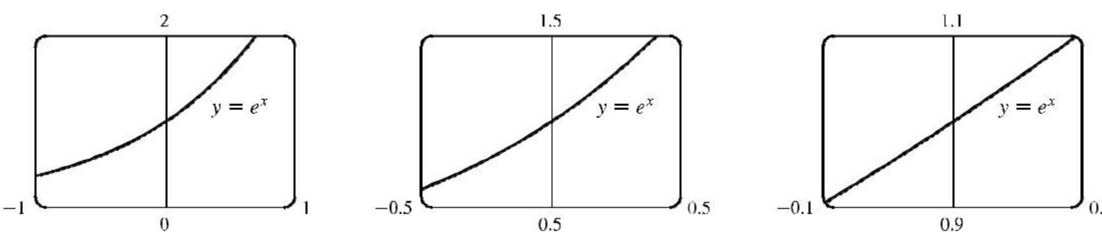
= 0 [by Exercise 65]

2.7 Derivatives and Rates of Change

(a) This is just the slope of the line through two points: = $\frac{\Delta}{\Delta} = \frac{( ) - (3)}{ - 3}$.

(b) This is the limit of the slope of the secant line as approaches : = $\lim_{\to 3} \frac{( ) - (3)}{ - 3}$.

2. The curve looks more like a line as the viewing rectangle gets smaller.



NOT FOR SALE

(a) (i) Using Definition 1 with ( ) = 4 − 2 and (1 3),

= lim ( ) − ( ) = lim (4 − 2) − 3 = lim −(2 − 4 + 3) = lim −(− 1)(− 3)

−→1−1→1−1→1−1→

lim (3 −) = 3 − 1 = 2

→

(ii) Using Equation 2 with ( ) = 4 − 2 and (1 3),

= lim (+) ( ) = lim (1+) (1) = lim 4(1+)−(1+)2 − 3

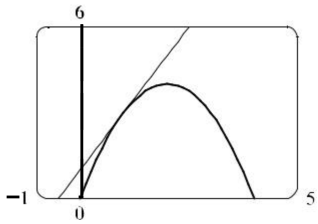
→0 →0 →0

= lim 4+4 −1−2 − 2−3 = lim − 2+2 = lim (− +2) = lim (+2)=2

→ → → →

(b) An equation of the tangent line is − ( ) = 0( )(−) ⇒ − (1) = 0(1)(− 1) ⇒ −3=2(−1), or = 2 + 1.

(c)



The graph of = 2 + 1 is tangent to the graph of = 4 − 2 at the point (1 3). Now zoom in toward the point (1 3) until the parabola and the tangent line are indistiguishable.

(a) (i) Using Definition 1 with ( ) = − 3 and (1 0),

= lim ( ) − 0 = lim − 3 = lim (1 − 2) = lim (1 +)(1 −)

→1 −1→1 −1−1 →1−1→1

= lim [(1+)]= 1(2)= 2

→

(ii) Using Equation 2 with ( ) = − 3 and (1 0),

= lim (+) ( ) = lim (1+) (1) = lim (1+)−(1+)3 − 0

→0 →0 →0

= lim 1+ −(1+3 +3 2+ 3) = lim − 3−3 2−2 = lim (− 2−3 −2)

→0 →0 →0

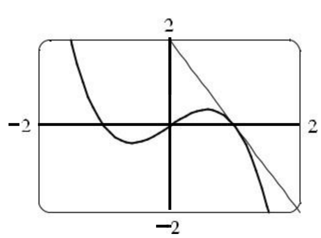
lim 2

→

(b) An equation of the tangent line is − ( ) = 0( )(−) ⇒ − (1) = 0(1)(− 1) ⇒ −0=−2(−1),

or = −2 + 2.

(c)



The graph of = −2 + 2 is tangent to the graph of = − 3 at the point (1 0). Now zoom in toward the point (1 0) until the cubic and the tangent line are indistinguishable.

NOT FOR SALE

5. Using (1) with $( ) = 4 - 3$ 2 and $(2 -4)$ [we could also use (2)],

$= \lim \frac{( ) - ( )}{-} = \lim \frac{4-3^2 - (-4)}{-} \lim \frac{-3^2+4+4}{-}$

$= \lim \frac{(-3 - 2)(- 2)}{-} = \lim (-3 - 2) = -3(2) - 2 = -8$

Tangent line: $- (-4) = -8(- 2) \Leftrightarrow +4=-8 +16 \Leftrightarrow =-8 +12.$

Using (2) with $( ) = {}^3 - 3 + 1$ and $(2\ 3)$,

$= \lim_{\to 0} \frac{(+) - ( )}{} = \lim_{\to 0} \frac{(2 +) - (2)}{} = \lim_{\to 0} \frac{(2 +)^3 - 3(2 +) + 1 - 3}{}$

$= \lim_{\to 0} \frac{8+12 +6^2+^3-6-3-2}{} = \lim_{\to 0} \frac{9+6^2+^3}{} = \lim_{\to 0} \frac{(9+6+^2)}{}$

$= \lim_{\to 0} (9 +6 + ^2)=9$

Tangent line: $- 3 = 9(- 2) \Leftrightarrow -3=9 -18 \Leftrightarrow =9 -15$

7. Using (1), $= \lim_{\to 1} \frac{\sqrt{} - \sqrt{1}}{- 1} = \lim_{\to 1} \frac{(\sqrt{} - 1)(\sqrt{} + 1)}{(-1)(\sqrt{} +1)} = \lim_{\to 1} \frac{- 1}{(-1)(\sqrt{} +1)} = \lim_{\to 1} \frac{1}{\sqrt{} +1} = \frac{1}{2}$.

Tangent line: $- 1 = \frac{1}{2}(-1) \Leftrightarrow = \frac{1}{2} + \frac{1}{2}$

Using (1) with $( ) = \frac{2 + 1}{+ 2}$ and $(1\ 1)$,

$= \lim_{\to} \frac{( ) - ( )}{-} = \lim_{\to 1} \frac{\frac{2 + 1}{+ 2} - 1}{- 1} = \lim_{\to 1} \frac{\frac{2 +1-(+2)}{+ 2}}{- 1} = \lim_{\to 1} \frac{- 1}{(-1)(+2)}$

$= \lim_{\to 1} \frac{1}{+ 2} = \frac{1}{1 + 2} = \frac{1}{3}$

Tangent line: $- 1 = \frac{1}{3}(-1) \Leftrightarrow - 1 = \frac{1}{3} - \frac{1}{3} \Leftrightarrow = \frac{1}{3} + \frac{2}{3}$

(a) Using (2) with $= ( ) = 3 + 4^2 - 2^3$,

$= \lim_{\to 0} \frac{(+) - ( )}{} = \lim_{\to 0} \frac{3 + 4(+)^2 - 2(+)^3 - (3 + 4^2 - 2^3)}{}$

$= \lim_{\to 0} \frac{3+4(^2+2 + ^2)-2(^3+3^2 +3^2+^3)-3-4^2+2^3}{}$

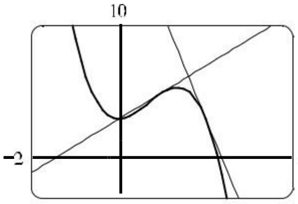
$= \lim_{\to 0} \frac{3+4^2+8 +4^2-2^3-6^2 -6^2-2^3-3-4^2+2^3}{}$

$= \lim_{\to 0} \frac{8 +4^2-6^2 -6^2-2^3}{} = \lim_{\to 0} \frac{(8 +4 -6^2-6 -2^2)}{}$

$\lim_{\to}$ 2 2 2

(b) At $(1\ 5)$: $= 8(1) - 6(1)^2 = 2$, so an equation of the tangent line is $- 5 = 2(- 1) \Leftrightarrow =2 +3.$

(c)



At (2 3): $= 8(2) - 6(2)^2 = -8$, so an equation of the tangent

line is $-3 = -8(-2) \Leftrightarrow = -8 + 19$.

10. (a) Using (1),

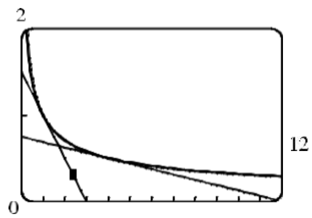
$$m = \lim_{x\to a}\frac{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{a}}}{x-a} = \lim_{x\to a}\frac{\frac{\sqrt{a}-\sqrt{x}}{\sqrt{ax}}}{x-a} = \lim_{x\to a}\frac{(\sqrt{a}-\sqrt{x})(\sqrt{a}+\sqrt{x})}{\sqrt{ax}\,(x-a)(\sqrt{a}+\sqrt{x})} = \lim_{x\to a}\frac{a-x}{\sqrt{ax}\,(x-a)(\sqrt{a}+\sqrt{x})}$$

$$= \lim_{x\to a}\frac{-1}{\sqrt{ax}\,(\sqrt{a}+\sqrt{x})} = \frac{-1}{\sqrt{a^2}\,(2\sqrt{a})} = \frac{-1}{2a^{3/2}} \text{ or } -\tfrac{1}{2}a^{-3/2}\ [a>0]$$

(b) At $(1,1)$: $m = -\frac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x-1) \Leftrightarrow y = -\frac{1}{2}x + \frac{3}{2}$.

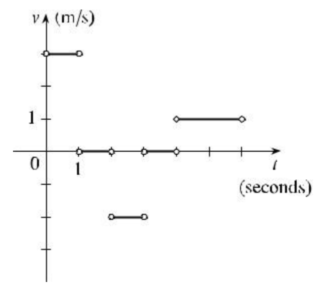
At $\left(4, \frac{1}{2}\right)$: $m = -\frac{1}{16}$, so an equation of the tangent line is $y - \frac{1}{2} = -\frac{1}{16}(x-4) \Leftrightarrow y = -\frac{1}{16}x + \frac{3}{4}$.

(c)



(a) The particle is moving to the right when s is increasing; that is, on the intervals $(0,1)$ and $(4,6)$. The particle is moving to the left when s is decreasing; that is, on the interval $(2,3)$. The particle is standing still when s is constant; that is, on the intervals $(1,2)$ and $(3,4)$.

(b) The velocity of the particle is equal to the slope of the tangent line of the graph. Note that there is no slope at the corner points on the graph. On the interval $(0,1)$ the slope is $\frac{3-0}{1-0} = 3$. On the interval $(2,3)$, the slope is $\frac{1-3}{3-2} = -2$. On the interval $(4,6)$, the slope is $\frac{3-1}{6-4} = 1$.



(a) **Runner A** runs the entire 100-meter race at the same velocity since the slope of the position function is constant. **Runner B** starts the race at a slower velocity than runner A, but finishes the race at a faster velocity.

The distance between the runners is the greatest at the time when the largest vertical line segment fits between the two graphs—this appears to be somewhere between 9 and 10 seconds.

The runners had the same velocity when the slopes of their respective position functions are equal—this also appears to be at about 9.5 s. Note that the answers for parts (b) and (c) must be the same for these graphs because as soon as the velocity for runner B overtakes the velocity for runner A, the distance between the runners starts to decrease.

Let $s(t) = 40t - 16t^2$.

$$v(2) = \lim_{t\to 2}\frac{s(t)-s(2)}{t-2} = \lim_{t\to 2}\frac{40t-16t^2-16}{t-2} = \lim_{t\to 2}\frac{-(16t^2-40t+16)}{t-2} = \lim_{t\to 2}\frac{-8(2t^2-5t+2)}{t-2}$$

$$= \lim_{t\to 2}\frac{-8(t-2)(2t-1)}{t-2} = -8\lim_{t\to 2}(2t-1) = -8(3) = -24$$

Thus, the instantaneous velocity when $t = 2$ is -24 ft/s.

14. (a) Let $( ) = 10 - 1\ 86\ ^2$.

$$(1) = \lim_{\to 0} \frac{(1+\ ) - (1)}{} = \lim_{\to 0} \frac{10(1+\ ) - 186(1+\ )^2 - (10 - 186)}{}$$

$$= \lim_{\to 0} \frac{10 + 10\ - 186(1 + 2\ + \ ^2) - 10 + 186}{}$$

$$= \lim_{\to 0} \frac{10 + 10\ - 186 - 372\ - 186\ ^2 - 10 + 186}{}$$

$$= \lim_{\to} \frac{628\ - 186\ ^2}{} = \lim_{\to} (6\ 28 - 186\ ) = 628$$

The velocity of the rock after one second is $6\ 28$ m s.

(b) $$( ) = \lim_{\to 0} \frac{(\ +\ ) - ( )}{} = \lim_{\to 0} \frac{10(\ +\ ) - 186(\ +\ )^2 - (10\ - 186\ ^2)}{}$$

$$= \lim_{\to 0} \frac{10\ + 10\ - 186(\ ^2 + 2\ \ + \ ^2) - 10\ + 186\ ^2}{}$$

$$= \lim_{\to 0} \frac{10\ + 10\ - 186\ ^2 - 372\ \ - 186\ ^2 - 10\ + 186\ ^2}{} = \lim_{\to 0} \frac{10\ - 372\ \ - 186\ ^2}{}$$

$$= \lim_{\to} \frac{(10 - 372\ - 186\ )}{} = \lim_{\to} (10 - 3\ 72\ - 186\ ) = 10 - 3\ 72$$

The velocity of the rock when $=$ is $(10 - 3\ 72\ )$ m s

(c) The rock will hit the surface when $= 0 \Leftrightarrow 10\ - 186\ ^2 = 0 \Leftrightarrow (10 - 1\ 86\ ) = 0 \Leftrightarrow = 0$ or $1\ 86 = 10$.

The rock hits the surface when $= 10\ 1\ 86 \approx 5\ 4$ s.

(d) The velocity of the rock when it hits the surface is $\frac{10}{1\ 86} = 10 - 372\ \frac{10}{1\ 86} = 10 - 20 = -10$ m s.

15. $$( ) = \lim_{\to 0} \frac{(+) - ()}{} = \lim_{\to 0} \frac{\frac{1}{(+)^2} - \frac{1}{^2}}{} = \lim_{\to 0} \frac{\frac{^2 - (\ +\ )^2}{^2(+)^2}}{} = \lim_{\to 0} \frac{^2 - (\ ^2 + 2\ \ + \ ^2)}{^2(+)^2}$$

$$= \lim_{\to 0} \frac{-(2\ \ + \ ^2)}{^2(+)^2} = \lim_{\to 0} \frac{-(2+)}{^2(+)^2} = \lim_{\to 0} \frac{-(2+)}{^2(+)^2} = \frac{-2}{^2\cdot ^2} = \frac{-2}{^3} \text{ m s}$$

So $(1) = \frac{-2}{1^3} = -2$ m s, $(2) = \frac{-2}{2^3} = -\frac{1}{4}$ m s, and $(3) = \frac{-2}{3^3} = -\frac{2}{27}$ m s.

16. (a) The average velocity between times and $+$ is

$$\frac{(\ +\ ) - ( )}{} = \frac{\frac{1}{2}(+)^2 - 6(\ +\ ) + 23 - \left(\frac{1}{2}\ ^2 - 6\ + 23\right)}{}$$

$$= \frac{\frac{1}{2}\ ^2 + \ \ + \frac{1}{2}\ ^2 - 6\ - 6\ + 23 - \frac{1}{2}\ ^2 + 6\ - 23}{} = \frac{\ \ + \frac{1}{2}\ ^2 - 6}{} = + \frac{1}{2}\ - 6 \text{ ft s}$$

[4 8]: $= 4$, $= 8 - 4 = 4$, so the average velocity is $4 + \frac{1}{2}(4) - 6 = 0$ ft s.

[6 8]: $= 6$, $= 8 - 6 = 2$, so the average velocity is $6 + \frac{1}{2}(2) - 6 = 1$ ft s.

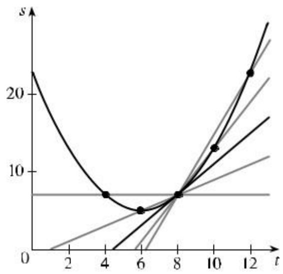
[8 10]: $= 8$, $= 10 - 8 = 2$, so the average velocity is $8 + \frac{1}{2}(2) - 6 = 3$ ft s.

[8 12]: $= 8$, $= 12 - 8 = 4$, so the average velocity is $8 + \frac{1}{2}(4) - 6 = 4$ ft s.

(b) $( ) = \lim_{\to 0} \frac{(+) - ( )}{} = \lim_{\to 0} + \frac{1}{2} - 6$

$= -6$, so $(8) = 2$ ft s.

(c)



17. $^0(0)$ is the only negative value. The slope at $= 4$ is smaller than the slope at $= 2$ and both are smaller than the slope at $=$ -2. Thus, $^0(0)$ 0 $^0(4)$ $^0(2)$ $^0(-2)$.

18. (a) On $[20\ 60]$: $\frac{(60) - (20)}{60 - 20} = \frac{700 - 300}{40} = \frac{400}{40} = 10$

(b) Pick any interval that has the same -value at its endpoints. $[0\ 57]$ is such an interval since $(0) = 600$ and $(57) = 600$.

(c) On $[40\ 60]$: $\frac{(60) - (40)}{60 - 40} = \frac{700 - 200}{20} = \frac{500}{20} = 25$

On $[40\ 70]$: $\frac{(70) - (40)}{70 - 40} = \frac{900 - 200}{30} = \frac{700}{30} = 23\frac{1}{3}$

Since 25 $23\frac{1}{3}$, the average rate of change on $[40\ 60]$ is larger.

$\frac{(40) - (10)}{40 - 10} = \frac{200 - 400}{30} = \frac{-200}{30} = -6\frac{2}{3}$

This value represents the slope of the line segment from $(10\ \ (10))$ to $(40\ \ (40))$.

(a) The tangent line at $= 50$ appears to pass through the points $(43\ 200)$ and $(60\ 640)$, so

$^0(50) \approx \frac{640 - 200}{60 - 43} = \frac{440}{17} \approx 26.$

The tangent line at $= 10$ is steeper than the tangent line at $= 30$, so it is larger in magnitude, but less in numerical value, that is, $^0(10)$ $^0(30)$.

The slope of the tangent line at $= 60$, $^0(60)$, is greater than the slope of the line through $(40\ \ (40))$ and $(80\ \ (80))$.

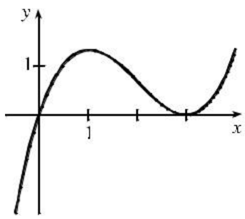
So yes, $^0(60)$ $\frac{(80) - (40)}{80 - 40}$.

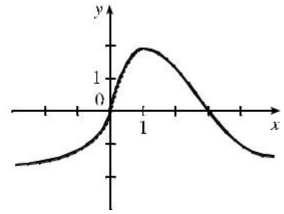
Since $(5) = -3$, the point $(5\ -3)$ is on the graph of . Since $^0(5) = 4$, the slope of the tangent line at $= 5$ is 4. Using the point-slope form of a line gives us $- (-3) = 4(- 5)$, or $= 4 - 23$.

For the tangent line $= 4 - 5$: when $= 2$, $= 4(2) - 5 = 3$ and its slope is 4 (the coefficient of). At the point of tangency, these values are shared with the curve $= ( )$; that is, $(2) = 3$ and $^0(2) = 4$.

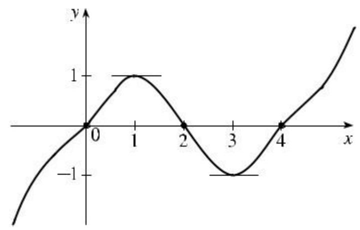
Since $(4\ 3)$ is on $= ( )$, $(4) = 3$. The slope of the tangent line between $(0\ 2)$ and $(4\ 3)$ is $\frac{1}{4}$, so $^0(4) = \frac{1}{4}$.

23. We begin by drawing a curve through the origin with a slope of 3 to satisfy $(0) = 0$ and $^0(0) = 3$. Since $_0(1) = 0$, we will round off our figure so that there is a horizontal tangent directly over $= 1$. Last, we make sure that the curve has a slope of -1 as we pass over $= 2$. Two of the many possibilities are shown.

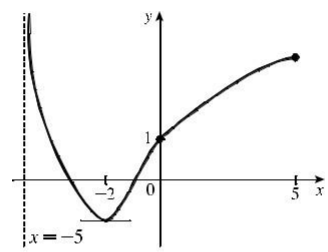




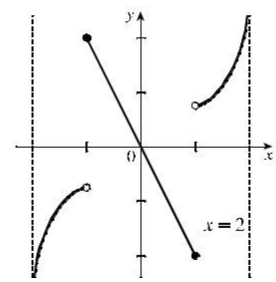
We begin by drawing a curve through the origin with a slope of 1 to satisfy $(0) = 0$ and $^0(0) = 1$. We round off our figure at $= 1$ to satisfy $^0(1) = 0$, and then pass through $(2\ 0)$ with slope -1 to satisfy $(2) = 0$ and $^0(2) = -1$. We round the figure at $= 3$ to satisfy $^0(3) = 0$, and then pass through $(4\ 0)$ with slope 1 to satisfy $(4) = 0$ and $^0(4) = 1$ Finally we extend the curve on both ends to satisfy $\lim_{\to\infty} (\ ) = \infty$ and $\lim_{\to -\infty} (\ ) = -\infty$.



We begin by drawing a curve through $(0\ 1)$ with a slope of 1 to satisfy $(0) = 1$ and $^0(0) = 1$. We round off our figure at $= -2$ to satisfy $^0(-2) = 0$. As $\to -5^+$, $\to \infty$, so we draw a vertical asymptote at $= -5$. As $\to 5^-$, $\to 3$, so we draw a dot at $(5\ 3)$ [the dot could be open or closed].



We begin by drawing an odd function (symmetric with respect to the origin) through the origin with slope -2 to satisfy $_0(0) = -2$. Now draw a curve starting at $= 1$ and increasing without bound as $\to 2^-$ since $\lim_{\to 2^-} (\ ) = \infty$. Lastly, reflect the last curve through the origin (rotate 180°) since is an odd function.



27. Using (4) with $(\ ) = 3\ ^2 - \ ^3$ and $= 1$,

$$^0(1) = \lim_{\to 0} \frac{(1+\ ) - (1)}{} = \lim_{\to 0} \frac{[3(1+\ )^2 - (1+\ )^3] - 2}{}$$

$$= \lim_{\to 0} \frac{(3 + 6\ + 3\ ^2) - (1 + 3\ + 3\ ^2 + \ ^3) - 2}{} = \lim_{\to 0} \frac{3\ - \ ^3}{} = \lim_{\to 0} \frac{(3 - \ ^2)}{}$$

$$= \lim_{\to 0} (3 - \ ^2) = 3 - 0 = 3$$

Tangent line: $- 2 = 3(\ - 1) \Leftrightarrow \ - 2 = 3\ - 3 \Leftrightarrow \ = 3\ - 1$

28. Using (5) with ( ) = $^4 - 2$ and = 1,

$$^0(1) = \lim_{\to 1} \frac{( ) - (1)}{- 1} = \lim_{\to 1} \frac{(^4 - 2) - (-1)}{- 1} = \lim_{\to 1} \frac{^4 - 1}{- 1} = \lim_{\to 1} \frac{(^2+1)(^2-1)}{- 1}$$

$$= \lim_{\to 1} \frac{(^2+1)(+1)(-1)}{- 1} = \lim_{\to 1} [(^2 + 1)(+ 1)] = 2(2) = 4$$

Tangent line: $- (-1) = 4(- 1) \Leftrightarrow +1 = 4 - 4 \Leftrightarrow = 4 - 5$

29. (a) Using (4) with ( ) = 5 $(1 + ^2)$ and the point (2 2), we have

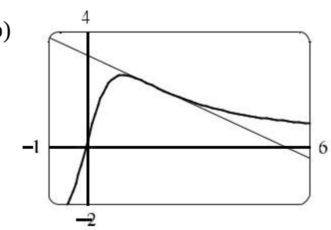
$$^0(2) = \lim_{\to 0} \frac{(2+) - (2)}{} = \lim_{\to 0} \frac{\frac{5(2+)}{1+(2+)^2} - 2}{}$$

$$= \lim_{\to 0} \frac{\frac{5 +10}{^2+4 +5} - 2}{} = \lim_{\to 0} \frac{\frac{5 +10-2(^2+4 +5)}{^2+4 +5}}{}$$

$$= \lim_{\to 0} \frac{-2 ^2-3}{(^2+4 +5)} = \lim_{\to 0} \frac{(-2 - 3)}{(^2+4 +5)} = \lim_{\to 0} \frac{-2 -3}{^2+4 +5} = \frac{-3}{5}$$

So an equation of the tangent line at (2 2) is $- 2 = -\frac{3}{5}(- 2)$ or $= -\frac{3}{5} + \frac{16}{5}$.

(b)



(a) Using (4) with ( ) = $4\ ^2 - \ ^3$, we have

$$^0( ) = \lim_{\to 0} \frac{(+)-()}{} = \lim_{\to 0} \frac{[4(+)^2-(+)^3]-(4 ^2- ^3)}{}$$

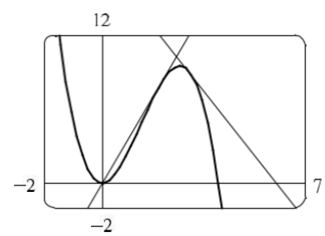
$$= \lim_{\to 0} \frac{4 ^2+8 +4 ^2-(^3+3 ^2 +3 ^2+ ^3)-4 ^2+ ^3}{}$$

$$= \lim_{\to 0} \frac{8 +4^2-3^2 -3 ^2- ^3}{} = \lim_{\to 0} \frac{(8 +4 -3 ^2-3 - ^2)}{}$$

lim →

At the point (2 8), $^0(2) = 16 - 12 = 4$, and an equation of the tangent line is $- 8 = 4(- 2)$, or $= 4$. At the point (3 9), $^0(3) = 24 - 27 = -3$, and an equation of the tangent line is $- 9 = -3(- 3)$, or $= -3 + 18$

(b)



Use (4) with ( ) = $3\ ^2 - 4 + 1$.

$$^0( ) = \lim_{\to 0} \frac{(+) - ( )}{} = \lim_{\to 0} \frac{[3(+)^2 - 4(+) + 1] - (3 ^2 - 4 + 1)]}{}$$

$$= \lim_{\to 0} \frac{3 ^2+6 +3 ^2-4 -4 +1-3 ^2+4 -1}{} = \lim_{\to 0} \frac{6 +3 ^2-4}{}$$

$$= \lim_{\to} \frac{(6 +3 -4)}{} = \lim_{\to} (6 + 3 - 4) = 6 - 4$$

NOT FOR SALE

Use (4) with $()=2^3+$.

$$0() = \lim \frac{(+)-()}{} = \lim \frac{[2(+)_3+(+)]-(2_3+)}{}$$

→0→0

$$= \lim_{\to 0} \frac{2^3+6^2+6^2+2^3++-2^3-}{} = \lim_{\to 0} \frac{6^2+6^2+2^3+}{}$$

$$= \lim_{\to 0} \frac{(6^2+6+2^2+1)}{} = \lim_{\to 0} (6^2+6+2^2+1) = 6^2+1$$

Use (4) with $()=(2+1)(+3)$.

$$0() = \lim_{\to 0} \frac{(+)-()}{} = \lim_{\to 0} \frac{\frac{2(+)+1}{(+)+3} - \frac{2+1}{+3}}{}$$

$$= \lim_{\to 0} \frac{(2+2+1)(+3)-(2+1)(++3)}{(++3)(+3)}$$

$$= \lim_{\to 0} \frac{(2^2+6+2+6++3)-(2^2+2+6+++3)}{(++3)(+3)}$$

$$= \lim_{\to 0} \frac{5}{(++3)(+3)} = \lim_{\to 0} \frac{5}{(++3)(+3)} = \frac{5}{(+3)^2}$$

Use (4) with $()=^{-2}=1^2$.

$$0() = \lim_{\to 0} \frac{(+)-()}{} = \lim_{\to 0} \frac{\frac{1}{(+)^2} - \frac{1}{2}}{} = \lim_{\to 0} \frac{\frac{2-(+)^2}{2(+)^2}}{}$$

$$= \lim_{\to 0} \frac{2-(2+2+2)}{2(+)^2} = \lim_{\to 0} \frac{-2-2}{2(+)^2} = \lim_{\to 0} \frac{(-2-)}{2(+)^2}$$

$$= \lim_{\to 0} \frac{-2-}{2(+)^2} = \frac{-2}{2(2)} = \frac{-2}{3}$$

Use (4) with $() = \sqrt{1-2}$.

$$0() = \lim_{\to 0} \frac{(+)-()}{} = \lim \frac{\sqrt{1-2(+)} - \sqrt{1-2}}{}$$

$$= \lim_{\to 0} \frac{\sqrt{1-2(+)} - \sqrt{1-2}}{} \cdot \frac{\sqrt{1-2(+)} + \sqrt{1-2}}{\sqrt{1-2(+)} + \sqrt{1-2}}$$

$$= \lim_{\to 0} \frac{1-2(+) - 1-2}{\left(\sqrt{1-2(+)} + \sqrt{1-2}\right)} = \lim_{\to 0} \frac{(1-2-2)-(1-2)}{\left(\sqrt{1-2(+)} + \sqrt{1-2}\right)}$$

$$= \lim \frac{-2}{} = \lim \frac{-2}{}$$

$$= \frac{-2}{\sqrt{1-2} + \sqrt{1-2}} = \frac{-2}{2\sqrt{1-2}} = \frac{-1}{\sqrt{1-2}}$$

NOT FOR SALE

36. Use (4) with $(\ ) = \dfrac{4}{\sqrt{1-}}$.

$$^{0}(\ ) = \lim_{\to 0} \frac{(\ +\ ) - (\ )}{} = \lim_{\to 0} \frac{\dfrac{4}{\sqrt{1-(\ +\ )}} - \dfrac{4}{\sqrt{1-}}}{}$$

$$= 4 \lim_{\to 0} \frac{\sqrt{1-} - \sqrt{1--}}{\sqrt{1--}\sqrt{1-}} = 4 \lim_{\to 0} \frac{\sqrt{1-} - \sqrt{1--}}{\sqrt{1--}\sqrt{1-}} \cdot \frac{\sqrt{1-} + \sqrt{1--}}{\sqrt{1-} + \sqrt{1--}}$$

$$= 4 \lim_{\to 0} \frac{(\sqrt{1-})^2 - (\sqrt{1--})^2}{\sqrt{1--}\sqrt{1-}(\sqrt{1-}+\sqrt{1--})} = 4 \lim_{\to 0} \frac{(1-)-(1--)}{\sqrt{1--}\sqrt{1-}(\sqrt{1-}+\sqrt{1--})}$$

$$= 4 \lim_{\to 0} \frac{1}{\sqrt{1--}\sqrt{1-}(\sqrt{1-}+\sqrt{1--})} = \frac{4}{\sqrt{1-}\sqrt{1-}(\sqrt{1-}+\sqrt{1-})}$$

$$= \frac{4}{(1-)(2\sqrt{1-})} = \frac{2}{(1-)^{1}(1-)^{1/2}} = \frac{2}{(1-)^{3/2}}$$

37. By (4), $\lim_{\to 0} \dfrac{\sqrt{9+}-3}{} = {}^{0}(9)$, where $(\ ) = \sqrt{\ }$ and $= 9$.

By (4), $\lim_{\to 0} \dfrac{^{-2+} - ^{-2}}{} = {}^{0}(-2)$, where $(\ ) =$ and $= -2$.

By Equation 5, $\lim_{\to 2} \dfrac{^{6} - 64}{-2} = {}^{0}(2)$, where $(\ ) = {}^{6}$ and $= 2$.

40. By Equation 5, $\lim_{\to 1/4} \dfrac{\frac{1}{} - 4}{-\frac{1}{4}} = {}^{0}(4)$, where $(\ ) = \frac{1}{}$ and $= \frac{1}{4}$.

By (4), $\lim_{\to 0} \dfrac{\cos(\ +\ ) + 1}{} = {}^{0}(\ )$, where $(\ ) = \cos$ and $=$.

Or: By (4), $\lim_{\to 0} \dfrac{\cos(\ +\ ) + 1}{} = {}^{0}(0)$, where $(\ ) = \cos(\ +\ )$ and $= 0$.

42. By Equation 5, $\lim_{\to 6} \dfrac{\sin - \frac{1}{2}}{-\ 6} = {}^{0}\left(\ 6\right)$, where $(\ ) = \sin$ and $= \ 6$.

43. $(4) = {}^{0}(4) = \lim_{\to 0} \dfrac{(4+\ ) - (4)}{} = \lim_{\to 0} \dfrac{80(4+\ ) - 6(4+\ )^2 - 80(4) - 6(4)^2}{}$

$$= \lim_{\to 0} \frac{(320+80\ -96-48\ -6\ ^2)-(320-96)}{} = \lim_{\to 0} \frac{32-6^2}{}$$

$= \lim (32-6\ )$
$= \lim (32 \quad 6\ ) =$
32 m/s

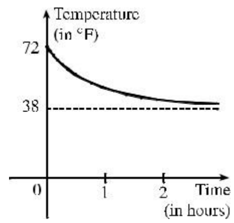
The speed when $= 4$ is $|32| = 32$ m s.

44. $(4) = {}^{0}(4) = \lim\limits_{\to 0} \dfrac{(4+\ )-(4)}{\ } = \lim\limits_{\to 0} \dfrac{\frac{10+}{4++1} - \frac{10}{4+1}}{\ } = \lim\limits_{\to 0} \dfrac{\frac{45}{5+} - 9}{\ }$

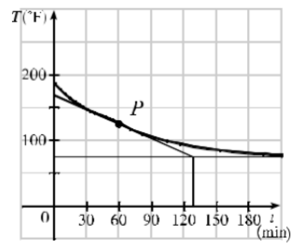
$= \lim\limits_{\to} \dfrac{45-9(5+\ )}{\ } = \lim\limits_{\to} \dfrac{-9}{\ } = \lim\limits_{\to} \dfrac{-9}{\ } = -\dfrac{9}{\ }$ m/s.

The speed when $= 4$ is $-\frac{\ }{5} = \frac{\ }{5}$ m s.

45. The sketch shows the graph for a room temperature of 72◦ and a refrigerator temperature of 38◦. The initial rate of change is greater in magnitude than the rate of change after an hour.



46. The slope of the tangent (that is, the rate of change of temperature with respect to time) at $= 1$ h seems to be about $\dfrac{75-168}{132-0} \approx -0\ 7$ ◦F min.



(a) (i) [1 0 2 0]: $\dfrac{(2)-(1)}{2-1} = \dfrac{0\ 18-0\ 33}{1} = -0\ 15 \dfrac{\text{mg/mL}}{\text{h}}$

[1 5 2 0]: $\dfrac{(2)-(1\ 5)}{2-1\ 5} = \dfrac{0\ 18-0\ 24}{0\ 5} = \dfrac{-0\ 06}{0\ 5} = -0\ 12 \dfrac{\text{mg/mL}}{\text{h}}$

[2 0 2 5]: $\dfrac{(2\ 5)-(2)}{2\ 5-2} = \dfrac{0\ 12-0\ 18}{0\ 5} = \dfrac{-0\ 06}{0\ 5} = -0\ 12 \dfrac{\text{mg/mL}}{\text{h}}$

[2 0 3 0]: $\dfrac{(3)-(2)}{3-2} = \dfrac{0\ 07-0\ 18}{1} = -0\ 11 \dfrac{\text{mg/mL}}{\text{h}}$

(b) We estimate the instantaneous rate of change at $= 2$ by averaging the average rates of change for [1 5 2 0] and [2 0 2 5]: $\dfrac{-0\ 12+(-0\ 12)}{2} = -0\ 12 \dfrac{\text{mg/mL}}{\text{h}}$. After 2 hours, the BAC is decreasing at a rate of 0 12 (mg mL) h.

(a) (i) [2006 2008]: $\dfrac{(2008)-(2006)}{2008-2006} = \dfrac{16{,}680-12{,}440}{2} = \dfrac{4240}{2} = 2120$ locations year

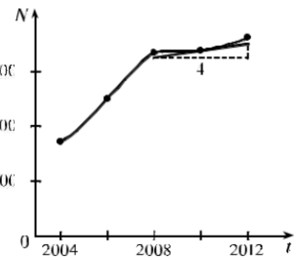
[2008 2010]: $\dfrac{(2010)-(2008)}{2010-2008} = \dfrac{16{,}858-16{,}680}{2} = \dfrac{178}{2} = 89$ locations year.

The rate of growth decreased over the period from 2006 to 2010.

[2010 2012]: $\dfrac{(2012)-(2010)}{2012-2010} = \dfrac{18{,}066-16{,}858}{2} = \dfrac{1208}{2} = 604$ locations year.

Using that value and the value from part (a)(ii), we have $\dfrac{89+604}{2} = \dfrac{693}{2} = 346\ 5$ locations year.

(c) The tangent segment has endpoints (2008 16,250) and (2012 17,500).



An estimate of the instantaneous rate of growth in 2010 is

$\frac{17{,}500 - 16{,}250}{2012 - 2008} = \frac{1250}{4} = 312\ 5$ locations/year.

(a) [1990 2005]: $\frac{84{,}077 - 66{,}533}{2005 - 1990} = \frac{17{,}544}{15} = 1169\ 6$ thousands of barrels per day per year. This means that oil consumption rose by an average of 1169 6 thousands of barrels per day each year from 1990 to 2005.

[1995 2000]: $\frac{76{,}784 - 70{,}099}{2000 - 1995} = \frac{6685}{5} = 1337$

[2000 2005]: $\frac{84{,}077 - 76{,}784}{2005 - 2000} = \frac{7293}{5} = 1458\ 6$

An estimate of the instantaneous rate of change in 2000 is $\frac{1}{2}(1337 + 1458\ 6) = 1397\ 8$ thousands of barrels per day per year.

(a) (i) [4 11]: $\frac{(11) - (4)}{11 - 4} = \frac{9\ 4 - 53}{7} = \frac{-43\ 6}{7} \approx -6\ 23$ RNA copies mL / day

[8 11]: $\frac{(11) - (8)}{11 - 8} = \frac{9\ 4 - 18}{3} = \frac{-8\ 6}{3} \approx -2\ 87$ RNA copies mL / day

[11 15]: $\frac{(15) - (11)}{15 - 11} = \frac{5\ 2 - 9\ 4}{4} = \frac{-4\ 2}{4} = -1\ 05$ RNA copies mL / day

[11 22]: $\frac{(22) - (11)}{22 - 11} = \frac{3\ 6 - 9\ 4}{11} = \frac{-5\ 8}{11} \approx -0\ 53$ RNA copies mL / day

(b) An estimate of $'(11)$ is the average of the answers from part (a)(ii) and (iii).

$'(11) \approx \frac{1}{2}[-2\ 87 + (-1\ 05)] = -1\ 96$ RNA copies mL / day.

$'(11)$ measures the instantaneous rate of change of patient 303's viral load 11 days after ABT-538 treatment began.

51. (a) (i) $\frac{\Delta}{\Delta} = \frac{(105) - (100)}{105 - 100} = \frac{6601\ 25 - 6500}{5} = \$20\ 25$ unit.

$\frac{\Delta}{\Delta} = \frac{(101) - (100)}{101 - 100} = \frac{6520\ 05 - 6500}{1} = \$20\ 05$ unit.

(b) $\frac{(100 +) - (100)}{} = \frac{5000 + 10(100 +) + 0\ 05(100 +)^2 - 6500}{} = \frac{20 + 0\ 05^2}{}$

$= 20 + 0\ 05 ,\ \ 6 = 0$

So the instantaneous rate of change is $\lim_{\to 0} \frac{(100 +) - (100)}{} = \lim_{\to 0} (20 + 0\ 05) = \$20$ unit.

NOT FOR SALE

52. $\Delta = (+) - () = 100{,}000\left(1 - \frac{+}{60}\right)^2 - 100{,}000\left(1 - \frac{}{60}\right)^2$ $+ \frac{}{3600} + \frac{}{3600}$

$= 100{,}000\ \ 1 - \frac{+}{30} + \frac{(+)^2}{3600} - 1 - \frac{}{30} + \frac{}{3600}\ = 100{,}000\ -\frac{}{30}$

$= \frac{100{,}000}{3600}\ (-120 + 2 +) = \frac{250}{9}\ (-120 + 2 +)$

Dividing Δ by and then letting $\to 0$, we see that the instantaneous rate of change is $\frac{500}{9}(- 60)$ gal min.

	Flow rate (gal min)	Water remaining ( ) (gal)
0	−3333 3	100 000
10	−2777 7	69 444 4
20	−2222 2	44 444 4
30	−1666 6	25 000
40	−1111 1	11 111 1
50	−5555	2 7777
60	0	0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

(a) $'( )$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.

After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17 ounce. So the cost of producing the 800th (or 801st) ounce is about \$17.

In the short term, the values of $'( )$ will decrease because more efficient use is made of start-up costs as increases. But eventually $'( )$ might increase due to large-scale operations.

(a) $'(5)$ is the rate of growth of the bacteria population when $= 5$ hours. Its units are bacteria per hour.

With unlimited space and nutrients, $'$ should increase as increases; so $'(5)\ '(10)$. If the supply of nutrients is limited, the growth rate slows down at some point in time, and the opposite may be true.

(a) $'(58)$ is the rate at which the daily heating cost changes with respect to temperature when the outside temperature is 58 ◦F. The units are dollars ◦F.

If the outside temperature increases, the building should require less heating, so we would expect $'(58)$ to be negative.

(a) $'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound. The units for $'(8)$ are pounds (dollars pound).

$'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

(a) $'( )$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are (mg L) ◦C.

For $= 16$◦C, it appears that the tangent line to the curve goes through the points (0 14) and (32 6). So

$'(16) \approx \frac{6 - 14}{32 - 0} = -\frac{8}{32} = -0\ 25$ (mg L) ◦C. This means that as the temperature increases past 16◦C, the oxygen solubility is decreasing at a rate of 0 25 (mg L) °C.

(a) $0(\ )$ is the rate of change of the maximum sustainable speed of Coho salmon with respect to the temperature. Its units are (cm s) °C.

(b) For $= 15^\circ\mathrm{C}$, it appears the tangent line to the curve goes through the points (10 25) and (20 32). So

$^0(15) \approx \frac{32 - 25}{20 - 10} = 0\ 7$ (cm s) °C. This tells us that at $= 15^\circ\mathrm{C}$, the maximum sustainable speed of Coho salmon is changing at a rate of 0.7 (cm s) °C. In a similar fashion for $= 25^\circ\mathrm{C}$, we can use the points (20 35) and (25 25) to obtain $^0(25) \approx \frac{25 - 35}{25 - 20} = -2$ (cm s) °C. As it gets warmer than 20°C, the maximum sustainable speed decreases rapidly.

Since $(\ ) = \sin(1\ )$ when $6= 0$ and $(0) = 0$, we have

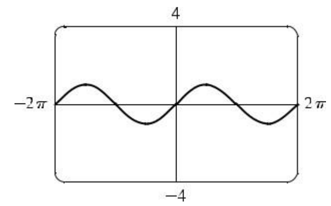
$_0(0) = \lim\limits_{\to 0} \frac{(0 + \ ) - (0)}{} = \lim\limits_{\to 0} \frac{\sin(1\ ) - 0}{} = \lim\limits_{\to 0} \sin(1\ )$. This limit does not exist since $\sin(1\ )$ takes the values -1 and 1 on any interval containing 0. (Compare with Example 2.2.4.)

Since $(\ ) = {}^2 \sin(1\ )$ when $6= 0$ and $(0) = 0$, we have

$_0(0) = \lim\limits_{\to 0} \frac{(0 + \ ) - (0)}{} = \lim\limits_{\to 0} \frac{{}^2 \sin(1\ ) - 0}{} = \lim\limits_{\to 0} \sin(1\ )$. Since $-1 \le \sin \frac{1}{} \le 1$, we have

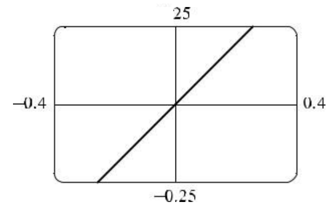
$-|\ | \le |\ | \sin \frac{1}{} \le |\ | \Rightarrow -|\ | \le \sin \frac{1}{} \le |\ |$. Because $\lim\limits_{\to 0} (-|\ |) = 0$ and $\lim\limits_{\to 0} |\ | = 0$, we know that

$\lim_{\to 0} \sin \frac{1}{} = 0$ by the Squeeze Theorem. Thus, ${}^{0}(0) = 0$.

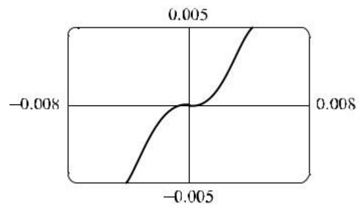


61. (a) The slope at the origin appears to be 1.

(b) The slope at the origin still appears to be 1.

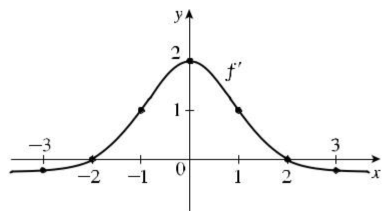


(c) Yes, the slope at the origin now appears to be 0.



2.8 The Derivative as a Function

1. It appears that is an odd function, so 0 will be an even function—that is, $^0(-) = {}^0( )$.

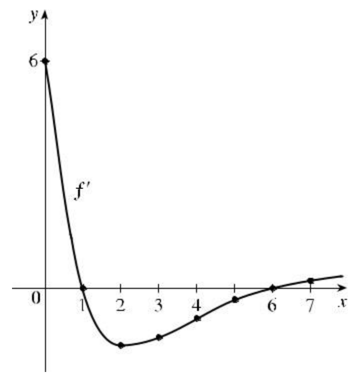


(a) 0(−3) ≈ −0 2

(b) 0(−2) ≈ 0 (c) ^{0}(−1) ≈ 1 (d) ^{0}(0) ≈ 2

(e) 0 (1) ≈ 1 (f) ^{0}(2) ≈ 0 (g) ^{0}(3) ≈ −0 2

2. Your answers may vary depending on your estimates.



(a) *Note:* By estimating the slopes of tangent lines on the graph of , it appears that 0(0) ≈ 6.

(b) ^{0}(1) ≈ 0

(c) ^{0}(2) ≈ −15 (d) ^{0}(3) ≈ −13 (e) ^{0}(4) ≈ −0 8

(f) ^{0}(5) ≈ −03 (g) ^{0}(6) ≈ 0 (h) ^{0}(7) ≈ 0 2

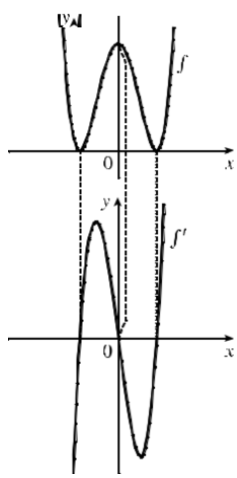
(a) 0 = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

(b) 0 = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c) 0 = I, since the slopes of the tangents to graph (c) are negative for 0 and positive for 0, as are the function values of graph I.

(d) 0 = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Hints for Exercises 4 –11: First plot -intercepts on the graph of 0 for any horizontal tangents on the graph of . Look for any corners on the graph of — there will be a discontinuity on the graph of 0. On any interval where has a tangent with positive (or negative) slope, the graph of 0 will be positive (or negative). If the graph of the function is linear, the graph of 0 will be a horizontal line.



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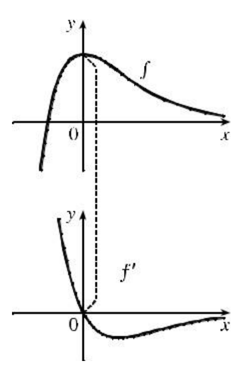
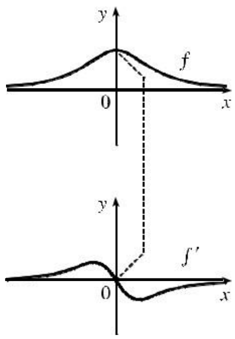


5.

6.



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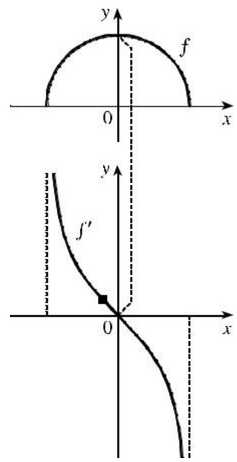

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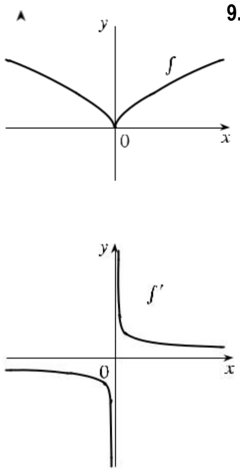
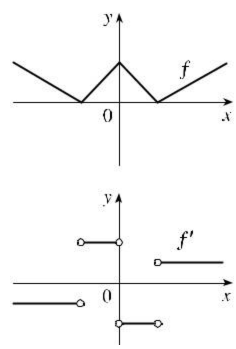
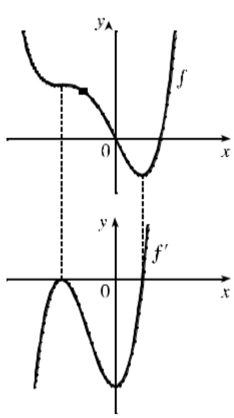


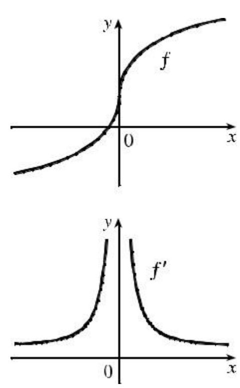
9.



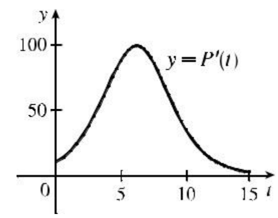


10.

11.

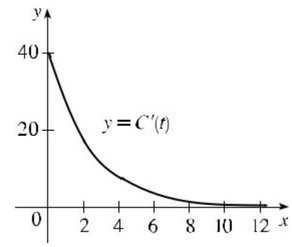


12. The slopes of the tangent lines on the graph of = ( ) are always positive, so the -values of = 0( ) are always positive. These values start out relatively small and keep increasing, reaching a maximum at about = 6. Then the -values of = 0( ) decrease and get close to zero. The graph of 0 tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.



13. (a) 0( ) is the instantaneous rate of change of percentage of full capacity with respect to elapsed time in hours.

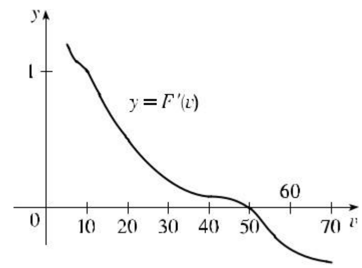
(b) The graph of 0( ) tells us that the rate of change of percentage of full capacity is decreasing and approaching 0.



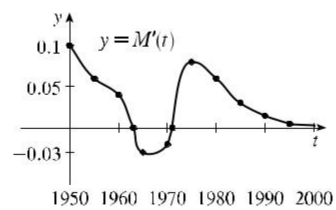
14. (a) 0( ) is the instantaneous rate of change of fuel economy with respect to speed.

(b) Graphs will vary depending on estimates of 0, but will change from positive to negative at about = 50.

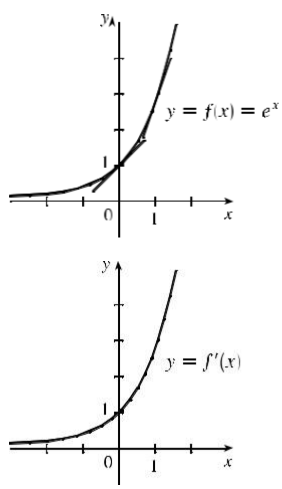
(c) To save on gas, drive at the speed where is a maximum and 0 is 0, which is about 50 mi h.



15. It appears that there are horizontal tangents on the graph of for = 1963 and = 1971. Thus, there are zeros for those values of on the graph of 0. The derivative is negative for the years 1963 to 1971.

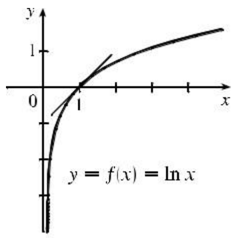


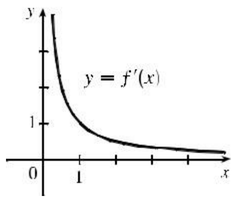
See Figure 3.3.1.



The slope at 0 appears to be 1 and the slope at 1 appears to be 2 7. As decreases, the slope gets closer to 0. Since the graphs are so similar, we might guess that 0( ) = .

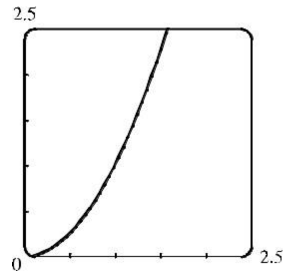
18.





As increases toward 1, $_0$( ) decreases from very large numbers to 1. As becomes large, $_0$( ) gets closer to 0. As a guess, ^{0}( ) = 1 ^{2} or ^{0}( ) = 1 makes sense.

19. (a) By zooming in, we estimate that $_0(0) = 0$, $_0$ $2\frac{1}{}$ = 1, $^0(1) = 2$, and $^0(2) = 4$.



(b) By symmetry, $^0(-\ ) = -\,^0(\ )$. So $^0\,-2\frac{1}{}$ = −1, $^0(-1) = -2$, and $^0(-2) = -4$.

(c) It appears that $_0$( ) is twice the value of , so we guess that $_0$( ) = 2 .

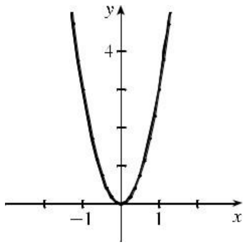
(d) $^0(\ ) = \lim\limits_{\to 0} \frac{(+)-()}{} = \lim\limits_{\to 0} \frac{(\ +\ )^2 - {}^2}{}$

$= \lim\limits_{\to 0} \frac{{}^2 + 2 + {}^2 - {}^2}{} = \lim\limits_{\to 0} \frac{2 + {}^2}{} = \lim\limits_{\to 0} \frac{(2+)}{} = \lim\limits_{\to 0} (2\ +\ ) = 2$

20. (a) By zooming in, we estimate that $_0(0) = 0$, $_0$ $2\frac{1}{}$ ≈ 0 75, $^0(1) \approx 3$, $^0(2) \approx 12$, and $^0(3) \approx 27$.

(b) By symmetry, $^0(-\ ) = {}^0(\ )$. So $^0\,-2\frac{1}{}$ ≈ 0 75, $^0(-1) \approx 3$, $^0(-2) \approx 12$, and $^0(-3) \approx 27$.

(c)



(d) Since $_0(0) = 0$, it appears that $_0$ may have the form $_0(\ ) = {}^2$. Using $^0(1) = 3$, we have = 3, so $^0(\ ) = 3\,{}^2$.

$^0(\ ) = \lim\limits_{\to 0} \frac{(\ +\ ) - ()}{} = \lim\limits_{\to 0} \frac{(\ +\ )^3 - {}^3}{} = \lim\limits_{\to 0} \frac{({}^3 + 3{}^2 + 3{}^2 + {}^3) - {}^3}{}$

$= \lim\limits_{\to 0} \frac{3{}^2 + 3{}^2 + {}^3}{} = \lim\limits_{\to 0} \frac{(3{}^2 + 3\ +\ {}^2)}{} = \lim\limits_{\to 0} (3{}^2 + 3\ +\ {}^2) = 3{}^2$

21. $^0( ) = \lim_{\to 0} \frac{(+)-()}{} = \lim_{\to 0} \frac{[3(+)-8]-(3-8)}{} = \lim_{\to 0} \frac{3+3-8-3+8}{}$

$= \lim_{\to 0} \frac{3}{} = \lim_{\to 0} 3 = 3$

Domain of = domain of $_0$ = R.

22. $^0( ) = \lim_{\to 0} \frac{(+)-()}{} = \lim_{\to 0} \frac{[(+)+]-(+)}{} = \lim_{\to 0} \frac{++--}{}$

$= \lim_{\to 0} \frac{}{} = \lim_{\to 0} =$

Domain of = domain of $_0$ = R.

23. $^0( ) = \lim_{\to 0} \frac{(+) - ()}{} = \lim_{\to 0} \frac{25(+)^2+6(+) - 25^2+6}{}$

$= \lim_{\to 0} \frac{25(^2+2+^2)+6+6-25^2-6}{} = \lim_{\to 0} \frac{25^2+5+25^2+6-25^2}{}$

$= \lim_{\to 0} \frac{5+25^2+6}{} = \lim_{\to 0} \frac{(5+25+6)}{} = \lim_{\to 0} (5+25+6)$

$=5+6$

Domain of = domain of $_0$ = R.

24. $^0() = \lim_{\to 0} \frac{(+) - ()}{} = \lim_{\to 0} \frac{4+8(+)-5(+)^2 - (4+8-5^2)}{}$

$= \lim_{\to 0} \frac{4+8+8-5(^2+2+^2)-4-8+5^2}{} = \lim_{\to 0} \frac{8-5^2-10-5^2+5^2}{}$

$= \lim_{\to} \frac{8-10-5^2}{} = \lim_{\to} \frac{(8-10-5)}{} = \lim_{\to} (8-10-5)$

$=8-10$

Domain of = domain of $_0$ = R.

$^0( ) = \lim_{\to 0} \frac{(+)-()}{} = \lim_{\to 0} \frac{[(+)^2-2(+)^3]-(^2-2^3)}{}$

$= \lim_{\to 0} \frac{^2+2+^2-2^3-6^2-6^2-2^3-^2+2^3}{}$

$= \lim_{\to 0} \frac{2+^2-6^2-6^2-2^3}{} = \lim_{\to 0} (2+-6^2-6-2^2)$

$= \lim_{\to 0} (2+-6^2-6-2^2) = 2-6^2$

Domain of = domain of $_0$ = R.

$^0( ) = \lim_{\to 0} \frac{(+) - ()}{} = \lim_{\to 0} \frac{\frac{1}{\sqrt{+}} - \frac{1}{\sqrt{}}}{} = \lim_{\to 0} \frac{\frac{\sqrt{}-\sqrt{+}}{\sqrt{+}\sqrt{}}}{} = \lim_{\to 0} \frac{\sqrt{}-\sqrt{+}}{\sqrt{}\sqrt{+}} \cdot \frac{\sqrt{}+\sqrt{+}}{\sqrt{}+\sqrt{+}}$

$= \lim_{\to 0} \frac{-(+)}{\sqrt{}\sqrt{+}(\sqrt{}+\sqrt{+})} = \lim_{\to 0} \frac{-}{\sqrt{}\sqrt{+}(\sqrt{}+\sqrt{+})} = \lim_{\to 0} \frac{-1}{\sqrt{}\sqrt{+}(\sqrt{}+\sqrt{+})}$

$= \frac{-}{} = \frac{-}{} = \frac{}{\sqrt{}\sqrt{}(\sqrt{}+\sqrt{})} \quad 2\sqrt{}$

Domain of = domain of 0 = $(0\ \infty)$.

27. $\lim \frac{(+)-()}{} = \lim \frac{\sqrt{9-(+)} - \sqrt{9-}}{} \cdot \frac{\sqrt{9-(+)}+\sqrt{9-}}{\sqrt{9-(+)}+\sqrt{9-}}$

$= \lim \frac{[9-(+)]-(9-)}{} = \lim \frac{-}{}$

$= \lim_{\to 0} \frac{-1}{\sqrt{9-(+)}+\sqrt{9-}} = \frac{-1}{2\sqrt{9-}}$

Domain of $= (-\infty\ 9]$, domain of ${}^0 = (-\infty\ 9)$.

28. ${}^0(\ ) = \lim_{\to 0} \frac{(+)-()}{} = \lim_{\to 0} \frac{\frac{(+)^2-1}{2(+)-3} - \frac{{}^2-1}{2-3}}{}$

$= \lim_{\to 0} \frac{[(+)^2-1](2-3)-[2(+)-3]({}^2-1)}{[2(+)-3](2-3)}$

$= \lim_{\to 0} \frac{({}^2+2\ +{}^2-1)(2-3)-(2+2-3)({}^2-1)}{[2(+)-3](2-3)}$

$= \lim_{\to 0} \frac{(2^3+4^2+2^2-2-3^2-6-3^2+3)-(2^3+2^2-3^2-2-2+3)}{(2+2-3)(2-3)}$

$= \lim_{\to 0} \frac{4^2+2^2-6-3^2-2^2+2}{(2+2-3)(2-3)} = \lim_{\to 0} \frac{(2^2+2-6-3+2)}{(2+2-3)(2-3)}$

$= \lim_{\to 0} \frac{2^2+2-6-3+2}{(2+2-3)(2-3)} = \frac{2^2-6+2}{(2-3)^2}$

Domain of $=$ domain of ${}^0 = (-\infty\ \frac{3}{2}) \cup (\frac{3}{2}\ \infty)$.

29. ${}^0(\ ) = \lim_{\to 0} \frac{(+)-()}{} = \lim_{\to 0} \frac{\frac{1-2(+)}{3+(+)} - \frac{1-2}{3+}}{}$

$= \lim_{\to 0} \frac{[1-2(+)](3+)-[3+(+)](1-2)}{[3+(+)](3+)}$

$= \lim_{\to 0} \frac{3+-6-2^2-6-2-(3-6+-2^2+-2)}{[3+(+)](3+)} = \lim_{\to 0} \frac{-6-}{(3++)(3+)}$

$= \lim_{\to 0} \frac{-7}{(3++)(3+)} = \lim_{\to 0} \frac{-7}{(3++)(3+)} = \frac{-7}{(3+)^2}$

Domain of $=$ domain of ${}^0 = (-\infty\ -3) \cup (-3\ \infty)$.

30. ${}^0(\ ) = \lim_{\to 0} \frac{(+)-()}{} = \lim_{\to 0} \frac{(+)^{32}-{}^{32}}{} = \lim_{\to 0} \frac{[(+)^{32}-{}^{32}][(+)^{32}+{}^{32}]}{[(+)^{32}+{}^{32}]}$

$= \lim_{\to 0} \frac{(+)^3-{}^3}{[(+)^{32}+{}^{32}]} = \lim_{\to 0} \frac{{}^3+3^2+3^2+{}^3-{}^3}{[(+)^{32}+{}^{32}]} = \lim_{\to 0} \frac{3^2+3+{}^2}{[(+)^{32}+{}^{32}]}$

$= \lim_{\to 0} \frac{3^2+3+{}^2}{(+)^{32}+{}^{32}} = \frac{3^2}{2^{32}} = \frac{3}{2}{}^{12}$

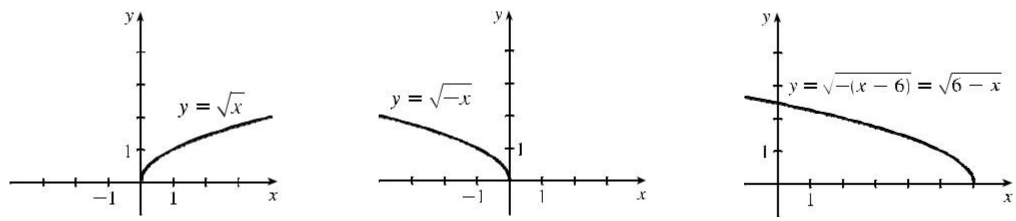
Domain of = domain of $_0$ = [0 ∞). Strictly speaking, the domain of $_0$ is (0 ∞) because the limit that defines $_0(0)$ does not exist (as a two-sided limit). But the right-hand derivative (in the sense of Exercise 64) does exist at 0, so in that sense one could regard the domain of 0 to be [0 ∞).

31. ( ) = lim →0 (+) − ( ) / − = lim →0 (+) 4 − 4 / = lim →0 4 +4 3 + 6 22 +4 3 + 4 − 4 /

= lim →0 4 3 +6 22 +4 3 + 4 / = lim →0 4 3 + 6 + 4 2 2 + 3 = 4 3

Domain of = domain of 0 = R.

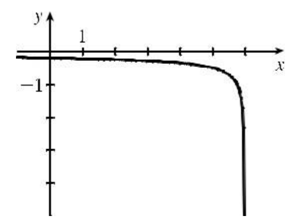
32. (a)



Note that the third graph in part (a) has small negative values for its slope, See the graph in part (d). 0; but as → 6−, 0 → −∞.

0( ) = lim →0 (+) − ( ) /

(d)



= lim →0 6 − (+) √ − √6 − · 6 (+)+√6 −

= lim [6−(+)]−(6−) = lim −

= lim →0 −1 / √6 − + √6 − = −1 / 2 √6 −

Domain of = (−∞ 6], domain of 0 = (−∞ 6).

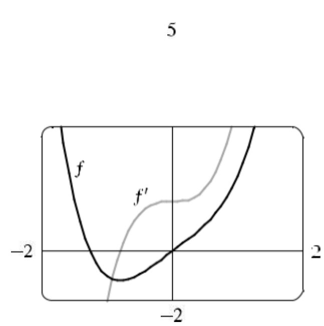
33. (a) 0( ) = lim →0 (+)−() = lim →0 [(+)4+2(+)]−(4+2)

= lim →0 4+4 3 +6 22 +4 3 + 4 +2 +2 − 4 −2

= lim →0 4 3 +6 22 +4 3 + 4 + 2 / = lim →0 (4 3 +6 2 +4 3 + +2) / 2

= lim →0 (4 3 + 6 2 + 4 2 + 3 + 2) = 4 3 + 2

(b) Notice that 0( ) = 0 when has a horizontal tangent, 0( ) is positive when the tangents have positive slope, and 0( ) is negative when the tangents have negative slope.

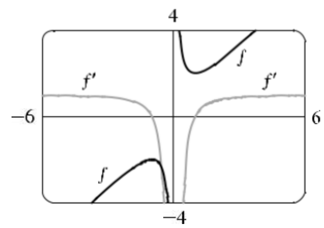


34. (a) $^0( ) = \lim_{\to 0} \frac{(+)-()}{} = \lim_{\to 0} \frac{[(+)+1(+)]-(+1)}{} = \lim_{\to 0} \frac{\frac{(+)^2+1}{+} - \frac{^2+1}{}}{}$

$= \lim_{\to 0} \frac{[(+)^2+1]-(+)(^2+1)}{(+)} = \lim_{\to 0} \frac{(^3+2\ ^2+\ ^2+)-(^3+ +\ ^2+)}{(+)}$

$= \lim_{\to} \frac{^2+\ ^2-}{(+)} = \lim_{\to} \frac{(\ ^2+\ -1)}{(+)} = \lim_{\to} \frac{^2+\ -1}{(+)} = \frac{^2-1}{^2}$, or $1 - \frac{1}{^2}$

(b) Notice that $^0( ) = 0$ when has a horizontal tangent, $^0( )$ is positive when the tangents have positive slope, and $^0( )$ is negative when the tangents have negative slope. Both functions are discontinuous at $= 0$.



(a) $^0( )$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent unemployed per year.

(b) To find $^0( )$, we use $\lim_{\to} \frac{(+)-()}{} \approx \frac{(+)-()}{}$ for small values of .

For 2003: $^0(2003) \approx \frac{(2004)-(2003)}{2004-2003} = \frac{55-60}{1} = -0\ 5$

For 2004: We estimate $^0(2004)$ by using $= -1$ and $= 1$, and then average the two results to obtain a final estimate.

$= 1 \Rightarrow\ ^0(2004) \approx \frac{(2003)-(2004)}{2003-2004} = \frac{60-55}{-1} = -05;$

$= 1 \Rightarrow\ ^0(2004) \approx \frac{(2005)-(2004)}{2005-2004} = \frac{51-55}{1} = -0\ 4.$

So we estimate that $^0(2004) \approx \frac{1}{2}[-0\ 5 + (-0\ 4)] = -0\ 45.$

	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
$^0( )$	−0 50	−0 45	−045	−025	060	2 35	190	−020	−075	−080

(a) $^0( )$ is the rate at which the number of minimally invasive cosmetic surgery procedures performed in the United States is changing with respect to time. Its units are thousands of surgeries per year.

(b) To find $^0( )$, we use $\lim_{\to} \frac{(+)-()}{} \approx \frac{(+)-()}{}$ for small values of .

For 2000: $^0(2000) \approx \frac{(2002)-(2000)}{2002-2000} = \frac{4897-5500}{2} = -301\ 5$

For 2002: We estimate $^0(2002)$ by using $= -2$ and $= 2$, and then average the two results to obtain a final estimate.

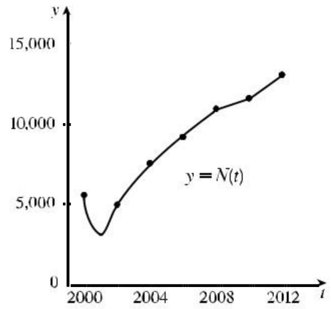
$= 2 \Rightarrow\ ^0(2002) \approx \frac{(2000)-(2002)}{2000-2002} = \frac{5500-4897}{-2} = -301\ 5$

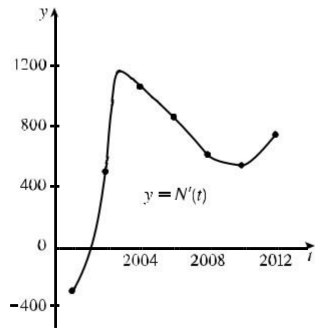
$= 2 \Rightarrow\ ^0(2002) \approx \frac{(2004)-(2002)}{2004-2002} = \frac{7470-4897}{2} = 1286\ 5$

So we estimate that $^0(2002) \approx \frac{1}{2}[-301\ 5 + 1286\ 5] = 492\ 5.$

	2000	2002	2004	2006	2008	2010	2012
$0()$	-301.5	492.5	1060.25	856.75	605.75	534.5	737

(c)

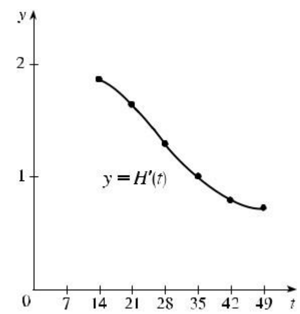




(d) We could get more accurate values for 0( ) by obtaining data for more values of .

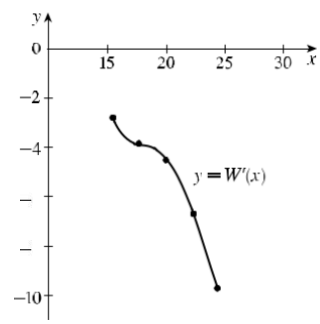
37. As in Exercise 35, we use one-sided difference quotients for the first and last values, and average two difference quotients for all other values.

	14	21	28	35	42	49
( )	41	54	64	72	78	83
0( )	$\frac{13}{7}$	$\frac{23}{14}$	$\frac{18}{14}$	$\frac{14}{14}$	$\frac{11}{14}$	$\frac{5}{7}$



As in Exercise 35, we use one-sided difference quotients for the first and last values, and average two difference quotients for all other values. The units for 0( ) are grams per degree (g ◦C).

	15 5	17 7	20 0	22 4	24 4
( )	37 2	31 0	19 8	9 7	−9 8
0( )	−2 82	−3 87	−4 53	−6 73	−9 75



(a)is the rate at which the percentage of the city's electrical power produced by solar panels changes with respect to time , measured in percentage points per year.

2 years after January 1, 2000 (January 1, 2002), the percentage of electrical power produced by solar panels was increasing at a rate of 3.5 percentage points per year.

is the rate at which the number of people who travel by car to another state for a vacation changes with respect to the price of gasoline. If the price of gasoline goes up, we would expect fewer people to travel, so we would expect to be negative.

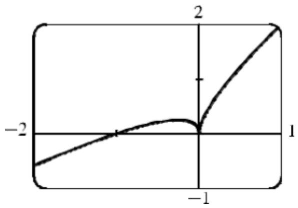
is not differentiable at $= -4$, because the graph has a corner there, and at $= 0$, because there is a discontinuity there.

is not differentiable at $= -1$, because there is a discontinuity there, and at $= 2$, because the graph has a corner there.

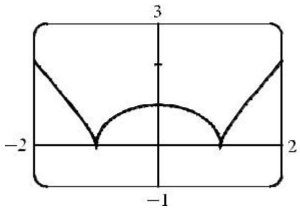
is not differentiable at $= 1$, because is not defined there, and at $= 5$, because the graph has a vertical tangent there.

is not differentiable at $= -2$ and $= 3$, because the graph has corners there, and at $= 1$, because there is a discontinuity there.

45. As we zoom in toward $(-1\ 0)$, the curve appears more and more like a straight line, so ( ) = + ∏is differentiable at = −1. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So is not differentiable at = 0.



46. As we zoom in toward $(0\ 1)$, the curve appears more and more like a straight line, so is differentiable at = 0. But no matter how much we zoom in toward (1 0) or (−1 0), the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So is not differentiable at = ±1.



Call the curve with the positive -intercept and the other curve . Notice that has a maximum (horizontal tangent) at = 0, but 6= ,0so cannot be the derivative of . Also notice that where is positive, is increasing. Thus, = and = 0. Now 0(−1) is negative since 0 is below the -axis there and 00(1) is positive since is concave upward at = 1. Therefore, 00(1) is greater than 0(−1).

Call the curve with the smallest positive -intercept and the other curve . Notice that where is positive in the first quadrant, is increasing. Thus, = and = 0. Now 0(−1) is positive since 0 is above the -axis there and 00(1) appears to be zero since has an inflection point at = 1. Therefore, 0(1) is greater than 00(−1).

= , = 0, = 00. We can see this because where has a horizontal tangent, = 0, and where has a horizontal tangent, = 0. We can immediately see that can be neither nor 0, since at the points where has a horizontal tangent, neither nor is equal to 0.

Where has horizontal tangents, only is 0, so 0 = . has negative tangents for 0 and is the only graph that is negative for 0, so 0 = . has positive tangents on R (except at = 0), and the only graph that is positive on the same domain is , so 0 = . We conclude that = , = 0, = 00, and = 000.

We can immediately see that is the graph of the acceleration function, since at the points where has a horizontal tangent, neither nor is equal to 0. Next, we note that = 0 at the point where has a horizontal tangent, so must be the graph of the velocity function, and hence, 0 = . We conclude that is the graph of the position function.

must be the jerk since none of the graphs are 0 at its high and low points. is 0 where has a maximum, so 0 = . is 0 where has a maximum, so 0 = . We conclude that is the position function, is the velocity, is the acceleration, and is the jerk.

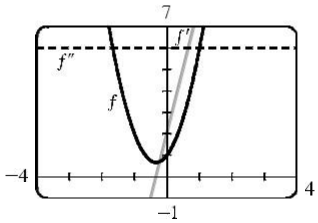
$$0(\ ) = \lim_{\to 0} \frac{(\ +\ ) - (\ )}{} = \lim_{\to 0} \frac{[3(\ +\ )^2 + 2(\ +\ ) + 1] - (3^2 + 2\ + 1)}{}$$

$$= \lim_{\to 0} \frac{(3^2 + 6\ + 3^2 + 2 + 2 + 1) - (3^2 + 2 + 1)}{} = \lim_{\to 0} \frac{6\ + 3^2 + 2}{}$$

$$= \lim_{\to 0} \frac{(6 + 3 + 2)}{} = \lim_{\to 0} (6\ + 3\ + 2) = 6\ + 2$$

$00(\ ) = \lim_{\to 0} \frac{0(+\ ) - 0()}{} = \lim_{\to 0} \frac{[6(\ +\ )+2] - (6\ +2)}{} = \lim_{\to 0} \frac{(6\ +6\ +2) - (6\ +2)}{}$

$= \lim_{\to 0} \frac{6}{} = \lim_{\to 0} 6 = 6$



We see from the graph that our answers are reasonable because the graph of 0 is that of a linear function and the graph of 00 is that of a constant function.

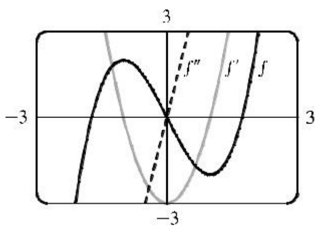
54. $0(\ ) = \lim_{\to 0} \frac{(\ +\ ) - (\ )}{} = \lim_{\to 0} \frac{[(\ +\ )^3 - 3(\ +\ )] - (\ ^3 - 3\ )}{}$

$= \lim_{\to 0} \frac{(\ ^3 + 3\ ^2\ + 3\ \ ^2 + \ ^3 - 3\ - 3\ ) - (\ ^3 - 3\ )}{} = \lim_{\to 0} \frac{3\ ^2\ + 3\ \ ^2 + \ ^3 - 3\ }{}$

$= \lim \frac{(3\ ^2 + 3\ + \ ^2 - 3)}{} = \lim (3\ ^2 + 3\ \ + \ ^2 - 3) = 3\ ^2 - 3$

$00(\ ) = \lim_{\to 0} \frac{0(+\ ) - 0()}{} = \lim_{\to 0} \frac{[3(\ +\ )^2 - 3] - (3\ ^2 - 3)}{} = \lim_{\to 0} \frac{(3\ ^2 + 6\ \ + 3\ ^2 - 3) - (3\ ^2 - 3)}{}$

$= \lim_{\to 0} \frac{6\ \ + 3\ ^2}{} = \lim_{\to 0} \frac{(6\ + 3\ )}{} = \lim_{\to 0} (6\ + 3\ ) = 6$



We see from the graph that our answers are reasonable because the graph of 0 is that of an even function (is an odd function) and the graph of 00 is that of an odd function. Furthermore, 0 = 0 when has a horizontal tangent and 00 = 0 when 0 has a horizontal tangent.

55. $0(\ ) = \lim_{\to 0} \frac{(\ +\ ) - (\ )}{} = \lim_{\to 0} \frac{2(\ +\ )^2 - (\ +\ )^3 - (2\ ^2 - \ ^3)}{}$

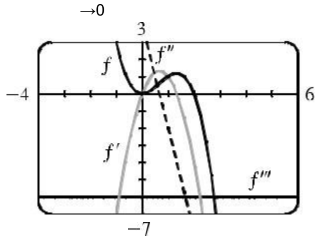
$= \lim \frac{(4\ \ + 2\ ^2 - 3\ ^2\ - 3\ \ ^2 - \ ^3)}{} = \lim (4\ + 2\ - 3\ ^2 - 3\ \ - \ ^2) = 4\ - 3\ ^2$

$00(\ ) = \lim_{\to 0} \frac{0(+) - 0()}{} = \lim_{\to 0} \frac{4(\ +\ ) - 3(\ +\ )^2 - (4\ - 3\ ^2)}{} = \lim_{\to 0} \frac{(4\ - 6\ \ - 3\ ^2)}{}$

$= \lim (4 - 6\ - 3\ ) = 4 - 6$

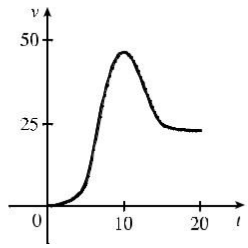
$000(\ ) = \lim \frac{00(\ +\ ) - 00(\ )}{} = \lim \frac{[4 - 6(\ +\ )] - (4 - 6\ )}{} = \lim \frac{-6\ }{} = \lim (\ 6) = \ 6$

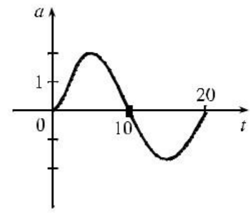
$(4)(\ ) = \lim_{\to 0} \frac{000(\ +\ ) - 000(\ )}{} = \lim_{\to 0} \frac{-6 - (-6)}{} = \lim_{\to 0} \frac{0}{} = \lim_{\to 0} (0) = 0$



The graphs are consistent with the geometric interpretations of the derivatives because 0 has zeros where has a local minimum and a local maximum, 00 has a zero where 0 has a local maximum, and 000 is a constant function equal to the slope of 00.

56. (a) Since we estimate the velocity to be a maximum at $= 10$, the acceleration is 0 at $= 10$.





Drawing a tangent line at $= 10$ on the graph of , appears to decrease by 10 ft s2 over a period of 20 s. So at $= 10$ s, the jerk is approximately $-10\ 20 = -0\ 5$ (ft s^2) s or ft s^3.

(a) Note that we have factored $-$ as the difference of two cubes in the third step.

$$0(\ ) = \lim_{\to} \frac{(\ ) - (\ )}{-} = \lim_{\to} \frac{13 - 13}{-} = \lim_{\to} \frac{13 - 13}{(13 - 13)(23 + 13 13 + 23)}$$

$$= \lim_{\to} \frac{1}{23 + 13\ 13 + 23} = \frac{1}{3\ 2\ 3} \text{ or } \tfrac{1}{3} -2\ 3$$

(b) $0(0) = \lim_{\to 0} \dfrac{(0 + \ ) - (0)}{} = \lim_{\to 0} \dfrac{\sqrt{3}\ - 0}{} = \lim_{\to 0} \dfrac{1}{2\ 3}$. This function increases without bound, so the limit does not exist, and therefore $0(0)$ does not exist.

(c) $\lim_{\to 0} |\ 0(\ )| = \lim_{\to 0} \dfrac{1}{} = \infty$ and is continuous at $= 0$ (root function), so has a vertical tangent at $= 0$.

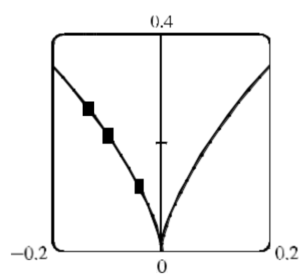
58. (a) $0(0) = \lim_{\to 0} \dfrac{(\ ) - (0)}{- 0} = \lim_{\to 0} \dfrac{2\ 3 - 0}{} = \lim_{\to 0} \dfrac{1}{1\ 3}$, which does not exist.

(b) $0(\ ) = \lim_{\to} \dfrac{(\ ) - (\ )}{-} = \lim_{\to} \dfrac{23 - 23}{-} = \lim_{\to} \dfrac{(13 - 13)(13 + 13)}{(13 - 13)(23 + 13\ 13 + 23)}$

$$= \lim_{\to} \frac{13 + 13}{23 + 1313 + 23} = \frac{2\ 1\ 3}{3\ 2\ 3} = \frac{2}{3\ 1\ 3} \text{ or } \tfrac{2}{3} -1\ 3$$

(c) $(\ ) = 2\ 3$ is continuous at $= 0$ and $\lim_{\to 0} |\ 0(\ )| = \lim_{\to 0} \dfrac{2}{3\ |\ |1\ 3} = \infty$. This shows that has a vertical tangent line at $= 0$.

(d)



59. $(\ ) = |\ - 6| = \begin{cases} - 6 & \text{if } - 6 \geq 6 \\ -(- 6) & \text{if } - 6 < 0 \end{cases} = \begin{cases} - 6 & \text{if } \geq 6 \\ 6 - & \text{if } < 6 \end{cases}$

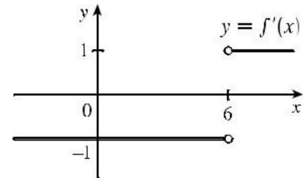
So the right-hand limit is $\lim_{\to 6^+} \dfrac{(\ ) - (6)}{- 6} = \lim_{\to 6^+} \dfrac{|\ - 6| - 0}{- 6} = \lim_{\to 6^+} \dfrac{- 6}{- 6} = \lim_{\to 6^+} 1 = 1$, and the left-hand limit is $\lim_{\to 6^-} \dfrac{(\ ) - (6)}{- 6} = \lim_{\to 6^-} \dfrac{|\ - 6| - 0}{- 6} = \lim_{\to 6^-} \dfrac{6 -}{- 6} = \lim_{\to 6^-} (-1) = -1$. Since these limits are not equal,

$^0(6) = \lim_{\to 6} \frac{(\ ) - (6)}{- 6}$ does not exist and is not differentiable at 6.

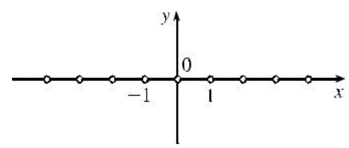
However, a formula for ^{0} is $^0(\ ) =$ 1 if 6

−1 if 6

Another way of writing the formula is $^0(\ ) = \frac{- 6}{| - 6|}$.



60. ( ) = [[ ]] is not continuous at any integer , so is not differentiable at by the contrapositive of Theorem 4. If is not an integer, then is constant on an open interval containing , so $_0(\ ) = 0$. Thus, $^0(\ ) = 0$, not an integer.

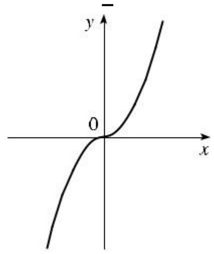


61. (a) ( ) = | | = ^{2} if ≥ 0; ^{2} if 0

(b) Since ( ) = ^{2} for ≥ 0, we have $^0(\ ) = 2$ for 0. [See Exercise 19(d).] Similarly, since ( ) = − ^{2} for 0, we have $^0(\ ) = -2$ for 0. At = 0, we have

$$_0(0) = \lim_{\to 0} \frac{(\ ) - (0)}{- 0} = \lim_{\to 0} \frac{|\ |}{} = \lim_{\to 0} |\ | = 0$$

So is differentiable at 0. Thus, is differentiable for all .

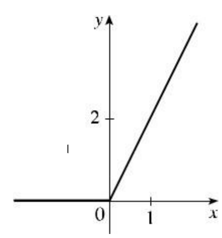


(c) From part (b), we have $_0(\ ) =$ 2 if ≥ 0; −2 if 0 = 2| |.

62. (a) | | = if ≥ 0; − if 0

so ( ) = + | | = 2 if ≥ 0; 0 if 0.

Graph the line = 2 for ≥ 0 and graph = 0 (the *x*-axis) for 0. is not differentiable at = 0 because the graph has a corner there, but is differentiable at all other values; that is, is differentiable on (−∞ 0) ∪ (0 ∞).



(c) ( ) = 2 if ≥ 0; 0 if 0 ⇒ $_0(\ ) =$ 2 if 0; 0 if 0

Another way of writing the formula is $^0(\ ) = 1 + \text{sgn}$ for 6= 0.

63. (a) If is even, then

$$_0(-\ ) = \lim_{\to 0} \frac{(- + \ ) - (-\ )}{} = \lim_{\to 0} \frac{[-(\ -\ )] - (-\ )}{}$$

$$= \lim_{\to 0} \frac{(-) - ()}{} = -\lim_{\to 0} \frac{(-) - ()}{-} \quad [\text{let } \Delta = -\ ]$$

$$= -\lim_{\Delta \to 0} \frac{(+\Delta) - ()}{\Delta} = -\ ^0(\ )$$

Therefore, $_0$ is odd.

(b) If is odd, then

$0(-) = \lim_{\to 0} \frac{(-+)-(-)}{} = \lim_{\to 0} \frac{[-(-)]-(-)}{}$

$= \lim_{\to 0} \frac{-(-)+()}{} = \lim_{\to 0} \frac{(-)-()}{-}$ [let Δ =]

$= \lim_{\Delta \to 0} \frac{(+\Delta)-()}{\Delta} = {}^{0}()$

Therefore, 0 is even.

64. (a) $^{)}(4) = \lim_{\to 0^-} \frac{(4+)-(4)}{} = \lim_{\to 0^-} \frac{5-(4+)-1}{}$

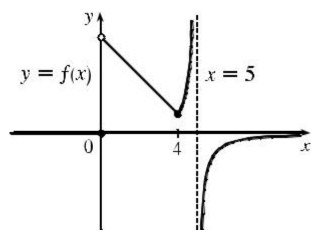
$= \lim_{\to 0^-} \frac{-}{} = 1$

and

$_+{}^{0}(4) = \lim_{\to} \frac{(4+)-(4)}{} = \lim_{\to} \frac{\frac{1}{5-(4+)} - 1}{}$

$= \lim_{\to 0^+} \frac{1-(1-)}{(1-)} = \lim_{\to 0^+} \frac{1}{1-} = 1$

(b)



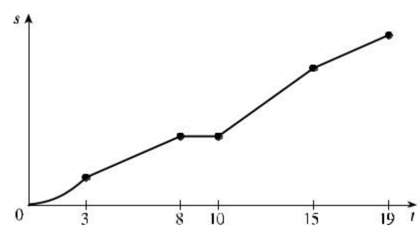
(c) $() = \begin{cases} 0 & \text{if } \le 0 \\ 5 - & \text{if } 0 4 \\ 1(5-) & \text{if } \ge 4 \end{cases}$

At 4 we have $\lim_{\to 4^-} () = \lim_{\to 4^-} (5 -) = 1$ and $\lim_{\to 4^+} () = \lim_{\to 4^+} \frac{1}{5-} = 1$, so $\lim_{\to 4} () = 1 = (4)$ and is continuous at 4. Since (5) is not defined, is discontinuous at 5. These expressions show that is continuous on the intervals $(-\infty, 0)$, $(0, 4)$, $(4, 5)$ and $(5, \infty)$. Since $\lim_{\to 0^+} () = \lim_{\to 0^+} (5 -) = 5 \ne 0 = \lim_{\to 0^-} ()$, $\lim_{\to 0} ()$ does not exist, so is discontinuous (and therefore not differentiable) at 0.

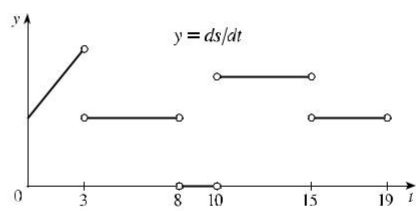
(d) From (a), is not differentiable at 4 since $_{-}{}^{0}(4) \ne {}_{+}{}^{0}(4)$, and from (c), is not differentiable at 0 or 5.

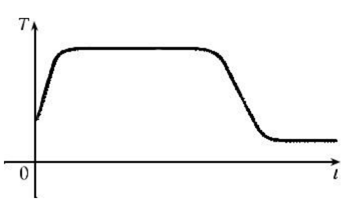
These graphs are idealizations conveying the spirit of the problem. In reality, changes in speed are not instantaneous, so the graph in (a) would not have corners and the graph in (b) would be continuous.

(a)



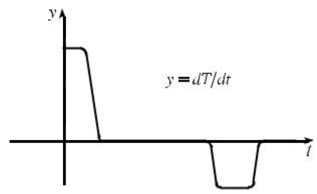
(b)

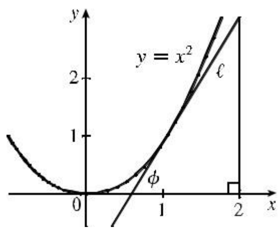




(a)

(b) The initial temperature of the water is close to room temperature because of the water that was in the pipes. When the water from the hot water tank starts coming out, is large and positive as increases to the temperature of the water in the tank. In the next phase, $= 0$ as the water comes out at a constant, high temperature. After some time, becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, is once again 0 as the water maintains its (cold) temperature.

(c) 

67. 

In the right triangle in the diagram, let Δ be the side opposite angle and Δ the side adjacent to angle . Then the slope of the tangent line is $= \Delta \Delta = \tan$. Note that 0. We know (see Exercise 19) that the derivative of $(\ ) = {}^2$ is ${}^0(\ ) = 2$. So the slope of the tangent to the curve at the point $(1\ 1)$ is 2. Thus, is the angle between 0 and $\frac{}{2}$ whose tangent is 2; that is, $= \tan^{-1} 2 \approx 63^\circ$.

2 Review

TRUE-FALSE QUIZ

False. Limit Law 2 applies only if the individual limits exist (these don't).

False. Limit Law 5 cannot be applied if the limit of the denominator is 0 (it is).

True. Limit Law 5 applies.

False. $\frac{{}^2 - 9}{- 3}$ is not defined when $= 3$, but $+ 3$ is.

True. $\lim_{\to 3} \frac{{}^2 - 9}{- 3} = \lim_{\to 3} \frac{(\ +3)(\ -3)}{(\ -3)} = \lim_{\to 3} (\ + 3)$

True. The limit doesn't exist since $(\ )\ \ (\ )$ doesn't approach any real number as approaches 5. (The denominator approaches 0 and the numerator doesn't.)

7. False. Consider $\lim_{\to 5} \frac{(\ -5)}{-5}$ or $\lim_{\to 5} \frac{\sin(\ -5)}{-5}$. The first limit exists and is equal to 5. By Example 2.2.3, we know that the latter limit exists (and it is equal to 1).

8. False. If $(\ ) = 1$, $(\ ) = -1$, and $= 0$, then $\lim_{\to 0} (\ )$ does not exist, $\lim_{\to 0} (\ )$ does not exist, but $\lim_{\to 0} [(\ ) + (\ )] = \lim_{\to 0} 0 = 0$ exists.

9. True. Suppose that $\lim_{\to} [(\ ) + (\ )]$ exists. Now $\lim_{\to} (\ )$ exists and $\lim_{\to} (\ )$ does not exist, but $\lim_{\to} \{\ (\ )] - \ \} - \lim_{\to} (\ )$ [by Limit Law 2], which exists, and we have a contradiction. Thus, $\lim_{\to} [(\ ) + (\ )]$ does not exist.

10. False. Consider $\lim_{\to 6} [\,(\,)\,(\,)] = \lim_{\to 6} \frac{1}{(\;-6)}$. It exists (its value is 1) but $(6) = 0$ and (6) does not exist,

so $(6)\,(6) \neq 1$.

11. True. A polynomial is continuous everywhere, so $\lim_{\to} (\,)$ exists and is equal to $(\,)$.

12. False. Consider $\lim_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{x^4}\right]$. This limit is $-\infty$ (not 0), but each of the individual functions approaches ∞.

13. True. See Figure 2.6.8.

14. False. Consider $f(x) = \sin x$ for $x \geq 0$. $\lim_{x\to\infty} f(x) \neq \pm\infty$ and f has no horizontal asymptote.

$= 1(x-1)$ if $x \neq 1$

15. False. Consider $f(x) = 2 -$ if $x = 1$

16. False. The function f must be *continuous* in order to use the Intermediate Value Theorem. For example, let

$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 3 \\ -1 & \text{if } x = 3 \end{cases}$ There is no number $c \in [0, 3]$ with $f(c) = 0$.

17. True. Use Theorem 2.5.8 with $a = 2$, $b = 5$, and $g(x) = 4x^2 - 11$. Note that $f(4) = 3$ is not needed.

18. True. Use the Intermediate Value Theorem with $a = -1$, $b = 1$, and $N = \pi$, since $3 < \pi < 4$.

True, by the definition of a limit with $\varepsilon = 1$.

20. False. For example, let $f(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$. Then $f(x) \leq 1$ for all x, but $\lim_{x\to 0} f(x) = \lim_{x\to 0}(x^2 + 1) = 1$.

False. See the note after Theorem 2.8.4.

True. $f'(r)$ exists $\Rightarrow$ f is differentiable at r $\Rightarrow$ f is continuous at r $\Rightarrow$ $\lim_{x\to r} f(x) = f(r)$.

False. $\frac{d^2y}{dx^2}$ is the second derivative while $\left(\frac{dy}{dx}\right)^2$ is the first derivative squared. For example, if $y = x$, then $\frac{d^2y}{dx^2} = 0$, but $\left(\frac{dy}{dx}\right)^2 = 1$.

True. $f(x) = x^{10} - 10x^2 + 5$ is continuous on the interval $[0, 2]$, $f(0) = 5$, $f(1) = -4$, and $f(2) = 989$. Since $-4 < 0 < 5$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^{10} - 10x^2 + 5 = 0$ in the interval $(0, 1)$. Similarly, there is a root in $(1, 2)$.

25. True. See Exercise 2.5.72(b).

26. False See Exercise 2.5.72(b).

NOT FOR SALE

EXERCISES

1. (a) (i) $\lim\limits_{\to 2^+} (\ ) = 3$ (ii) $\lim\limits_{\to -3^+} (\ ) = 0$

(iii) $\lim\limits_{\to -3} (\ )$ does not exist since the left and right limits are not equal. (The left limit is -2.)

$\lim\limits_{\to 4} (\ ) = 2$

$\lim (\ ) = \infty$ (vi) $\lim\limits_{\to 2^-} (\ ) = -\infty$

(vii) $\lim\limits_{\to\infty} (\ ) = 4$ (viii) $\lim\limits_{\to -\infty} (\ ) = -1$

The equations of the horizontal asymptotes are $= -1$ and $= 4$.

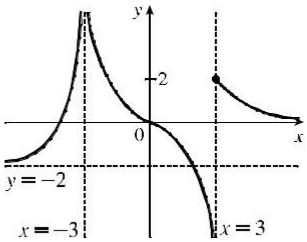
The equations of the vertical asymptotes are $= 0$ and $= 2$.

is discontinuous at $= -3, 0, 2$, and 4. The discontinuities are jump, infinite, infinite, and removable, respectively.

2. $\lim\limits_{\to -\infty} (\ ) = -2$, $\lim\limits_{\to\infty} (\ ) = 0$, $\lim\limits_{\to -3} (\ ) = \infty$,

$\lim\limits_{\to 3^-} (\ ) = -\infty$, $\lim\limits_{\to 3^+} (\ ) = 2$,

is continuous from the right at 3



Since the exponential function is continuous, $\lim\limits_{\to 1} {}^{3-} = {}^{1-1} = {}^{0} = 1$.

4. Since rational functions are continuous, $\lim\limits_{\to 3} \dfrac{^2 - 9}{^2 + 2 - 3} = \dfrac{3^2 - 9}{3^2 + 2(3) - 3} = \dfrac{0}{12} = 0$.

5. $\lim\limits_{\to -3} \dfrac{^2 - 9}{^2 + 2 - 3} = \lim\limits_{\to -3} \dfrac{(\ +3)(\ -3)}{(\ +3)(\ -1)} = \lim\limits_{\to -3} \dfrac{\ -3}{\ -1} = \dfrac{-3-3}{-3-1} = \dfrac{-6}{-4} = \dfrac{3}{2}$

6. $\lim\limits_{\to 1^+} \dfrac{^2 - 9}{^2 + 2 - 3} = -\infty$ since $^2 + 2 - 3 \to 0^+$ as $\to 1^+$ and $\dfrac{^2 - 9}{^2 + 2 - 3} < 0$ for $1 < < 3$.

7. $\lim\limits_{\to 0} \dfrac{(\ -1)^3 + 1}{} = \lim\limits_{\to 0} \dfrac{^3 - 3\,^2 + 3 - 1 + 1}{} = \lim\limits_{\to 0} \dfrac{^3 - 3\,^2 + 3}{} = \lim\limits_{\to 0} (\,^2 - 3 + 3) = 3$

Another solution: Factor the numerator as a sum of two cubes and then simplify.

$\lim\limits_{\to 0} \dfrac{(\ -1)^3 + 1}{} = \lim\limits_{\to 0} \dfrac{(\ -1)^3 + 1^3}{} = \lim\limits_{\to 0} \dfrac{[(\ -1) + 1]\,[(\ -1)^2 - 1(\ -1) + 1^2]}{}$

$= \lim\limits_{\to 0} [(\ -1)^2 - \ + 2] = 1 + 0 + 2 = 3$

8. $\lim\limits_{\to 2} \dfrac{^2 - 4}{^3 - 8} = \lim\limits_{\to 2} \dfrac{(\ +2)(\ -2)}{(\ -2)(\,^2 + 2 + 4)} = \lim\limits_{\to 2} \dfrac{\ +2}{^2 + 2 + 4} = \dfrac{2 + 2}{4 + 4 + 4} = \dfrac{4}{12} = \dfrac{1}{3}$

9. $\lim\limits_{\to 9} \dfrac{\sqrt{\ }}{} = \infty$ since $(\ -9) \to 0$ as $\to 9$ and $\dfrac{\sqrt{\ }}{} > 0$ for $\neq 9$.

10. $\lim_{\to 4^+} \frac{4-}{|4-|} = \lim_{\to 4^+} \frac{4-}{-(4-)} = \lim_{\to 4^+} \frac{1}{-1} = 1$ −

11. $\lim_{\to 1} \frac{{}^4-1}{{}^3+5{}^2-6} = \lim_{\to 1} \frac{({}^2+1)({}^2-1)}{({}^2+5-6)} = \lim_{\to 1} \frac{({}^2+1)(+1)(-1)}{(+6)(-1)} = \lim_{\to 1} \frac{({}^2+1)(+1)}{(+6)} = \frac{2(2)}{1(7)} = \frac{4}{7}$

12. $\lim_{\to 3} \frac{\sqrt{+6}-}{3\;\;3^2} = \lim_{\to 3} \frac{\sqrt{+6}-}{{}^2(\;\;3)} \cdot \frac{\sqrt{+6}+}{\sqrt{+6}+} = \lim_{\to 3} \frac{(+6)\;\;{}^2}{{}^2(\;\;3)\sqrt{+6}+}$

− − $2-$

$= \lim_{\to 3} \frac{+6-{}^2}{{}^2(\;\;3)\sqrt{+6}+} = \lim_{\to 3} \frac{-(-\;-6)}{{}^2(\;\;3)\sqrt{+6}+} = \lim_{\to 3} \frac{-(-3)(+\;\;2)}{{}^2(\;\;3)\sqrt{+6}+}$

$= \lim_{\to 3} \frac{-}{{}^2\sqrt{+6}+} = \frac{-}{9(3+3)} = -\frac{}{54}$

Since is positive, $\sqrt{{}^2} = |\,| =$. Thus,

$\lim_{\to\infty} \frac{\sqrt{{}^2\;\;9}}{} = \lim_{\to\infty} \frac{\sqrt{{}^2\;\;9}}{} = \lim_{\to\infty} \frac{\sqrt{1\;\;9}}{} = \frac{\sqrt{1\;\;0}}{} = \frac{1}{2}$

Since is negative, $\sqrt{{}^2} = |\,| = -$. Thus,

$\lim_{\to-\infty} \frac{\sqrt{{}^2\;\;9}}{2\;\;6} = \lim_{\to-\infty} \frac{\sqrt{}\;\;\sqrt{9}}{(2\;\;6)(\;\;)} = \lim_{\to-\infty} \frac{\sqrt{1-9}}{2+6} = \frac{1-0}{2+0} = -\frac{1}{2}$

15. Let $= \sin$. Then as $\to$ $-$ $+$, $\sin \to 0$ $+$, so $\to 0$ $+$. Thus, $\lim_{\to}$ $\ln(\sin\;) = \lim_{\to 0^+} \ln = -\infty$.

16. $\lim_{\to-\infty} \frac{1-2{}^2-{}^4}{5+\;\;3{}^4} = \lim_{\to-\infty} \frac{(1-2{}^2-{}^4)\;{}^4}{(5+\;\;3{}^4)\;{}^4} = \lim_{\to-\infty} \frac{1\;{}^4-2\;{}^2-1}{5\;{}^4+1\;{}^3\;\;3} = \frac{0-0-1}{0+0\;\;3} = \frac{-1}{-3} = \frac{1}{3}$

17. $\lim_{\to\infty} \left(\sqrt{{}^2+4+1}\;-\;\right) = \lim_{\to\infty} \frac{\sqrt{{}^2+4+1}\;-}{1} \cdot \frac{\sqrt{{}^2+4+1}+}{\sqrt{{}^2+4+1}+} = \lim_{\to\infty} \frac{({}^2+4+1)-{}^2}{\sqrt{{}^2+4+1}+}$

$= \lim_{\to\infty} \frac{(4+1)\;\;}{(\sqrt{{}^2+4+1}+)\;\;}$ divide by $= \sqrt{{}^2}$ for 0

$= \lim_{\to\infty} \frac{4+1}{\sqrt{1+4+1\;{}^2}+1} = \frac{4+0}{\sqrt{1+0+0}+1} = \frac{4}{2} = 2$

18. Let $= -{}^2 = (1-)$. Then as $\to\infty$, $\to -\infty$, and $\lim_{\to\infty}$ $-{}^2 = \lim_{\to-\infty} = 0$.

19. Let $= 1$. Then as $\to 0$ $+$, $\to\infty$. $\lim \tan^{-1}(1\;) = \lim_{\to\infty} \tan^{-1} = \frac{}{2}$.

20. $\lim_{\to 1} \left(\frac{1}{-1} + \frac{1}{{}^2-3+2}\right) = \lim_{\to 1} \left(\frac{1}{-1} + \frac{1}{(-1)(-2)}\right) = \lim_{\to 1} \left(\frac{-2}{(-1)(-2)} + \frac{1}{}\right)$

= lim − 1 = (−1)(−2)

lim 1 = 1 =

1

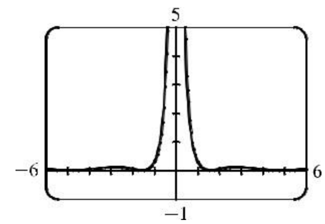
→1 (−1)(−2) →1 − 2 1 − 2 −

NOT FOR SALE

21. From the graph of $= \cos^2 2$ $= 0$ is the horizontal

$= 0$, it appears that 0 $(\cos\ )^2$ 1

asymptote and is the vertical asymptote. Now $\leq$ $\leq$ $\Rightarrow$



$\frac{0}{2} \leq \frac{\cos^2}{2} \frac{1}{2} \Rightarrow 0 \leq \frac{\cos^2}{2} \leq \frac{1}{2}$. But $\lim_{\to\pm\infty} 0 = 0$ and

$\lim_{\to\pm\infty} \frac{1}{2} = 0$, so by the Squeeze Theorem, $\lim_{\to\pm\infty} \frac{\cos^2}{2} = 0$.

Thus, $= 0$ is the horizontal asymptote. $\lim_{\to 0} \frac{\cos^2}{2} = \infty$ because $\cos^2 \to 1$ and $^2 \to 0^+$ as $\to 0$, so $= 0$ is the vertical asymptote.

From the graph of $= ( ) = \sqrt{2 + + 1} - \sqrt{2 -}$, it appears that there are 2 horizontal asymptotes and possibly 2 vertical asymptotes. To obtain a different form for , let's multiply and divide it by its conjugate.

$$_1( ) = \sqrt{2 + + 1} - \sqrt{2 -} \quad \frac{\sqrt{2 + + 1} + \sqrt{2 -}}{\sqrt{2 + + 1} + \sqrt{2 -}} = \frac{(2 + + 1) - (2 -)}{\sqrt{2 + + 1} + \sqrt{2 -}}$$

$$= \frac{2 + 1}{\sqrt{2 + + 1} + \sqrt{2 -}}$$

Now

$$\lim_{\to\infty} {}_1( ) = \lim_{\to\infty} \frac{2 + 1}{\sqrt{2 + + 1} + \sqrt{2 -}}$$

$$= \lim_{\to\infty} \frac{2 + (1\ )}{\sqrt{1 + (1\ ) + (1\ ^2)} + \sqrt{1 - (1\ )}} \quad [\text{since } \sqrt{2} = \text{ for } 0]$$

$$= \frac{2}{1 + 1} = 1,$$

so $= 1$ is a horizontal asymptote. For 0, we have $\sqrt{2} = | | = -$, so when we divide the denominator by , with 0, we get

$$\frac{\sqrt{2 + + 1} + \sqrt{2 -}}{} = -\frac{\sqrt{2 + + 1} + \sqrt{2 -}}{} = -\left[\sqrt{1 + \frac{1}{} + \frac{1}{2}} + \sqrt{1 - \frac{1}{}}\right]$$

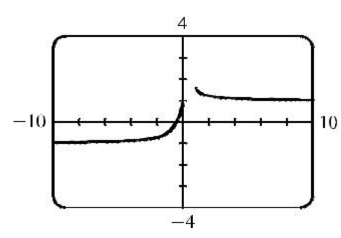
Therefore,

$$\lim_{\to -\infty} {}_1( ) = \lim_{\to -\infty} \frac{2 + 1}{\sqrt{2 + + 1} + \sqrt{2 -}} = \lim_{\to\infty} \frac{2 + (1\ )}{-\left[\sqrt{1 + (1\ ) + (1\ ^2)} + \sqrt{1 - (1\ )}\right]}$$

$$= \frac{2}{-(1 + 1)} = -1$$

so $= -1$ is a horizontal asymptote.

The domain of is $(-\infty\ 0] \cup [1\ \infty)$. As $\to 0^-$, $( ) \to 1$, so



$= 0$ is *not* a vertical asymptote. As $\to 1$, $() \to 3$, so 1 is *not* a vertical asymptote and hence there are no vertical asymptotes.

Since $2 \ - 1 \leq (\ ) \leq {}^2$ for $0 3$ and $\lim_{\to 1} (2 \ - 1) = 1 = \lim_{\to 1} {}^2$, we have $\lim_{\to 1} (\ ) = 1$ by the Squeeze Theorem.

24. Let $(\ ) = -{}^2$, $(\ ) = {}^2 \cos 1 \ {}^2$ and $(\ ) = {}^2$. Then since $\cos 1 \ {}^2$ ≤ 1 for $6= ,0$ we have

$(\ ) \leq (\ ) \leq (\ )$ for $6= ,0$ and so $\lim_{\to} (\ ) = \lim_{\to} (\ ) = 0$ $\lim_{\to} (\ ) = 0$ by the Squeeze Theorem.

25. Given 0, we need 0 such that if $0 \ | - 2|$, then $|(14 - 5\ ) - 4|$. But $|(14 - 5\ ) - 4|$ $\Leftrightarrow$

$|-5 \ + 10|$ $\Leftrightarrow$ $|-5||-2|$ $\Leftrightarrow | \ - 2|5$. So if we choose $= \ 5$, then $0 \ | - 2| \Rightarrow$ $|(14 - 5\ ) - 4|$. Thus, $\lim_{\to 2}(14 - 5\ ) = 4$ by the definition of a limit.

26. Given 0 we must find 0 so that if $0 \ | - 0|$, then $|\sqrt[3]{\ } - 0|$. Now $|\sqrt[3]{\ } - 0| = |\sqrt[3]{\ }|$ $\Rightarrow$ $|\ | = |\sqrt[3]{\ }|^3$. So take $= {}^3$. Then $0 \ | - 0| = |\ | {}^3 \Rightarrow |\sqrt[3]{\ } - 0| = |\sqrt[3]{\ }| = \sqrt[3]{|\ |} \sqrt[3]{{}^3} =$.

Therefore, by the definition of a limit, $\lim_{\to 0} \sqrt[3]{\ } = 0$.

Given 0, we need 0 so that if $0 \ | - 2|$, then ${}^2 - 3 \ - (-2)$. First, note that if $| - 2| \ 1$, then

$-1 \ - 2 \ 1$, so $0 \ - 1 \ 2 \Rightarrow | - 1| \ 2$. Now let $= \min\{\ 2 \ 1\}$. Then $0 \ | - 2|$ $\Rightarrow$

${}^2 - 3 - (2-2) = |(-2)(-1)| = | -2||-1| \ (\ 2)(2) =$.

Thus, $\lim_{\to} ({}^2 - 3\ ) = 2$ by the definition of a limit.

28. Given 0, we need 0 such that if $0 - 4$, then $2 \sqrt{\ - 4}$. This is true $\Leftrightarrow$ $\sqrt{\ - 4} \ 2 \Leftrightarrow$

$- 4 \ 4 \ {}^2$. So if we choose $= 4 \ {}^2$, then $0 \ - 4$ $\Rightarrow 2 \sqrt{\ - 4}$. So by the definition of a limit,

$\lim_{\to 4^+} 2 \sqrt{\ - 4} = \infty$.

29. (a) $(\ ) = \sqrt{-}$ if 0, $(\ ) = 3 -$ if $0 \leq \ 3$, $(\ ) = (\ - 3)^2$ if 3.

(i) $\lim_{\to 0^+} (\ ) = \lim_{\to 0^+} (3 \ - \ ) = 3$

(ii) $\lim_{\to 0^-} (\ ) = \lim_{\to 0^-} \sqrt{-} = 0$

(iii) Because of (i) and (ii), $\lim_{\to 0} (\ )$ does not exist.

(iv) $\lim_{\to 3^-} (\ ) = \lim_{\to 3^-} (3 - \ ) = 0$

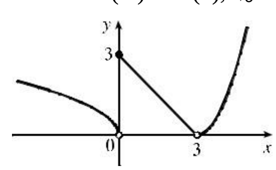
(v) $\lim_{\to 3^+}$ $\lim_{\to 3^+}$ $-$

(vi) Because of (iv) and (v), $\lim_{\to 3}$.

(b) is discontinuous at 0 since $\lim_{\to 0} (\ )$ does not exist.

is discontinuous at 3 since (3) does not exist.

(c)



30. (a) $(\ ) = 2 \ - {}^2$ if $0 \leq \ \leq 2$, $(\ ) = 2 -$ if $2 \leq 3$, $(\ ) = \ - 4$ if $3 4$, $(\ ) =$ if ≥ 4.

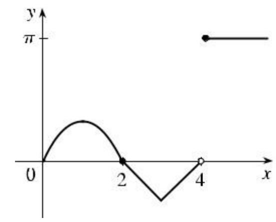
Therefore, $\lim_{\to 2^-} (\ ) = \lim_{\to 2^-} (2 \ - {}^2) = 0$ and $\lim_{\to 2^+} (\ ) = \lim_{\to 2^+} (2 \ - \ ) = 0$. Thus, $\lim_{\to 2} (\ ) = 0 = (2)$,

$\lim (\ ) = \lim (2$ $) = 1$ $\lim (\ ) =$ $\lim$ $($ $4) = 1$

so is continuous at 2. $\to 3^-$ $\to 3^-$ $-$ $-$ and $\to 3^+$ $\to 3^+$ $-$ $-$. Thus,

$\lim_{3} (\,) = \quad - \quad 1 = (3) \quad$, so is continuous at 3.

(b)



$\lim_{\to 4^-} (\,) = \lim_{\to 4^-} (\quad - 4) = 0$ and $\lim_{\to 4^+} (\,) = \lim_{\to 4^+} \quad = \quad$.

Thus, $\lim_{\to 4} (\,)$ does not exist, so is discontinuous at 4. But $\lim_{\to 4^+} (\,) = \quad = (4)$, so is continuous from the right at 4.

$\sin$ and are continuous on $\mathbb{R}$ by Theorem 2.5.7. Since is continuous on $\mathbb{R}$, $\sin$ is continuous on $\mathbb{R}$ by Theorem 2.5.9. Lastly, is continuous on $\mathbb{R}$ since it's a polynomial and the product $\sin$ is continuous on its domain $\mathbb{R}$ by Theorem 2.5.4.

${}^2 - 9$ is continuous on $\mathbb{R}$ since it is a polynomial and $\sqrt{}$ is continuous on $[0\ \infty)$ by Theorem 2.5.7, so the composition $\sqrt{{}^2 - 9}$ is continuous on $\{\ \mid {}^2 - 9 \geq 0\} = (-\infty\ -3] \cup [3\ \infty)$ by Theorem 2.5.9. Note that ${}^2 - 2 \neq 0$ on this set and so the quotient function $(\,) = \dfrac{\sqrt{{}^2 - 9}}{{}^2 - 2}$ is continuous on its domain, $(-\infty\ -3] \cup [3\ \infty)$ by Theorem 2.5.4.

$(\,) = {}^5 - {}^3 + 3 - 5$ is continuous on the interval $[1\ 2]$, $(1) = -2$, and $(2) = 25$. Since $-2 < 0 < 25$, there is a number in $(1\ 2)$ such that $(\,) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation ${}^5 - {}^3 + 3 - 5 = 0$ in the interval $(1\ 2)$.

$(\,) = \cos\sqrt{} - + 2$ is continuous on the interval $[0\ 1]$, $(0) = 2$, and $(1) \approx -0\ 2$. Since $-0\ 2 < 0 < 2$, there is a number in $(0\ 1)$ such that $(\,) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos\sqrt{} - + 2 = 0$, or $\cos\sqrt{} = - 2$, in the interval $(0\ 1)$.

(a) The slope of the tangent line at $(2\ 1)$ is

$$\lim_{\to 2} \frac{(\,) - (2)}{- 2} = \lim_{\to 2} \frac{9 - 2\,{}^2 - 1}{- 2} = \lim_{\to 2} \frac{8 - 2\,{}^2}{- 2} = \lim_{\to 2} \frac{-2({}^2 - 4)}{- 2} = \lim_{\to 2} \frac{-2(\ - 2)(\ + 2)}{- 2}$$
$$= \lim_{\to 2} [-2(\ + 2)] = -2 \cdot 4 = -8$$

(b) An equation of this tangent line is $- 1 = -8(\ - 2)$ or $= -8\ + 17$.

For a general point with -coordinate , we have

$$= \lim_{\to} \frac{2(1-3\,) - 2(1-3\,)}{-} = \lim_{\to} \frac{2(1-3\,) - 2(1-3\,)}{(1-3\,)(1-3\,)(\ -\ )} = \lim_{\to} \frac{6(\ -\ )}{(1-3\,)(1-3\,)(\ -\ )}$$
$$= \lim_{\to} \frac{6}{(1-3\,)(1-3\,)} = \frac{6}{(1-3\,)^2}$$

For $= 0$, $= 6$ and $(0) = 2$, so an equation of the tangent line is $- 2 = 6(\ - 0)$ or $= 6 + 2$ For $= -1$, $= \frac{3}{8}$ and $(-1) = \frac{1}{2}$, so an equation of the tangent line is $- \frac{1}{2} = \frac{3}{8}(\ + 1)$ or $= \frac{3}{8} + \frac{7}{8}$.

37. (a) $= (\,) = 1 + 2 + {}^2\ 4$. The average velocity over the time interval $[1\ 1 +\ ]$ is

$$_{\text{ave}} = \frac{(1+\,) - (1)}{(1+\,) - 1} = \frac{1 + 2(1+\,) + (1+\,)^2\ 4 - 13\ 4}{} = \frac{10 + {}^2}{} = \frac{10 +}{}$$

[continued]

So for the following intervals the average velocities are:

(i) [1 3]: = 2, ave = (10 + 2) 4 = 3 m s (ii) [1 2]: = 1, ave = (10 + 1) 4 = 2 75 m s

[1 1 5]: = 0 5, ave = (10 + 0 5) 4 = 2 625 m s (iv) [1 1 1]: = 0 1, ave = (10 + 0 1) 4 = 2 525 m s

When = 1, the instantaneous velocity is $\lim\limits_{\to 0} \frac{(1 + \ ) - (1)}{} = \lim\limits_{\to 0} \frac{10 + }{4} = \frac{10}{4} = 2\ 5$ m s.

(a) When increases from 200 in^3 to 250 in^3, we have Δ = 250 − 200 = 50 in^3, and since = 800 ,

Δ = (250) − (200) = $\frac{800}{250} - \frac{800}{200}$ = 3 2 − 4 = −0 8 lb in^2. So the average rate of change

is $\frac{\Delta}{\Delta} = \frac{-0\ 8}{50} = -0\ 016 \frac{\text{lb in}^2}{\text{in}^3}$.

(b) Since = 800 , the instantaneous rate of change of with respect to is

$$\lim_{\to 0} \frac{\Delta}{\Delta} = \lim_{\to 0} \frac{(+) - ()}{} = \lim_{\to 0} \frac{800\ (\ +\ ) - 800}{} = \lim_{\to 0} \frac{800[\ - (\ +\ )]}{(\ +\ )}$$

$$= \lim_{\to 0} \frac{-800}{} = -\frac{800}{^2}$$

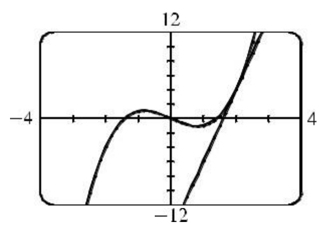
which is inversely proportional to the square of .

39. (a) $^0(2) = \lim\limits_{\to 2} \frac{(\ ) - (2)}{- 2} = \lim\limits_{\to 2} \frac{^3 - 2 - 4}{- 2}$

$= \lim\limits_{\to 2} \frac{(-2)(^2 + 2 + 2)}{- 2} = \lim\limits_{\to 2} (^2 + 2 + 2) = 10$

(b) − 4 = 10(− 2) or = 10 − 16

(c)



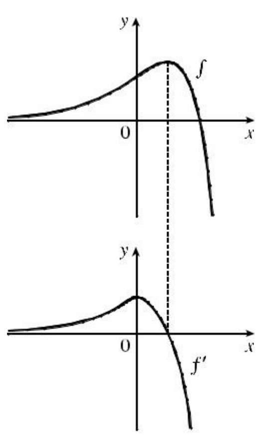
40. 2^6 = 64, so ( ) = ^{6} and = 2.

(a) 0( ) is the rate at which the total cost changes with respect to the interest rate. Its units are dollars (percent per year).

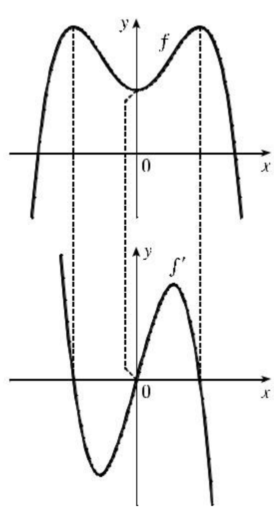
The total cost of paying off the loan is increasing by $1200 (percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately $1200.

As increases, increases. So 0( ) will always be positive.

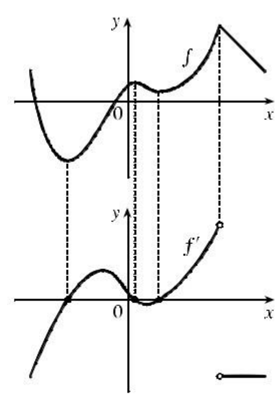
42.



43.



44.



NOT FOR SALE

45. (a) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{3 - 5(x+h)} - \sqrt{3-5x}}{h} \cdot \dfrac{\sqrt{3-5(x+h)} + \sqrt{3-5x}}{\sqrt{3-5(x+h)} + \sqrt{3-5x}}$

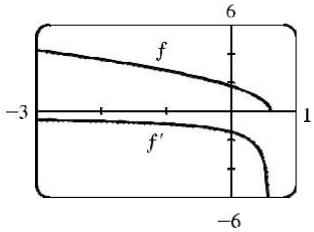
$= \lim_{h\to 0} \dfrac{[3 - 5(x+h)] - (3-5x)}{h\left(\sqrt{3-5(x+h)} + \sqrt{3-5x}\right)} = \lim_{h\to 0} \dfrac{-5}{\sqrt{3-5(x+h)} + \sqrt{3-5x}} = \dfrac{-5}{2\sqrt{3-5x}}$

(b) Domain of f: (the radicand must be nonnegative) $3 - 5x \geq 0 \Rightarrow$

$5x \leq 3 \Rightarrow x \in \left(-\infty, \frac{3}{5}\right]$

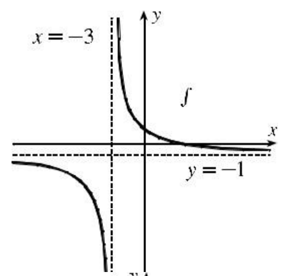
Domain of f': exclude $\frac{3}{5}$ because it makes the denominator zero;

$x \in \left(-\infty, \frac{3}{5}\right)$

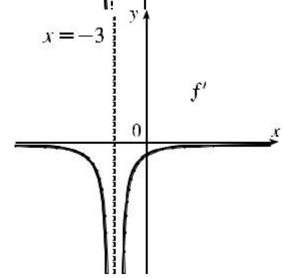


(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.

46. (a) As $x \to \pm\infty$, $f(x) = (4 - x)/(3 + x) \to -1$, so there is a horizontal asymptote at $y = -1$. As $x \to -3^+$, $f(x) \to \infty$, and as $x \to -3^-$, $f(x) \to -\infty$. Thus, there is a vertical asymptote at $x = -3$.



(b) Note that f is decreasing on $(-\infty, -3)$ and $(-3, \infty)$, so f' is negative on those intervals. As $x \to \pm\infty$, $f' \to 0$. As $x \to -3^-$ and as $x \to -3^+$, $f' \to -\infty$.



(c) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{4-(x+h)}{3+(x+h)} - \dfrac{4-x}{3+x}}{h} = \lim_{h\to 0} \dfrac{(3+x)[4-(x+h)] - (4-x)[3+(x+h)]}{h[3+(x+h)](3+x)}$

$= \lim_{h\to 0} \dfrac{(12 - 3x - 3h + 4x - x^2 - hx) - (12 + 4x + 4h - 3x - x^2 - hx)}{h[3+(x+h)](3+x)}$

$= \lim_{h\to 0} \dfrac{-7h}{h[3+(x+h)](3+x)} = \lim_{h\to 0} \dfrac{-7}{[3+(x+h)](3+x)} = -\dfrac{7}{(3+x)^2}$

(d) The graphing device confirms our graph in part (b).

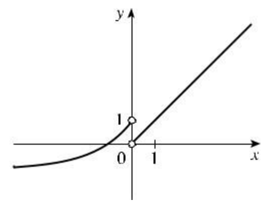
f is not differentiable: at $x = -4$ because f is not continuous, at $x = -1$ because f has a corner, at $x = 2$ because f is not continuous, and at $x = 5$ because f has a vertical tangent.

The graph of a has tangent lines with positive slope for $x < 0$ and negative slope for $x > 0$, and the values of c fit this pattern, so c must be the graph of the derivative of the function for a. The graph of c has horizontal tangent lines to the left and right of the x-axis and b has zeros at these points. Hence, b is the graph of the derivative of the function for c. Therefore, a is the graph of f, c is the graph of f', and b is the graph of f''.

NOT FOR SALE

49. Domain: $(-\infty\ 0) \cup (0\ \infty)$; $\lim\limits_{\to 0^-} (\ ) = 1$; $\lim\limits_{\to 0^+} (\ ) = 0$;

$0(\ )\quad 0$ for all in the domain; $\lim\limits_{\to -\infty} {}^0(\ ) = 0$; $\lim\limits_{\to \infty} {}^0(\ ) = 1$



(a) $0(\ )$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (% yr).

(b) To find ${}^0(\ )$, we use $\lim\limits_{\to} \dfrac{(+)-()}{}$ $\dfrac{(+)-()}{}$ for small values of .

For 1950: ${}^0(1950) \approx \dfrac{(1960)-(1950)}{1960-1950} = \dfrac{357-311}{10} = 046$

For 1960: We estimate ${}^0(1960)$ by using $=-10$ and $=10$, and then average the two results to obtain a final estimate.

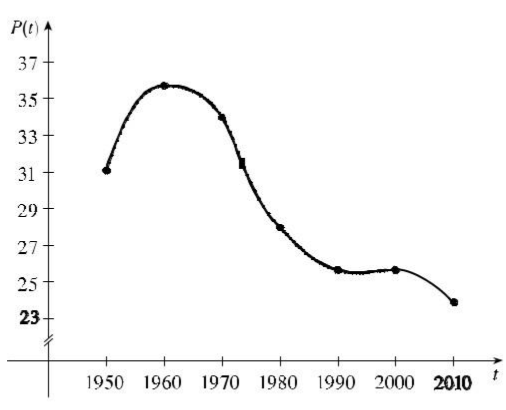
$= \ \ 10 \Rightarrow {}^0(1960) \approx \dfrac{(1950)-(1960)}{1950-1960} = \dfrac{311-357}{-10} = 046$

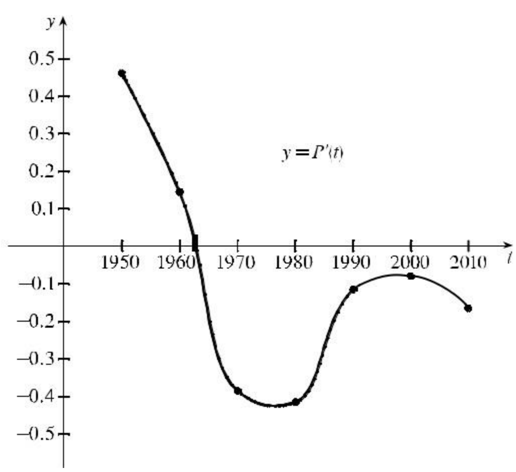
$= 10 \Rightarrow {}^0(1960) \approx \dfrac{(1970)-(1960)}{1970-1960} = \dfrac{340-357}{10} = -0\ 17$

So we estimate that ${}^0(1960) \approx \frac{1}{2}[0\ 46 + (-0\ 17)] = 0\ 145$.

	1950	1960	1970	1980	1990	2000	2010
${}^0(\ )$	0 460	0 145	−0 385	−0 415	−0 115	−0 085	−0 170

(c)





We could get more accurate values for $0(\ )$ by obtaining data for the mid-decade years 1955, 1965, 1975, 1985, 1995, and 2005.

$0(\ )$ is the rate at which the number of US \$20 bills in circulation is changing with respect to time. Its units are billions of bills per year. We use a symmetric difference quotient to estimate $0(2000)$.

$0(2000) \approx \dfrac{(2005)-(1995)}{2005-1995} = \dfrac{5\ 77-4\ 21}{10} = 0\ 156$ billions of bills per year (or 156 million bills per year).

(a) Drawing slope triangles, we obtain the following estimates: $^0(1950) \approx \frac{1\ 1}{10} = 0\ 11$, $^0(1965) \approx \frac{-\ 16}{10} = -0\ 16$,

and $^0(1987) \approx \frac{0\ 2}{10} = 0\ 02$.

The rate of change of the average number of children born to each woman was increasing by $0\ 11$ in 1950, decreasing by $0\ 16$ in 1965, and increasing by $0\ 02$ in 1987.

There are many possible reasons:

In the baby-boom era (post-WWII), there was optimism about the economy and family size was rising.

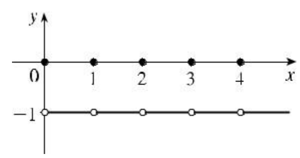
In the baby-bust era, there was less economic optimism, and it was considered less socially responsible to have a large family.

In the baby-boomlet era, there was increased economic optimism and a return to more conservative attitudes.

53. $|\ (\ )| \le (\ ) \Leftrightarrow -(\ ) \le (\ ) \le (\ )$ and $\lim_{\to} (\ ) = 0 = \lim_{\to} -(\ )$.

Thus, by the Squeeze Theorem, $\lim_{\to} (\ ) = 0$.

54. (a) Note that is an even function since $(\ ) = (-\ )$. Now for any integer ,



$[[\ ]] + [[-\ ]] = - = 0$, and for any real number which is not an integer,
$[[\ ]] + [[-\ ]] = [[\ ]] + (-[[\ ]] - 1) = -1$. So $\lim_{\to} (\ )$ exists (and is equal to -1)

for all values of .

(b) is discontinuous at all integers.

PROBLEMS PLUS

1. Let $= \sqrt[6]{\ }$, so $= {}^6$. Then $\to 1$ as $\to 1$, so

$$\lim_{\to 1} \frac{\sqrt[3]{\ } - 1}{\sqrt{\ } - 1} = \lim_{\to 1} \frac{{}^2 - 1}{{}^3 - 1} = \lim_{\to 1} \frac{(\ -1)(\ +1)}{(\ -1)({}^2 + \ +1)} = \lim_{\to 1} \frac{\ +1}{{}^2 + \ +1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}.$$

Another method: Multiply both the numerator and the denominator by $(\sqrt{\ } +1)\left(\sqrt[3]{\ }^{\,2} + \sqrt[3]{\ } + 1\right)$.

2. First rationalize the numerator: $\lim_{\to 0} \frac{\sqrt{\ +\ } - 2}{\ } \cdot \frac{\sqrt{\ +\ } + 2}{\sqrt{\ +\ } + 2} = \lim_{\to 0} \frac{\ +\ - 4}{\ (\sqrt{\ +\ } + 2)}$. Now since the denominator approaches 0 as $\to 0$, the limit will exist only if the numerator also approaches 0 as $\to 0$. So we require that $(0) + \ - 4 = 0 \Rightarrow \ = 4$. So the equation becomes $\lim_{\to 0} \frac{\ }{\sqrt{\ +4} + 2} = 1 \Rightarrow \frac{\ }{\sqrt{4} + 2} = 1 \Rightarrow \ = 4$.

Therefore, $\ = \ = 4$.

For $-\frac{1}{2} < \ < \frac{1}{2}$, we have $2\ - 1 < 0$ and $2\ + 1 > 0$, so $|2\ - 1| = -(2\ - 1)$ and $|2\ + 1| = 2\ + 1$.

Therefore, $\lim_{\to} \frac{|2\ -1| - |2\ +1|}{\ } = \lim_{\to} \frac{-(2\ -1) - (2\ +1)}{\ } = \lim_{\to} \frac{-4\ }{\ } = \lim_{\to} (-4) = -4.$

Let $\ $ be the midpoint of $\ $, so the coordinates of $\ $ are $\left(\frac{1}{2}\ , \frac{1}{2}\ ^2\right)$ since the coordinates of $\ $ are $(\ , \ ^2)$. Let $= (0, \ )$.

Since the slope $= \frac{\ ^2}{\ } = \ $, $= -\frac{1}{\ }$ (negative reciprocal). But $= \frac{\frac{1}{2}\ ^2 - \ }{\frac{1}{2}\ - 0} = \frac{\ ^2 - 2\ }{\ }$, so we conclude that

$-1 = \ ^2 - 2\ \Rightarrow 2\ = \ ^2 + 1 \Rightarrow \ = \frac{1}{2}\ ^2 + \frac{1}{2}$. As $\to 0$, $\to \frac{1}{2}$ and the limiting position of $\ $ is $\left(0, \frac{1}{2}\right)$.

5. (a) For $0 < \ < 1$, $[[\ ]] = 0$, so $\frac{[[\ ]]}{\ } = 0$, and $\lim_{\to 0^+} \frac{[[\ ]]}{\ } = 0$. For $-1 < \ < 0$, $[[\ ]] = -1$, so $\frac{[[\ ]]}{\ } = \frac{-1}{\ }$, and

$\lim_{\to 0^-} \frac{[[\ ]]}{\ } = \lim_{\to 0^-} \frac{-1}{\ } = \infty$. Since the one-sided limits are not equal, $\lim_{\to 0} \frac{[[\ ]]}{\ }$ does not exist.

(b) For $0 < \ < 1$, $\frac{1}{\ } - 1 \le [[1/\ ]] \le \frac{1}{\ } \Rightarrow \ \left(\frac{1}{\ } - 1\right) \le \ [[1/\ ]] \le \ \left(\frac{1}{\ }\right) \Rightarrow 1 - \ \le \ [[1/\ ]] \le 1$.

As $\to 0^+$, $1 - \ \to 1$, so by the Squeeze Theorem, $\lim_{\to 0^+} \ [[1/\ ]] = 1$.

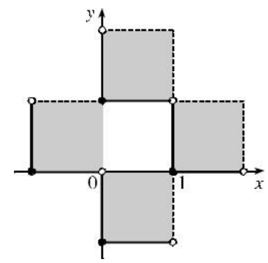
For $-1 < \ < 0$, $\frac{1}{\ } - 1 \le [[1/\ ]] \le \frac{1}{\ } \Rightarrow \ \left(\frac{1}{\ } - 1\right) \ge \ [[1/\ ]] \ge \ \left(\frac{1}{\ }\right) \Rightarrow 1 - \ \ge \ [[1/\ ]] \ge 1$.

As $\to 0^-$, $1 - \ \to 1$, so by the Squeeze Theorem, $\lim_{\to 0^-} \ [[1/\ ]] = 1$.

Since the one-sided limits are equal, $\lim_{\to 0} \ [[1/\ ]] = 1$.

NOT FOR SALE

6. (a) $[[\ ]]^2 + [[\ ]]^2 = 1$. Since $[[\ ]]^2$ and $[[\ ]]^2$ are positive integers or 0, there are only 4 cases:



Case (i): $[[\ ]] = 1$, $[[\ ]] = 0 \Rightarrow 1 \quad \le \ 2$ and $0 \le \ 1$

Case (ii): $[[\ ]] = -1$, $[[\ ]] = 0 \Rightarrow -1 \le 0$ and $0 \le 1$

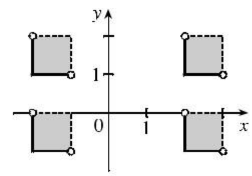
Case (iii): $[[\ ]] = 0$, $[[\ ]] = 1 \Rightarrow 0 \quad \le \ 1$ and $1 \le \ 2$

Case (iv): $[[\ ]] = 0$, $[[\ ]] = -1 \Rightarrow 0 \quad \le \ 1$ and $-1 \le \ 0$

(b) $[[\ ]]^2 - [[\ ]]^2 = 3$. The only integral solution of $^2 - {}^2 = 3$ is $= \pm 2$ and $= \pm 1$. So the graph is

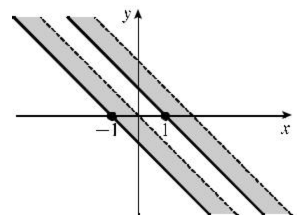
$\{(\ \ )\,|\,[[\ ]]=\pm 2,\ [[\ ]]=\pm 1\} = (\ \ )\qquad {}_2 \le 3$ or $\quad 2 \quad 1$.

$\le \qquad\qquad - \le$



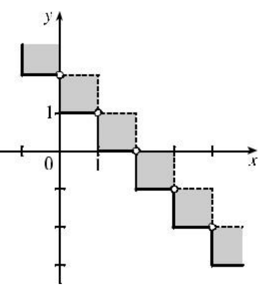
(c) $[[\ + \ ]] = 1 \Rightarrow [[\ + \ ]] = \pm 1 \qquad \Rightarrow 1 \le + 2$

or $-1 \le \ + 0$



(d) For $\le \qquad + 1$, $[[\ ]] = \ $. Then $[[\ ]] + [[\ ]] = 1 \quad \Rightarrow [[\ ]] = 1 - \qquad \Rightarrow$

$1 - \quad \le \qquad 2 - \ $. Choosing integer values for produces the graph.

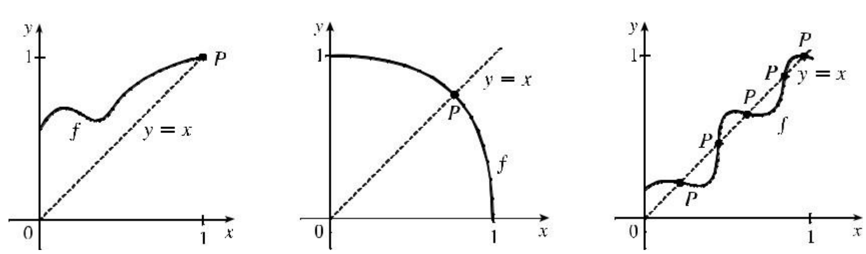


is continuous on $(-\infty\ )$ and $(\ \infty)$. To make continuous on $\mathbb{R}$, we must have continuity at . Thus,

$$\lim_{\to^+} (\ ) = \lim_{\to^-} (\ ) \quad \Rightarrow \quad \lim_{\to^+} {}^2 = \lim_{\to^-} (\ + 1) \quad \Rightarrow \quad {}^2 = \ + 1 \quad \Rightarrow \quad {}^2 - \ - \quad 1 = 0 \quad \Rightarrow$$

[by the quadratic formula] $= \quad 1 \pm \sqrt{5}\ 2 \approx 1\ 618$ or $-0\ 618$.

(a) Here are a few possibilities:



The "obstacle" is the line = (see diagram). Any intersection of the graph of with the line = constitutes a fixed point, and if the graph of the function does not cross the line somewhere in $(0\ 1)$, then it must either start at $(0\ 0)$ (in which case 0 is a fixed point) or finish at $(1\ 1)$ (in which case 1 is a fixed point).

Consider the function ( ) = ( ) − , where is any continuous function with domain [0 1] and range in [0 1]. We shall prove that has a fixed point. Now if (0) = 0 then we are done: has a fixed point (the number 0), which is what we are trying to prove. So assume (0) 6= 0. For the same reason we can assume that (1) 6= 1. Then (0) = (0) 0 and (1) = (1) − 1 0. So by the Intermediate Value Theorem, there exists some number in the interval (0 1) such that ( ) = ( ) − = 0. So ( ) = , and therefore has a fixed point.

9. → lim [()+()]=2 ⇒→ lim ( ) + lim→ ( )=2 **(1)**

lim [()−()]=1 lim ( ) − lim ( )=1 **(2)**

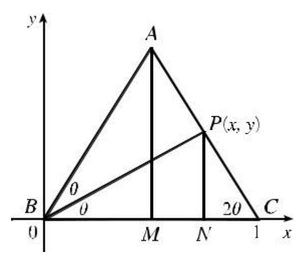
→ Adding equations (1) and (2) gives us → → 2 lim ( ) = 3 ⇒ lim ( ) = $\frac{3}{}$. From equation (1), ( ) lim $\frac{1}{}$. Thus,

→ → →

lim [( ) ( )] = lim ( ) lim ( ) = $\frac{3}{}$ $\frac{1}{2}$ = $\frac{3}{}$.

→ → →

10. (a) *Solution 1:* We introduce a coordinate system and drop a perpendicular



1 −

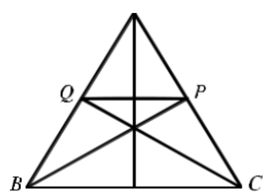
we get —— = tan 2 = 2 tan / 1 − tan = 2() / 1−() . After a bit of

1 − 1 2 2 2

simplification, this becomes 1 − = ^{2} − ^{2} ⇔ ^{2} = (3 −2).

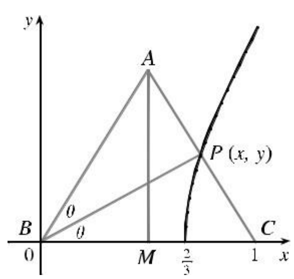
As the altitude decreases in length, the point will approach the -axis, that is, → 0, so the limiting location of must be one of the roots of the equation (3 − 2) = 0. Obviously it is not = 0 (the point can never be to the left of the altitude , which it would have to be in order to approach 0) so it must be 3 − 2 = 0, that is, = $\frac{2}{3}$.

Solution 2: We add a few lines to the original diagram, as shown. Now note that ∠ = ∠ (alternate angles; k by symmetry) and similarly ∠ = ∠ . So ∆ and ∆ are isosceles, and the line segments , and are all of equal length. As | | → 0, and approach points on the base, and the point is seen to approach a position two-thirds of the way between and , as above.



(b) The equation 2 = (3 − 2) calculated in part (a) is the equation of the curve traced out by . Now as | | → ∞, 2 → , → , → 1, and since tan = , → 1. Thus, only traces out the

2 4



11. (a) Consider $(\ ) = (\ +180^\circ) - (\ )$. Fix any number . If $(\ ) = 0$, we are done: Temperature at = Temperature at $+ 180^\circ$. If $(\ )\quad 0$, then $(\ + 180^\circ) = (\ + 360^\circ) - (\ + 180^\circ) = (\ ) - (\ + 180^\circ) = -(\ )\quad 0$.
Also, is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, has a zero on the interval [$+ 180^\circ$]. If $(\ )\quad 0$, then a similar argument applies.

(b) Yes. The same argument applies.

(c) The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.

12. $^0(\ ) = \lim_{\to 0} \dfrac{(\ +\ )\ \ (\ )}{-} = \lim_{\to 0} \dfrac{(+)(+)\ \ (\ )}{-} = \lim_{\to 0} \dfrac{(\ +\ )\ \ (\ )}{-} + \ (\ +\ )$

$= \lim_{\to 0} \dfrac{(+)-()}{} + \lim_{\to 0} (\ +\ ) = {}^0(\ ) + (\ )$

because is differentiable and therefore continuous.

13. (a) Put $= 0$ and $= 0$ in the equation: $(0 + 0) = (0) + (0) + 0^2 \cdot 0 + 0 \cdot 0^2 \Rightarrow (0) = 2\,(0)$.
Subtracting (0) from each side of this equation gives $(0) = 0$.

(b) $^0(0) = \lim_{\to 0} \dfrac{(0+\ )\ \ (0)}{-} = \lim_{\to 0} \dfrac{(0) + (\ ) + 0^2\ + 0\ ^2 - (0)}{} = \lim_{\to 0} \dfrac{(\ )}{} = \lim_{\to 0} \dfrac{(\ )}{} = 1$

(c) $^0(\ ) = \lim_{\to 0} \dfrac{(\ +\ )\ \ (\ )}{-} = \lim_{\to 0} \dfrac{(\ ) + (\ ) + {}^2 + {}^2 - (\ )}{} = \lim_{\to 0} \dfrac{(\ ) + {}^2 + {}^2}{}$

$= \lim_{\to 0} \dfrac{(\ )}{} + {}^2 + = 1 + {}^2$

We are given that $|(\ )| \leq {}^2$ for all . In particular, $|(0)| \leq 0$, but $|\ | \geq 0$ for all . The only conclusion is that $(0) = 0$. Now $\dfrac{-(0)}{0} = \dfrac{(\ )}{} = \dfrac{|\ |}{|\ |} \leq \dfrac{|\ |}{} = |\ | \Rightarrow -|\ | \leq \dfrac{-(0)}{0} \leq |\ |$.

But $\lim_{\to}(\ ) = 0 = \lim_{\to}$, so by the Squeeze Theorem, $\lim_{\to} \dfrac{(\ )\ (0)}{-0} = 0$. So by the definition of a derivative, is differentiable at 0 and, furthermore, $^0(0) = 0$.

Limits and Derivatives

2.1 The Tangent and Velocity Problems

SUGGESTED TIME AND EMPHASIS

$\frac{1}{2}$–1 class Essential material

POINTS TO STRESS

The tangent line viewed as the limit of secant lines.

The concepts of average versus instantaneous velocity, described numerically, visually, and in physical terms.

The tangent line as the line obtained by "zooming in" on a smooth function; local linearity.

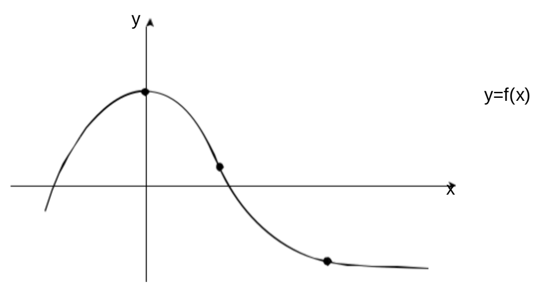
Approximating the slope of the tangent line using slopes of secant lines.

QUIZ QUESTIONS

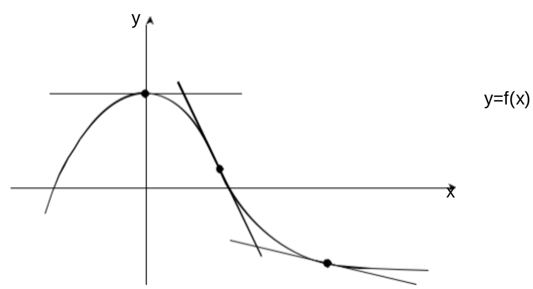
TEXT QUESTION Geometrically, what is "the line tangent to a curve" at a particular point?

ANSWER There are different correct ones. Examples include the best linear approximation to a curve at a point, or the result of repeated "zooming in" on a curve.

DRILL QUESTION Draw the line tangent to the following curve at each of the indicated points:



ANSWER



MATERIALS FOR LECTURE

Point out that if a car is driving along a curve, the headlights will point along the direction of the tangent line.

Discuss the phrase "instantaneous velocity." Ask the class for a definition, such as, "It is the limit of average velocities." Use this discussion to shape a more precise definition of a limit.

Illustrate that many functions such as x^2 and x 2 sin x look locally linear, and discuss the relationship of this property to the concept of the tangent line. Then pose the question, "What does a secant line to a linear function look like?"

Show that the slopes of the tangent lines to $f\ x\ {}^3\ x$ and $g\ x\ x$ are not defined at x 0. Note that f has a tangent line (which is vertical), but g does not (it has a cusp). The absolute value function can be explored graphically.

WORKSHOP/DISCUSSION

Estimate slopes from discrete data, as in Exercises 2 and 7.

Estimate the slope of $y \quad \frac{3}{1\ x^2}$ at the point $1 \quad \frac{3}{2}$ using the graph, and then numerically. Draw the tangent line to this curve at the indicated point. Do the same for the points 0 3 and $2 \quad \frac{3}{5}$.

ANSWER 1 5, 0, 0 48

Draw tangent lines to the curve $y \quad \sin \frac{1}{x}$ at $x \quad \frac{1}{2\pi}$ and $x \quad \frac{1}{\pi\ 2}$. Notice the difference in the quality of the tangent line approximations.

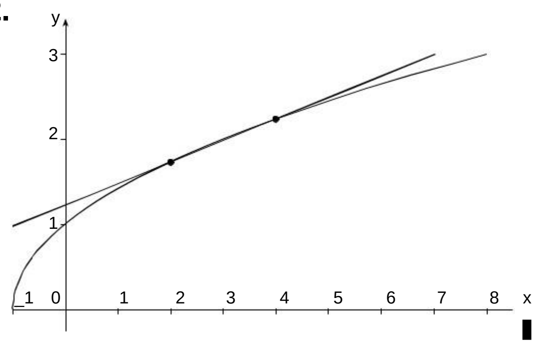
GROUP WORK 1: WHAT'S THE PATTERN?

The students will not be able to do Problem 3 from the graph alone, although some will try. After a majority of them are working on Problem 3, announce that they can do this numerically.

If they are unable to get Problem 6, have them repeat Problem 4 for x 15, and again for x 0.

ANSWERS

1, 2.



3. 23 0268, 5 2 0236, 4238

4 5 3 5 0 250, 0 4 0 250

4. $\frac{1}{4}$ is a good estimate.

5. $\frac{1}{6}$ is a good estimate.

6. $\frac{1}{2\ a\ 1}$

GROUP WORK 2: SLOPE PATTERNS

When introducing this activity, it may be best to fill out the first line of the table with your students, or to estimate the slope at x 1. If a group finishes early, have them try to justify the observations made in the last part of Problem 2.

ANSWERS

1. (a) 0, 0 2, 0 4, 0 6 (b) 11 5

(a) Estimating from the graph gives that the function is increasing for x3 2, decreasing for 3 2 x 3 2, and increasing for x 3 2.

The slope of the tangent line is positive when the function is increasing, and the slope of the tangent line is negative when the function is decreasing.

The slope of the tangent line is zero somewhere between x3 2 and 3 1, and somewhere between x 3 1 and 3 2. The graph has a local maximum at the first point and a local minimum at the second.

The tangent line approximates the curve worst at the maximum and the minimum. It approximates best at x 0, where the curve is “straightest,” that is, at the point of inflection.

HOMEWORK PROBLEMS

CORE EXERCISES 1, 5, 9

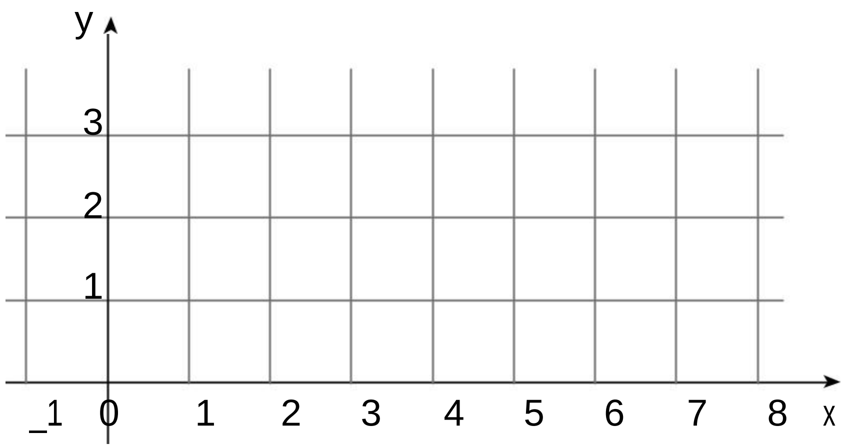
SAMPLE ASSIGNMENT 1, 3, 5, 9

EXERCISE	D	A	N	G
3				

GROUP WORK 1, SECTION 2.1
What's the Pattern?

Consider the function $f\,x$ $\frac{1}{x}$.

Carefully sketch a graph of this function on the grid below.



Sketch the secant line to f between the points with x-coordinates x 2 and x 4.

Sketch the secant lines to f between the pairs of points with the following x-coordinates, and compute their slopes:

(a) x 2 and x 3 (b) x 3 and x 4 (c) x 2 5 and x 3 5 (d) x 2 8 and x 3 2

Using the slopes you’ve found so far, estimate the slope of the tangent line at $x = 3$.

Repeat Problem 4 for $x = 8$.

6. Based on Problems 4 and 5, guess the slope of the tangent line at any point $x = a$, for a 1.

GROUP WORK 2, SECTION 2.1
Slope Patterns

(a) Estimate the slope of the line tangent to the curve y0 1x^2, where x 0, 1, 2, 3. Use your information to fill in the following table:

	Ā Ä	ÈÄ	` Ā↠Ā
	lope of tangent line		
0			
1			
2			
3			

You should notice a pattern in the above table. Using this pattern, estimate the slope of the line tangent to y 0 1x^2 at the point x 57 5.

Consider the function f x 0 1x^3 3x.

On what intervals is this function increasing? On what intervals is it decreasing?

On what interval or intervals is the slope of the tangent line positive? On what interval or intervals is the slope of the tangent line negative? What is the connection between these questions and part (a)?

Where does the slope of the tangent line appear to be zero? What properties of the graph occur at these points?

Where does the tangent line appear to approximate the curve the best? The worst? What properties of the graph seem to make it so?

SUGGESTED TIME AND EMPHASIS

1 class Essential material

POINTS TO STRESS

The various meanings of "limit" (descriptive, numeric, graphic), both finite and infinite. Note that algebraic manipulations are not yet emphasized.

The geometric and limit definitions of vertical asymptotes.

The advantages and disadvantages of using a calculator to compute a limit.

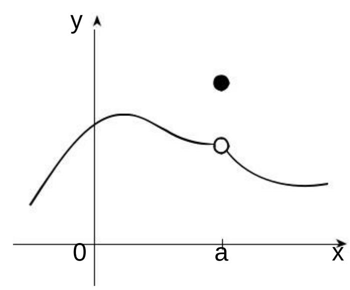
QUIZ QUESTIONS

TEXT QUESTION What is the difference between the statements "$f\ a\ \ L$" and "$\lim_{x\ a} f\ x\ \ L$"?

ANSWER The first is a statement about the value of f at the point $x\ a$ the second is a statement about the values of f at points near, but not equal to, $x\ a$.

DRILL QUESTION The graph of a function f is shown below. Are the following statements about f true or false? Why?

(a) $x\ \ a$ is in the domain of f (b) $\lim_{x\ a} f\ x$ exists (c) $\lim_{x\ a} f\ x$ is equal to $\lim_{x\ a} f\ x$



ANSWER

True, because f is defined at $x\ \ a$.

True, because as x gets close to a, $f\ x$ approaches a value.

True, because the same value is approached from both directions.

MATERIALS FOR LECTURE

Present the "motivational definition of limit": *We say that* $\lim_{x\ a} f\ x\ \ L$ *if as x gets close to a*, $f\ x$ *gets close to L*, and then lead into the definition in the text. (Note that limits will be defined more precisely in Section 2.4.)

Describe asymptotes verbally, and then give graphic and limit definitions. If foreshadowing horizontal asymptotes, note that a function *can* cross a horizontal asymptote. Perhaps foreshadow the notion of slant asymptotes, which are covered later in the text.

Discuss how we can rephrase the last section's concept as "the slope of the tangent line is the limit of the slopes of the secant lines as $x\ \ 0$".

Stress that if $\lim_{x\ a} f\ x$ and $\lim_{x\ a} g\ x$, then we still don't know anything about $\lim_{x\ a} f\ x\ g\ x$.

WORKSHOP/DISCUSSION

Explore the greatest integer function $f\ x\ [[x]]$ on the interval [1 2] in terms of left-hand and right-hand limits.

Explore $\lim_{x \to 0}\ 2\ \ 2^{1\ x}$, looking carefully at $2^{1\ x}$ from both sides for small x. Discuss in graphical, numerical,

ANSWER The left-hand limit is 2, because $2^{1\ x}$ vanishes.
The right-hand limit is .

Discuss a limit such as $\lim_{x \to 3} \dfrac{x\ \ 3}{4x\ \ 2}$. When can you "just plug in the numbers"?

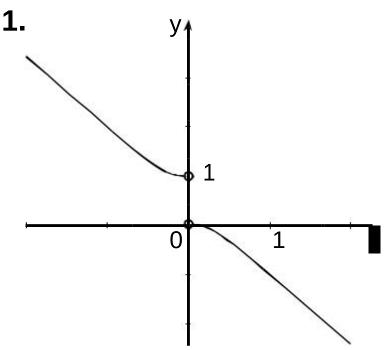
GROUP WORK 1: AN INTERESTING FUNCTION

Introduce this activity with a review of the concepts of left- and right-hand limits. Also make sure that the students can articulate that when a denominator gets small, the function gets large, and vice versa. (This will be needed for the second question in Problems 2 and 3.) When a group is done, inform them that one of them will be chosen at random to discuss the answer with the class, so all should be able to describe their results.

When they graph $y \dfrac{1}{1\ \ 2^{1\ x}}$, they are expected to use graphing technology. When they are all finished, have a different person present the solution to each part.

ANSWERS

1.



2. The limit is 0. When x is small and positive, $1\ x$ is large and positive, so $1\ \ 2^{1\ x}$ is large and negative. Therefore its reciprocal is very small and negative, approaching zero.

3. The limit is 1. When x is small and negative, $1\ x$ is large and negative, and $1\ \ 2^{1\ x}$ is very close to 1.

4. The limit doesn't exist because the left- and right-hand limits are not equal.

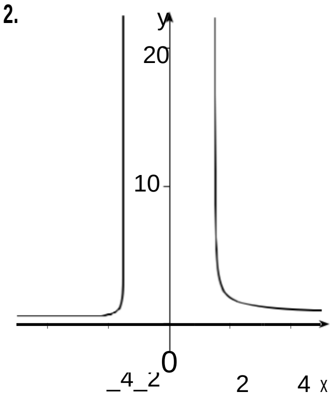
GROUP WORK 2: INFINITE LIMITS

After the students are finished, Problem 2 can be used to initiate a discussion of left and right hand limits, and of the precise definition of a vertical asymptote, as presented in the text. In addition, Section 2.6 can be foreshadowed by asking the students to explore the behavior of $\dfrac{3x^2\ \ \ 4x\ \ \ 5}{\sqrt{16\ x^4\ 81}}$ for large positive and large negative values of x, both on the graph and numerically. If there is time, the students can be asked to analyze the asymptotes of $f\ x$ sec x and the other trigonometric functions.

ANSWERS

Answers will vary. The main thing to check is that there are vertical asymptotes at 2 and at $\frac{3}{2}$.

2.



There are vertical asymptotes at $x1\ 5$.

GROUP WORK 3: THE SHAPE OF THINGS TO COME

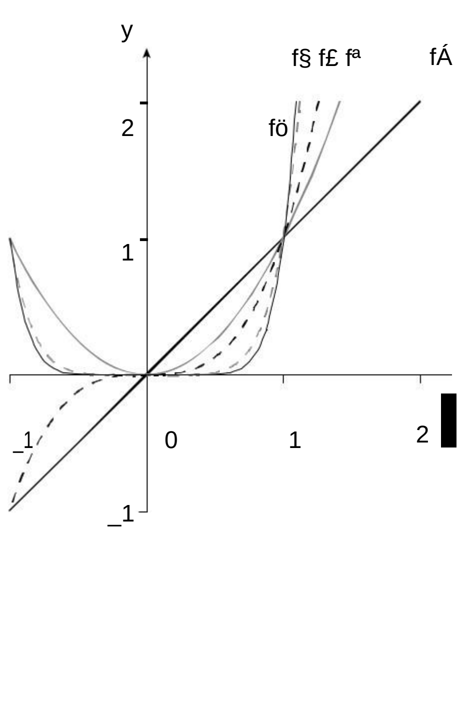
This activity foreshadows concepts that will be discussed later, but can be introduced now. The idea is to show the students that the concept of “limit” can get fairly subtle, and that care is needed. The second page anticipates Section 2.6, and the third page anticipates Section 2.4. Pages 2 and 3 are independent of each other; either or both can be used. Problem 4 on page 3 is a little tricky and can be omitted if desired.

ANSWERS

PAGE 1

$2^1\ \ 2,\ 06^2\ \ 036,\ 08^3\ \ 0512,0^4\ \ 0,\ 101^8\ \ 10829$

2.



3. $f_n\ 1\quad 1$ for all n.

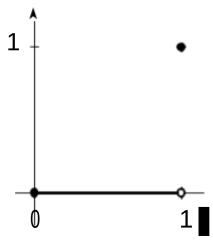
4. The curves all go through the origin.

PAGE 2

1. (a) 0 (b) 1 (c) 1

2. (a) 0 (b) 0 (c) 0 (d) 1

The function $g(x)$ is important in real analysis. Its graph looks like this:



4. (a) 1 (b) 0

PAGE 3

1. $\frac{1}{10}$ (or any positive number less than $\frac{1}{10}$) **2.** Estimates will vary.

3. $\frac{\sqrt{101}}{10} - 1 \approx 0.0049876$ (or any positive number less than 0.0049876)

4. Yes, the problems could have been done with any smaller positive number.

5. The students can be forgiven for not answering this question. It will be fully answered in Section 2.4. The short answer: Let a be the "small number you can name." Then we have shown that we can always find a small interval about x such that $x^2 - 0 < a$. A similar argument can be made for the second part. The

main idea here is to set up ideas

GROUP WORK 4: WHY CAN'T WE JUST TRUST THE TABLE?

This activity was inspired by the article "An Introduction to Limits" from *College Mathematics Journal*, January 1997, page 51, and extends Example 4.

Put the students into groups and give each group two different digits between 1 and 9, and then let them proceed with the problems in the handout.

ANSWERS

The answer, of course, depends on the starting digit:

x	$\sin\frac{\pi}{x}$
0 1	0
0 01	0
0 001	0
0 0001	0
0 00001	0
0 000001	0

x	$\sin\frac{\pi}{x}$
0 2	0
0 02	0
0 002	0
0 0002	0
0 00002	0
0 000002	0

x	$\sin\frac{\pi}{x}$
0 3	$\frac{3}{2}$
0 03	$-^{3}$
0 003	$\frac{}{2}$
0 0003	$\frac{3}{2}$
0 00003	$\frac{3}{2}$
0 000003	$\frac{3}{2}$

x	$\sin\frac{\pi}{x}$
0 4	1
0 04	0
0 004	0
0 0004	0
0 00004	0
0 000004	0

x	$\sin\frac{\pi}{x}$
0 5	0
0 05	0
0 005	0
0 0005	0
0 00005	0
0 000005	0

x	$\sin\frac{\pi}{x}$
0 6	$\frac{3}{2}$
0 06	$\frac{3}{2}$
0 006	$\frac{3}{2}$
0 0006	$\frac{3}{2}$
0 00006	$\frac{3}{2}$
0 000006	$\frac{3}{2}$

x	$\sin\frac{\pi}{x}$
0 7	0 974927912
0 07	0 781831482
0 007	0 433883739
0 0007	0 974927912
0 00007	0 781831482
0 000007	0 433883739

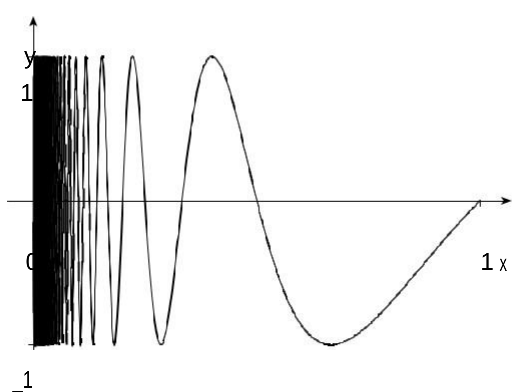
x	$\sin\frac{\pi}{x}$
0 8	$\frac{2}{}$
0 08	1
0 008	0
0 0008	0
0 00008	0
0 000008	0

x	$\sin\frac{\pi}{x}$
0 9	0 342020143
0 09	0 342020143
0 009	0 342020143
0 0009	0 342020143
0 00009	0 342020143
0 000009	0 342020143

Answers will vary.

Answers will vary.

There is no limit.



HOMEWORK PROBLEMS

CORE EXERCISES 1, 5, 8, 11, 19, 33, 50

SAMPLE ASSIGNMENT 1, 5, 7, 8, 11, 16, 17, 19, 33, 42, 50, 53

EXERCISE	D	A	N	G
1				
5				
7				
8				
11				
16				
17				
19				
33				
42				
50				
53				

GROUP WORK 1, SECTION 2.2

An Interesting Function

1. Create a graph of the function $y = \frac{1}{1 + 2^{1/x}}$, $-2 \le x \le 2$.

2. Estimate $\lim_{x \to 0} \frac{1}{1 + 2^{1/x}}$ from the graph. Back up your estimate by looking at the function, and discussing why your estimate is probably correct.

3. Estimate $\lim_{x \to 0} \frac{1}{1 + 2^{1/x}}$ from the graph. Back up your estimate by looking at the function, and discussing why your estimate is probably correct.

4. Does $\lim_{x \to 0} \frac{1}{1 + 2^{1/x}}$ exist? Justify your answer.

GROUP WORK 2, SECTION 2.2

Infinite Limits

Draw an odd function which has the lines $x \quad 2$ and $x \, \frac{3}{2}$ among its vertical asymptotes.

2. Analyze the vertical asymptotes of $\dfrac{3x^2 \quad 4x \quad 5}{16x^4 \quad 81}$.

GROUP WORK 3, SECTION 2.2
The Shape of Things to Come

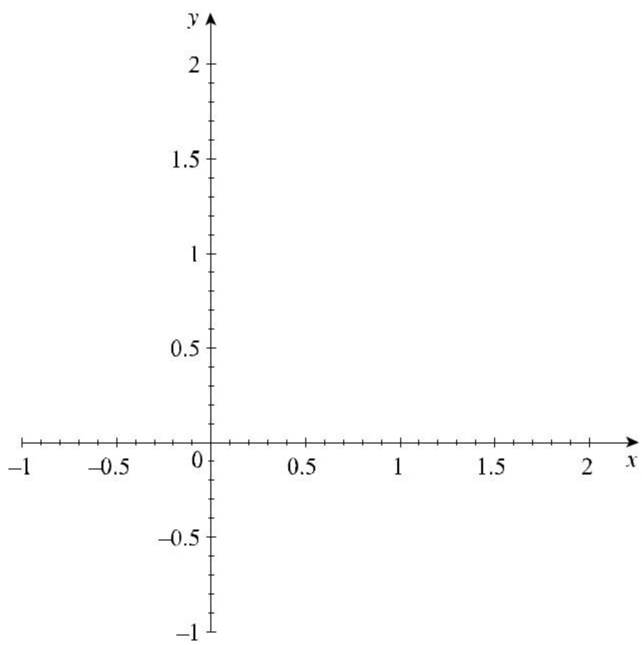
In this activity we are going to explore a set of functions:

$$f_1(x) = x$$
$$f_2(x) = x^2$$
$$f_3(x) = x^3$$
$$\vdots$$
$$f_n(x) = x^n, \ n \text{ any positive integer}$$

To start with, let's practice the new notation. Compute the following:

$f_1(2) =$ ______ $f_2(0.6) =$ ______ $f_3(0.8) =$ ______ $f_4(0) =$ ______ $f_8(1.01) =$ ______

Sketch the functions f_1, f_2, f_3, f_6, and f_8 on the set of axes below.



3. The number 0 plays a special role, since $f_n(0) = 0^n = 0$ for all positive integers n. Find another number $a \neq 0$ such that $f_n(a) = a$ for all positive integers n.

4. We know that $\lim_{x \to 0} f_n(x) = 0$ for all positive integers n. How is this fact reflected on your graphs above?

GROUP WORK 3, SECTION 2.2

The Shape of Things to Come: Approaching Infinity

Using what you know about limits, compute the following quantities:

(a) $\lim_{x \quad 0} f_3 \;\; x$

(b) $\lim_{x \quad 1} f_4 \;\; x$

(c) $\lim_{x \quad 1} f_{15} \;\; x$

2. Using what you know about limits, compute the following quantities:

(a) $\lim_{n} f_n \;\; \frac{1}{2}$.

(b) $\lim_{n} f_n \;\; 0\;99$

(c) $\lim_{n} f_n \;\; x$, where $x \qquad 1$

(d) $\lim_{n} f_n \;\; 1$

3. Let $g\, x \qquad \lim_{n} f_n \;\; x$ for $0 \qquad x \qquad 1$. Sketch $g\, x$, paying particular attention to $g\;1$ and values of x close to 1.

4. Are the following quantities defined? If so, what are they? If not, why not?

(a) $\lim_{n} \quad \lim_{x \;\; 1} f_n \;\; x$

(b) $\lim_{x \;\; 1} \quad \lim_{n} f_n \;\; x$

GROUP WORK 3, SECTION 2.2

The Shape of Things to Come: The Nitty-Gritty

By definition, $\lim f_2 \; x \quad 0$” means that by taking x very close to zero, we can make $\; x_2 \, 0$ smaller than any small number you can name.

1. Find a number δ 0 such that if δ x δ, then $f_2 \; x \quad \frac{1}{100}$.

2. Use a graph to find a number δ 0 such that if δ x 1 δ, then $x^2 \quad 1 \quad \frac{}{100}$.

3. Now use algebra to find a number δ 0 such that if δ x 1 δ, then $x^2 \quad 1 \quad \frac{}{100}$.

When constructing this problem, $\frac{1}{100}$ was used as an arbitrary smallish number. Could you have done the previous problems if we replaced $\frac{1}{100}$ by $\frac{1}{10{,}000}$? How about $\frac{1}{1{,}000{,}000}$?

5. Reread the first sentence on this page. How do your answers to Problems 1 and 4 show that $\lim_{x \quad 0} f_2 \; x$ 0? Do your answers to Problems 2, 3, and 4 show that $\lim_{x \quad 1} x^2 \quad 1$ 0? Why?

GROUP WORK 4, SECTION 2.2
Why Can't We Just Trust the Table?

We are going to investigate $\lim_{x \to 0} \sin \frac{\pi}{x}$. We will take values of x closer and closer to zero, and see what value the function approaches.

1. Your teacher has given you a digit — let's call it d. Fill out the following table. If, for example, your digit is 3, then you would compute $\sin \frac{\pi}{03}$, $\sin \frac{\pi}{003}$, $\sin \frac{\pi}{0003}$, $\sin \frac{\pi}{00003}$, etc.

x	$\sin \frac{\pi}{x}$
0 d	
0 0d	
0 00d	
0 000d	
0 0000d	
0 00000d	

2. What is $\lim_{x \to 0} \sin \frac{\pi}{x}$?

3. Now fill out the table with a different digit.

x	$\sin \frac{\pi}{x}$
0 d	
0 0d	
0 00d	
0 000d	
0 0000d	
0 00000d	

Do you get the same result?

4. What is $\lim_{x \to 0} \sin \frac{\pi}{x}$?

2.3 Calculating Limits Using the Limit Laws

SUGGESTED TIME AND EMPHASIS

1 class Essential material

POINTS TO STRESS

- The algebraic computation of limits: manipulating algebraically, examining left- and right-hand limits, using the limit laws to break monstrous functions into pieces, and analyzing the pieces.
- The evaluation of limits from graphical representations.
- Examples where limits don't exist (using algebraic and graphical approaches).
- The computation of limits when the limit laws do not apply, and the use of direct substitution property when they do.

QUIZ QUESTIONS

TEXT QUESTION In Example 4, why isn't $\lim_{x\to 1} g(x) = \pi$?

ANSWER Because the limit isn't affected by the function when $x = 1$ only when x is near 1.

DRILL QUESTION If $a \neq 0$, find $\lim_{x\to a} \frac{x^2 - 2ax + a^2}{x^2 - a^2}$.

(A) $\frac{1}{2a}$ (B) $\frac{1}{2a^2}$ (C) $-\frac{1}{2a^2}$ (D) 0 (E) Does not exist

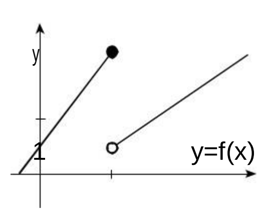
ANSWER (D)

MATERIALS FOR LECTURE

- Discuss why $\lim_{x\to 0} [[x]] \sin x = 0$ is not a straightforward application of the Product Law.
- Have the students determine the existence of $\lim_{x\to 0} \frac{|x|}{x}$ and determine why we cannot compute $\lim_{x\to 0} \frac{1}{x}$.
- Use the Squeeze Theorem to show that $\lim_{x\to 0} x^2 [[x]] = 0$.

WORKSHOP/DISCUSSION

- Compute some limits of quotients, such as $\lim_{x\to 2} \frac{x^2-4}{x-2}$, $\lim_{x\to 0} \frac{x^3-8}{x-2}$, $\lim_{x\to 3} \frac{x^3-8}{x-2}$, and $\lim_{x\to 2} \frac{x^3-8}{x-2}$, always attempting to plug values in first.
- Have the students check if $\lim_{x\to 5} \frac{x-5}{|x-5|}$ exists, and then compute left- and right-hand limits. Then check $\lim_{x\to 5} \frac{(x-5)^2}{|x-5|}$.
- Do some subtle product and quotients, such as $\lim_{x\to 0} \sqrt{x}\sin x$ and $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}$.
- Present some graphical examples, such as $\lim_{x\to 0} f(x)$ and $\lim_{x\to 2} f(x)$ in the graph below.



2 x

GROUP WORK 1: EXPLORING LIMITS

Have the students work on this activity in groups. Problem 2 is more conceptual than Problem 1, but makes an important point about the sums and products of limits.

ANSWERS

1. (a) (i) Does not exist (ii) Does not exist (iii) 4 (iv) Does not exist

(b) (i) Does not exist (ii) 1 (c) (i) 0 (ii) Does not exist

(a) Both quantities exist.

Each quantity may or may not exist.

GROUP WORK 2: FIXING A HOLE

This activity foreshadows concepts used later in the discussion of continuity, in addition to giving the students practice in taking limits. After the activity, point out that mathematicians use the word "puncture" as well as "hole".

ANSWERS

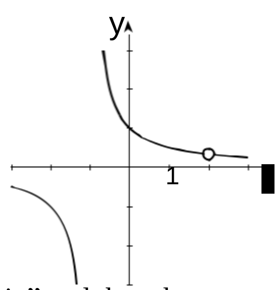
No, yes, yes, no

x x1 2

Does not exist, $\frac{1}{3}$

4.

1



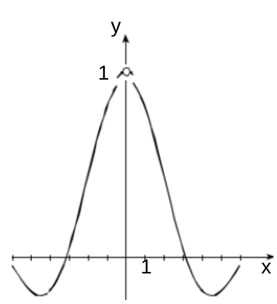
One of the discontinuities can be "filled in" and the other cannot.

A "hole" is an x-value at which the function is not defined, yet the left- and right-hand limits exist. *Or*:

A "hole" is an x-value where the function is undefined, yet the function is defined near x.

Or: A "hole" is an x-value at which we can add a point to the function and thus make it continuous there.

6.



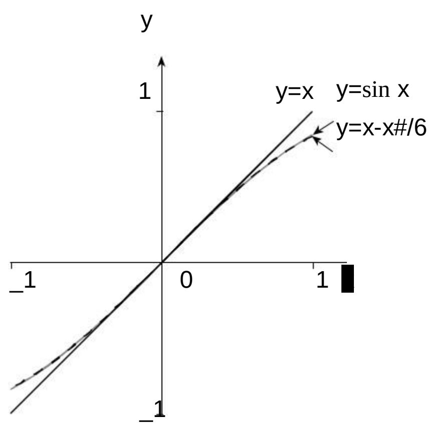
g has a hole at x 0.

GROUP WORK 3: THE SQUEEZE THEOREM

This activity gives an informal graphical way to show that $\lim_{x\;\;0}\frac{\sin x}{x}\;\;1$. A more careful geometric argument is given in Section 3.3.

ANSWERS

1, 2.



3. For $x\;\;0$, $\sin x\;\;x$ $\quad\frac{\sin x}{x}\;\;1.$

For $x\;\;0$, $\sin x\;\;x$ $\quad\frac{\sin x}{x}\;\;1$

(reversing the second inequality because $x\;\;0$).

4. $f\;x1\;\;\frac{x^2}{6}$

5. The Squeeze Theorem now gives $\lim_{x\,0}\frac{\sin x}{x}\;\;1.$

HOMEWORK PROBLEMS

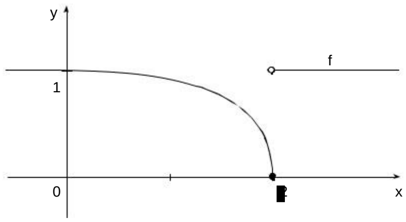
CORE EXERCISES 2, 5, 18, 50, 51, 60

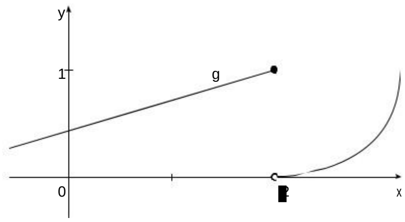
SAMPLE ASSIGNMENT 2, 5, 10, 18, 32, 35, 47, 50, 51, 60, 61

EXERCISE	D	A	N	G
2				
10				
47				
50				

GROUP WORK 1, SECTION 2.3
Exploring Limits

Given the functions *f* and *g* (defined visually below) and *h* and *j* (defined algebraically), compute each of the following limits, or state why they don't exist:





$h(x) = \dfrac{x^2 \quad 4}{x \quad 2}$

$j(x) = \begin{cases} 0 & \text{if } x \quad 2 \\ 1 & \text{if } x \quad 2 \end{cases}$

(a) (i) $\lim_{x \to 2} f(x)$ (ii) $\lim_{x \to 2} g(x)$ (iii) $\lim_{x \to 2} h(x)$ (iv) $\lim_{x \to 2} j(x)$

(b) (i) $\lim_{x \to 2} \left[g(x) \quad h(x) \right]$ (ii) $\lim_{x \to 2} \left[f(x) \quad j(x) \right]$

(c) (i) $\lim_{x \to 2} f(x)\, g(x)$ (ii) $\lim_{x \to 2} f(x)\, j(x)$

2. (a) In general, if $\lim_{x \to a} m(x)$ exists and $\lim_{x \to a} n(x)$ exists, is it true that $\lim_{x \to a} [m(x) \quad n(x)]$ exists? How about $\lim_{x \to a} [m(x)\, n(x)]$? Justify your answers.

(b) In general, if $\lim_{x \to a} m(x)$ does not exist and $\lim_{x \to a} n(x)$ does not exist, is it true that $\lim_{x \to a} [m(x) \quad n(x)]$ does not exist? How about $\lim_{x \to a} [m(x)\, n(x)]$? Compare these with your answers to part (a).

GROUP WORK 71, SECTION

Fixing a Hole

Consider $f\ x x^2\ \dfrac{2}{x\ \ 2}$.

Is $f\ x$ defined for $x 1$? For $x\ \ 0$? For $x\ \ 1$? For $x\ \ 2$?

What is the domain of f?

3. Compute $\lim_{x 1} \dfrac{x\ \ 2}{x^2\ \ x\ \ 2}$ and $\lim_{x\ 2} \dfrac{x\ \ 2}{x^2\ \ x\ \ 2}$. Notice that one limit exists, and one does not.

4. Graph $y\ x^2\ \dfrac{x\ \ 2}{x\ 2}$. There are two x-values that are not in the domain of f. Later, we will call these "discontinuities". Geometrically, what is the difference between the two discontinuities?

5. We say that $f\ x$ has one *hole* in it. Where do you think that the hole is? Define "hole" in this context.

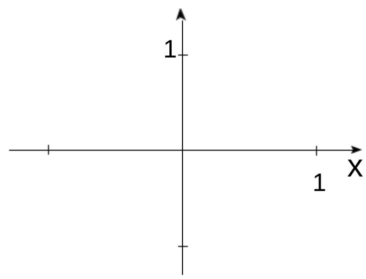
6. The function $g\ x\ \dfrac{\sin x}{x}$ is not defined at $x\ \ 0$. Sketch this function. Does it have a hole at $x\ \ 0$?

GROUP WORK 3, SECTION 2.3

The Squeeze Theorem

In this activity, we take a graphical approach to computing $\lim_{x \quad 0} \dfrac{\sin x}{x}$.

Using a graphing calculator, show that if $0 \quad x \quad 1$, then $x \quad \frac{x^3}{6} \sin x \qquad x$. Give a rough sketch of the three functions over the interval [0 1] on the graph below.



2. Again using a graphing calculator, show that if $1 \, x \, 0$, then $x \, \frac{x^3}{6} \sin x \, x$. If you have not done so already, add these portions of the three functions to your graph above.

Explain why $\dfrac{\sin x}{x} \, 1$ for $1 \, x \, 1$, $x \, 0$. Use the inequalities in parts 1 and 2 to help you.

4. Again using parts 1 and 2, can you find a function $f\,x$ with $f\,x \qquad \dfrac{\sin x}{x}$ on $1 \quad x \quad 1$, $x \quad 0$ such that $\lim_{x\,0} f\,x \quad 1$?

5. Using parts 3 and 4, compute $\lim_{x \quad 0} \dfrac{\sin x}{x}$.

2.4 The Precise Definition of a Limit

SUGGESTED TIME AND EMPHASIS

1–$1\frac{1}{2}$ classes Optional material

POINTS TO STRESS

The geometry of the ε-δ definition, what the notation means, and how it relates to the geometry.

The “narrow range” definition of a limit, as defined below.

Extending the precise definition to one-sided and infinite limits.

QUIZ QUESTIONS

TEXT QUESTION Example 1 finds a number δ such that $x^3 \quad 5x \quad 6 \quad 2 \quad 0\,2$ whenever $x \quad 1 \quad$ δ. Why does this *not* prove that $\lim\limits_{x \quad 1} x^3 \quad 5x \quad 6 \quad 2$?

ANSWER It is not a proof because 0 2; a proof would hold for *all* ε.

DRILL QUESTION Let $f\,x\ 5x\ 2$. Find δ such that $f\,x\ 12\ 0\,01$ whenever δ $x\ 2$ δ. ANSWER δ 0 002 works, as does any smaller δ.

MATERIALS FOR LECTURE

The “narrow range” definition of limit may be covered as a way of introducing the ε–δ definition to the students in a familiar numerical context. *We say that* $\lim\limits_{x \quad a} f\,x \quad L$ *if for any y-range centered at L there is an x-range centered at a such that the graph is “trapped” in the window*— that is, does not go off the top or the bottom of the window. The transition to the traditional definition can now be made easier by observing that the width of the *y*-range is 2ε and the width of the *x*-range is 2δ. If the students are familiar with graphing calculators, this definition can be illustrated with setting different viewing windows for a particular graph.

Make sure the students understand that limit proofs, as described in the book, are two-step processes. The act of finding δ is separate from writing the proof that the students’ choice of δ works in the limit definition. This fact is stated clearly in the text, but it is a novel enough idea that it should be reinforced.

Discuss how close x needs to be to 4, first to ensure that $\dfrac{1}{x\ 4^2}$ 1000, and then so that $\dfrac{1}{x\ 4}$ 20 000. Then argue intuitively that $\lim\limits_{x \quad 4} \dfrac{1}{x\ 4}$.

Using the formal definition of limit, show that neither 1 nor 1 is the limit of $h\,x \quad \begin{cases} 1 & \text{if } x \quad 0 \\ 1 & \text{if } x \quad 0 \end{cases}$ as x goes to 0. Emphasize that although this result is obvious from the graph, the idea is to see how the definition works using a function that is easy to work with.

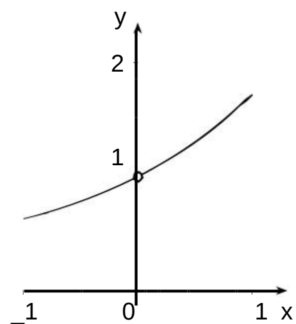
WORKSHOP/DISCUSSION

Estimate how close x must be to 0 to ensure that $\sin x\, x$ is within 0 03 of 1. Then estimate how close x must be to 0 to ensure that $\sin x\, x$ is within 0 001 of 1. Describe what you did in terms of the definition of a limit.

Return to the interesting function $f\, x \; 1 \; 2 \frac{1}{1 \; x}$ from Group Work 1 in Section 2.2, and describe why the right- and left-hand limits exist at x 0, but the limit does not exist.

Discuss why $f\; x$ [[x]] does not have a limit at x 0, first using the "narrow range" definition of limit, and then possibly the ε-δ definition of limit.

Then discuss why $\lim_{x \to 0} \frac{e \quad 1}{x}$ 1, using the "narrow range" definition of limit and a graph like the one at right.



Find, numerically or algebraically, a δ 0 such that if 0 x 0 δ, then x^3 0 10^{3}. Similarly,

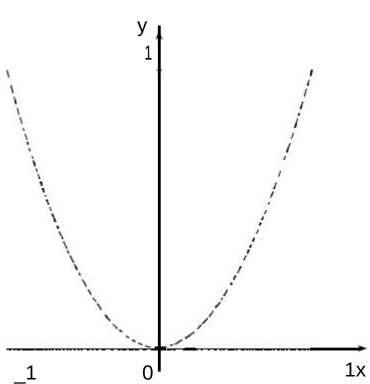
compute a δ 0 such that if 0 x 2 δ, then x^3 8 10^{3}.

GROUP WORK 1: A JITTERY FUNCTION

This activity can be done in several ways. After they have worked for a while, perhaps ask one group to try to solve it using the Squeeze Theorem, another to solve it using the "narrow range" definition of limit, and a third to solve it using the ε-δ definition of limit. They should show why their method works for Problem 2, and fails for Problem 3.

ANSWERS

1.



2. $\lim_{x \to 0} f\, x$ 0. Choose ε with ε 0. Let δ $\sqrt{\varepsilon}$. Now if δ x δ, then x^2 ε, regardless of whether x is rational or irrational. This can also be shown using the Squeeze Theorem and the fact that 0 $f\; x x^2$, and then using the Limit Laws to compute $\lim_{x\to 0} 0$ and $\lim_{x \to 0} x^2$.

3. It does not exist. Assume that $\lim_{x \to 1} f\, x L$. Choose ε $\frac{1}{10}$. Now, whatever your choice of δ, there are some x-values in the interval 1 δ 1 δ with $f\; x$0, so L must be less than $\frac{1}{10}$. But there are also values of x in the interval with $f\, x$ 10^{2} , so L must be greater than $\frac{1}{10}$. So L cannot exist. The "narrow range" definition of limit can also be used to solve this problem.

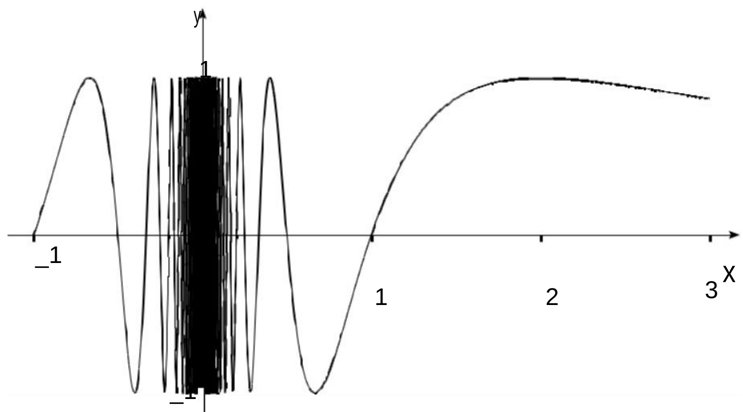
We can conjecture that the limit does not exist by applying the reasoning from Problem 3.

GROUP WORK 2: THE DIRE WOLF COLLECTS HIS DUE

The students will not be able to do Problem 1 with any kind of accuracy. Let them discover for themselves how deceptively difficult it is, and then tell them that they should do the best that they can to show what is happening as x goes to zero. Ask them to compare their result with $\lim_{x \to 0} x \sin \frac{\pi}{x}$. If a group finishes early, pass out the supplementary problems.

ANSWERS

1.



2. (a) 1, 1, 1 (b) 1 (c) 0 (d) A function must approach only one number for the limit to exist.

ANSWERS TO SUPPLEMENTARY PROBLEMS

The length of the boundary is infinite. There are infinitely many wiggles, each adding at least 2 to the total perimeter length.

The area is finite. It is less than the area of the rectangle defined by $0 \quad x \quad 1$, $2 \quad y \quad 1$.

Answers will vary.

GROUP WORK 3: INFINITY IS VERY BIG

The precise definition of infinite limits is similar to the standard definition, but it is different enough that most students need a little practice before they can grasp it.

ANSWERS

1. $x \quad 0\,001$

2. (a) Choose M. Now let $\delta = \frac{1}{\sqrt{M}}$. If $0 \quad x \quad \frac{1}{\sqrt{M}}$, then $\frac{1}{x^2} \quad M$. Values of δ less than $\frac{1}{\sqrt{M}}$ work, too.

(b) $\frac{1}{x}$ is large negative for small negative values of x, and large positive for small positive values of x.

GROUP WORK 4: THE SIGNIFICANCE OF THE "FOR EVERY"

The purpose of this activity is to allow the students to discover that rigor in mathematics is often necessary and useful. Problem 1 is designed to lead the students to make a false assumption about the third function, $h(x)$. Problem 2 should dispel that assumption.

This activity is longer than it appears. Allow the students plenty of time to do the first three questions, which should help them to internalize and understand the formal definition of a limit. Closure is important to ensure that the "punchline" isn't lost in the algebra.

When the students are finishing up, it is *crucial* to pass out Problem 2. This part asks them to look at the functions a third time, with ε 0 01 Make sure that the students remember to check values of $h\ x$ for 0 and for x 0. Finish up by having them draw a graph of $h\ x$

NOTE If time is limited, allow the students to find a δ that works from looking at graphs, as opposed to finding the largest possible δ algebraically.

ANSWERS

PART 1

1. (a) δ $\frac{1}{4}$ (b) δ $\frac{1}{2}$ (c) Any δ will work.
2. δ $\frac{1}{20}$, δ $\frac{1}{10}$, any δ will work. $h\ x$ $\frac{1}{25}$ 0 08 if x 0 which is always less than $\frac{1}{10}$.

Students may or may not see the wrinkle in $h\ x$ at this point.

PART 2

δ $\frac{1}{}$, δ $\frac{1}{}$, no δ will work. $h\ x$ $\frac{1}{}$ 0 08 if x 0 which is always greater than 0 01.

HOMEWORK PROBLEMS

CORE EXERCISES 3, 7, 28, 42

SAMPLE ASSIGNMENT 3, 7, 28, 33, 37, 41, 42, 44

EXERCISE	D	A	N	G
3				

GROUP WORK 1, SECTION 2.4
A Jittery Function

Not all functions that occur in mathematics are simple combinations of the "toolkit" functions usually seen in calculus. Consider this function:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$$

It is obvious that you can't graph this function in the same literal way that you would graph $y = \cos x$, but it is useful to have some idea of what this function looks like. Try to sketch the graph of $y = f(x)$.

2. Does $\lim_{x \to 0} f(x)$ exist? If so, what is its value? If not, why not? Make sure to justify your answer carefully.

3. Does $\lim_{x \to 1} f(x)$ exist? Carefully justify your answer.

4. What do you conjecture about $\lim_{x \to a} f(x)$ if $a \neq 0$?

GROUP WORK 2, SECTION 2.4
The Dire Wolf Collects his Due

In this activity we will explore a function that is particularly loved by mathematicians everywhere, sin π x .

Sketch the graph of y sin π x on the interval [1 3].

It appears that this function is not defined at x 0 does not have a limit at x 0 and in fact, does not even have a right-hand limit.

Evaluate sin π x at x $\frac{2}{1}$, $\frac{2}{5}$, and $\frac{2}{9}$.

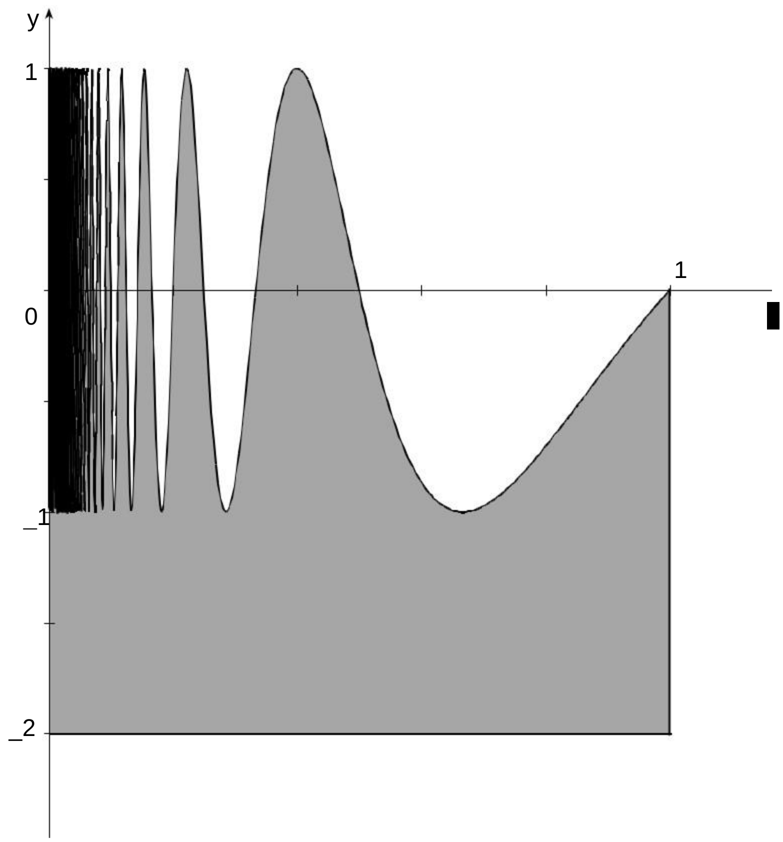
Evaluate sin π x for x $\frac{2}{4n^{\ }\ 1}$, n a positive integer, using the pattern from part (a).

Evaluate sin π x for x $\frac{1}{1}$, $\frac{1}{2}$, and $\frac{1}{3}$. Using this pattern, evaluate sin π x for x $\frac{1}{n}$, n a positive integer.

(d) Give an argument to show that $\lim_{x \quad 0}$ sin π x does not exist.

GROUP WORK 2, SECTION 2.4
The Dire Wolf Collects his Due (Supplementary Problems)

Consider the region bounded on the bottom by the line y 2 on the left by the line x 0, on the right by the line x 1, and on top by the graph of y sin π x as shown:



Is the length of the boundary of this region finite or infinite? Justify your answer.

Is the area of this region finite or infinite? Justify your answer.

Do you think this result is as interesting as we do? Why or why not?

GROUP WORK 3, SECTION 2.4
Infinity is Very Big

For what values of x near 0 is it true that $\frac{1}{x^2} > 1{,}000{,}000$?

2. The precise definition of $\lim_{x \to a} f(x) = \infty$ states that for every positive number M, no matter how large, there is a corresponding positive number δ such that $f(x) > M$ whenever $0 < |x - a| < \delta$.

(a) Use this definition to prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

(b) Why is it not true that $\lim_{x \to 0} \frac{1}{x} = \infty$? Give reasons for your answer.

GROUP WORK 4, SECTION 2.4

The Significance of the “ For Every” (Part 1)

Consider the following functions:

$$f\ \ x\ \ 2x\ \ 3 \qquad\qquad g\ x\ \ \frac{x^2\ \ 4}{x\ \ 2} \qquad\qquad h\ \ x\ \ \frac{x}{25x}$$

We want to try to prove the following statements:

$$\lim_{x\ 1} f\ \ x\quad 5 \qquad\qquad \lim_{x\ 2} g\ x\quad 4 \qquad\qquad \lim_{x\ 0} h\ x\quad \frac{1}{25}$$

Notice that these are not obvious statements, since $g\ 2$ and $h\ 0$ are both undefined.

We start with $\varepsilon\ \ \frac{1}{2}$.

Can you find a number δ with the property that, when $x\ 1\ \delta$, $f\ x\ 5\ \frac{1}{2}$? Illustrate your answer with a graph, and prove it algebraically.

(b) Can you find a number δ with the property that, when $x\quad 2\quad \delta$, $g\ x\quad 4\quad \frac{1}{2}$?

(c) Can you find a number δ with the property that, when $x\ \ 0\ \ \delta$, $h\ x\quad \frac{1}{\ }\quad \frac{1}{\ }$?

We now have some reason to believe that the above statements are true. But just having “some reason to believe” isn’t enough for mathematicians. Repeat the previous problem for $\varepsilon\ \frac{1}{10}$.

Now, what do you believe about these limits?

GROUP WORK 4, SECTION 2.4

The Significance of the ‘‘ For Every’’ (Part 2)

Try the three limits again, this time for $\varepsilon \ \frac{1}{100}$. Make sure that when you are trying to verify the condition x x_0 δ, you check values of x_0 x and x_0 x. Do you wish to change your answer to Problem 3 from Part 1?

2.5 Continuity

SUGGESTED TIME AND EMPHASIS

$1\text{–}1\frac{1}{2}$ classes Essential material

POINTS TO STRESS

The graphical and mathematical definitions of continuity, and the basic principles.

Examples of discontinuity.

The Intermediate Value Theorem: mathematical statement, graphical examples, and applied examples.

QUIZ QUESTIONS

TEXT QUESTION The text says that y tan x is discontinuous at x 2 . This would seem to contradict Theorem 7. Does it? Why or why not?

ANSWER It does not; tan x is indeed continuous at every point in its domain, but x 2 is not in its domain.

DRILL QUESTION Assume that f 1 5, and f 3 5. Does there have to be a value of x, between 1 and 3, such that f x 0?

ANSWER No, there does not. Only if the function is continuous does the IVT indicate that there must be such a value.

MATERIALS FOR LECTURE

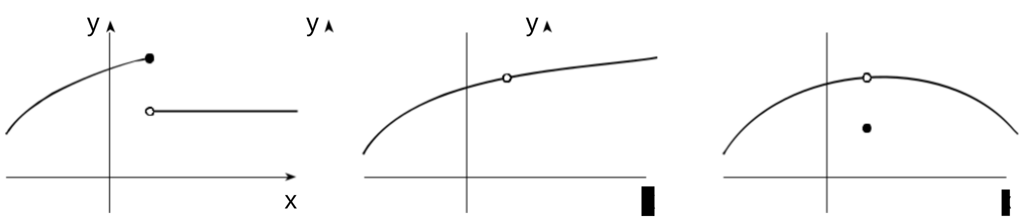
Discuss the idea of continuity at a point, continuity on an interval, and the basic types of discontinuities. Note that the statement "f is continuous at x a" is implicitly saying three things:

1. f a exists.

2. $\lim_{x \quad a} f \quad x$ exists.

The two quantities are equal.

To show that all three statements are important to continuity, have the students come up with examples where the first holds and the second does not, the second holds and the first does not, and where the first two hold and the third does not. Examples are sketched below.



Some students tend to believe that all piecewise functions are discontinuous at the border points. Examine

the function $f \quad x \quad \begin{array}{ll} x^2 & \text{if } x \quad 1 \\ \ln x \quad 1 & \text{if } 1 \quad x \quad e \\ x & \text{if } x \quad e \end{array}$ at the points x 1 and x e. This would be a good

time to point out that the function x is continuous everywhere, including at x 0

Start by stating the basic idea of the Intermediate Value Theorem (IVT) in broad terms. (Given a function on an interval, the function hits every y-value between the starting and ending y-values.) Then attempt to translate this statement into precise mathematical notation. Show that this process reveals some flaws in our original statement that have to be corrected (the interval must be closed; the function must be continuous.)

To many students the IVT says something trivial to the point of uselessness. It is important to show examples where the IVT is used to do non-trivial things.

Example: A graphing calculator uses the IVT when it graphs a function. A pixel represents a starting and ending y-value, and it is assumed that all the intermediate values are there. This is why graphing calculators are notoriously bad at graphing discontinuous functions.

Example: Assume a circular wire is heated. Use the IVT to show that there exist two diametrically opposite points with the same temperature.

ANSWER Let $f x$ be the difference between the temperature at a point x and the temperature at the point opposite x. f is a difference of continous functions, and is thus continuous itself. If $f x$ 0, then $f x$ 0, so by the IVT there must exist a point at which f 0.

Example: Show that there exists a number whose cube is one more than the number itself. (This is Exercise 69.)

ANSWER Let $f\ x x^3 \quad x \quad 1.$ f is continuous, and $f\ 0 \quad 0$ and $f\ 2 \quad 0$. So by the IVT, there exists an x with $f\ x \quad 0$.

Have the students look at the function $f\ x$ 0 x irrational; $\frac{1}{q}$ x $\frac{p}{q}$, where p and q are integers, q is positive, and the fraction is in lowest terms

This function, discovered by Riemann, has the property that it is continuous where x is irrational, and not continuous where x is rational.

WORKSHOP/DISCUSSION

Indicate why $f x$ csc x is continuous everywhere on its domain, but is not continuous everywhere. Then discuss the continuity of $g x e^{\csc x}$, and why all the discontinuities of g are removable.

0 if x is rational Ask

f x is irrational If the group activity "A Jittery Function" was assigned, revisit $f x_2$

the students to guess if this function is continuous at x 0. Many will not believe that it is. Now look at it using the definition of continuity. They should agree that f 0 0. In the activity it was shown that $\lim_{x \quad 0} f x$ existed and was equal to 0. So, this function is continuous at x 0. A sketch such as the one found in the answer to that group work may be helpful.

Present the following scenario: two ice fishermen are fishing in the middle of a lake. One of them gets up at 6:00 P.M. and wanders back to camp along a scenic route, taking two and a half hours to get there. The second one leaves at 7:00 P.M., and walks to camp along a direct route, taking one hour to get there. Show that there was a time where they were equidistant from camp.

Revisit Exercise 5 in Section 2.2, discussing why the function is discontinuous.

Show that $f x\ 1\ 2\frac{1}{1\ x}$ is not continuous. (This is the same function used in "An Interesting Function", Group Work 1 in Section 2.2.)

GROUP WORK 1: EXPLORING CONTINUITY

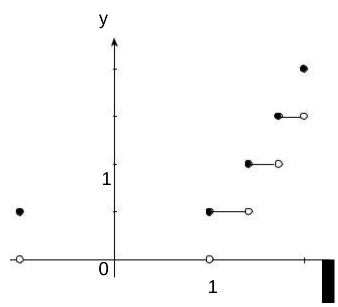
Warm the students up by having them graph 2^x without their calculators, and asking where it is continuous. The first problem is appropriate for all classes. Problem 2 assumes the students have previously seen the activity "A Jittery Function". If they have not, skip it and go directly to Problem 3. Before handing out Problem 3, make sure that the students recall the definition of the greatest integer ("floor") function y [[x]]. After this activity, discuss the continuity of [[x]] at integer and at non-integer values. Problem 4 is intended for classes with a more theoretical bent.

ANSWERS

c 4, m 5

2. (b) 0 (c) 0 (d) It is continuous because f 0 $\lim_{x\to 0} f\ x$.

3. (a)



(b) All values except a1, 2, 3, 2

(c) $\lim_{x\to 0} x^2$ 0; $\lim_{x\to 2} x^2$ does not exist because the left- and right-hand limits are different.

4. (a) The fact that f is continuous implies that $\lim_{x\to a} f\ x\ f\ a$ for all a. Then, by the Limit Laws,

$$\lim_{x\to a} h\,x \quad \lim_{x\to a} f\ x^2 \quad \lim_{x\to a} f\ x \quad 2 \quad f\ a^2 \quad h\,a\,.$$

False. For example, let $f\ x$ 1 if x 0 1 if x 0

GROUP WORK 2: THE AREA FUNCTION

This activity is designed to reinforce the notion of continuity by presenting it in an unfamiliar context. It will also ease the transition to area functions in Chapter 5. It is important that this activity be well set up. Do Problem 1 with the students, making sure to compute a few values of $A\ r$ and to sketch it. The students should try to answer Problems 2 and 3 using their intuition and the definition of continuity. It may be desirable to have the students restrict themselves to r 0. Note that in this activity, one can "prove" continuity by looking at the actual formulas for $A\ r$ and $B\ r$, but that the goal of the activity is that the students understand intuitively why both area functions are continuous.

Students may disagree on the answer to Problem 3. If you are fortunate enough to have groups that have reached opposite conclusions, break up one or more of them, and have representatives go to other groups to try to convince them of the error of their ways.

ANSWER Yes to all three questions. For all r, $A\ r$, $B\ r$, and $C\ r$ exist; and $\lim_{x\to r} A\ x A\ r$, $\lim_{x\to r} B\ x B\ r$, and $\lim_{x\to r} C\ x$ $C\ r$. (The limits can be shown to exist by looking at the left- and right-hand limits.)

GROUP WORK 3: THE TWIN PROBLEM

When students see this problem, there is a good chance that they will disagree among themselves about the answer. Let them argue for a while. Ideally, they will come up with the idea of using the Intermediate Value Theorem to prove that Dr. Stewart was correct. If they don't, this may need to be given to them as a hint. Another hint they may need is that the Intermediate Value Theorem deals with a single continuous function, whereas the problem is talking about two functions, Stewart's temperature and Shasta's temperature. They will have to figure out a way to find a single function that they can use. Encourage them to write up a solution to the exact degree of rigor that will be expected of them on homework and exams; this is a good opportunity to convey the course's expectations to the students.

ANSWER Let $S\ t$ and $O\ t$ be Dr. Stewart's and Shasta's temperatures at time t. Now let $T\ t\ S\ t\ O\ t$. $T\ t$ is continuous (being a difference of continuous functions), T 0 0 (Dr. Stewart is warmer at first), and $T\ f$ 0 (where f represents the end of the vacation; Shasta is warmer at the end). Therefore, by the IVT, there exists a time a at which $T\ a$ 0 and hence $S\ a\ O\ a$. Notice that most students who try to argue that the conclusion is false (using things such as stasis chambers and exceeding the speed of light) are really trying to construct a scenario where the continuity of the temperature function is violated.

GROUP WORK 4: SWIMMING TO THE SHORE

Emphasize to the students that they are not trying to *find* x, but simply trying to prove its existence. As in the Twin Problem, a first hint might be to use the IVT, and a second could be to find a single continuous function of x.

It is probably best to do this activity after the students have seen the solution to the ice fisherman problem above, or the Twin Problem.

ANSWER Let $D\ x\ d\ P_x\ A\ d\ P_x\ B$. $D\ x$ is continuous, $D\ A$ 0, and $D\ B$ 0. Therefore, by the IVT, there is a place where $D\ x$ 0.

HOMEWORK PROBLEMS

CORE EXERCISES 4, 7, 10, 12, 24, 44, 53, 67

SAMPLE ASSIGNMENT 4, 7, 10, 12, 15, 19, 24, 25, 40, 44, 53, 67, 73

EXERCISE	D	A	N	G
7				
19				

GROUP WORK 1, SECTION 2.5

Exploring Continuity

1. Are there values of c and m that make $h(x) = \begin{cases} cx^2 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ x^3 + mx & \text{if } x > 1 \end{cases}$ continuous at $x = 1$? Find c and m, or explain why they do not exist.

2. Recall the function $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$

(a) Do you believe that $f(x)$ is continuous at $x = 0$? Why or why not?

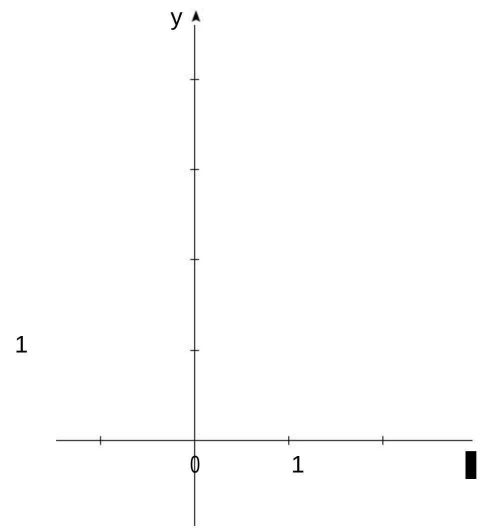
(b) What is $f(0)$?

(c) What is $\lim_{x \to 0} f(x)$?

(d) Use parts (b) and (c) either to revise your answer to part (a), or to prove that your answer is correct.

Consider the function $h\ x x^2$

Sketch the graph of the function for 1 x 2.



(b) For what values of a, 1 a 2, is $\lim\limits_{x\ \ a} h\ x$ $h\ a$?

(c) Compute $\lim\limits_{x\ \ 0} x^2$ and $\lim\limits_{x\ \ 2} x^2$, if they exist. Explain your answers.

We know that the function $g\ x x^2$ is continuous everywhere.

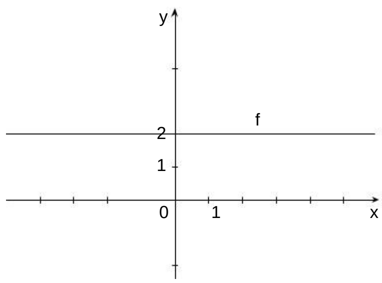
Show that if f is continuous everywhere, then $h\ x\ f\ x^{\ 2}$ is continuous everywhere, using a limit argument.

Is it true or false that if $h\ x\ f\ x^{\ 2}$ is continuous everywhere, then f is continuous everywhere? If it is true, prove it. If it is false, give a counterexample.

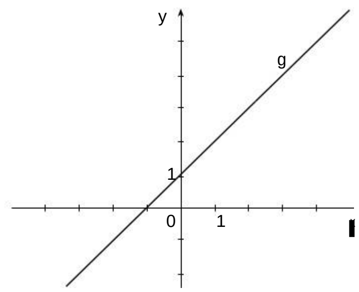
GROUP WORK 2, SECTION 2.5

The Area Function

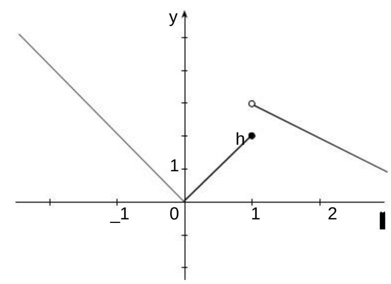
The following are graphs of $y \quad f \; x$, $y \quad g \; x$, and $y \quad h \; x$:



$f \; x 2$



$g \; x x \quad 1$



$h \; x \qquad 2x \qquad \text{if } x \quad 1$

$\qquad\qquad x \quad 4 \qquad \text{if } x \quad 1$

Let $A \; r$ be the area enclosed by the x-axis, the y-axis, the graph of the function f , and the line $x \; r$. Would you conjecture that $A \; r$ is continuous at every point in the domain of f ? Why or why not?

Let $B \; r$ be the area enclosed by the x-axis, the y-axis, the graph of the function g, and the line $x \; r$. Would you conjecture that $B \; r$ is continuous at every point in the domain of g? Why or why not?

Let $C \; r$ be the area enclosed by the x-axis, the y-axis, the graph of the function h, and the line $x \; r$. Would you conjecture that $C \; r$ is continuous at every point in the domain of h? Why or why not?

GROUP WORK 3, SECTION 2.5
The Twin Problem

There is a bit of trivia about the author of your textbook, Dr. James Stewart, that few people know. He has an evil twin sister named Shasta. Although he loves his sister dearly, she dislikes him and tries to be different from him in all things.

Last winter, they both went on vacation. Dr. Stewart went to Hawaii. Shasta had planned on going to Aruba, but she decided against it. She hates her brother so much that she was afraid there would be a chance that they might be experiencing the same temperature at the same time, and that prospect was distasteful to her. So she decided to vacation in northern Alaska.

After a few days, Dr. Stewart received a call: “This is Shasta. I am cold and uncomfortable here. That’s good, since you are undoubtedly warm and comfortable, and I want us to be different. But I’m not sure why I should be the one in northern Alaska. I think we should switch places for the last half of our trip.”

“It is only fair,” he agreed.

So they each traveled again. Dr. Stewart took a trip from Hawaii to Alaska, while Shasta took a trip from Alaska to Hawaii. They each traveled their own different routes, perhaps stopping at different places along the way. Eventually, they had reversed locations. Dr. Stewart was shivering in Alaska; Shasta was in Hawaii, warm and happy. She received a call from her brother.

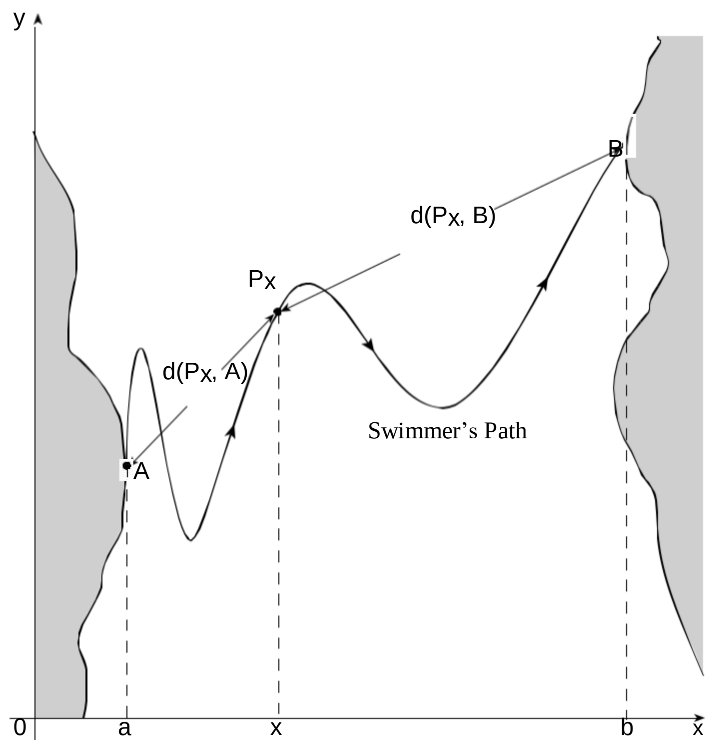
“Hi, Shasta. Guess what? At some time during our travels, we were experiencing exactly the same temperature at the same time. So HA!”

Is Dr. Stewart right? Has Good triumphed over Evil? He would try to write out a proof of his statement, but his hands are too frozen to grasp his pen. Help him out. Either prove him right, or prove him wrong, using mathematics.

GROUP WORK 4, SECTION 2.5
Swimming to the Shore

A swimmer crosses a river starting at point A and ending at point B, following the path shown below. Prove that for some value x, the swimmer's distance $d\,P_x\,A$ from A is the same as the distance $d\,P_x\,B$ from

.



2.6 Limits at Infinity; Horizontal Asymptotes

SUGGESTED TIME AND EMPHASIS

1 class Essential material(This material may also be covered after Section 4.2.)

POINTS TO STRESS

The geometric and limit definitions of horizontal asymptotes, particularly as they pertain to rational functions.

The computation of infinite limits.

The technique and the dangers of using calculators to check limits (both numerically and graphically).

QUIZ QUESTIONS

TEXT QUESTION To evaluate the limit at infinity of a rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. Why must we do such a thing? ANSWER By doing this division, we make the denominator approach a finite value as $x \to \infty$. Now we can take the limit of the numerator, and easily divide it by the limit of the denominator.

DRILL QUESTION Compute $\lim_{x \to \infty} \frac{1 \quad x^2 \quad 2x^3}{5x \quad 3x \quad 5}$.

ANSWER 2

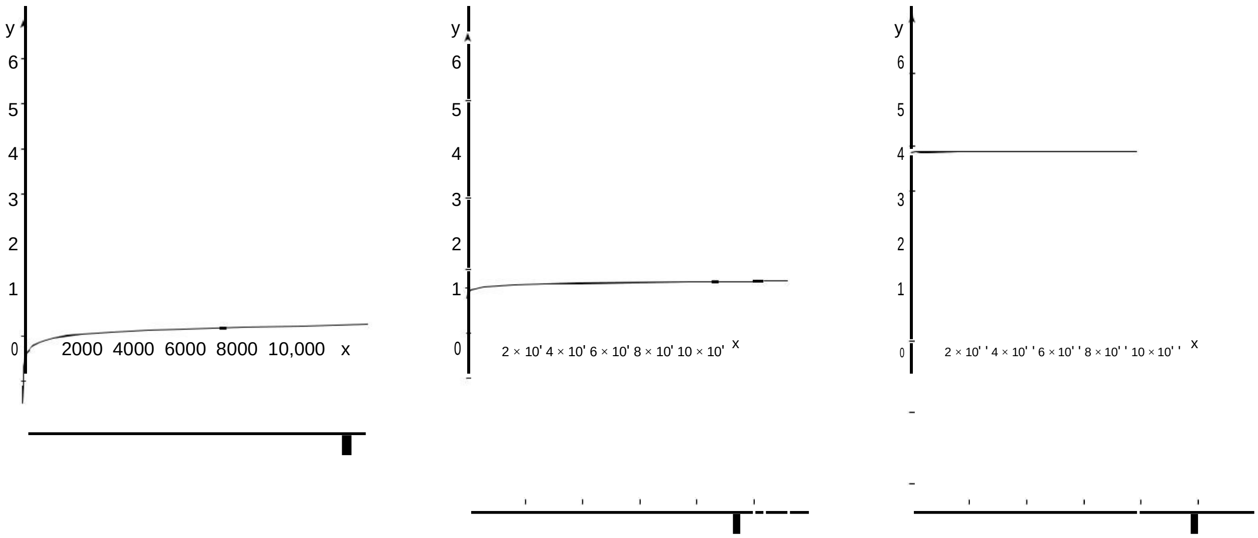
MATERIALS FOR LECTURE

Describe asymptotes verbally and then give graphical and limit definitions. Note that a function *can* cross its horizontal asymptote. Explain the difference between the definitions of $\lim_{x \to a} f(x) = L$ and $\lim_{x \to \infty} f(x) = L$, emphasizing that in one case we choose a small δ and in the other, a large value N. Perhaps include a description of slant asymptotes.

Ask students if a function can be bounded but not have a horizontal asymptote. Does $\sin x$ have a horizontal asymptote? What about $\frac{\sin x}{x}$? How is $\frac{\sin x}{x}$ different?

$\overline{100}$ $\overline{x}$

Examine $\lim_{x} \ln \ln x$ on a graphing calculator, first by plugging in large numbers, then by examining the graph. Then show that this limit is, in fact, infinity. If teaching an advanced class, one might try to "prove" that this limit is the expected 5 429 using epsilons and deltas, and see how the attempt fails. (NOTE 5 429 is $\ln \ln 10^{99}$, which is what a student would come up with by plugging very large numbers into a calculator.)



$y \quad \ln \ln x$ $\qquad$ $y \quad \ln \ln x$ $\qquad$ $y \quad \ln \ln x$

Discuss rates of growth. For large values of x, $3^x \quad 2^x \quad x^3 \quad x^2 \quad x \quad \ln x \quad \ln \ln x$, even though they all approach infinity. (An advanced class can discuss the even larger x^x.) Point out that functions such as $0\ 85^x$ and x^{2} don't go to infinity. Note that for values of x near zero, $x\ x^2\ x^3$, although all approach zero. Point out that as x approaches 0, a^x approaches 1 and $\log_a x$ approaches .

WORKSHOP/DISCUSSION

Compute the limits of $y\ \frac{5}{x2\ 25}$ as x , and as x . Graph the function. Also perhaps review how to find the limits as x 5.

Graph $y \quad \frac{2x^3 \quad 16}{x^3 \quad 27}$, after calculating limits as x 3 and as x.

Calculate $\lim_{x} \quad e^x$. Show the students how to find a domain for x such that $e^x \quad 0\ 001$ for all x in that domain.

Examine $\lim_{x} \frac{[[x]]}{x}$ and $\lim_{x} \frac{[[x]]}{x}$.

GROUP WORK 1: TO INFINITY AND BEYOND

This activity is intended to develop the students' intuition about infinite limits. While they should justify their answers, it is important that they also get some feel for how limits as x behave.

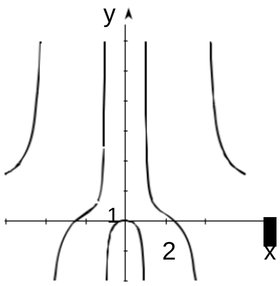
ANSWERS **1.** (a) $y \quad \frac{1}{3}$ (b) None (c) $y \quad 6$ 2. 0 3. 4. 0

GROUP WORK 2: INFINITE LIMITS

This activity is too long to be done in a 50-minute session. Pick and choose problems. It is more important to have good introduction and closure on each part than to have all of them worked out. Problem 4 is an extension of Exercise 55.

ANSWERS

Answers will vary.



Vertical: x 3 2, horizontal: y 3 4

Answers will vary. Possible answers: (a) $f\ xx^2$,

$g\ xx$ (b) $f\ xx$, $g\ xx^2$ (c) $f\ xx$,

$g\ xx$ (d) $f\ xx^2$ 42, $g\ xx^2$

4. (a) $\frac{a_m}{b_n}$ (b) 0 (c) or

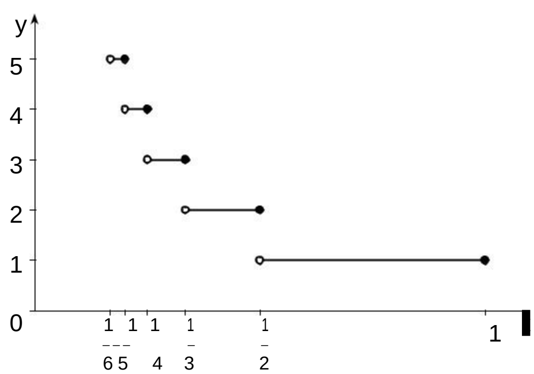
GROUP WORK 3: I AM THE GREATEST

Before handing this activity out, make sure the students know the definition of the greatest integer function, and can sketch its graph.

ANSWERS

This can be done from the graph, or using the definition. (Choose ε 0, then let δ 0 2.)

(a)



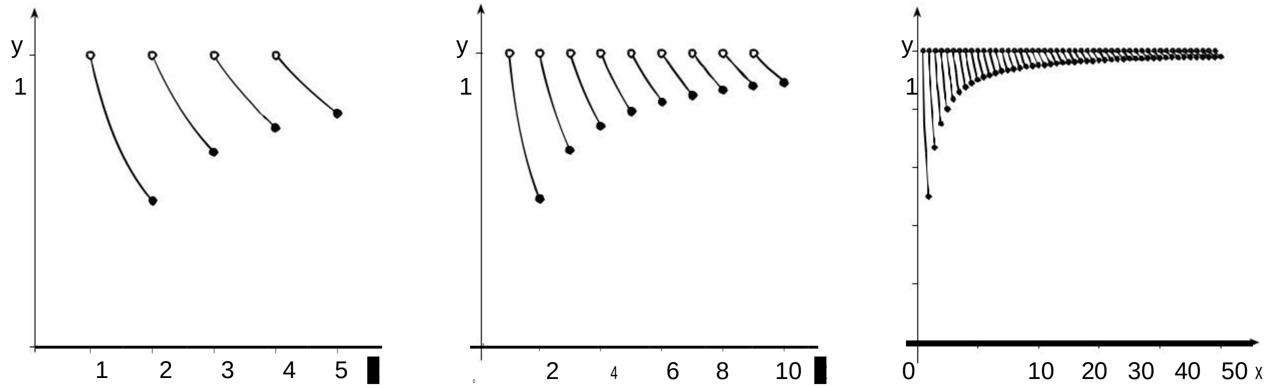
(b) Lower bound $\frac{n}{n\ \ 1}$, upper bound 1

Use the Squeeze Theorem, taking the limits of the bounds as n 0.

0

When x 1, $\frac{10}{x}$

5. $\lim_{x \to \infty} \frac{[[x]]}{x} = 1$. This can be seen by a similar bounding argument to the one above. If you use this activity, it is a good idea to show the graph to your students, for it is a truly pretty thing:



The tops of the lines are at $y = 1$ and the bottoms trace out the curve $y = 1 - 1/x$.

HOMEWORK PROBLEMS

CORE EXERCISES 3, 10, 49, 55, 56, 71, 77

SAMPLE ASSIGNMENT 3, 10, 12, 18, 44, 49, 51, 55, 56, 59, 65, 68, 71, 77, 81

EXERCISE	D	A	N	G
3				
10				
12				
18				
44				
49				
51				
55				
56				
59				
65				
68				
71				
77				
81				

To Infinity and Beyond

Describe the horizontal asymptotes, if any, of the following functions.

Ā Ȁ ÈȀ ` Ā↠Ā Ā Ā

$$f\ x \quad \frac{x^2 \quad 2}{3x^4 \quad x^2 \quad 5}$$

(b) $$\frac{2x^5 \quad 2x^3 \quad 18}{x^4 \quad x^2 \quad x \quad 2} \qquad f \quad x$$

(c) $$f\ x \quad \frac{2x^5 \quad 2x^3 \quad 18}{x^4 \quad 3x^3 \quad x \quad 2} \quad 2x$$

2. Find $\lim_{x} x^{25} e^{x}$.

3. Find $\lim_{x} \frac{x}{\ln x}$.

4. Find $\lim_{x} \frac{\cos x}{\ln \ln x}$.

GROUP WORK 2, SECTION 2.6

Infinite Limits

Draw an even function which has the lines y 1, x4, and x1 among its asymptotes.

2. Describe all vertical and horizontal asymptotes of $f\ x\ \dfrac{3x^2\ \ 4x\ \ 5}{16x^4\ \ 81}$.

3. Find formulas for two functions, f and g, such that $_x\lim$ $f\ x_x\lim\ g\ x$ and

(a) $\lim\limits_{x}\ f\ x\ \ \ g\ x$

(b) $\lim\limits_{x}\ f\ x\ \ \ g\ x$

(c) $\lim\limits_{x}\ f\ x\ \ \ g\ x\ \ \ 0$

(d) $\lim\limits_{x}\ f\ x\ \ \ g\ x\ \ \ 42$

Let $P\ x\ a_m\ x^m\ a_1x\ a_0$, and $Q\ x\ b_n\ x^n\ b_1x\ b_0$ be polynomials of degree m and n, respectively.

(a) Find $_x\lim \dfrac{P\ x}{Q\ \ x}$ if $m\ \ n$.

(b) Find $\lim\limits_{x} \dfrac{P\ x}{Q\ x}$ if $m\ \ n$.

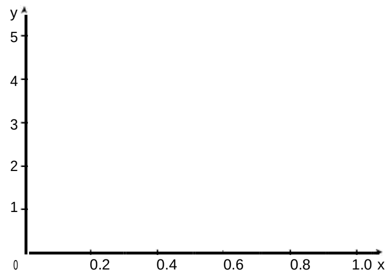
(c) Find $\lim\limits_{x} \dfrac{P\ x}{Q\ x}$ if $m\ \ n$.

GROUP WORK 3, SECTION 2.6

I Am the Greatest

1. Show that $\lim_{x \to 0} \frac{[[x]]}{x} = 0$

2. (a) Sketch a graph of $\left[\left[\frac{1}{x}\right]\right]$ on the axes below.



(b) If $\frac{1}{n+1} < x \le \frac{1}{n}$ find upper and lower bounds for the expression $x\left[\left[\frac{1}{x}\right]\right]$.

(c) Use the estimates above to show that $\lim_{x \to 0} x\left[\left[\frac{1}{x}\right]\right] = 1$.

$$\frac{1}{x}$$

$x \quad 0$

4. $\lim x \quad \frac{1}{} \quad 0.$

Show that $x \qquad x$

5. Compute $\lim \frac{1}{x} [[x]]$ Justify your reasoning.

x

2.7 Derivatives and Rates of Change

SUGGESTED TIME AND EMPHASIS

1–2 classes   Essential material

POINTS TO STRESS

1. The slope of the tangent line as the limit of the slopes of secant lines (visually, numerically, algebraically).

2. Physical examples of instantaneous rates of change (velocity, reaction rate, marginal cost, and so on) and their units.

3. The derivative notations $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ and $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

Using f' to write an equation of the tangent line to a curve at a given point.

Using f' as an approximate rate of change when working with discrete data.

QUIZ QUESTIONS

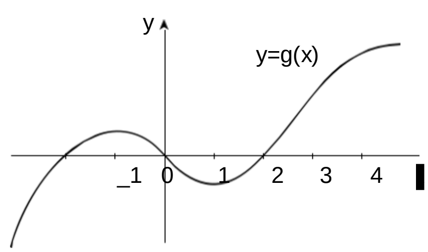
TEXT QUESTION Why is it necessary to take a limit when computing the slope of the tangent line? ANSWER There are several possible answers here. Examples include the following:

By definition, the slope of the tangent line is the limit of the slopes of secant lines.

You don't know where to draw the tangent line unless you pick two points very close together. The idea is to get them thinking about this question.

DRILL QUESTION For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

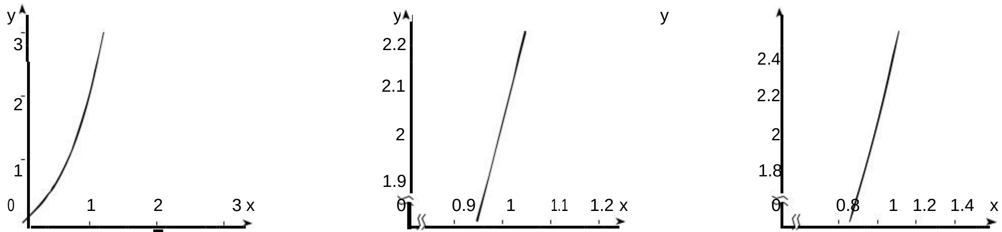
$0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$



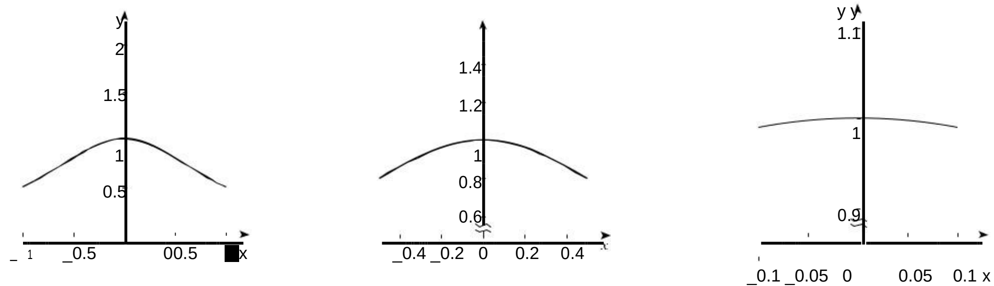
ANSWER $g'(0) < 0 < g'(4) < g'(-2) < g'(2)$

MATERIALS FOR LECTURE

Review the geometry of the tangent line, and the concept of "locally linear". Estimate the slope of the line tangent to $y \quad x^3 \quad x$ at $1\ 2$ by looking at the slopes of the lines between $x \quad 0\ 9$ and $x \quad 1\ 1$, $0\ 99$ and $x\ 1\ 01$, and so forth. Illustrate these secant lines on a graph of the function, redrawing the figure when necessary to illustrate the "zooming in" process.



Similarly examine $y \quad \frac{1}{\ }$ at $0\ 1$.



If "A Jittery Function" was covered in Section 2.4, look at $f\ x \quad \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$ Poll the class: Is there a tangent line at $x\ 0$? Then examine what happens if you look at the limits of the secant lines.

Have students estimate the slope of the tangent line to $y \sin x$ at various points. Foreshadow the concept of concavity by asking them some open-ended questions such as the following: What happens to the function when the slope of the tangent is increasing? Decreasing? Zero? Slowly changing?

Discuss how physical situations can be translated into statements about derivatives. For example, the budget deficit can be viewed as the derivative of the national debt. Describe the units of derivatives in real world situations. The budget deficit, for example, is measured in billions of dollars per year. Another example: if $s\ d$ represents the sales figures for a magazine given d dollars of advertising, where s is the number of magazines sold, then $s\ \ d$ is in magazines per dollar spent. Describe enough examples to make the pattern evident.

Note that the text shows that if $f\ x\ x^2\ 8x\ 9$, then $f\ a\ 2a\ 8$. Thus, $f\ 55\ 102$ and $f\ 100\ 192$. Demonstrate that these quantities cannot be easily estimated from a graph of the function. Foreshadow the treatment of a as a variable in Section 2.8.

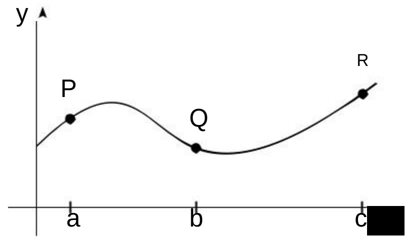
If a function models discrete data and the quantities involved are orders of magnitude larger than 1, we can use the approximation $f\ x\ f\ x\ 1\ f\ x$. (That is, we can use $h\ 1$ in the limit definition of the derivative.) For example, let $f\ t$ be the total population of the world, where t is measured in years since 1800. Then f

211 is the world population in 2011, *f* 212 is the total population in 2012, and *f* 211 is approximately the change in population from 2011 to 2012. In business, if *f n* is the total cost of producing *n* objects, *f n* approximates the cost of producing the *n* 1 th object.

WORKSHOP/DISCUSSION

"Thumbnail" derivative estimates: graph a function on the board and have the class call out rough values of the derivative. Is it larger than 1? About 1? Between 0 and 1? About 0? Between 1 and 0? About 1? Smaller than 1? This is good preparation for Group Work 2 ("Oiling Up Your Calculators").

Draw a function like the following, and first estimate slopes of secant lines between $x \quad a$ and $x \quad b$, and between $x \quad b$ and $x \quad c$. Then order the five quantities $f \; a$, $f \; b$, $f \; c$, m_{PQ} , and m_{QR} in decreasing order. [Answer: $f \quad b \quad m_{PQ} \quad m_{QR} \quad f \quad c \quad f \quad a$.]

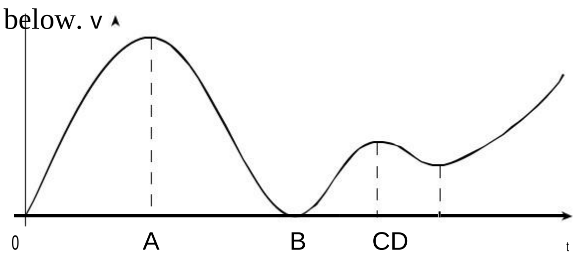


Start the following problem with the students: A car is travelling down a highway away from its starting location with distance function $d\ t\ 8\ t^3\ 6t^2\ 12t$, where t is in hours, and d is in miles.

How far has the car travelled after 1, 2, and 3 hours?

What is the average velocity over the intervals [0 1], [1 2], and [2 3]?

Consider a car's velocity function described by the graph below.



Ask the students to determine when the car was stopped.

Ask the students when the car was accelerating (that is, when the velocity was increasing). When was the car decelerating?

Ask the students to describe what is happening at times A, C, and D in terms of both velocity and acceleration. What is happening at time B?

Estimate the slope of the tangent line to $y \quad \sin x$ at $x \quad 1$ by looking at the following table of values.

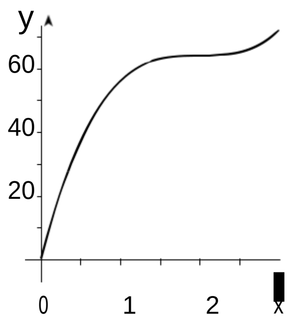
x	$\sin x$	$\dfrac{\sin x \quad \sin 1}{x \quad 1}$
0	0	0 841471
0 5	0 4794	0 724091
0 9	0 7833	0 581441
0 99	0 8360	0 544501
0 999	0 8409	0 540723
1 0001	0 8415	0 540260
1 001	0 8420	0 539881

Demonstrate some sample computations similar to Example 4, such as finding the derivative of $t1 \quad t$ at $t \quad 3$, or of $g\ xx \quad x^2$ at $x \quad 1$.

GROUP WORK 1: FOLLOW THAT CAR

Start this problem by giving the students the function $d\ t\ 8\ t^3\ 6t^2\ 12t$ and having them sketch its graph. Ask them how far the car has traveled after 1, 2, and 3 hours, and then show them how to compute the average velocity for [0 1], [1 2], and [2 3].

ANSWERS



It appears to stop at $t \quad 2$.

8 mi h, 2 mi h, 0 08 mi h

0 mi h. This is where the car stops.

GROUP WORK 2: OILING UP YOUR CALCULATORS

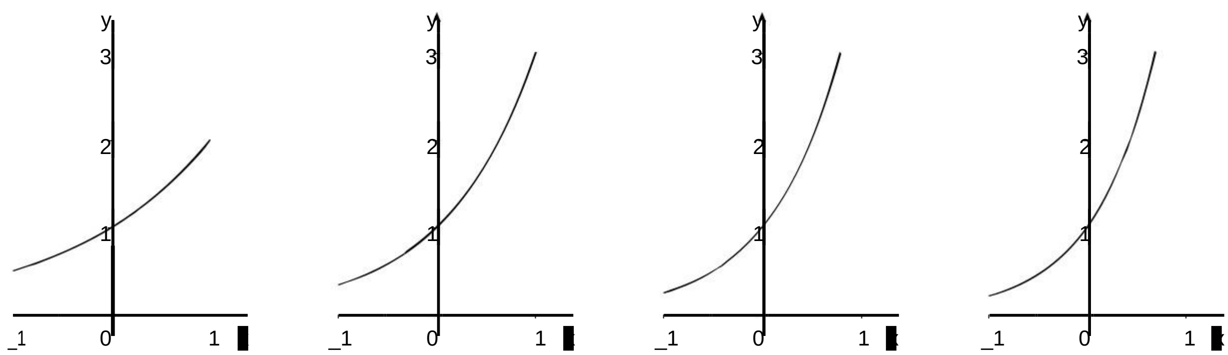
As long as the students have the ability to estimate the slope of a curve at a point, this is a good time to hint at the uniqueness of e as the base of an exponential function.

ANSWERS

1. If the students do this numerically, they should be able to get some pretty good estimates of ln 3 1 098612. If they use graphs, they should be able to get 1 1 as an estimate.

0 7 is a good estimate from a graph, and ln 2 0 693147 is attainable numerically.

As a increases, the slope of the curve at $x \quad 0$ is increasing, as can be seen below.



The slope is less than 1 at $a \quad 2$ and greater than 1 at $a \quad 3$. Now apply the Intermediate Value Theorem.

The students are estimating e and should get 2 72 at a minimum level of accuracy.

GROUP WORK 3: CONNECT THE DOTS

Closure is particularly important on this activity. At this point in the course, many students will have the impression that all reasonable estimates are equally valid, so it is crucial that students discuss Problem 4. If there is student interest, this table can generate a rich discussion. Can A ever be negative? What would that mean in real terms? What would A mean in real terms in this instance?

ANSWERS

A 3500 0 06 % $ It is likely to be an overestimate, because the function lies below its tangent line near p 3500.

After spending $3500, consumer approval is increasing at the rate of about 0 06 % for every additional dollar spent.

Percent per dollar

A $3550 0 06 % $. This is a better estimate because the same figures now give a two-sided approximation of the limit of the difference quotient.

GROUP WORK 4: DERIVATIVES AND INVERSES

If inverse functions were covered, this activity is an excellent way for students to synthesize the two concepts, and to gain intuition and understanding about what the derivative means in a real-world context.

ANSWERS

f^{1} is the time at which a given number of centimeters of rain have fallen. The domain is from 0 cm to the maximum total rainfall. The range is from midnight to the end of the storm.

(a) At 5:00 A.M., 2 cm of rain has fallen.

5 cm of rain has fallen at 2:00 A.M.

At 5 A.M., the rain is falling at the rate of 0 5 cm h.

After 5 cm of rain has fallen, time is passing at a rate of one half hour per centimeter of rainfall.

HOMEWORK PROBLEMS

CORE EXERCISES 3, 7, 13, 14, 18, 23, 29, 33, 59

SAMPLE ASSIGNMENT 3, 7, 11, 13, 14, 17, 18, 23, 29, 33, 37, 47, 49, 54, 59

EXERCISE	D	A	N	G
3				
11				
17				
18				
23				
29				

GROUP WORK 1, SECTION 2.7
Follow that Car

Here, we continue with the analysis of the distance $d\ t\ 8\ t^3\ 6t^2\ 12t$ of a car, where d is in miles and t is in hours.

Draw a graph of $d\ t$ from $t\ \ 0$ to $t\ \ 3$.

Does the car ever stop?

What is the average velocity over [1 3]? over [1 5 2 5]? over [1 9 2 1]?

Estimate the instantaneous velocity at $t\ \ 2$. Give a physical interpretation of your answer.

GROUP WORK 2, SECTION 2.7
Oiling Up Your Calculators

Use your calculator to graph $y\ 3^{x}$. Estimate the slope of the line tangent to this curve at $x\ 0$ using a method of your choosing.

Use your calculator to graph $y\ 2^{x}$. Estimate the slope of the line tangent to this curve at $x\ 0$ using a method of your choosing.

It is a fact that, as a increases, the slope of the line tangent to $y\ a^{x}$ at $x\ 0$ also increases in a continuous way. Geometrically, why should this be the case?

Prove that there is a special value of a for which the slope of the line tangent to $y\ a^{x}$ at $x\ 0$ is 1.

By trial and error, find an estimate of this special value of a, accurate to two decimal places.

GROUP WORK 3, SECTION 2.7
Connect the Dots

A company does a study on the effect of production value p of an advertisement on its consumer approval rating A. After interviewing eight focus groups, they come up with the following data:

Production Value	**Consumer Approval**
\$1000	32%
\$2000	33%
\$3000	46%
\$3500	55%
\$3600	61%
\$3800	65%
\$4000	69%
\$5000	70%

Assume that $A\ \ p$ gives the consumer approval percentage as a function of p.

Estimate $A\ \ \$3500$. Is this likely to be an overestimate or an underestimate?

Interpret your answer to Problem 1 in real terms. What does your estimate of $A\ \ \$3500$ tell you?

What are the units of $A\ \ p$?

Estimate $A\ \ \$3550$. Is your estimate better or worse than your estimate of $A\ \ \$3500$? Why?

GROUP WORK 4, SECTION 2.7
Derivatives and Inverses

Let $f\ t$ be the number of centimeters of rainfall that has fallen on my porch since midnight, where t is the time in hours.

Describe the inverse function $f\ ^{1}$ in words. What are the domain and range of $f\ ^{1}$?

Interpret the following in practical terms. Include units in your answers.

$f\ 5 \quad 2$

$f\ ^{1}\ 5 \quad 2$

$f\ 5 \quad 0\,5$

$f\ ^{1}\ 5 \quad 0\,5$

WRITING PROJECT Early Methods for Finding Tangents

The history of calculus is a fascinating and too-often neglected subject. Most people who study history never see calculus, and vice versa. We recommend assigning this section as extra credit to any motivated class, and possibly as a required group project, especially for a class consisting of students who are not science or math majors.

The students will need clear instructions detailing what their final result should look like. For example, recommend a page or two about Fermat's or Barrow's life and career, followed by two or three technical pages describing the alternate method of finding tangent lines as in the project's directions, and completed by a final half page of meaningful conclusion.

2.8 The Derivative as a Function

SUGGESTED TIME AND EMPHASIS

2 classes Essential material

POINTS TO STRESS

The concept of a differentiable function interpreted visually, algebraically, and descriptively.

Obtaining the derivative function f by first considering the derivative at a point x, and then treating x as a variable.

How a function can fail to be differentiable.

Sketching the derivative function given a graph of the original function.

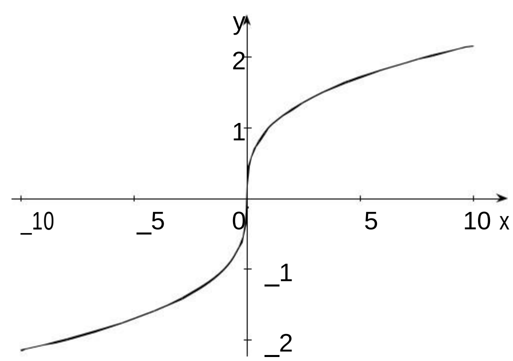
Second and higher derivatives

QUIZ QUESTIONS

TEXT QUESTION The previous section discussed the derivative $f\ a$ for some function f . This section discusses the derivative $f\ x$ for some function f . What is the difference, and why is it significant enough to merit separate sections?

ANSWER a is considered a constant, x is considered a variable. So $f\ a$ is a number (the slope of the tangent line) and $f\ x$ is a function.

DRILL QUESTION Consider the graph of $f\ x = \sqrt[3]{x}$. Is this function defined at $x\ 0$? Continuous at x 0? Differentiable at x 0? Why?



ANSWER It is defined and continuous, but not differentiable because it has a vertical tangent.

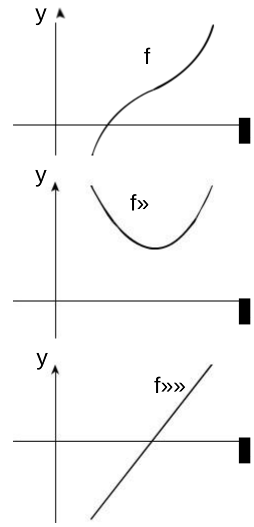
MATERIALS FOR LECTURE

Ask the class this question: “If you were in a car, blindfolded, ears plugged, all five senses neutralized, what quantities would you still be able to perceive?” (Answers: They could feel the second derivative of motion, acceleration. They could also feel the third derivative of motion, “jerk”.) Many students incorrectly add velocity to this list. Stress that acceleration is perceived as a force (hence F ma) and that “jerk” causes the uncomfortable sensation when the car stops suddenly.

Review definitions of differentiability, continuity, and the existence of a limit.

Sketch f from a graphical representation of $f\ x\ x^{2}\ 4$, noting where f does not exist. Then sketch f from the graph of f . Point out that differentiability implies continuity, and not vice versa.

Examine graphs of f and f aligned vertically as shown. If you wish to foreshadow f, add its graph below.
Discuss what it means for f to be positive, negative or zero.
Then discuss what it means for f to be increasing, decreasing or constant.



If the group work "A Jittery Function" was covered in Section 2.4, then examine the differentiability of

$$f\ x \quad \begin{matrix} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{matrix}$$

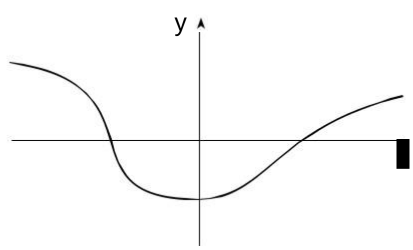
at x 0 and elsewhere, if you have not already done so.

Show that if $f\,x\;x^4\;x^2\;x\;1$, then $f^5\;x\;0$. Conclude that if $f\,x$ is a polynomial of degree m, then $f^{m\;1}\;x\;0$.

WORKSHOP/DISCUSSION

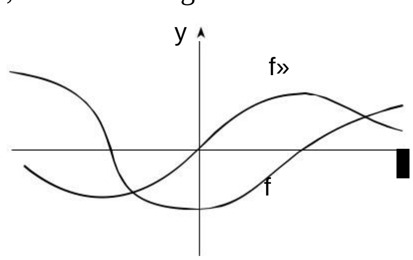
Estimate derivatives from the graph of $f\,x$ sin x. Do this at various points, and plot the results on the blackboard. See if the class can recognize the graph as a graph of the cosine curve.

Given the graph of f below, have students determine where f has a horizontal tangent, where f is positive, where f is negative, where f is increasing (this may require some additional discussion), and where f is decreasing. Then have them sketch the graph of f.



TEC has more exercises of this type using a wide variety of functions.

ANSWER There is a horizontal tangent near x 0. f is positive to the right of 0, negative to the left. f is increasing between the x-intercepts, and decreasing outside of them.



Compute $f\ x$ and $g\ x$ if $f\ x\quad x^2\quad x\ 2$ and $g\,x\quad x^2\quad x\ 4$. Point out that $f\ x\quad g\ x$ and discuss why the constant term is not important. Next, compute $h\ x$ if $h\,x\quad x^2\quad 2x\ 2$. Point out that the graph of $h\ x$ is just the graph of $f\ x$ shifted up one unit, so the linear term just shifts derivatives. TEC contains more explorations on how the coefficients in polynomials and other functions affect first and second derivatives.

Consider the function $f\ x \quad x$ Show that it is not differentiable at 0 in two ways: (it has a by inspection cusp); and by computing the left- and right-hand limits of $f\ x$ at $x \quad 0$ ($\lim_{x\ 0} f\ x$, $\lim_{x\ 0} f \quad x$).

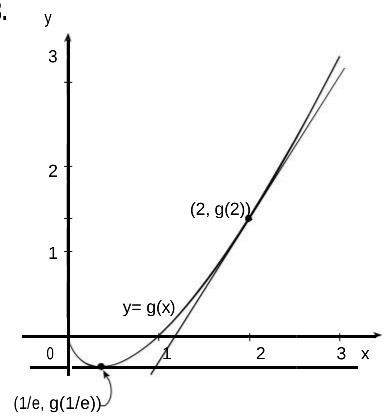
TEC can be used to develop students' ability to look at the graph of a function and visualize the graph of that function's derivative. The key feature of this module is that it allows the students to mark various features of the derivative *directly on the graph of the function* (for example, where the derivative is positive or negative). Then, after using this information and sketching a graph of the derivative, they can view the actual graph of the derivative and check their work.

GROUP WORK 1: TANGENT LINES AND THE DERIVATIVE FUNCTION

This simple activity reinforces that although we are moving to thinking of the derivative as a function of x, it is still the slope of the line tangent to the graph of f.

ANSWERS

1, 3.



$y \quad \ln 2 \quad 1\ x \quad 2 \quad 2 \ln 2$ or $y \quad \ln 2 \quad 1\ x \quad 2$

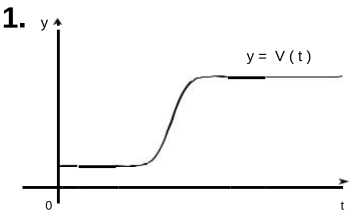
$y 1\ e$

GROUP WORK 2: THE REVENGE OF ORVILLE REDENBACHER

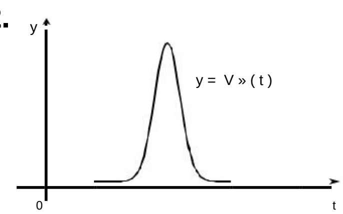
In an advanced class, or a class in which one group has finished far ahead of the others, ask the students to repeat the activity substituting "$D\ t$, the density function" for $V\ t$.

ANSWERS

1.

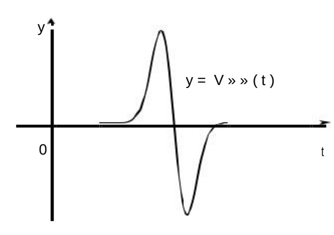


2.



Units are cm s.

3.



When the second derivative crosses the x-axis, the first derivative has a maximum,

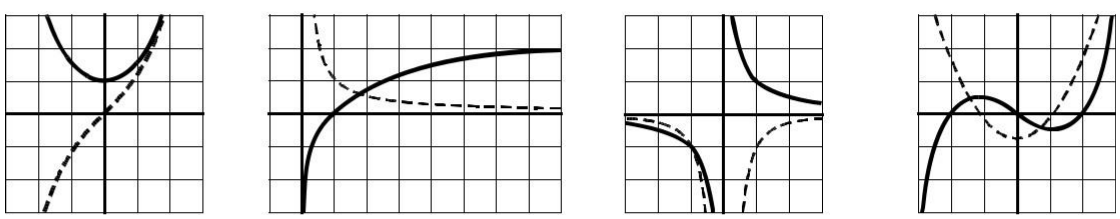
meaning the popcorn is
expanding the fastest.

GROUP WORK 3: THE DERIVATIVE FUNCTION

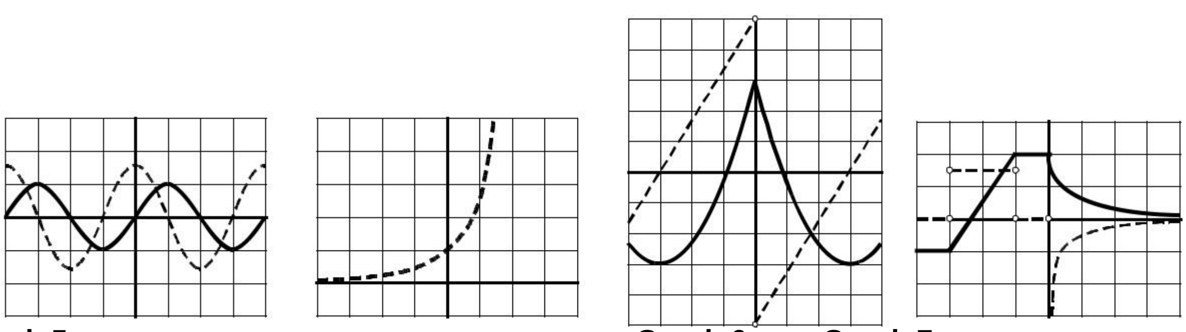
Give each group of between three and five students the picture of all eight graphs. They are to sketch the derivative functions by first estimating the slopes at points, and plotting the values of $f\ x$. Each group should also be given a large copy of one of the graphs, perhaps on acetate. When they are ready, with this information they can draw the derivative graph on the same axes. For closure, project their solutions on the wall and point out salient features. Perhaps the students will notice that the derivatives turn out to be positive when their corresponding functions are increasing. Concavity can even be introduced at this time. Large copies of the answers are provided, in case the instructor wishes to overlay them on top of students' answers for reinforcement. Note that the derivative of graph 6 ($y\ e^{x}$) is itself. Also note that the derivative of graph 1 (y cosh x) is *not* a straight line. Leave at least 15 minutes for closure. The whole activity should take about 45–60 minutes, but it is really, truly worth the time.

If a group finishes early, have them discuss where *f* is increasing and where it is decreasing. Also show that where *f* is increasing, *f* is positive, and where *f* is decreasing, *f* is negative.

ANSWER (larger answer graphs are included after the group work)



Graph 1 **Graph 2** **Graph 3** **Graph 4**



Graph 5 **Graph 6** **Graph 7** **Graph 8**

HOMEWORK PROBLEMS

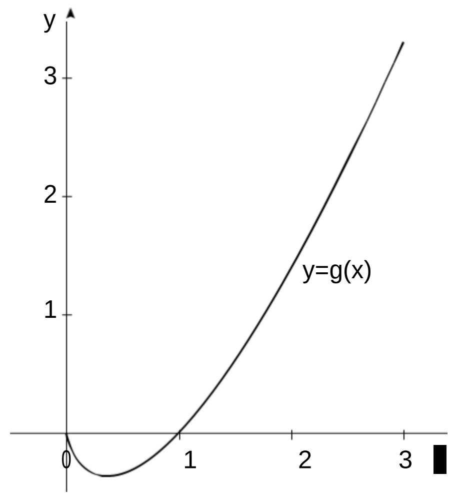
CORE EXERCISES 1, 3, 13, 16, 19, 28, 39, 42, 49

SAMPLE ASSIGNMENT 1, 3, 13, 14, 16, 17, 19, 22, 28, 36, 39, 42, 49, 61, 63

EXERCISE	D	A	N	G
1				
3				
13				
14				
16				
17				
19				
22				
28				
36				
39				
42				
49				
61				
63				

Tangent Lines and the Derivative Function

The following is a graph of $g(x) = x \ln x$.



It is a fact that the derivative of this function is $g'(x) = \ln x + 1$.

Sketch the line tangent to $g(x)$ at $x = 2$ on the graph above.

Find an equation of the tangent line at $x = 2$.

3. Now sketch the line tangent to $g(x)$ at $x = \frac{1}{e} \approx 0.368$.

4. Find an equation of the tangent line at $x = \frac{1}{e}$.

GROUP WORK 2, SECTION 2.8
The Revenge of Orville Redenbacher

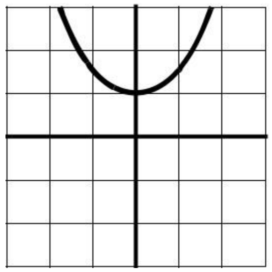
Consider a single kernel of popcorn in a microwave oven. Let $V\,t$ be the volume in cm^3 of the kernel at time t seconds. Draw a graph of $V\,t$, including as much detail as you can, up to the time that the kernel is taken from the oven.

Now sketch a graph of the derivative function $V\ \ t$. What are the units of $V\ \ t$?

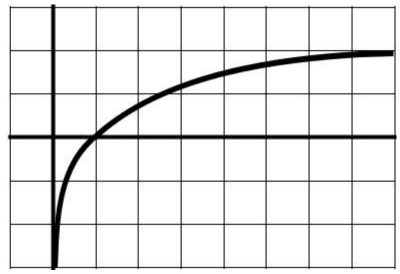
3. Finally, sketch a graph of $V\ \ \ t$. What does it mean when this graph crosses the x-axis?

GROUP WORK 3, SECTION 2.8
The Derivative Function

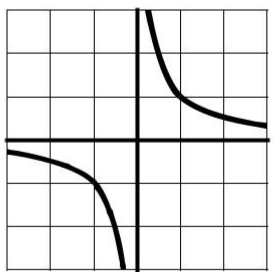
The graphs of several functions f are shown below. For each function, estimate the slope of the graph of f at various points. From your estimates, sketch graphs of f .



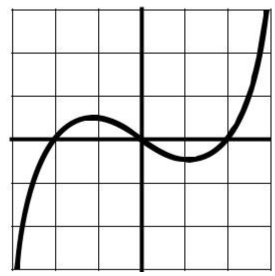
Graph 1



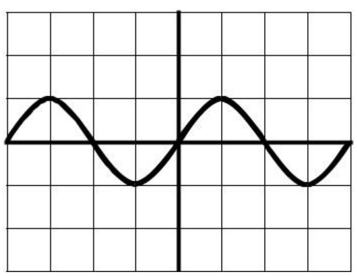
Graph 2



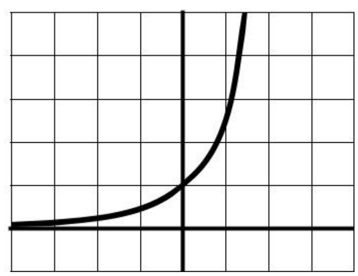
Graph 3



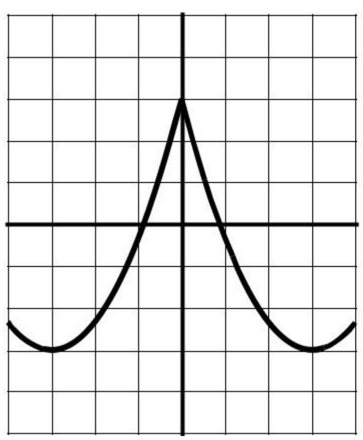
Graph 4



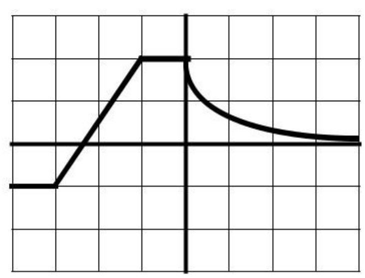
Graph 5



Graph 6

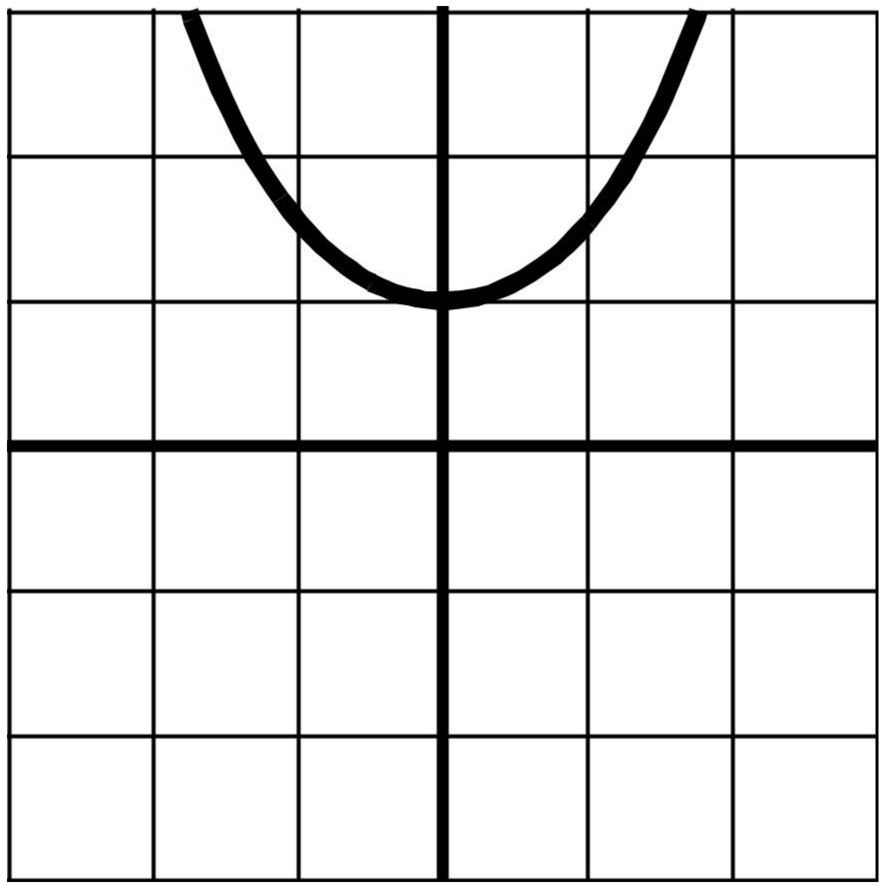


Graph 7

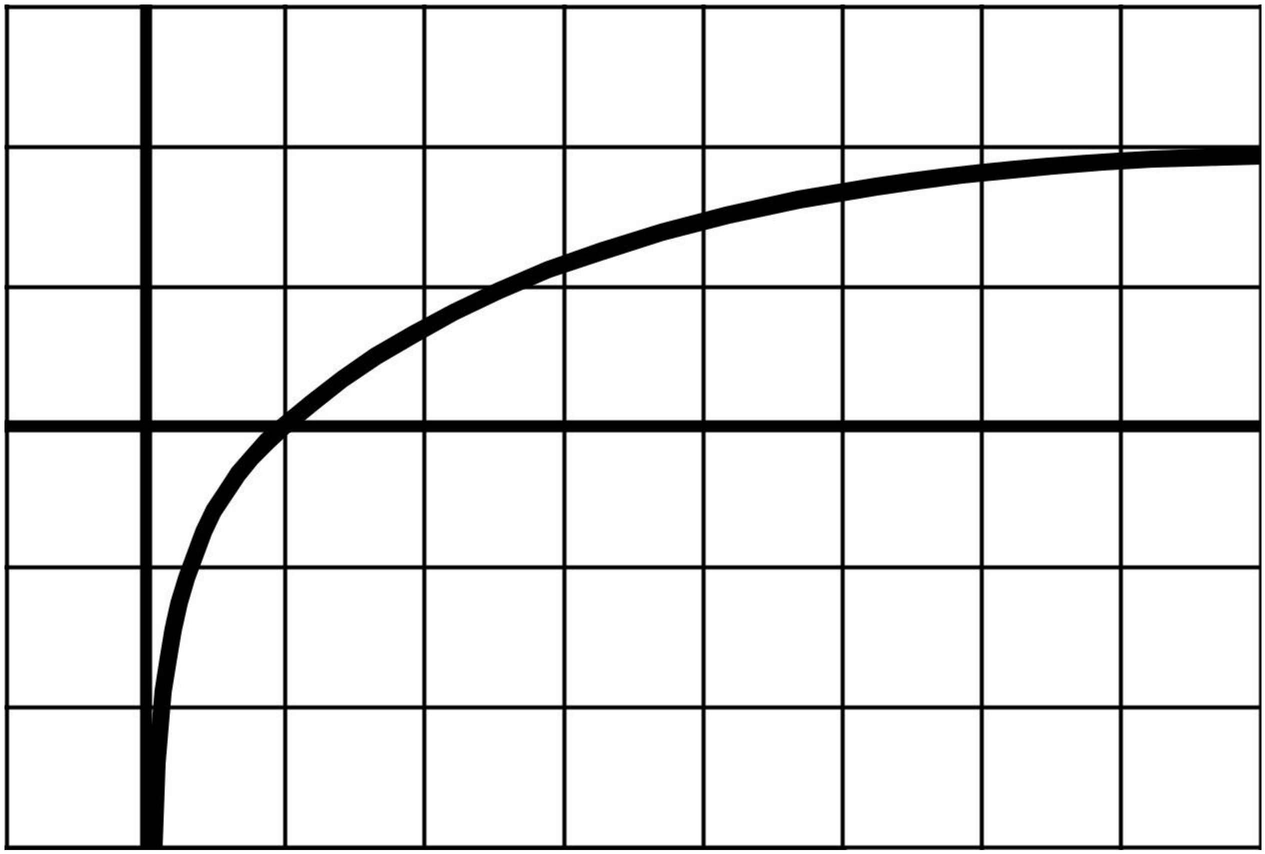


Graph 8

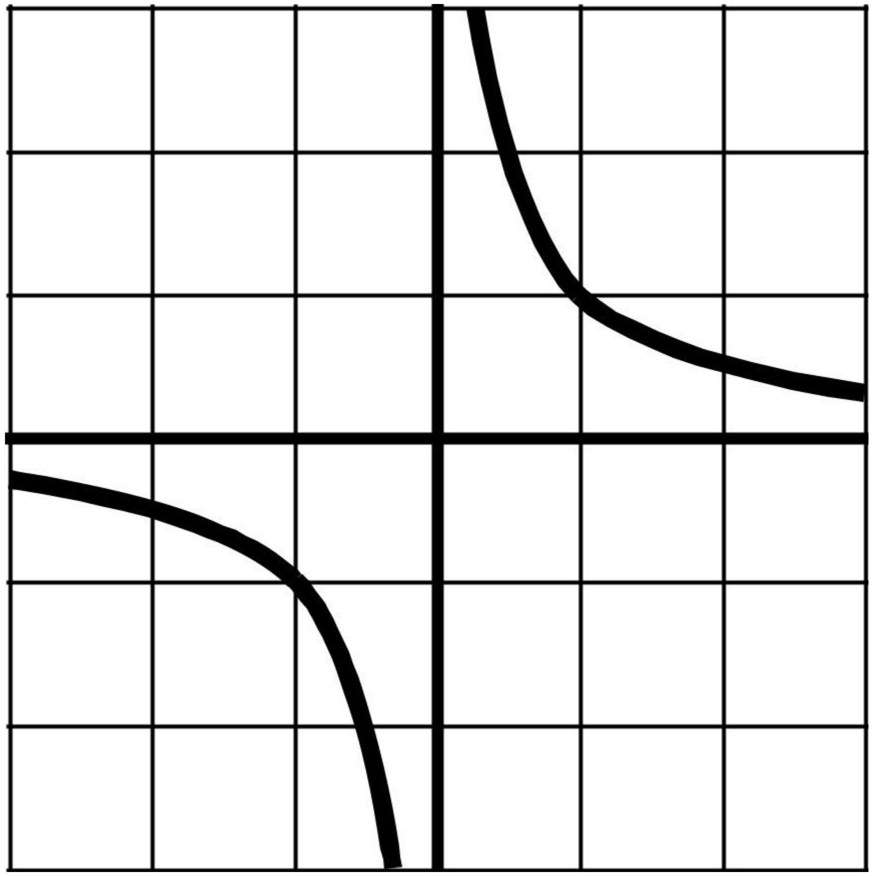
Graph 1



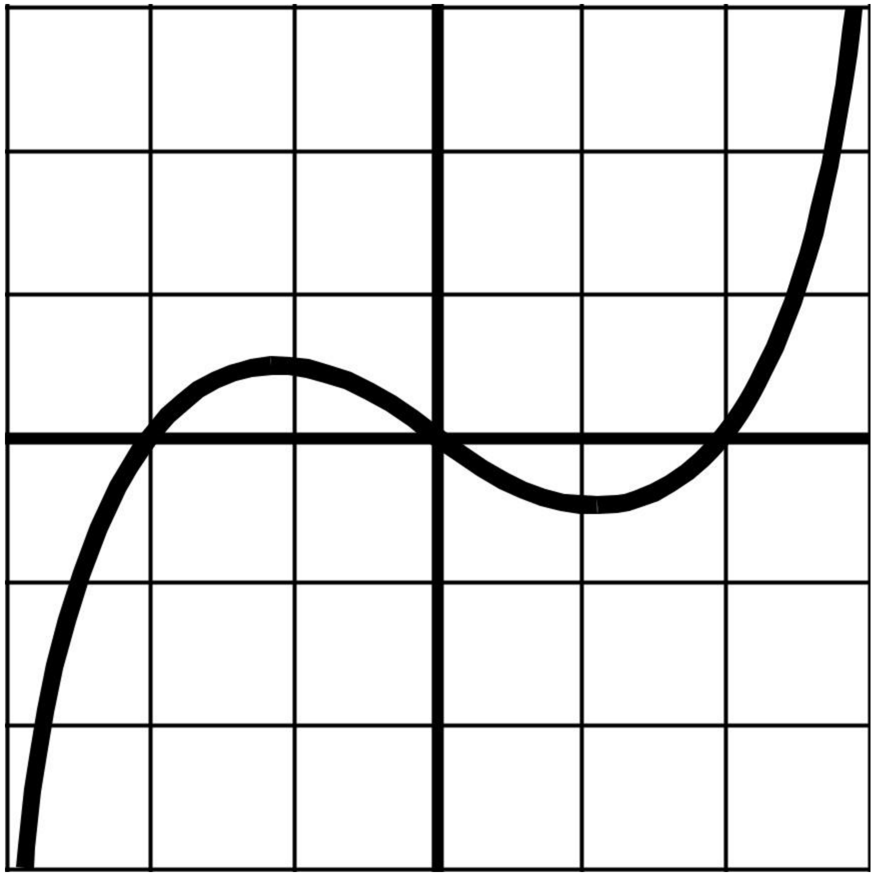
Graph 2



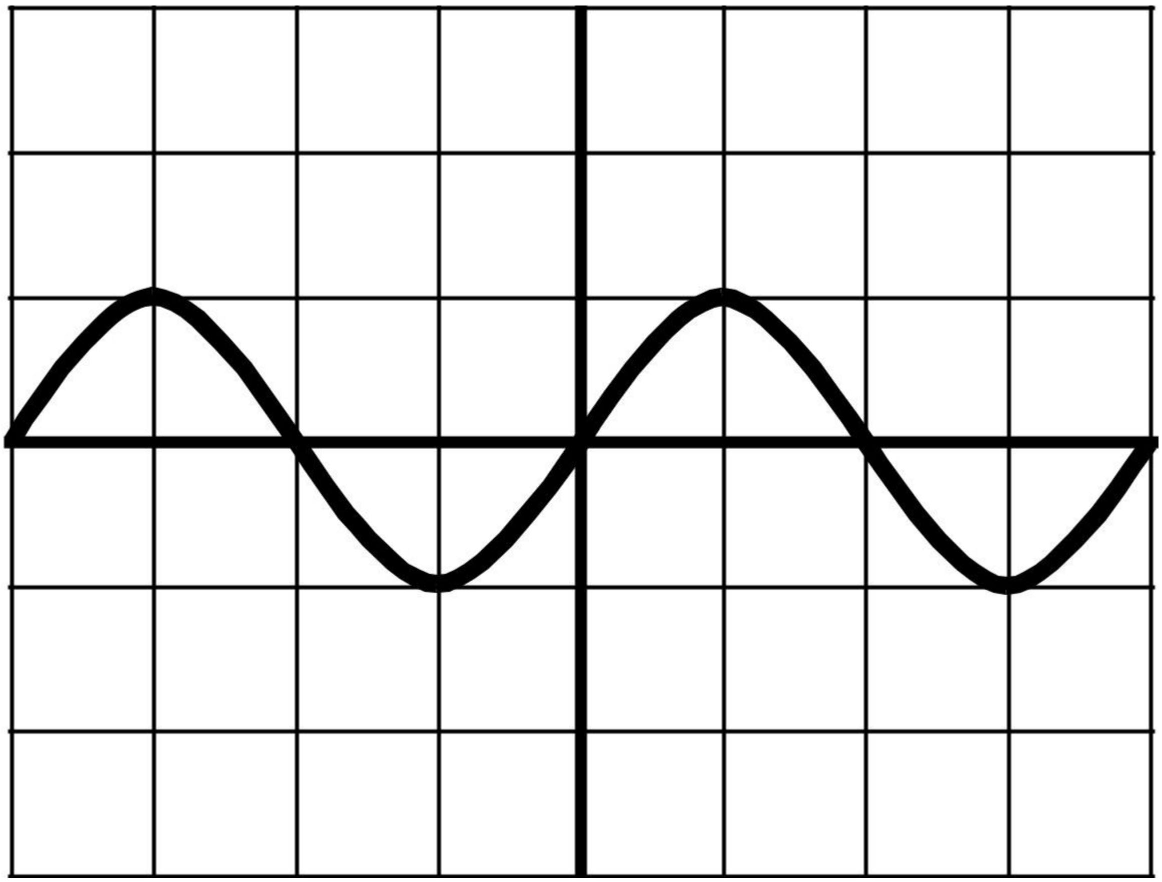
Graph 3



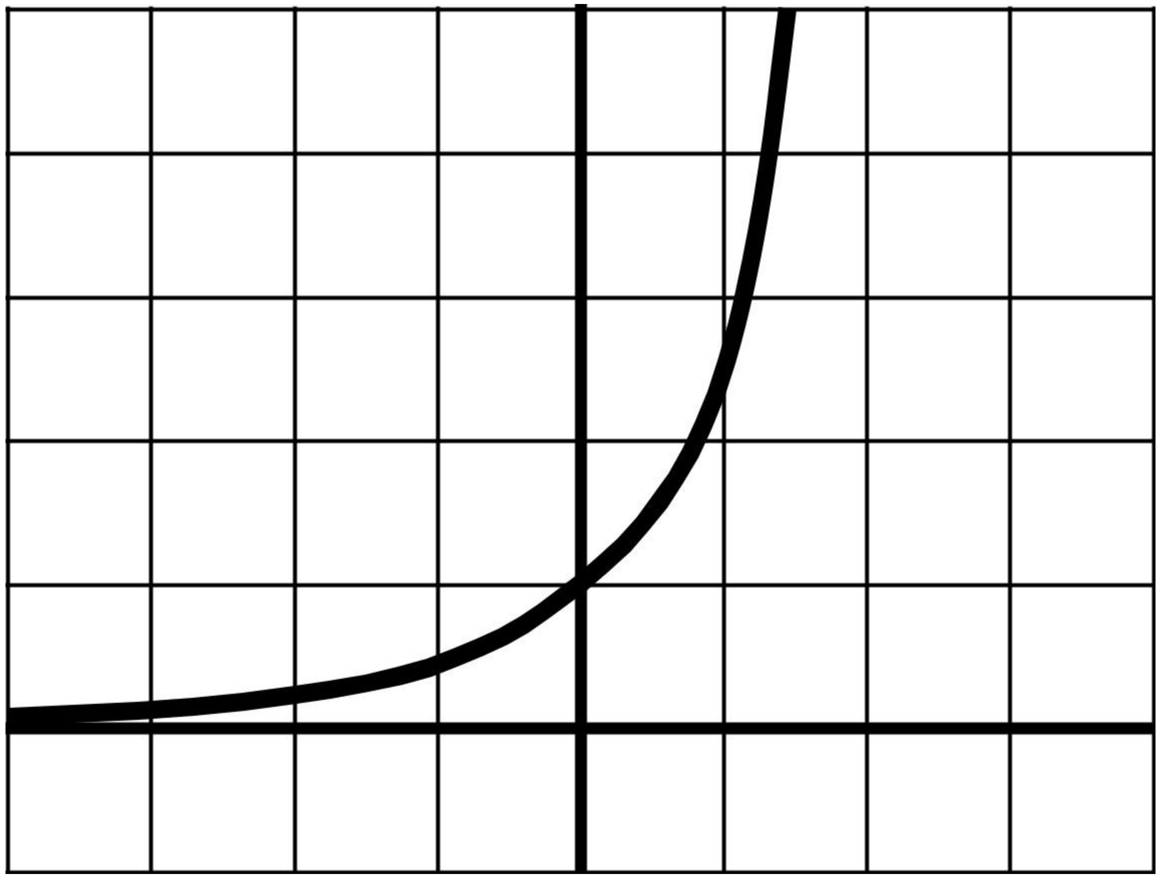
Graph 4



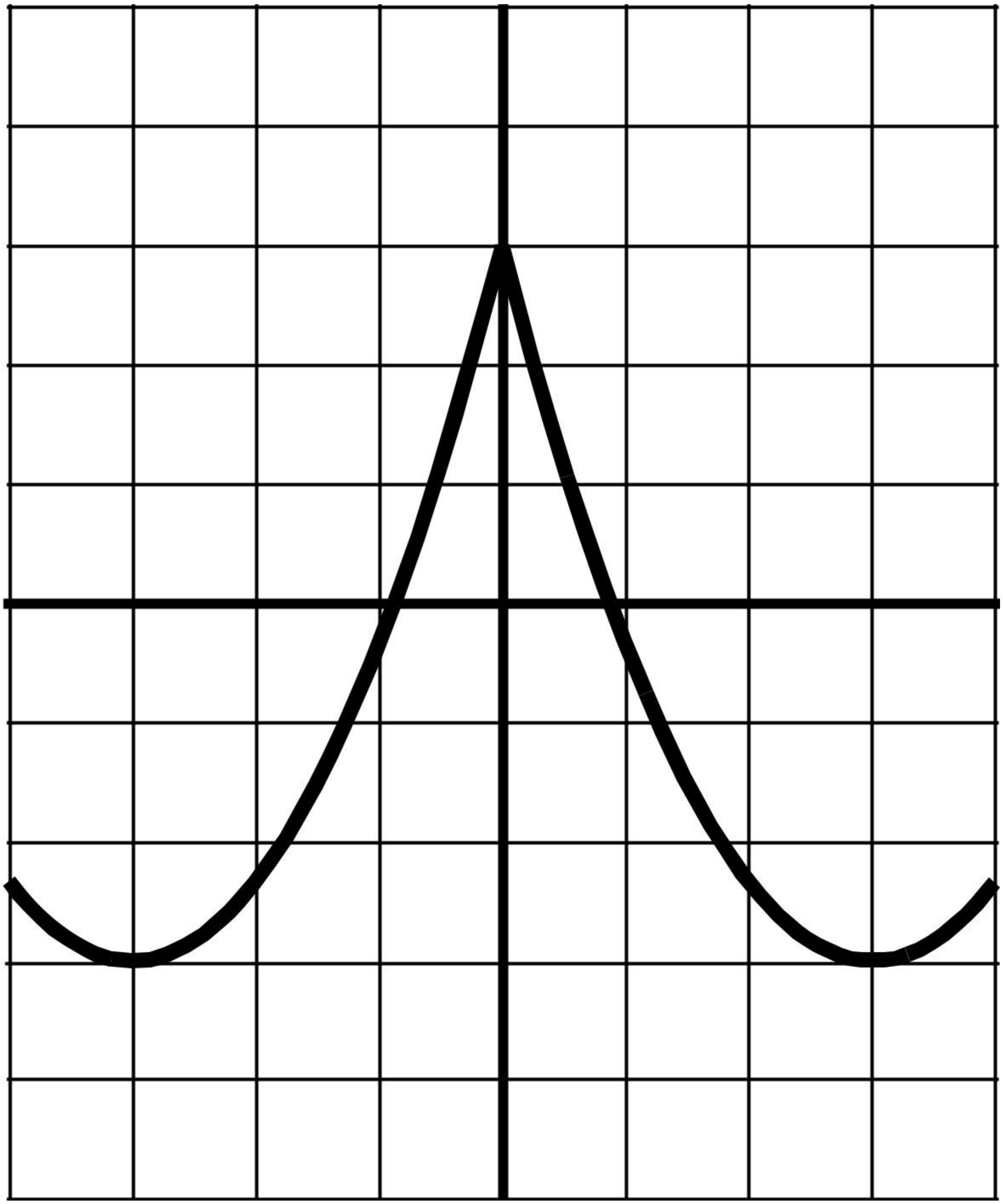
Graph 5



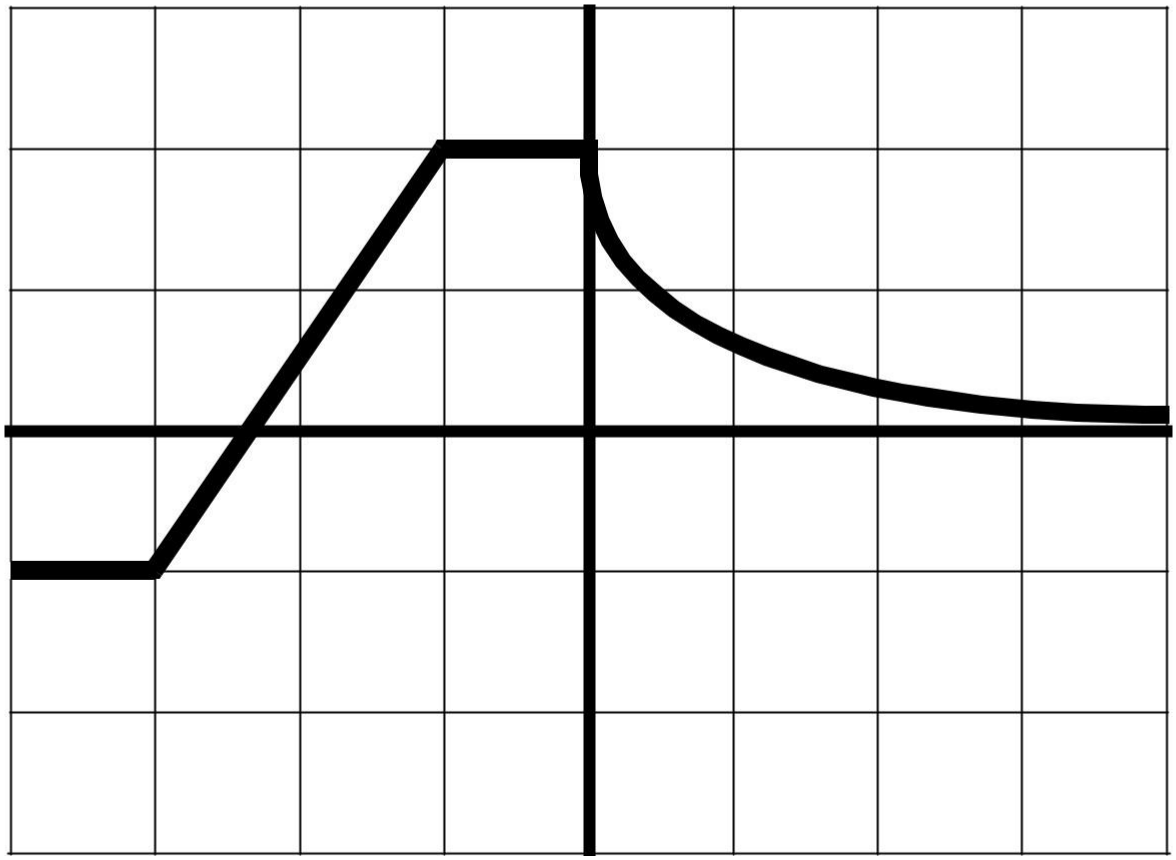
Graph 6



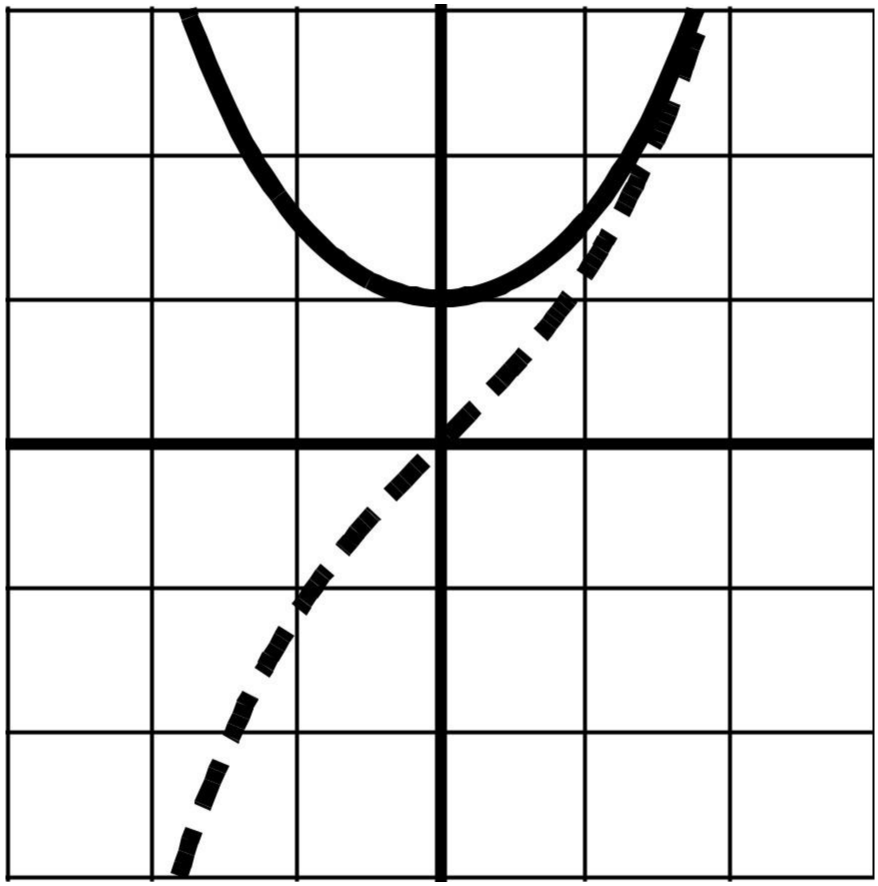
Graph 7



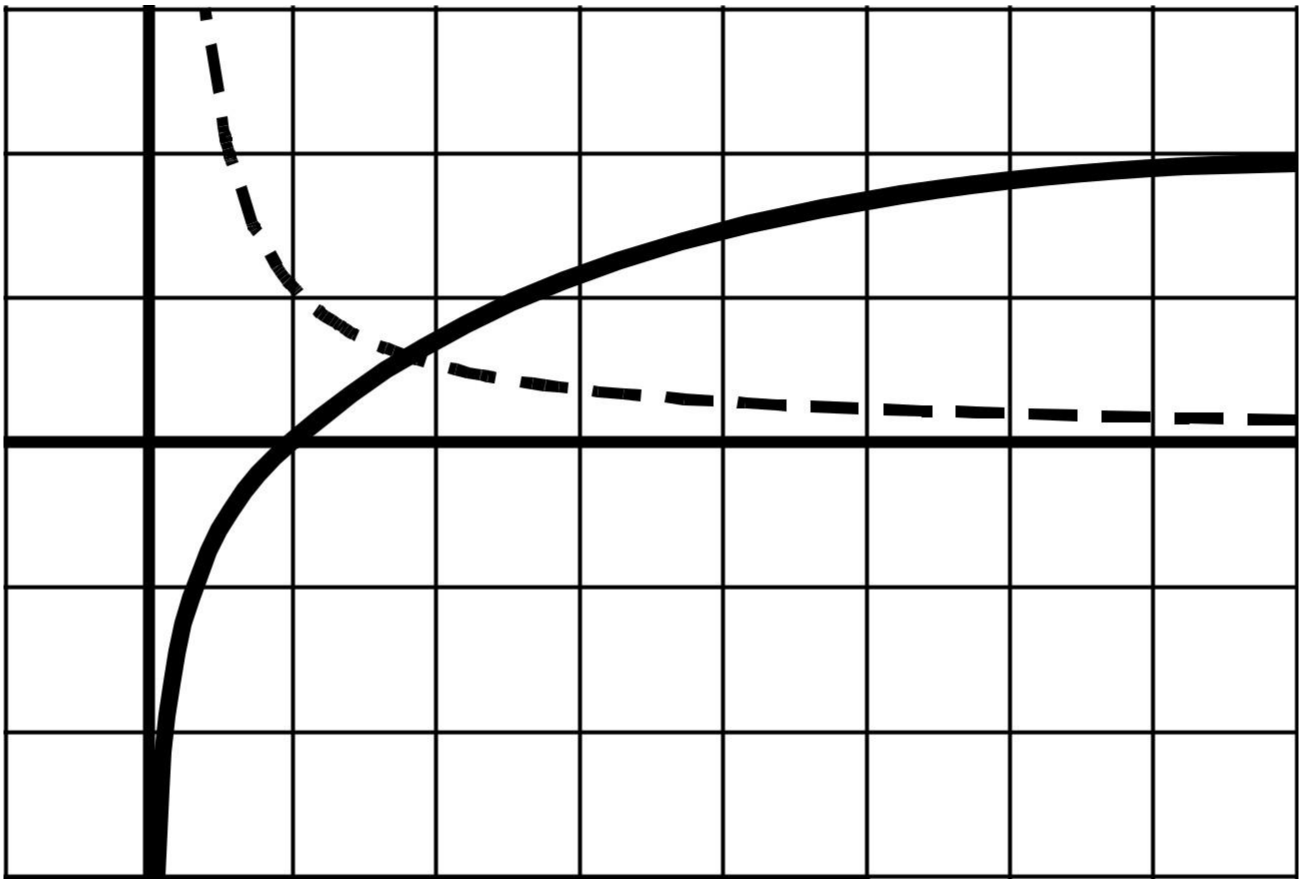
Graph 8



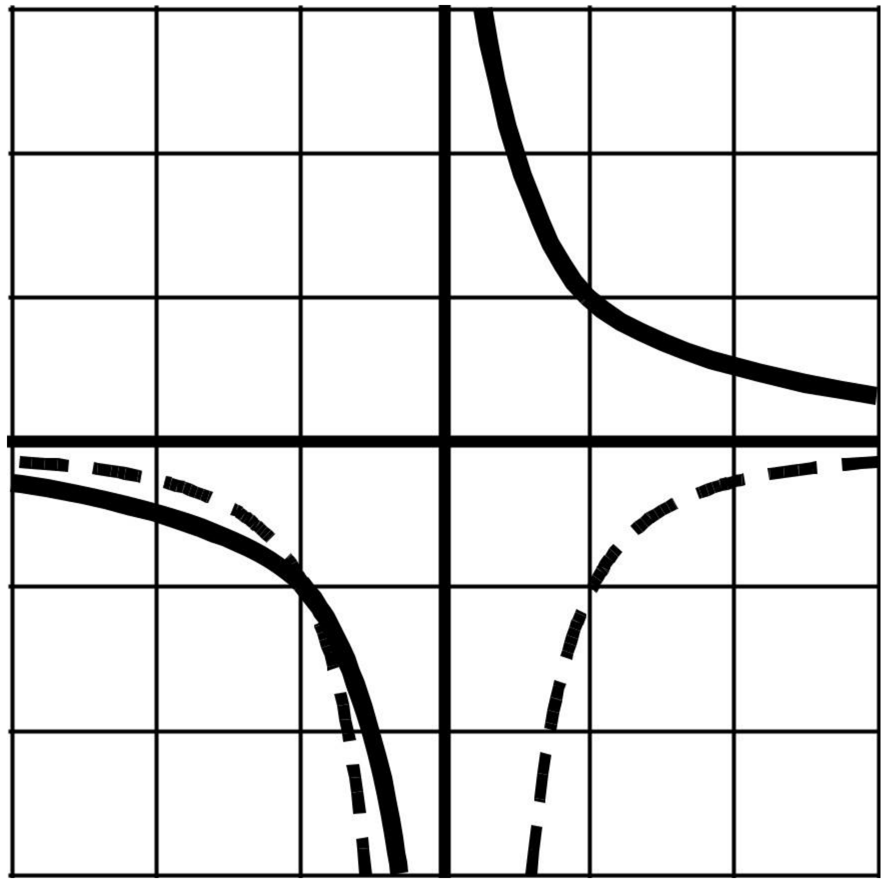
Answer 1



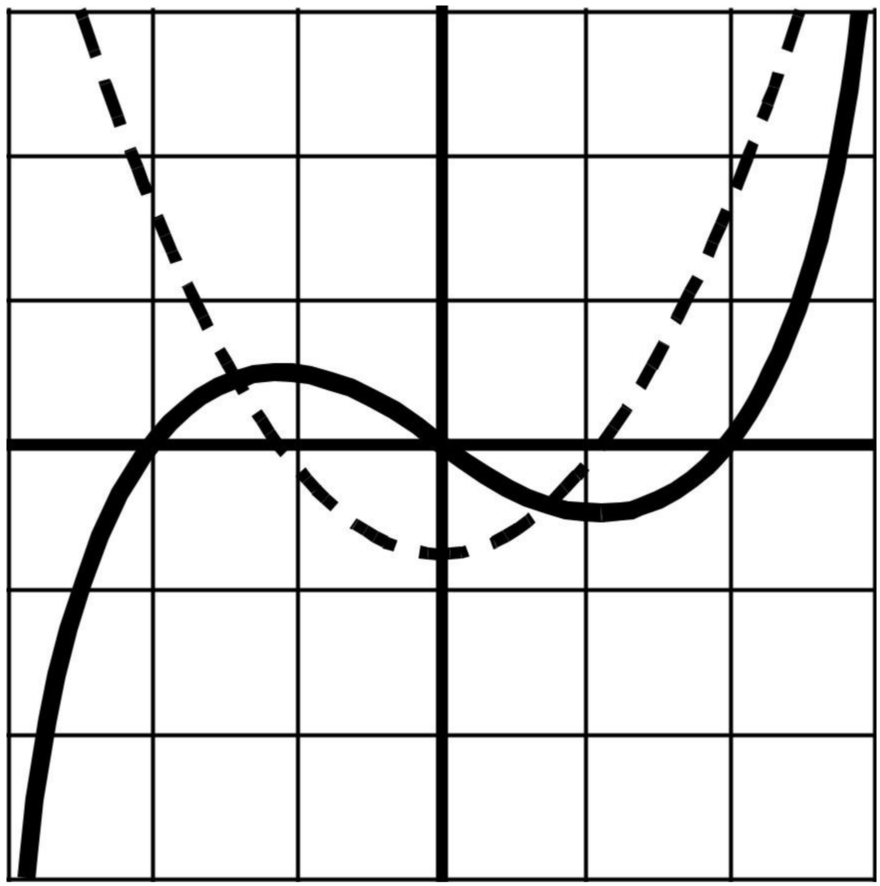
Answer 2



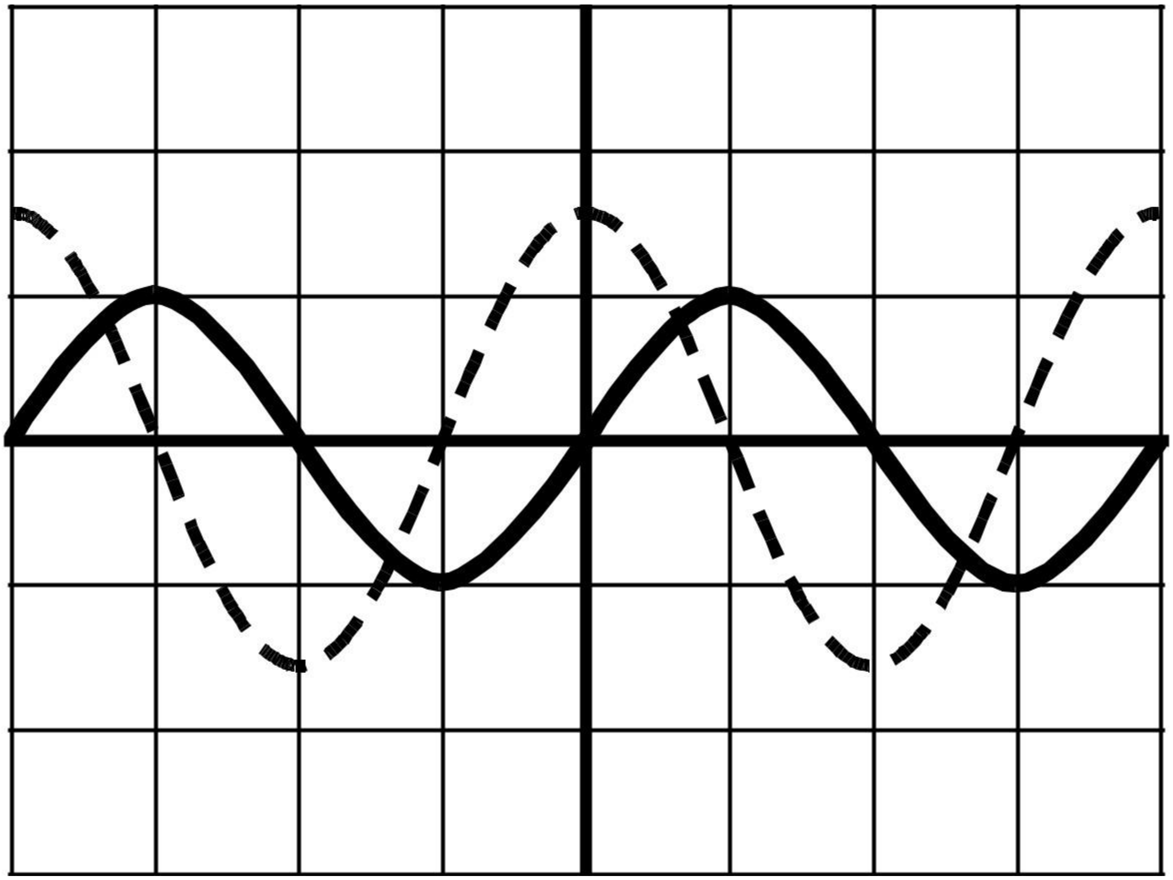
Answer 3



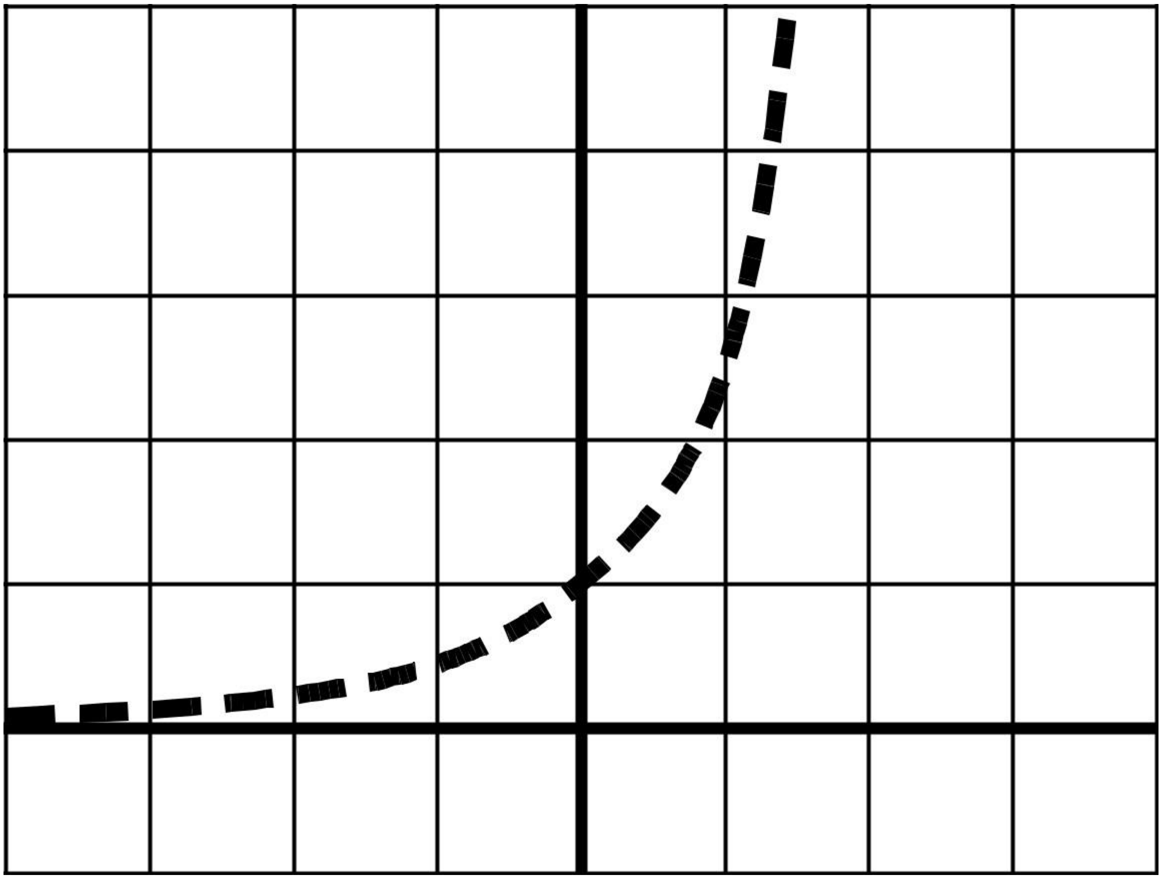
Answer 4



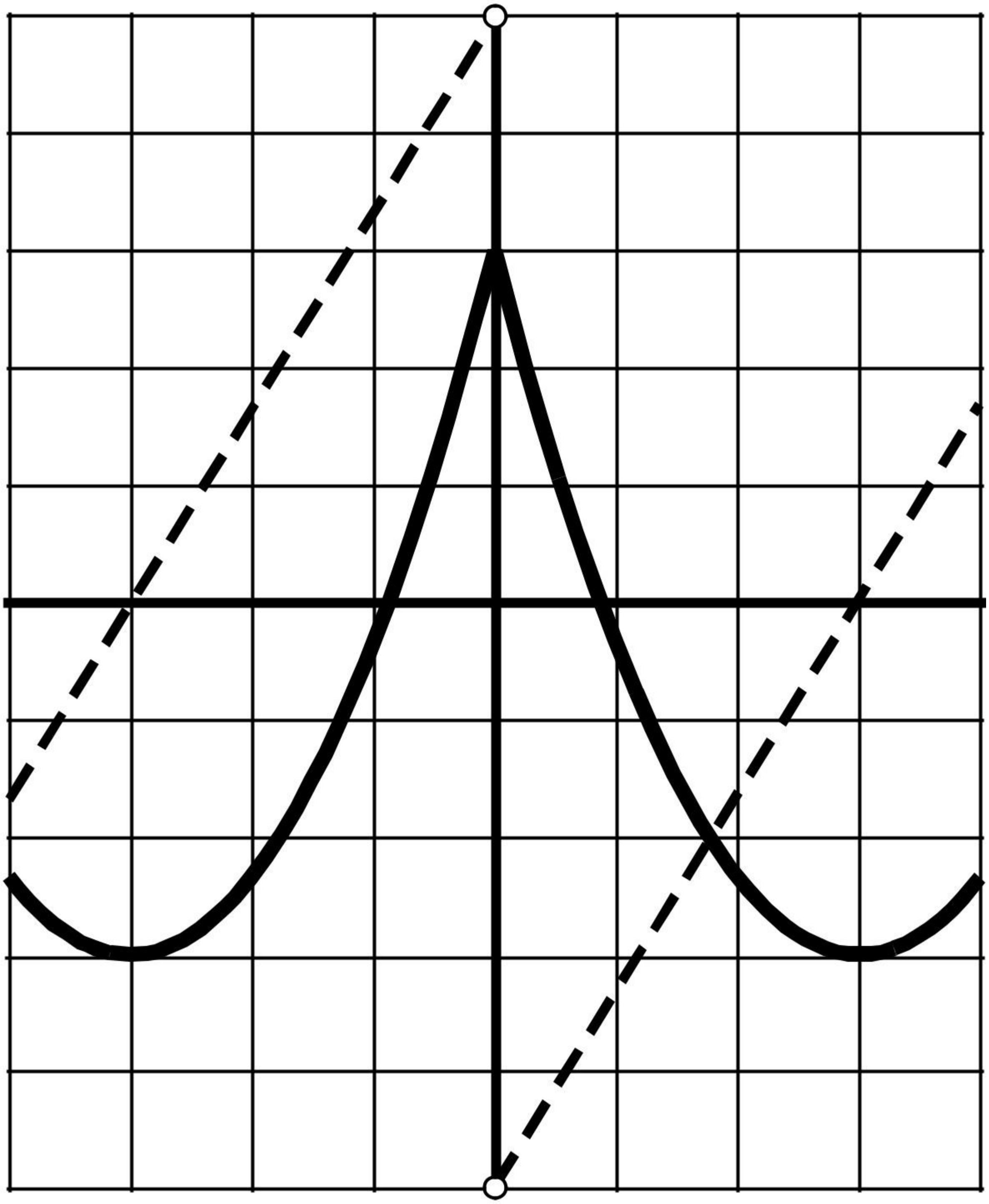
Answer 5



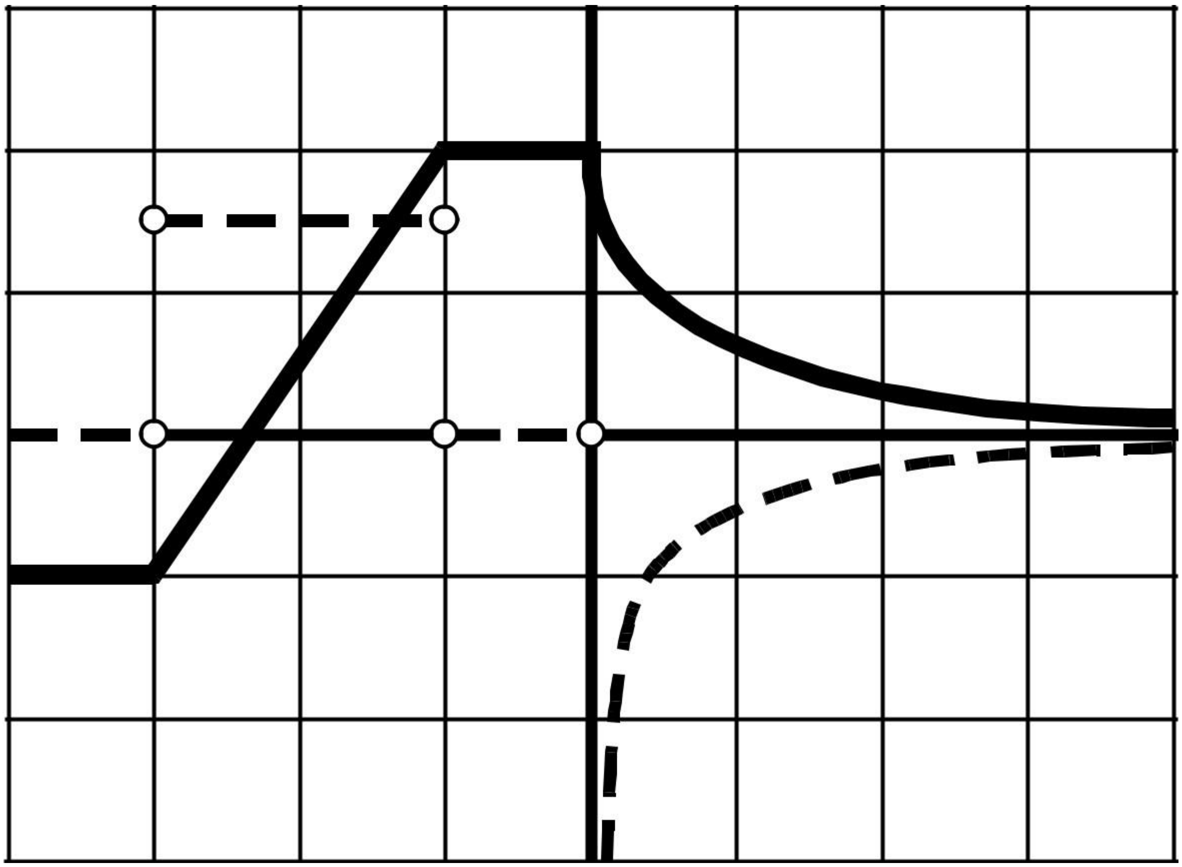
Answer 6



Answer 7



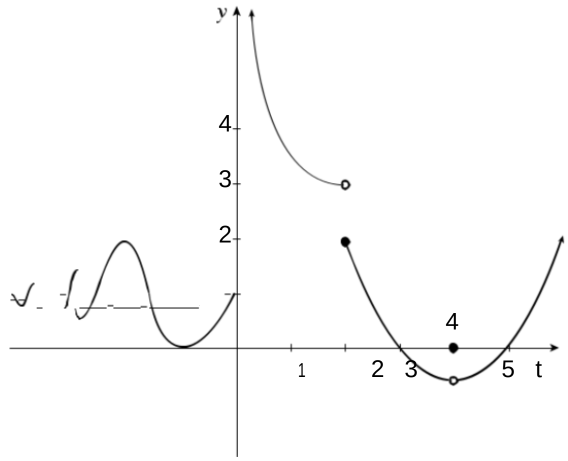
Answer 8



SAMPLE EXAM

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. Consider the following graph of f .



(a) What is $\lim_{t\ 0} f\ t$? $\lim_{t\ 0} f\ t$? $\lim_{t\ 2} f\ t$? $\lim_{t} f\ t$?

(b) For what values of x does $\lim_{t\ x} f\ t$ exist?

Does f have any vertical asymptotes? If so, where?

Does f have any horizontal asymptotes? If so, where?

For what values of x is f discontinuous?

Find values for a and b that will make f continuous everywhere, if

$f\ x$ $ax\ \ b$ if $2\ \ x\ \ 5$

$3x\ \ 1$ if $x\ \ 2$

Find the vertical and horizontal asymptotes for $f\ xa\ ^{1}\ \ x\ ^{1}\ ^{1}$, where a is a positive number.

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Consider the function $f\ x x2\ \ 3x\ \ 4$ •

What is the domain of f?

(b) Compute $\lim_{x\ \ 4} f\ \ x$, if this limit exists.

Is f continuous at x 4? Explain your answer by either proving that f is continuous at x 4 or telling how to modify f to make it continuous.

Let f be a continuous function such that f 1 1 and f 1 1. Classify the following statements as

(A) Always true
(B) Never true, or
(C) True in some cases, false in others.

Justify your answers.

$f\ 0\ \ \ 0$

For some x with $1\ \ x\ \ 1, f\ \ x\ \ \ 0$

For all x with $1 \quad x \quad 1 \quad 1 \quad f\ x \quad 1$

Given any y in [1 1], then $y \quad f\ x$ for some x in [1 1].

If $x1$ or $x \quad 1$, then $f\ x1$ or $f\ x \quad 1$.

$f\ x1$ for $x \quad 0$ and $f\ x \quad 1$ for $x \quad 0$.

Consider the function $f\ x \qquad \begin{matrix} x \quad 2 & \text{if } x \quad 1 \\ 2 & \\ 2x & \text{if } x \quad 1 \end{matrix}$

(a) Let $L \quad \lim_{x \quad 0} f \quad x$. Find L.

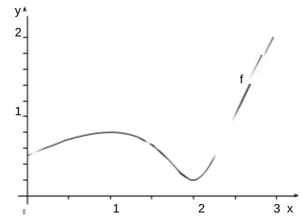
Find a number δ 0 so that if $0 \quad x \quad$ δ, then $f\ x \quad L \quad 0\ 01$.

Show that f does not have a limit at 1.

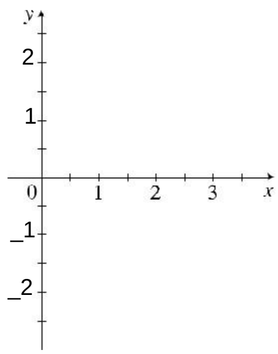
(d) Explain what would go wrong if you tried to show that $\lim_{x\ 1} f\ x1$ using the ε-δ definition.

HINT Try ε $\frac{1}{2}$.

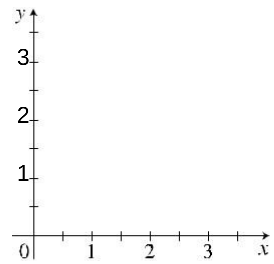
Let f be the function whose graph is given below.



Sketch a plausible graph of f .



Sketch a plausible graph of a function F such that F f and F 0 1.



Suppose that the line tangent to the graph of y f x at x 3 passes through the points 2 3 and 4 1 .

Find f 3 .

Find f 3 .

What is the equation of the line tangent to f at 3?

9. Give examples of functions $f\ x$ and $g\ x$ with $\lim_{x} f\ x \lim_{x} g\ x$ and

(a) $\lim_{x} \frac{f\ x}{g\ x}$.

(b) $\lim_{x} \frac{f\ x}{g\ x}$ 6

(c) $\lim_{x} \frac{f\ x}{g\ x}$. 0

(d) Is it possible to have $\lim_{x} \frac{f\ x}{g\ x}$ 1? Either give an example or explain why it is not possible.

Each of the following limits represent the derivative of a function f at some point a. State a formula for f and the value of the point a.

(a) $\lim_{h\ 0} \frac{3\ h^2\ 9}{h}$

(b) $\lim_{x\ 1} \frac{2^x\ 2}{x\ 1}$

(c) $\lim_{x\ 3} \frac{x\ 1^{3\,2}\ 8}{x\ 3}$

(d) $\lim_{h\ 0} \frac{\sin \pi\ 2\ \ h0}{h}$

11. Let

$$f\,x \quad \begin{cases} 3 \quad x & \text{if } x \quad 1 \\ & \text{if } 1 \quad x \quad 3 \\ 27\,x & \text{if } x \quad 3 \end{cases}$$

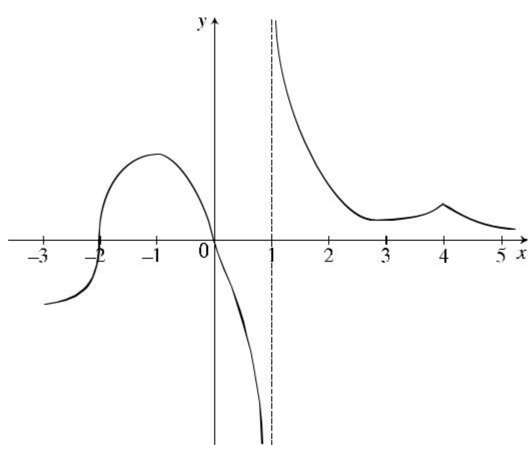
(a) Evaluate each limit, if it exists.

(i) $\lim_{x \; 1} f\,x$ (ii) $\lim_{x \; 1} f\,x$ (iii) $\lim_{x \; 1} f\,x$ (iv) $\lim_{x \; 3} f\,x$

(v) $\lim_{x \; 3} f\,x$ (vi) $\lim_{x \; 3} f\,x$ (vii) $\lim_{x \; 9} f\,x$ (viii) $\lim_{x 6} f\,x$

Where is f discontinuous?

The graph of $f\,x$ is given below. For which value(s) of x is $f\,x$ not differentiable? Justify your answer(s).



A bicycle starts from rest and its distance travelled is recorded in the following table at one-second intervals.

t (s)	0	1	2	3	4	5	6
d (ft)	0	10	24	42	63	84 5	107

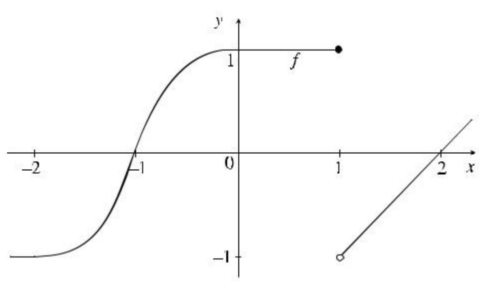
Estimate the speed after 2 seconds.

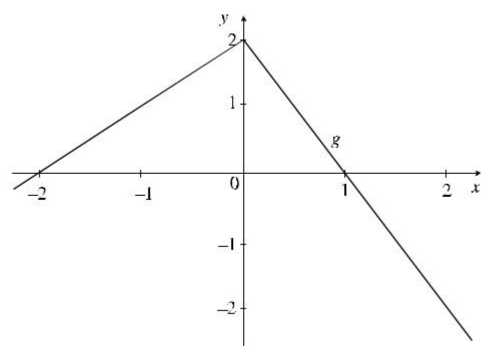
Estimate the speed after 5 seconds.

Estimate the speed after 6 seconds.

Can we determine if the cyclist's speed is constantly increasing? Explain.

Referring to the graphs given below, find each limit.





(a) $\lim_{x\ 0}$ $f\ x$ $g\ x$ (b) $\lim_{x\ 1}$ $g\ x$ $f\ x$ (c) $\lim_{x1}$ $\dfrac{g\ x}{f\ x}$ $g\ x$ (d) $\lim_{x\ 2}$ x $g\ x$

(e) $\lim_{x1}$ $g\ x$ $f\ x$ (f) $\lim_{x\ 1}$ x $f\ x$ $\lim_{1}$ $f\ x$ x

15. Draw a graph of $f\ x$ ln ln x

(a) Over the range [2 10].

(b) Over the range [2 100].

(c) What is $\lim_{x}$ ln ln x?

SAMPLE EXAM SOLUTIONS

1. (a) $\lim_{t 0} f\ t$, $\lim_{t\ 0} f\ t 1$, $\lim_{t\ 2} f\ t\ \ 3$, t $\lim f\ t 1$

(b) $t\ \ x\ f\ \ t$ exists for all x except $x\ \ 0$ and $x\ \ 2$.

There is a vertical asymptote at $x\ \ 0$.

There is a horizontal asymptote at $y\ \ 1$.

f is discontinuous at $x\ \ 0$, 2, and 4.

Solve 3 2 1 2a b and 5^2 5a b to get a 6, b5.

3. Taking $x$lim $f\ x$ gives a horizontal asymptote at ya Algebraic simplification gives a vertical asymptote at xa The function is undefined at x0, but there is no asymptote there because $x$lim0 $f\ x$ 0.

4. $f\ \ x\ \ x\ \ 4\ x \quad \frac{x\ \ 4}{1}$

(a) The domain is all values of x except $x\ \ 1$ and x4.

(b) Algebraic simplification gives a limit of $\frac{1}{5}$.

(c) f is not continuous at x4, for it is not defined there. It can be modified by defining $f\ \ 4$ to be $\frac{1}{5}$.

5. (a) C. True for $f\ \ xx$, untrue for $f\ \ xx^2\ \ x\ \ 1$

(b) A. True by the Intermediate Value Theorem

(c) C. True for $f\ \ xx$, untrue for $f\ \ xx^2\ \ x\ \ 1$

(d) A. True by the Intermediate Value Theorem

(e) C. True for $f\ \ xx$, untrue for $f\ \ xx^2\ \ x\ \ 1$

(f) B. $\lim_{x\ 0} f\ \ x$ does not exist, contradicting the continuity of f.

6. (a) $L\ \ 0$

(b) Let δ be any number greater than zero and less than $\frac{0\ 01}{2}$. δ 0 07 works, for example.
The left hand limit is 2, and the right hand limit is 1.

Choose ε $\frac{1}{2}$. We now need a δ such that $f\ x$ 1 $\frac{1}{2}$ for all x with x 1 δ But if x 1, as x approaches 1, $f\ x$ approaches 2, and $f\ x$ 1 approaches 1, which is greater than $\frac{1}{2}$.

(a) Answers will vary. Look for:

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eros at 1 and 2

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positive for x [0 1 and 2 3]

Ā Ä ÈÄ Ā↠Ā Ā
negative for x 1 2

Ā Ä ÈÄ Ā↛Ā Ā
flattens out for x 2 5

Answers will vary. Look for

$F\ 0 \quad 1$

F always increasing

(ii) F is never perfectly flat

(iv) F is closest to being flat at $x \quad 2$

(v) F is concave up for $x \quad 0\ 1$ and $x \quad 2\ 3$

(vi) F is concave down for $x \quad 1\ 2$

8. (a) $\frac{3}{2}\ \frac{1}{4} \quad \frac{2}{3}$

The equation of the tangent line is $y \quad 3\frac{2}{3}\ x \quad 2$, so $f\ 3 \quad \frac{2}{3}\ 3 \quad 2 \quad 3 \quad \frac{1}{3}$.

The equation of the tangent line is $y \quad 3\frac{2}{3}\ x \quad 2$.

Answers will vary; the following are samples only.

$f\ x x^2,\ g\ x x$

$f\ x 6x,\ g\ x x$

$f\ x x,\ g\ x x^2$

(d) This is not possible. For $\lim_x \frac{f\ x}{g\ x}$ 1, either f or g would have to be negative for large x This contradicts the assumption that $\lim_x f\ x \lim_x g\ x$.

10. (a) $f\ x x^2,\ a \quad 3$

(b) $f\ x 2^x,\ a \quad 1$

(c) $f\ x x \quad 1^{3\ 2},\ a \quad 3$

(d) $f\ x \sin \pi x,\ a \quad 2$

11. (a) (i) 2 (ii) 1 (iii) Does not exist (iv) 9 (v) 9 (vi) 9 (vii) 3 (viii) 3

f is discontinuous at $x \quad 1$.

f isn't differentiable at x 1, because it is not continuous there; at x 2 because it has a vertical tangent there; and at x 4, because it has a cusp there.

(a) Answers will vary. One good answer would be to compute the average speed between 1 and 2 (14 ft/s) and the average speed between 2 and 3 (18 ft/s) and average them to get 16 ft/s. This is also the answer obtained by computing the average speed between 1 and 3.

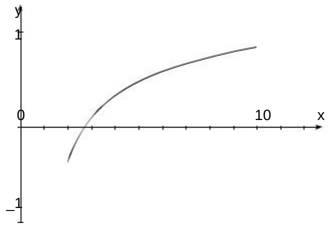
Answers will vary. Using reasoning similar to the previous part, we get an estimate of 22 ft/s, but it could be argued that a number closer to 22 5 would be more accurate.

Answers will vary. The average speed between $t \quad 5$ and $t \quad 6$ is 22 5 ft/s

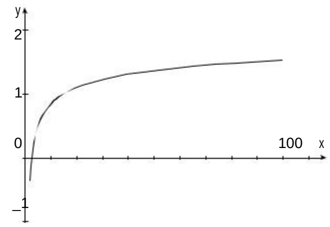
Since we are given information only about the cyclist's position at one-second intervals, we cannot determine if the speed is constantly increasing.

14. (a) $\frac{1}{2}$ (b) 0 (c) Does not exist (d) 4 (e) 1 (f) 2 (g) 0

15. (a)



(b)



(c) $\lim_{x} \ln \ln x$