1 ☐ FUNCTIONS AND LIMITS

1.1 Four Ways to Represent a Function

The functions ( ) = + $\sqrt{}$2 − and ( ) = + $\sqrt{}$2 − give exactly the same output values for every input value, so and are equal.

( ) = 2 − = (− 1) = for − 1 6= ,0so and [where ( ) =] are not equal because (1) is undefined and
−1−1
(1) = 1.

(a) The point (1 3) is on the graph of , so (1) = 3.

When = −1, is about −0 2, so (−1) ≈ −0 2.

( ) = 1 is equivalent to = 1 When = 1, we have = 0 and = 3.

A reasonable estimate for when = 0 is = −0 8.

The domain of consists of all -values on the graph of . For this function, the domain is −2 ≤ ≤ 4, or [−2 4]. The range of consists of all -values on the graph of . For this function, the range is −1 ≤ ≤ 3, or [−1 3].

(f) As increases from −2 to 1, increases from −1 to 3. Thus, is increasing on the interval [−2 1].

(a) The point (−4 −2) is on the graph of , so (−4) = −2. The point (3 4) is on the graph of , so (3) = 4.

We are looking for the values of for which the -values are equal. The -values for and are equal at the points (−2 1) and (2 2), so the desired values of are −2 and 2.

( ) = −1 is equivalent to = −1. When = −1, we have = −3 and = 4.

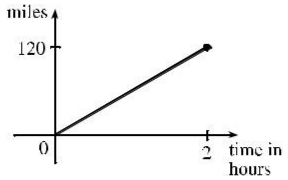
As increases from 0 to 4, decreases from 3 to −1. Thus, is decreasing on the interval [0 4].

The domain of consists of all -values on the graph of . For this function, the domain is −4 ≤ ≤ 4, or [−4 4]. The range of consists of all -values on the graph of . For this function, the range is −2 ≤ ≤ 3, or [−2 3].

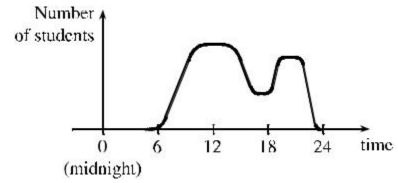
(f) The domain of is [−4 3] and the range is [0 5 4].

From Figure 1 in the text, the lowest point occurs at about $(\ ) = (12\ -85)$. The highest point occurs at about $(17\ 115)$. Thus, the range of the vertical ground acceleration is $-85 \leq \leq 115$. Written in interval notation, we get $[-85\ 115]$.

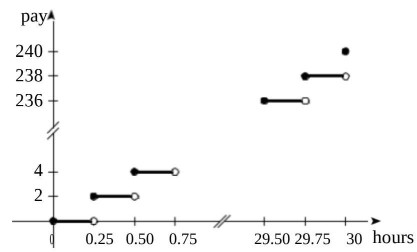
6. *Example 1:* A car is driven at 60 mi h for 2 hours. The distance traveled by the car is a function of the time . The domain of the function is { | 0 ≤ ≤ 2}, where is measured in hours. The range of the function is { | 0 ≤ ≤ 120}, where is measured in miles.



Example 2: At a certain university, the number of students on campus at any time on a particular day is a function of the time after midnight. The domain of the function is { | 0 ≤ ≤ 24}, where is measured in hours. The range of the function is { | 0 ≤ ≤ }, where is an integer and is the largest number of students on campus at once.



Example 3: A certain employee is paid $8 00 per hour and works a maximum of 30 hours per week. The number of hours worked is rounded down to the nearest quarter of an hour. This employee's gross weekly pay is a function of the number of hours worked . The domain of the function is [0 30] and the range of the function is {0 2 00 4 00 238 00 240 00}.



No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.

Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is [−2 2] and the range is [−1 2].

Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is [−3 2] and the range is [−3 −2) ∪ [−1 3].

No, the curve is not the graph of a function since for = 0, ±1, and ±2, there are infinitely many points on the curve.

(a) When = 1950, ≈ 13 8°C, so the global average temperature in 1950 was about 13 8°C.

When = 14 2°C, ≈ 1990.

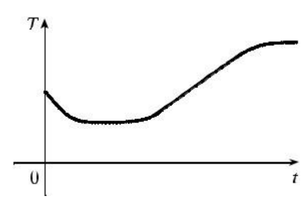
The global average temperature was smallest in 1910 (the year corresponding to the lowest point on the graph) and largest in 2005 (the year corresponding to the highest point on the graph).

When = 1910, ≈ 13 5°C, and when = 2005, ≈ 14 5°C. Thus, the range of is about [13 5, 14 5].

(a) The ring width varies from near 0 mm to about 1 6 mm, so the range of the ring width function is approximately [0 1 6].

According to the graph, the earth gradually cooled from 1550 to 1700, warmed into the late 1700s, cooled again into the late 1800s, and has been steadily warming since then. In the mid-19th century, there was variation that could have been associated with volcanic eruptions.

13. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.

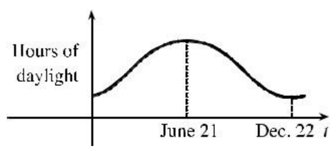


Runner A won the race, reaching the finish line at 100 meters in about 15 seconds, followed by runner B with a time of about 19 seconds, and then by runner C who finished in around 23 seconds. B initially led the race, followed by C, and then A. C then passed B to lead for a while. Then A passed first B, and then passed C to take the lead and finish first. Finally, B passed C to finish in second place. All three runners completed the race.

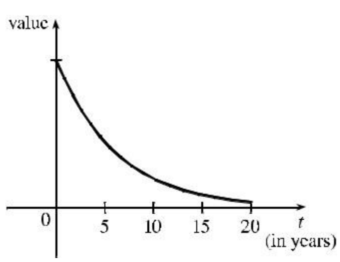
(a) The power consumption at 6 AM is 500 MW which is obtained by reading the value of power when = 6 from the graph. At 6 PM we read the value of when = 18 obtaining approximately 730 MW

The minimum power consumption is determined by finding the time for the lowest point on the graph, = 4 or 4 AM. The maximum power consumption corresponds to the highest point on the graph, which occurs just before = 12 or right before noon. These times are reasonable, considering the power consumption schedules of most individuals and businesses.

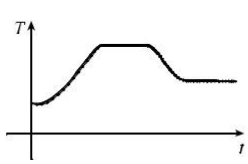
16. The summer solstice (the longest day of the year) is around June 21, and the winter solstice (the shortest day) is around December 22. (Exchange the dates for the southern hemisphere.)



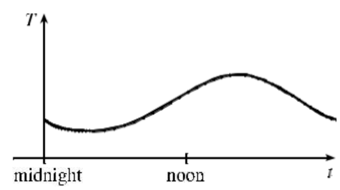
The value of the car decreases fairly rapidly initially, then somewhat less rapidly.



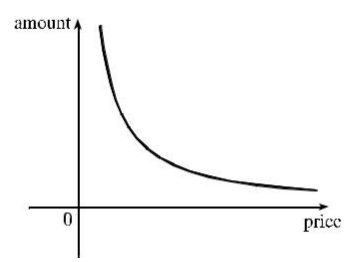
The temperature of the pie would increase rapidly, level off to oven temperature, decrease rapidly, and then level off to room temperature.



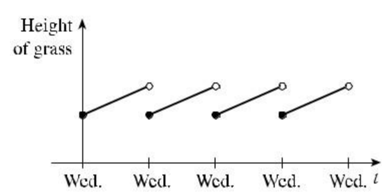
17. Of course, this graph depends strongly on the geographical location!



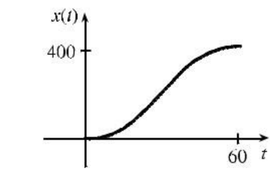
As the price increases, the amount sold decreases.



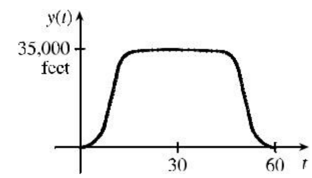
21.



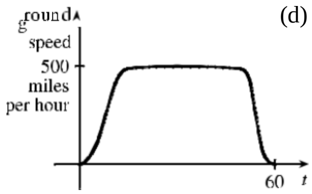
22. (a)



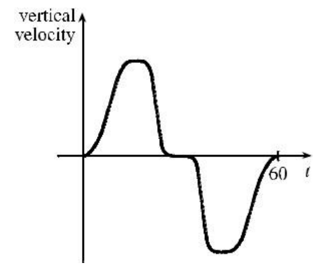
(b)



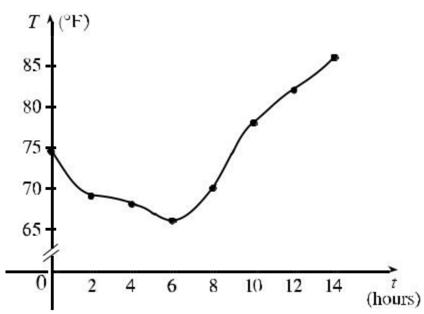
(c)



(d)

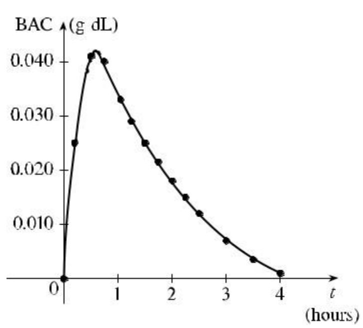


23. (a)



(b) 9:00 AM corresponds to $=9$. When $=9$, the temperature is about 74°F.

24. (a)



(b) The blood alcohol concentration rises rapidly, then slowly decreases to near zero. Note that the BAC in this exercise is measured in g dL, not percent.

$(\ )=3\ ^2-\ +2$ $(2)=3(2)^2-2+2=12-2+2=12$

$(-2)=3(-2)^2-(-2)+2=12+2+2=16$ $(\ )=3\ ^2-\ +2$

$(-\ )=3(-\ )^2-(-\ )+2=3\ ^2+\ +2$

$(\ +1)=3(\ +1)^2-(\ +1)+2=3(\ ^2+2\ +1)-\ -1+2=3\ ^2+6\ +3-\ +1=3\ ^2+5\ +4$ $2\ (\ )=2\cdot\ (\ )=2(3\ ^2-\ +2)=6\ ^2-2\ +4$

$(2\ )=3(2\ )^2-(2\ )+2=3(4\ ^2)-2\ +2=12\ ^2-2\ +2\ (\ ^2)=3(\ ^2)^2-($
$^2)+2=3(\ ^4)-\ ^2+2=3\ ^4-\ ^2+2$

$[()]^2=3_2-\quad +2^2=3_2-\quad +2\quad 3_2-\quad +2$

$9\ ^4-3\ ^3+6\ ^2-3\ ^3+\ ^2-2\ +6\ ^2-2\ +4=9\ ^4-6\ ^3+13\ ^2-4\ +4\ (\ +\ )=3(\ +\ )^2-(\ +\ )+2=3(\ ^2+2\ +\ ^2)-\ -$

$+2=3\ ^2+6\ +3\ ^2-\ -\ +2$

A spherical balloon with radius $+ 1$ has volume $(+ 1) = \frac{4}{3}\ (+ 1)^3 = \frac{4}{3}\ (\ ^3 + 3\ ^2 + 3 + 1)$. We wish to find the amount of

air needed to inflate the balloon from a radius of to $+ 1$. Hence, we need to find the difference

$(\ +1)-(\ )=\frac{4}{3}\ (\ ^3+3\ ^2+3\ +1)-\frac{4}{3}\ \ ^3=\frac{4}{3}\ (3\ ^2+3\ +1).$

$(\ ) = 4 + 3 - \ ^2$, so $(3 + \ ) = 4 + 3(3 + \ ) - (3 + \ )^2 = 4 + 9 + 3 - (9 + 6 + \ ^2) = 4 - 3 - \ ^2$, and $\underline{(3 + \ ) - (3)} = \underline{(4 - 3 - \ ^2) - 4} = \underline{(-3 - \ )} = -3 - \ .$

$(\ ) = \ ^3$, so $(\ +\ ) = (\ +\ )^3 = \ ^3 + 3\ ^2\ + 3\ \ ^2 + \ ^3$,

and $\underline{(+)-()} = \underline{(^3+3^2+3\ ^2+\ ^3)-\ ^3} = \underline{(3\ ^2+3\ \ +\ ^2)} = 3\ ^2+3\ \ +\ ^2.$

29. $\underline{()-()} = \frac{\frac{1}{}-\frac{1}{}}{} = \frac{\frac{-}{}}{} = \frac{-}{(\ \ )} = \frac{-1(-)}{(\ \ )} = \frac{1}{-}$

30. $\frac{(\ )-(1)}{-1} = \frac{\frac{+3}{+1}-2}{-1} = \frac{\frac{+3-2(\ +1)}{+1}}{-1} = \frac{+3-2\ -2}{(\ +1)(\ -1)}$

$= \frac{-\ +1}{(\ +1)(\ -1)} = \frac{-(\ -1)}{(\ +1)(\ -1)} = -\frac{1}{+1}$

31. $(\ ) = (\ + 4)\ (\ ^2 - 9)$ is defined for all except when $0 = \ ^2 - 9 \Leftrightarrow 0 = (\ + 3)(\ - 3) \Leftrightarrow \ = -3$ or 3, so the domain is $\{\ \in \mathbb{R}\mid\ 6=-3\ 3\} = (-\infty\ -3) \cup (-3\ 3) \cup (3\ \infty)$.

32. $(\ ) = (2\ ^3 - 5)\ (\ ^2 + \ - 6)$ is defined for all except when $0 = \ ^2 + \ - 6 \Leftrightarrow 0=(\ +3)(\ -2) \Leftrightarrow$ $= -3$ or 2, so the domain is $\{\ \in \mathbb{R}\mid\ 6=-3\ 2\} = (-\infty\ -3) \cup (-3\ 2) \cup (2\ \infty)$.

33. $(\ ) = \sqrt[3]{2\ -1}$ is defined for all real numbers. In fact $\sqrt[3]{(\ )}$, where $(\ )$ is a polynomial, is defined for all real numbers.

Thus, the domain is $\mathbb{R}$ or $(-\infty\ \infty)$.

34. $()=\sqrt{3-}\ -\sqrt{2+}$ is defined when $3 - \ \geq 0 \Leftrightarrow\ \leq 3$ and $2 +\ \geq 0 \Leftrightarrow\ \geq -2$. Thus, the domain is

$-2 \leq\ \leq 3$, or $[-2\ 3]$.

35. $(\ )=1\ \sqrt[4]{\ ^2-5}$ is defined when $\ ^2 - 5\ 0 \Leftrightarrow (\ - 5)\ \ 0$. Note that $\ ^2 - 5\ 6= 0$ since that would result in

division by zero. The expression $(\ - 5)$ is positive if 0 or 5. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty\ 0) \cup (5\ \infty)$.

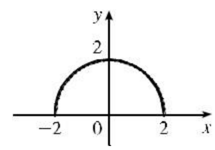
36. ( ) = $\dfrac{+1}{1+\frac{1}{+1}}$ is defined when + 1 6= 0[6=−1] and 1 + $\dfrac{1}{+1}$ 6= .0Since 1 + $\dfrac{1}{+1}$ = 0 ⇔

$\dfrac{1}{+1}$ = −1 ⇔ 1 = − −1 ⇔ = −2, the domain is { | 6=−2, 6=−1} = (−∞ −2) ∪(−2 −1) ∪(−1 ∞).

37. ( ) = √ 2 − √ is defined when ≥ 0 and 2 − √ ≥ 0. Since 2 − √ ≥ 0 ⇔ 2 ≥ √ ⇔ √ ≤ 2 ⇔

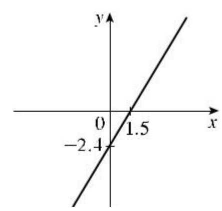
≤ ≤ 4, the domain is [0 4].

38. ( ) = √4 − 2. Now = √4 − 2 ⇒ 2 = 4 − 2 ⇔ 2 + 2 = 4, so



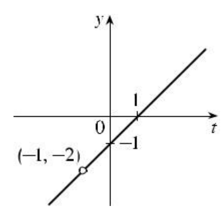
the graph is the top half of a circle of radius 2 with center at the origin. The domain is | 4 − 2 ≥ 0 = | 4 ≥ 2 = { | 2 ≥ | |} = [−2 2]. From the graph, the range is 0 ≤ ≤ 2, or [0 2].

39. The domain of ( ) = 1 6 − 2 4 is the set of all real numbers, denoted by R or



(−∞ ∞). The graph of *f* is a line with slope 1 6 and *y*-intercept −2 4.

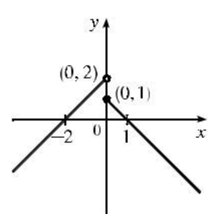
40. Note that ( ) = $\dfrac{2-1}{+1}$ = $\dfrac{(+1)(-1)}{+1}$ = − 1 for + 1 6 = ,0i.e., 6 − = 1.



The domain of is the set of all real numbers except −1. In interval notation, we have (−∞ −1) ∪ (−1 ∞). The graph of is a line with slope 1, -intercept −1, and a hole at = −1.

41. ( )= 1 − if0
+ 2 if0 ≥

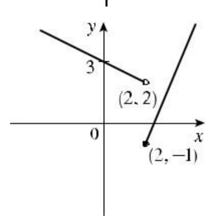
(−3) = −3 + 2 = −1, (0) = 1 − 0 = 1, and (2) = 1 − 2 = −1.



42. ( )= 2 − 2 5 if2

3 − $\frac{1}{2}$ if2 ≥

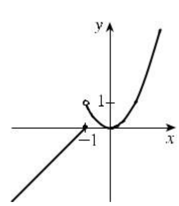
(−3) = 3 − $\frac{1}{2}$(−3) = $\frac{9}{2}$, (0) = 3 − $\frac{1}{2}$(0) = 3,



+ 1 if ≤ −1

43. ()=2 if−1

(−3) = −3 + 1 = −2, (0)= 0^2 = 0, and (2) = 2^2 = 4.

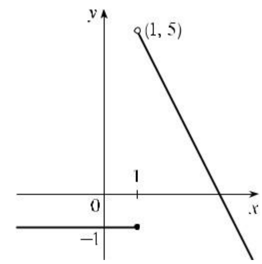


44. ( )= 7 2 if1

−1 if ≤ 1

−

(−3) = −1, (0) = −1, and (2) = 7 − 2(2) = 3.



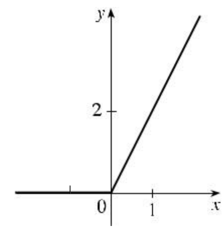
45. | |= − if0

if ≥ 0

so | | 0 if0

()=+ = 2 if ≥ 0

Graph the line = 2 for ≥ 0 and graph = 0 (the -axis) for 0

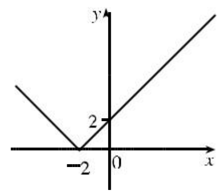


46. ( )=| +2|=−(+2) if + 2 0

+ 2 if + 2 ≥ 0

= − − 2 if −2

+ 2 if ≥ −2



47. ()=|1−3|= (1 3) if 1 3 0

1 − 3 if 1 − 3 ≥ 0

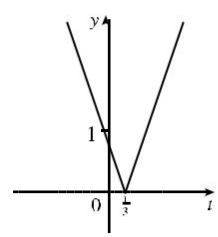
− − −

1

1 − 3 if ≤ 3

−

3

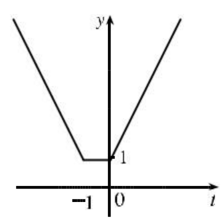


if ≥ 0

48. ||= − if0 and

+ 1 if + 1 ≥ 0 + 1 if ≥ −1

| +1|= −(+ 1) if + 1 0 = − − 1 if −1

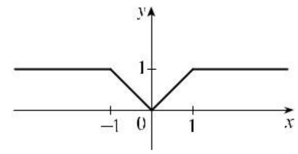


so ()= + + 1 = +(+1) if ≥ 1 0 2 + 1 if ≥ 1 0

+(+1) if ≤ 0 = 1 if ≤ 0

− + (− − 1) if −1 −2 − 1 if −1

49. To graph ( ) = 1| | if ≤ 1, if | | 1, graph = | | (Figure 16) for −1 ≤ ≤ 1 and graph = 1 for 1 and for −1.



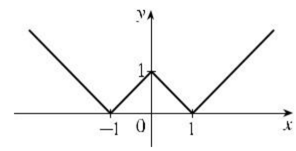
We could rewrite *f* as ( ) = 1 if 1; − if −1 ≤ 0; if 0 ≤ ≤ 1; 1 if 1.

50. ( ) = ||−1 = (| − 1) if | | − 1 ≥ 0; −(| | − 1) if | | − 1 0

= | | − 1 if | | ≥ 1; −| | + 1 if | | 1

= − 1 if | | ≥ 1 and ≥ 0; − − 1 if | | ≥ 1 and 0; − + 1 if | | 1 and ≥ 0; − (−) + 1 if | | 1 and 0

= − 1 if ≥ 1; − − 1 if ≤ −1; − + 1 if 0 ≤ 1; + 1 if −1 0



Recall that the slope of a line between the two points (1 1) and (2 2) is = (2 − 1)/(2 − 1) and an equation of the line connecting those two points is − 1 = (− 1). The slope of the line segment joining the points (1 −3) and (5 7) is (7 − (−3))/(5 − 1) = 5/2, so an equation is − (−3) = 5/2 (− 1). The function is ( ) = 5/2 − 11/2, 1 ≤ ≤ 5.

52. The slope of the line segment joining the points (−5 10) and (7 −10) is (−10 − 10)/(7 − (−5)) = −5/3, so an equation is − 10 = −5/3 [− (−5)]. The function is ( ) = −5/3 + 5/3, −5 ≤ ≤ 7.

53. We need to solve the given equation for . + (− 1)^2 = 0 ⇔ (− 1)^2 = − ⇔ − 1 = ±√− ⇔ = 1 ± √−. The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want ( ) = 1 − √−. Note that the domain is ≤ 0.

54. ^{2} + (− 2)^2 = 4 ⇔ (− 2)^2 = 4 − ^{2} ⇔ − 2 = ±√(4 − ^{2}) ⇔ = 2 ± √(4 − ^{2}). The top half is given by the function ( ) = 2 + √(4 − ^{2}), −2 ≤ ≤ 2.

For 0 ≤ ≤ 3, the graph is the line with slope −1 and -intercept 3, that is, = − + 3. For 3 ≤ 5, the graph is the line with slope 2 passing through (3 0); that is, − 0 = 2(− 3), or = 2 − 6. So the function is

( ) = − + 3 if 0 ≤ ≤ 3; 2 − 6 if 3 5

56. For −4 ≤ ≤ −2, the graph is the line with slope −3/2 passing through (−2 0); that is, − 0 = −3/2 [− (−2)], or = −3/2 − 3. For −2 2, the graph is the top half of the circle with center (0 0) and radius 2. An equation of the circle

is $^2 + ^2 = 4$, so an equation of the top half is $= \sqrt{4 - 2}$. For $2 \le \le 4$, the graph is the line with slope $\frac{3}{2}$ passing through (2 0); that is, $- 0 = \frac{3}{2}(- 2)$, or $= \frac{3}{2} - 3$. So the function is

0= √42 if 22

$-\frac{3}{2} - 3$ if $-4 \le \le -2$

57. Let the length and width of the rectangle be and . Then the perimeter is $2 + 2 = 20$ and the area is= .

$\frac{20-2}{2}$

lengths are positive, the domain of is 0 10. If we further restrict to be larger than , then 5 10 would be the domain.

Let the length and width of the rectangle be and . Then the area is = 16, so that = 16 . The perimeter is = 2 + 2 , so ( ) = 2 + 2(16) = 2 + 32 , and the domain of is 0, since lengths must be positive quantities. If we further restrict to be larger than , then 4 would be the domain.

Let the length of a side of the equilateral triangle be . Then by the Pythagorean Theorem, the height of the triangle satisfies $^2 + \frac{1}{2} \; ^2 = \; ^2$, so that $^2 = \; ^2 - \frac{1}{4} \; ^2 = \frac{3}{4} \; ^2$ and $= \frac{\sqrt{3}}{2}$. Using the formula for the area of a triangle,

$= \frac{1}{2}$ (base)(height), we obtain $( ) = \frac{1}{2}( ) \; \frac{\sqrt{3}}{2} \; = \frac{\sqrt{3}}{4} \; ^2$, with domain 0.

Let the length, width, and height of the closed rectangular box be denoted by , , and , respectively. The length is twice the width, so = 2 . The volume of the box is given by = . Since = 8, we have 8 = (2) ⇒

$8=2^2 \Rightarrow = \frac{8}{2 \; 2} = \frac{4}{2}$, and so $= ( ) = \frac{4}{2}$.

Let each side of the base of the box have length , and let the height of the box be . Since the volume is 2, we know that $2 = \; ^2$, so that $= 2 \; ^2$, and the surface area is $= \; ^2 + 4$. Thus, $( ) = \; ^2 + 4 \; (2 \; ^2) = \; ^2 + (8 \;)$, with domain 0.

The area of the window is $= \; + \frac{1}{2} \; \frac{1}{2} \; ^2 = \; + \; 8^2$, where is the height of the rectangular portion of the window.

The perimeter is $= 2 + + \frac{1}{2} = 30 \Leftrightarrow 2 = 30 - - \frac{1}{2} \Leftrightarrow = \frac{1}{4}(60 - 2 -)$. Thus,

$( ) = \frac{60 - 2 - 4}{4} + \frac{2}{8} = 15 - \frac{}{2} - \frac{}{4} + \frac{}{8} = 15 - \frac{}{8} - \frac{}{8} = 15 - \frac{+4}{8}$.

Since the lengths and must be positive quantities, we have 0 and 0. For 0, we have 2 0 ⇔ $30 - - \frac{1}{2} \; 0 \Leftrightarrow 60 \; 2 + \Leftrightarrow \; \frac{60}{2 +}$. Hence, the domain of is $0 \; \frac{60}{2 +}$.

The height of the box is and the length and width are = 20 − 2 , = 12 − 2 . Then =and so

$( ) = (20-2)(12-2)( ) = 4(10-)(6-)( ) = 4 (60-16 + \; ^2) = 4 \; ^3 - 64 \; ^2 + 240$.

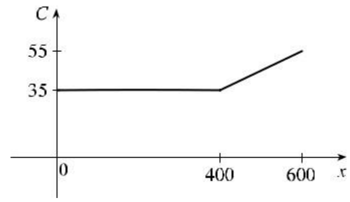
The sides , , and must be positive. Thus, 0 ⇔ 20 − 2 0 ⇔ 10;

0 ⇔ 12 − 2 0 ⇔ 6; and 0. Combining these restrictions gives us the domain 0 6.

64. We can summarize the monthly cost with a piecewise defined function.

( ) = 35 +010(− 400) if400

35 if 0 ≤ ≤ 400

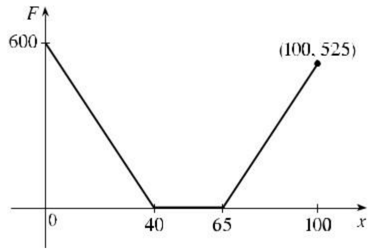


65. We can summarize the amount of the fine with a piecewise defined function.

15(40 −) if 0 ≤ 40

()= 0 15(− 65) if 40 ≤ ≤ 65

if 65



66. For the first 1200 kWh, ( ) = 10 + 0 06 .

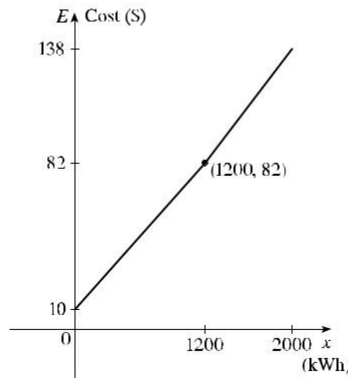
For usage over 1200 kWh, the cost is

( ) = 10 + 0 06(1200) + 0 07(− 1200) = 82 + 0 07(− 1200).

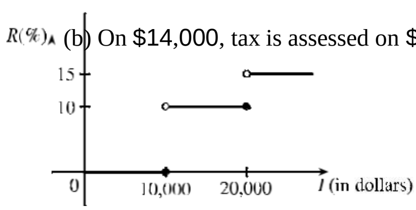
Thus,

10+006 if 0 ≤ ≤ 1200

()= 82+007(− 1200) if1200



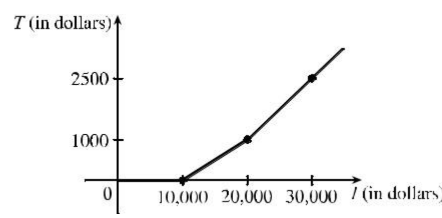
67. (a)



(b) On $14,000, tax is assessed on $4000, and 10%($4000) = $400. On $26,000, tax is assessed on $16,000, and

10%($10,000) + 15%($6000) = $1000 + $900 = $1900.

(c) As in part (b), there is $1000 tax assessed on $20,000 of income, so the graph of is a line segment from (10,000 0) to (20,000 1000). The tax on $30,000 is $2500, so the graph of for 20,000 is the ray with initial point (20,000 1000) that passes through (30,000 2500).



One example is the amount paid for cable or telephone system repair in the home, usually measured to the nearest quarter hour. Another example is the amount paid by a student in tuition fees, if the fees vary according to the number of credits for which the student has registered.

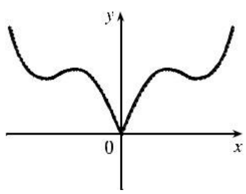
is an odd function because its graph is symmetric about the origin. is an even function because its graph is symmetric with respect to the -axis.

is not an even function since it is not symmetric with respect to the -axis. is not an odd function since it is not symmetric about the origin. Hence, is *neither* even nor odd. is an even function because its graph is symmetric with respect to the -axis.

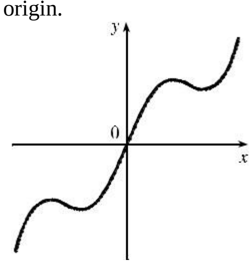
(a) Because an even function is symmetric with respect to the -axis, and the point (5 3) is on the graph of this even function, the point (−5 3) must also be on its graph.

Because an odd function is symmetric with respect to the origin, and the point (5 3) is on the graph of this odd function, the point (−5 −3) must also be on its graph.

72. (a) If is even, we get the rest of the graph by reflecting about the -axis.



(b) If is odd, we get the rest of the graph by rotating 180° about the origin.

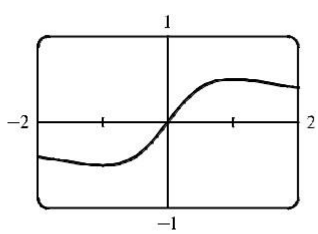


( )= 2+1.

( ) = – = – = = ().

– (−)2+1 2 + 1 − 2 + 1 −

Since (−) = − ( ), is an odd function.

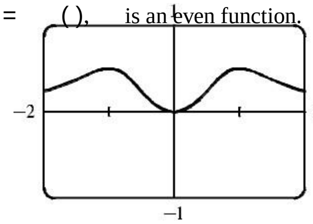


()= 2. 4 + 1

(−)2 2

(−)= = = (). (−)4+1 4+1

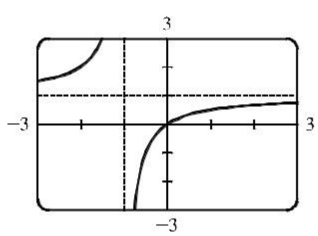
Since (−) = ( ), is an even function.



75. ( )= , so ( ) = – = . . .

+ 1 – + 1 1

Since this is neither ( ) nor − ( ), the function is neither even nor odd.

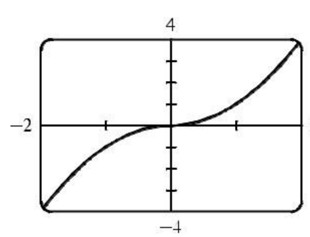


()= ||.

(−)=(−)|− |=(−)| |=−(| |)

− ( )

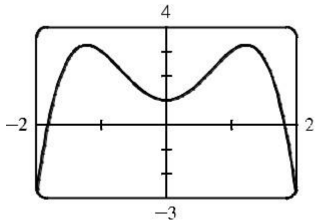
Since (−) = − ( ), is an odd function.



( )=1+3 ^{2}– ^{4}.

(−) = 1+3(−)^{2}−(−)^{4} = 1+3 ^{2}− ^{4} = ( ). Since (−) =

( ), is an even function.

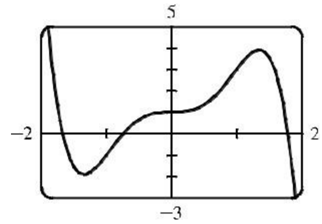


( ) = 1 + 3 ^{3} − ^{5}, so

(−) =1+3(−)^{3}−(−)^{5} =1+3(− ^{3})−(− ^{5})

1−3 ^{3}+ ^{5}

Since this is neither ( ) nor − ( ), the function is neither even nor odd.



(i) If and are both even functions, then (−) = ( ) and (−) = ( ). Now

(+)(−) = (−) + (−) = ( ) + ( ) = (+)( ), so + is an *even* function.

If and are both odd functions, then (−) = − ( ) and (−) = − ( ). Now

(+)(−) = (−) + (−) = − ( ) + [− ( )] = −[( ) + ( )] = −(+)( ), so + is an *odd* function.

If is an even function and is an odd function, then (+)(−) = (−) + (−) = ( ) + [− ( )] = ( ) − ( ), which is not (+)( ) nor −(+)( ), so + is *neither* even nor odd. (Exception: if is the zero function, then + will be *odd*. If is the zero function, then + will be *even*.)

(i) If and are both even functions, then (−) = ( ) and (−) = ( ). Now

( )(−) = (−) (−) = ( ) ( ) = ( )( ), so is an *even* function.

If and are both odd functions, then (−) = − ( ) and (−) = − ( ). Now

( )(−) = (−) (−) = [− ( )][− ( )] = ( ) ( ) = ( )( ), so is an *even* function.

If is an even function and is an odd function, then

( )(−) = (−) (−) = ( )[− ( )] = −[( ) ( )] = −( )( ), so is an *odd* function.

1.2 Mathematical Models: A Catalog of Essential Functions

(a) ( ) = log$_{2}$ is a logarithmic function.

(b) ( ) = $\sqrt[4]{\ }$ is a root function with = 4.

(c) ( ) = $\dfrac{2^{3}}{1-{}^{2}}$ is a rational function because it is a ratio of polynomials.

( ) = 1 − 1 1 + 2 54 ^{2} is a polynomial of degree 2 (also called a *quadratic function*).

( ) = 5 is an exponential function.

(f) ( ) = sin cos^{2} is a trigonometric function.

(a) = is an exponential function (notice that is the *exponent*).

= is a power function (notice that is the *base*).

= $^2(2 - ^3) = 2\ ^2 - ^5$ is a polynomial of degree 5.

= tan − cos is a trigonometric function.

= (1 +) is a rational function because it is a ratio of polynomials.

(f) = $\sqrt{\ }$ 3 − 1 (1 + $\sqrt{\ }$ 3) is an algebraic function because it involves polynomials and roots of polynomials.

We notice from the figure that and are even functions (symmetric with respect to the -axis) and that is an odd function (symmetric with respect to the origin). So (b) = ^{5} must be . Since is flatter than near the origin, we must have

= ^{8} matched with and (a) = ^{2} matched with .

(a) The graph of = 3 is a line (choice).

= 3 is an exponential function (choice).

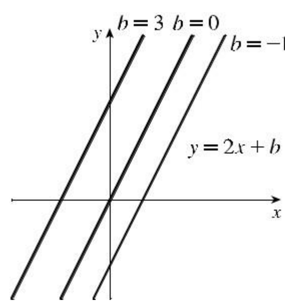
= ^{3} is an odd polynomial function or power function (choice).

= $\sqrt{\ }$ 3 = 1 3 is a root function (choice).

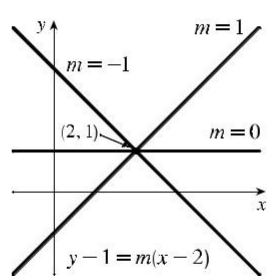
5. The denominator cannot equal 0, so 1 − sin 6= 0 ⇔ sin 6= 1 ⇔ 6=2 + 2 . Thus, the domain of ( ) = cos 1 − sin | 6=2 + 2 , an integer .

6. The denominator cannot equal 0, so 1 − tan 6= 0 ⇔ tan 6= 1 6=4 + . The tangent function is not defined 1 + , 6=2 + , an integer .

if 6=2 + . Thus, the domain of ( ) = 1 − tan is | 6=4



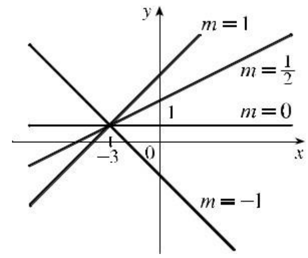
(a) An equation for the family of linear functions with slope 2 is = ( ) = 2 + , where is the -intercept.



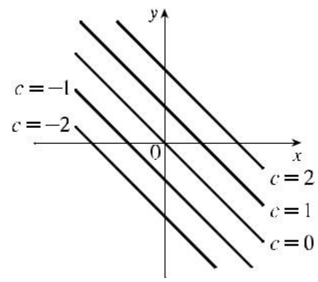
(b) (2) = 1 means that the point (2 1) is on the graph of . We can use the point-slope form of a line to obtain an equation for the family of linear functions through the point (2 1). − 1 = (− 2), which is equivalent to = + (1 − 2) in slope-intercept form.

To belong to both families, an equation must have slope = 2, so the equation in part (b), = + (1 − 2), becomes = 2 − 3. It is the *only* function that belongs to both families.

All members of the family of linear functions ( ) = 1 + (+ 3) have graphs that are lines passing through the point (−3 1).



9. All members of the family of linear functions ( ) = − have graphs that are lines with slope −1. The -intercept is .



The vertex of the parabola on the left is (3 0), so an equation is = (− 3)^2 + 0. Since the point (4 2) is on the parabola, we'll substitute 4 for and 2 for to find . 2 = (4 − 3)^2 ⇒ = 2, so an equation is ( ) = 2(− 3)^2. The -intercept of the parabola on the right is (0 1), so an equation is = ^{2} + + 1. Since the points (−2 2) and (1 −2 5) are on the parabola, we'll substitute −2 for and 2 for as well as 1 for and −2 5 for to obtain two equations with the unknowns and .

(−2 2): 2=4 −2 +1 ⇒4−2=1 **(1)**

(1 −2 5): −25= + +1 ⇒ + =−35 **(2)**

2 · **(2)** + **(1)** gives us 6 = −6 ⇒ = −1. From **(2)**, −1 + = −3 5 ⇒ = −2 5, so an equation is ( ) = − ^{2} − 2 5 + 1.

Since (−1) = (0) = (2) = 0, has zeros of −1, 0, and 2, so an equation for is ( ) = [− (−1)](− 0)(− 2), or ( ) = (+ 1)(− 2). Because (1) = 6, we'll substitute 1 for and 6 for ( ).

6 = (1)(2)(−1) ⇒ −2 = 6 ⇒ = −3, so an equation for is ( ) = −3 (+ 1)(− 2).

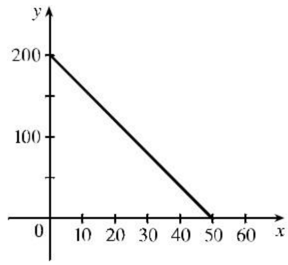
(a) For = 0 02 + 8 50, the slope is 0 02, which means that the average surface temperature of the world is increasing at a rate of 0 02 °C per year. The -intercept is 8 50, which represents the average surface temperature in °C in the year 1900.

(b) = 2100 − 1900 = 200 ⇒ = 0 02(200) + 8 50 = 12 50 °C

(a) = 200, so = 0 0417 (+ 1) = 0 0417(200)(+ 1) = 8 34 + 8 34. The slope is 8 34, which represents the change in mg of the dosage for a child for each change of 1 year in age.

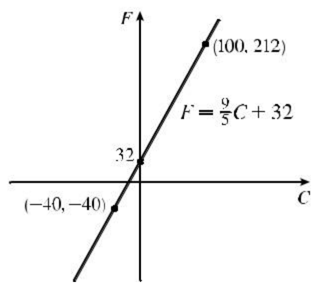
For a newborn, = 0, so = 8 34 mg.

14. (a)



(b) The slope of -4 means that for each increase of 1 dollar for a rental space, the number of spaces rented *decreases* by 4. The -intercept of 200 is the number of spaces that would be occupied if there were no charge for each space. The -intercept of 50 is the smallest rental fee that results in no spaces rented.

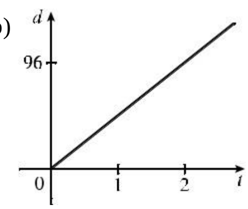
15. (a)



(b) The slope of $\frac{9}{5}$ means that increases $\frac{9}{5}$ degrees for each increase of 1°C. (Equivalently, increases by 9 when increases by 5 and decreases by 9 when decreases by 5.) The -intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0.

16. (a) Let = distance traveled (in miles) and = time elapsed (in hours). At $= 0$, $= 0$ and at $= 50$ minutes $= 50 \cdot \frac{1}{60} = \frac{5}{6}$ h, $= 40$. Thus we have two points: $(0\ 0)$ and $\frac{5}{6}\ 40$, so $= \frac{40-0}{\frac{5}{6}-0} = 48$ and so $= 48$.

(b)



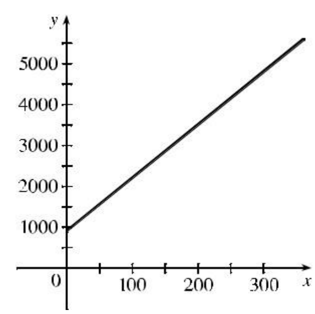
(c) The slope is 48 and represents the car's speed in mi h.

17. (a) Using in place of and in place of , we find the slope to be $\frac{2-1}{2-1} = \frac{80-70}{173-113} = \frac{10}{60} = \frac{1}{6}$. So a linear equation is $-80 = \frac{1}{6}(-173) \Leftrightarrow -80 = \frac{1}{6} - \frac{173}{6} \Leftrightarrow = \frac{1}{6} + \frac{307}{6}$ $\frac{307}{6} = 51\frac{1}{6}$.

The slope of $\frac{1}{6}$ means that the temperature in Fahrenheit degrees increases one-sixth as rapidly as the number of cricket chirps per minute. Said differently, each increase of 6 cricket chirps per minute corresponds to an increase of 1°F.

When $= 150$, the temperature is given approximately by $= \frac{1}{6}(150) + \frac{307}{6} = 76\ 16\ ^{\circ}\text{F} \approx 76\ ^{\circ}\text{F}$.

18. (a) Let denote the number of chairs produced in one day and the associated cost. Using the points $(100\ 2200)$ and $(300\ 4800)$, we get the slope $\frac{4800-2200}{300-100} = \frac{2600}{200} = 13$. So $-2200 = 13(-100) \Leftrightarrow$ $= 13 + 900$.



(b) The slope of the line in part (a) is 13 and it represents the cost (in dollars) of producing each additional chair.

The -intercept is 900 and it represents the fixed daily costs of operating the factory.

19. (a) We are given $\frac{\text{change in pressure}}{10 \text{ feet change in depth}} = \frac{4\ 34}{10} = 0\ 434$. Using for pressure and for depth with the point $(\quad) = (0\ 15)$, we have the slope-intercept form of the line, $= 0\ 434 + 15$.

(b) When = 100, then 100 = 0 434 + 15 ⇔ 0 434 = 85 ⇔ = 0 ^{85}434 ≈ 195 85 feet. Thus, the pressure is 100 lb in^2 at a depth of approximately 196 feet.

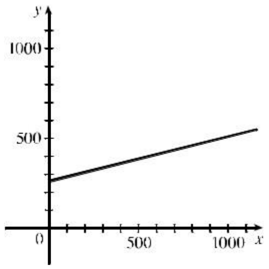
(a) Using in place of and in place of , we find the slope to be $\frac{2-1}{2-1} = \frac{460-380}{800-480} = \frac{80}{320} = \frac{1}{4}$.

So a linear equation is $-460 = \frac{1}{4}(-800)$ ⇔ $-460 = \frac{1}{4} - 200$ ⇔ $= \frac{1}{4} + 260$.

(b) Letting = 1500 we get = $\frac{1}{4}$ (1500) + 260 = 635.

The cost of driving 1500 miles is $635.

(c)



The slope of the line represents the cost per mile, $0 25.

(d) The -intercept represents the fixed cost, $260.

(e) A linear function gives a suitable model in this situation because you have fixed monthly costs such as insurance and car payments, as well as costs that increase as you drive, such as gasoline, oil, and tires, and the cost of these for each additional mile driven is a constant.

(a) The data appear to be periodic and a sine or cosine function would make the best model. A model of the form ( ) = cos( ) + seems appropriate.

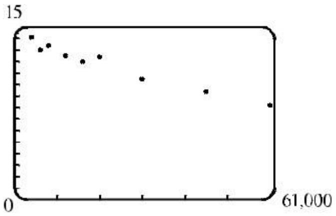
The data appear to be decreasing in a linear fashion. A model of the form ( ) =+ seems appropriate.

(a) The data appear to be increasing exponentially. A model of the form ( ) = · or ( ) = · + seems appropriate.

The data appear to be decreasing similarly to the values of the reciprocal function. A model of the form ( ) =seems appropriate.

Exercises 23– 28: Some values are given to many decimal places. These are the results given by several computer algebra systems — rounding is left to the reader.

23. (a)

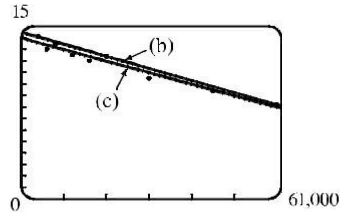


A linear model does seem appropriate.

(b) Using the points (4000 14 1) and (60,000 8 2), we obtain

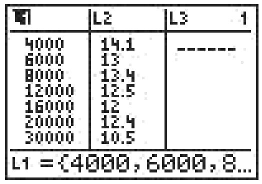
$- 141 = \frac{82-141}{60\,000 - 4000}(- 4000)$ or, equivalently,

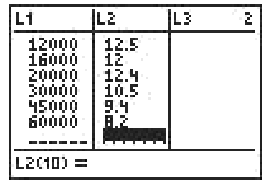
≈ −0 000105357 + 14 521429.



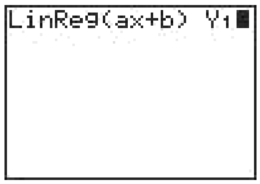
Using a computing device, we obtain the least squares regression line = −0 0000997855 + 13 950764. The following commands and screens illustrate how to find the least squares regression line on a TI-84 Plus.

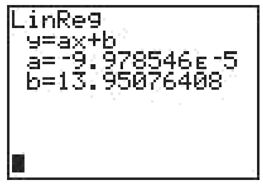
Enter the data into list one (L1) and list two (L2). Press STAT 1 to enter the editor.

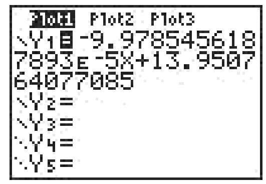




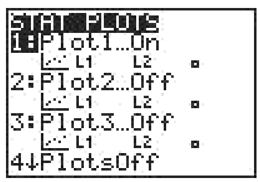
Find the regression line and store it in Y1. Press 2nd QUIT STAT ▸ 4 VARS ▸ 1 1 ENTER

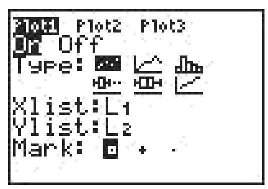




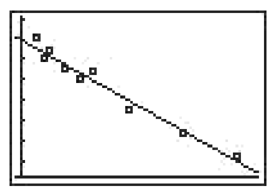


Note from the last figure that the regression line has been stored in Y1 and that Plot1 has been turned on (Plot1 is highlighted). You can turn on Plot1 from the Y= menu by placing the cursor on Plot1 and pressing ENTER or by pressing 2nd STAT PLOT 1 ENTER .





Now press ZOOM 9 to produce a graph of the data and the regression line. Note that choice 9 of the ZOOM menu automatically selects a window that displays all of the data.

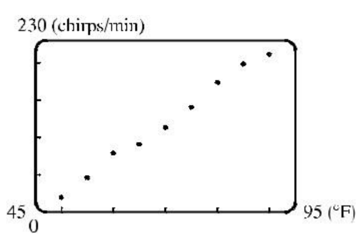


When = 25,000, ≈ 11 456; or about 11 5 per 100 population.

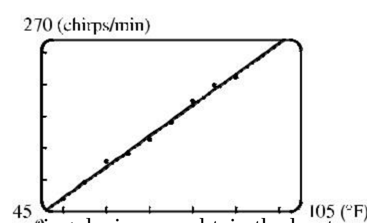
When = 80,000, ≈ 5 968; or about a 6% chance.

(f) When = 200,000, is negative, so the model does not apply.

24. (a)



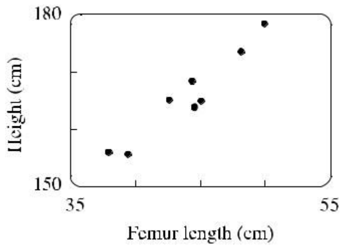
(b)



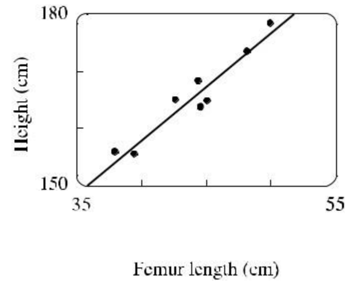
Using a computing device, we obtain the least squares regression line = 4 856 − 220 96.

(c) When = 100°F, = 264 7 ≈ 265 chirps min.

25. (a)



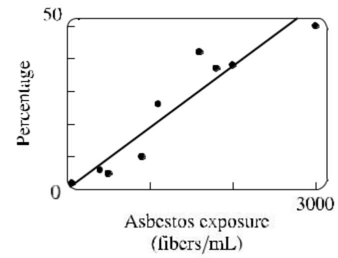
(b) Using a computing device, we obtain the regression line = 1 88074 + 82 64974.



(c) When = 53 cm, ≈ 182 3 cm.

(a) Using a computing device, we obtain the regression line = 0 01879 + 0 30480.

(b) The regression line appears to be a suitable model for the data.

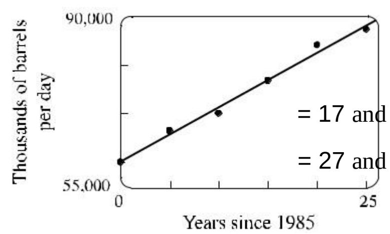


(c) The -intercept represents the percentage of laboratory rats that develop lung tumors when *not* exposed to asbestos fibers.

27. (a) See the scatter plot in part (b). A linear model seems appropriate.

(b) Using a computing device, we obtain the regression line

= 1116 64 + 60,188 33.



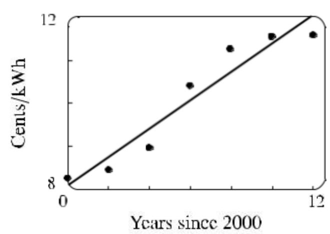
(c) For 2002, = 17 and ≈ 79,171 thousands of bar

For 2012, = 27 and ≈ 90,338 thousands of bar

(a) See the scatter plot in part (b). A linear model seems appropriate.

(b) Using a computing device, we obtain the regression line

= 0 33089 + 8 07321.



(c) For 2005, = 5 and ≈ 9 73 cents kWh. For 2013, = 13 and ≈ 12

37 cents kWh.

If is the original distance from the source, then the illumination is ( ) = $^{-2}$ = ^{2}. Moving halfway to the lamp gives us an illumination of $\frac{1}{2}$ = $\frac{1}{2}$ $^{-2}$ = (2)^{2} = 4(^{2}), so the light is 4 times as bright.

(a) If = 60, then = 0 7 $^{0\ 3}$ ≈ 2 39, so you would expect to find 2 species of bats in that cave.

(b) = 4 ⇒ 2 4=07 ^{03} ⇒ $\frac{40}{7}$ = 310 ⇒= $\left(\frac{40}{7}\right)^{10\ 3}$ ≈ 333 6, so we estimate the surface area of the cave to be 334 m .

(a) Using a computing device, we obtain a power function =, where ≈ 3 1046 and ≈ 0 308.

If = 291, then =≈ 17 8, so you would expect to find 18 species of reptiles and amphibians on Dominica.

(a) = 1 000 431 227 $^{1\ 499\ 528\ 750}$

The power model in part (a) is approximately $= {}^{1\ 5}$. Squaring both sides gives us ${}^{2} = {}^{3}$, so the model matches Kepler's Third Law, ${}^{2} = {}^{3}$.

1.3 New Functions from Old Functions

(a) If the graph of is shifted 3 units upward, its equation becomes $= () + 3$.

If the graph of is shifted 3 units downward, its equation becomes $= () - 3$.

If the graph of is shifted 3 units to the right, its equation becomes $= (- 3)$.

If the graph of is shifted 3 units to the left, its equation becomes $= (+ 3)$.

If the graph of is reflected about the -axis, its equation becomes $= - ()$.

(f) If the graph of is reflected about the -axis, its equation becomes $= (-)$.

If the graph of is stretched vertically by a factor of 3, its equation becomes $= 3\,()$.
If the graph of is shrunk vertically by a factor of 3, its equation becomes $= \frac{1}{3}\,()$.

(a) To obtain the graph of $= () + 8$ from the graph of $= ()$, shift the graph 8 units upward.

To obtain the graph of $= (+ 8)$ from the graph of $= ()$, shift the graph 8 units to the left.

To obtain the graph of $= 8\,()$ from the graph of $= ()$, stretch the graph vertically by a factor of 8.

To obtain the graph of $= (8\ )$ from the graph of $= ()$, shrink the graph horizontally by a factor of 8.

To obtain the graph of $= - () - 1$ from the graph of $= ()$, first reflect the graph about the -axis, and then shift it 1 unit downward.

(f) To obtain the graph of $= 8\,(\frac{1}{8}\ )$ from the graph of $= ()$, stretch the graph horizontally and vertically by a factor of 8.

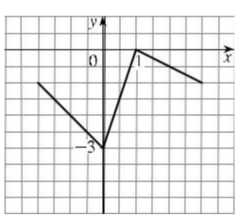
(a) (graph 3) The graph of is shifted 4 units to the right and has equation $= (- 4)$.
(graph 1) The graph of is shifted 3 units upward and has equation $= () + 3$.

(graph 4) The graph of is shrunk vertically by a factor of 3 and has equation $= \frac{1}{3}\,()$.

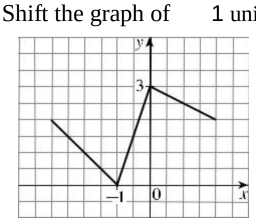
(graph 5) The graph of is shifted 4 units to the left and reflected about the -axis. Its equation is $= - (+ 4)$.

(graph 2) The graph of is shifted 6 units to the left and stretched vertically by a factor of 2. Its equation is $= 2\,(+ 6)$.

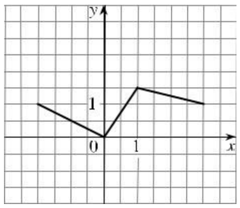
4. (a) $= () - 3$: Shift the graph of 3 units down.



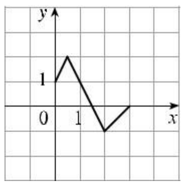
(b) $= (+ 1)$: Shift the graph of 1 unit to the left.



$= \frac{1}{2}$ ( ): Shrink the graph of vertically by a factor of 2.

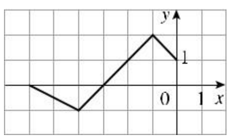


(a) To graph = (2) we shrink the graph of horizontally by a factor of 2.



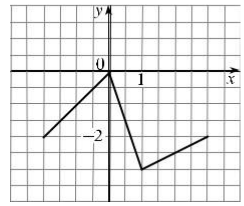
The point (4 −1) on the graph of corresponds to the point $\frac{1}{2} \cdot 4$ −1 = (2 −1).

To graph = (−) we reflect the graph of about the -axis.

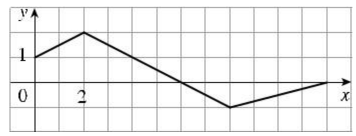


The point (4 −1) on the graph of corresponds to the point (−1 · 4 −1) = (−4 −1).

(d) = − ( ): Reflect the graph of about the -axis.

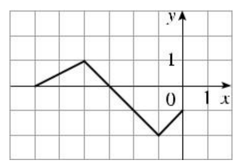


To graph = $\frac{1}{2}$ we stretch the graph of horizontally by a factor of 2.



The point (4 −1) on the graph of corresponds to the point (2 · 4 −1) = (8 −1).

To graph = − (−) we reflect the graph of about the -axis, then about the -axis.



The point (4 −1) on the graph of corresponds to the point (−1 · 4 −1 · −1) = (−4 1).

The graph of = ( ) = $\sqrt{3 - {}^2}$ has been shifted 2 units to the right and stretched vertically by a factor of 2. Thus, a function describing the graph is

$$=2\ (\ -2)=2\ \sqrt{3(\ -2)-(\ -2)^2} =2\ \sqrt{3\ -6-(\ ^2-4\ +4)}=2\ \sqrt{-\ ^2+7\ -10}$$

The graph of = ( ) = $\sqrt{3 - {}^2}$ has been shifted 4 units to the left, reflected about the -axis, and shifted downward 1 unit.

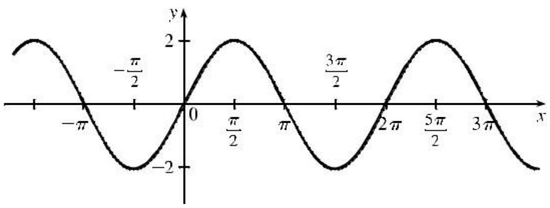
Thus, a function describing the graph is

$$= \underbrace{-1 \cdot}_{\substack{\text{reflect} \\ \text{about -axis}}} \underbrace{(\ +4)}_{\substack{\text{shift} \\ \text{4 units left}}} \underbrace{-\ 1}_{\substack{\text{shift} \\ \text{1 unit left}}}$$

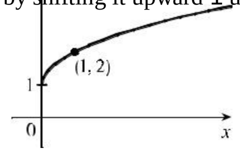
This function can be written as

$$=-\ (\ +4)-1=-\sqrt{3(\ +4)-(\ +4)^2}-1 =-\sqrt{3\ +12-(\ ^2+8\ +16)}-1=-\sqrt{-\ ^2-5\ -4}-1$$

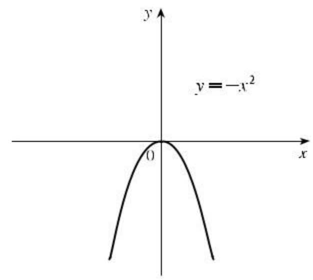
(a) The graph of = 2 sin can be obtained from the graph of = sin by stretching it vertically by a factor of 2.



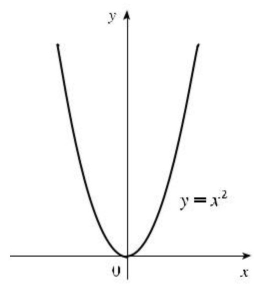
The graph of = 1 + √ can be obtained from the graph of = √ by shifting it upward 1 unit.

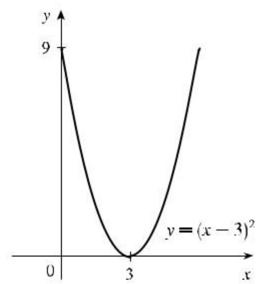


9. = − 2: Start with the graph of = 2 and reflect about the -axis.

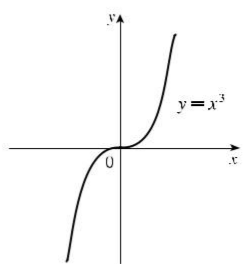


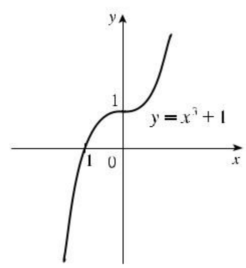
10. = (− 3)^2: Start with the graph of = 2 and shift 3 units to the right.



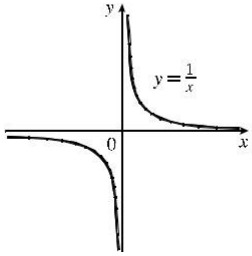


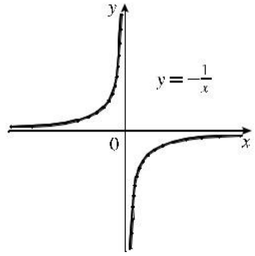
11. = 3 + 1: Start with the graph of = 3 and shift upward 1 unit.

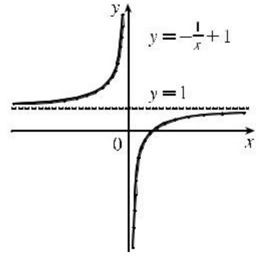




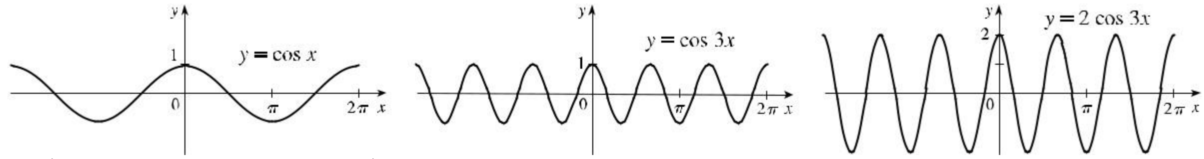
= 1 − $\frac{1}{}$ = −$\frac{1}{}$ + 1: Start with the graph of = $\frac{1}{}$, reflect about the -axis, and shift upward 1 unit.



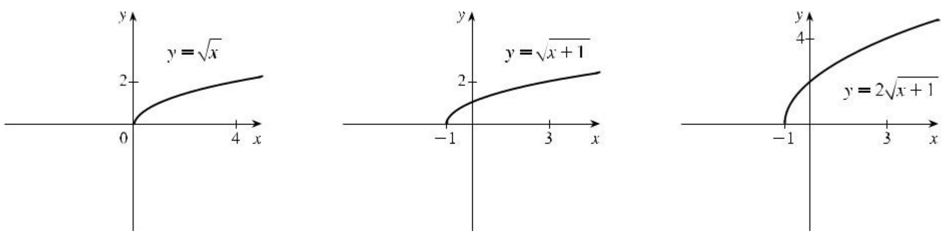




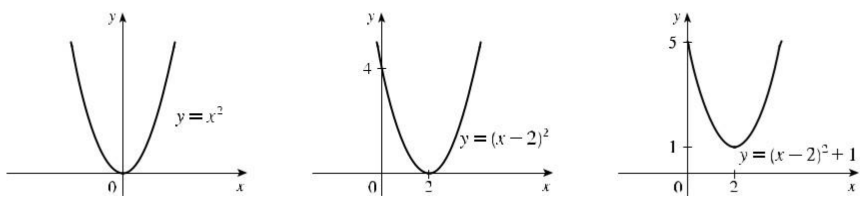
$= 2 \cos 3$: Start with the graph of $= \cos$, compress horizontally by a factor of 3, and then stretch vertically by a factor of 2.



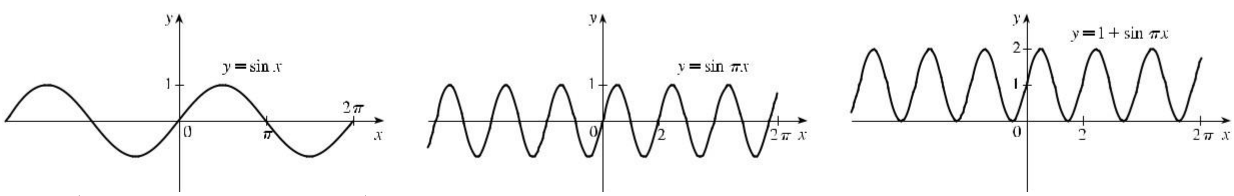
$= 2\sqrt{} + 1$: Start with the graph of $= \sqrt{}$, shift 1 unit to the left, and then stretch vertically by a factor of 2.



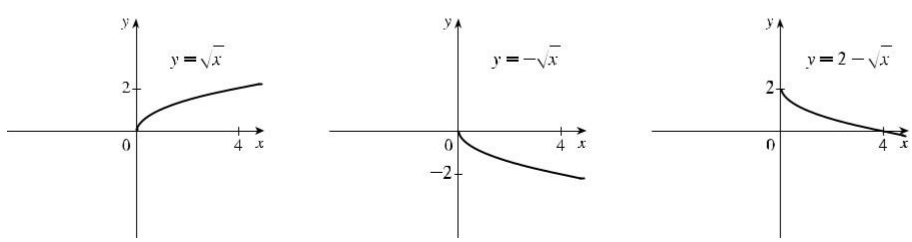
$= {}^2 - 4 + 5 = ({}^2 - 4 + 4) + 1 = (- 2)^2 + 1$: Start with the graph of $= {}^2$, shift 2 units to the right, and then shift upward 1 unit.



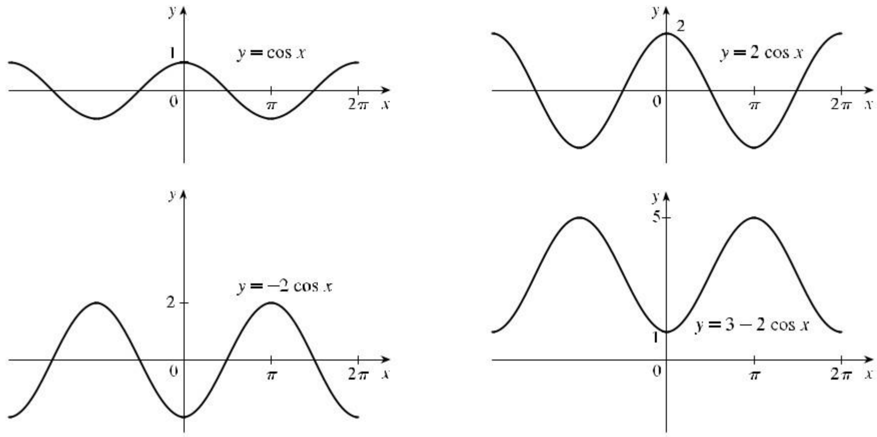
$= 1 + \sin$: Start with the graph of $= \sin$, compress horizontally by a factor of , and then shift upward 1 unit.



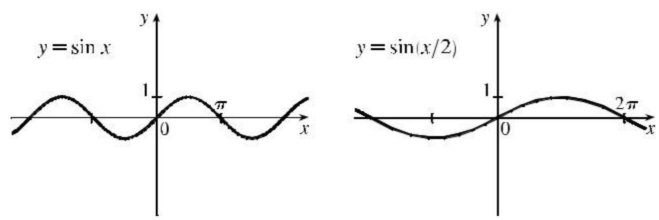
$= 2 - \sqrt{}$: Start with the graph of $= \sqrt{}$, reflect about the -axis, and then shift 2 units upward.



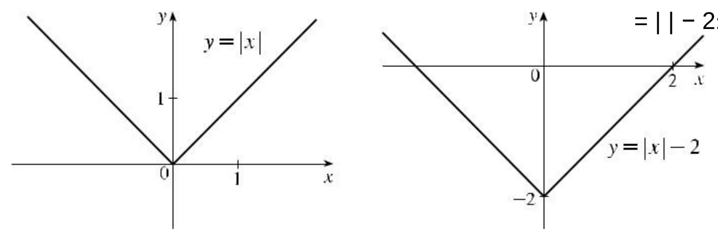
= 3 − 2 cos : Start with the graph of = cos , stretch vertically by a factor of 2, reflect about the -axis, and then shift 3 units upward.



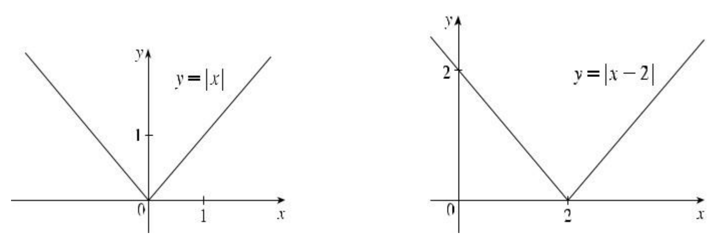
19. = sin(2): Start with the graph of = sin and stretch horizontally by a factor of 2.



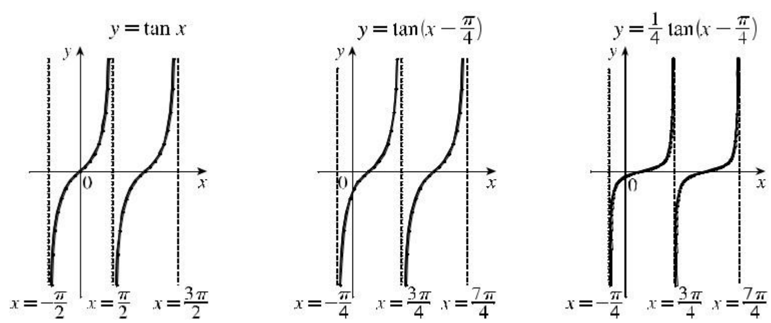
20. = | | − 2: Start with the grap = | | and shift 2 units downward.



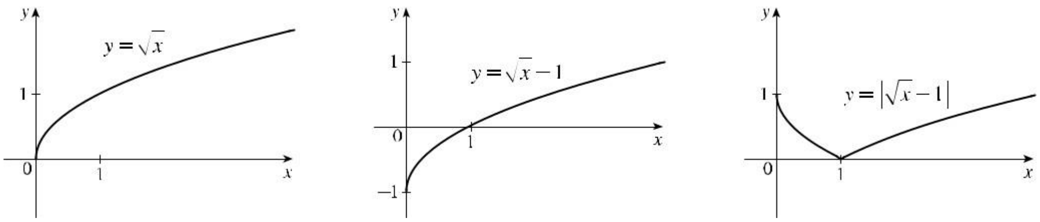
21. = | − 2|: Start with the graph of = | | and shift 2 units to the right.



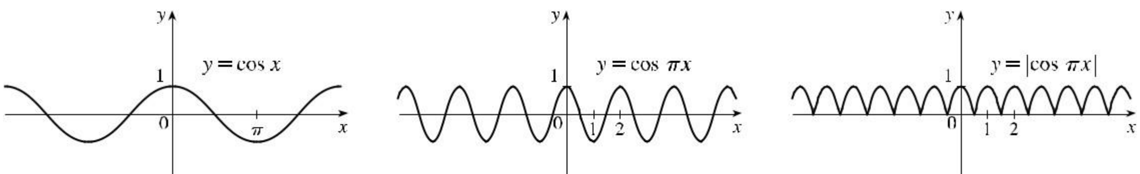
22. = 4 1 tan(− 4): Start with the graph of = tan , shift 4 units to the right, and then compress vertically by a factor of 4.



$= |\sqrt{\ } - 1|$: Start with the graph of $= \sqrt{\ }$, shift it 1 unit downward, and then reflect the portion of the graph below the -axis about the -axis.



$= |\cos\ |$: Start with the graph of $= \cos$, shrink it horizontally by a factor of , and reflect all the parts of the graph below the -axis about the -axis.



This is just like the solution to Example 4 except the amplitude of the curve (the 30°N curve in Figure 9 on June 21) is $14 - 12 = 2$. So the function is $(\ ) = 12 + 2 \sin \frac{2}{365}(\ - 80)$. March 31 is the 90th day of the year, so the model gives $(90) \approx$ 12 34 h. The daylight time (5:51 AM to 6:18 PM) is 12 hours and 27 minutes, or 12 45 h. The model value differs $\frac{12\ 45 - 12\ 34}{12\ 45}$

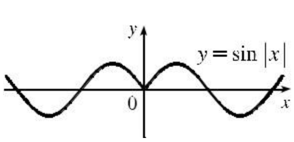
Using a sine function to model the brightness of Delta Cephei as a function of time, we take its period to be 5 4 days, its amplitude to be 0 35 (on the scale of magnitude), and its average magnitude to be 4 0. If we take $= 0$ at a time of average brightness, then the magnitude (brightness) as a function of time in days can be modeled by the formula $(\ ) = 4\ 0 + 0\ 35 \sin \frac{2}{5\ 4}$.

The water depth ( ) can be modeled by a cosine function with amplitude $\frac{12 - 2}{2} = 5$ m, average magnitude $\frac{12 + 2}{2} = 7$ m, and period 12 hours. High tide occurred at time 6:45 AM ($= 6\ 75$ h), so the curve begins a cycle at time $= 6\ 75$ h (shift 6.75 units to the right). Thus, $(\ ) = 5 \cos \frac{2}{12}(\ - 6\ 75) + 7 = 5 \cos \frac{}{6}(\ - 6\ 75) + 7$, where is in meters and is the number of hours after midnight.

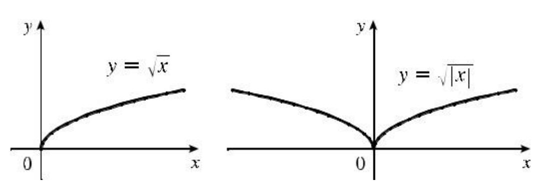
The total volume of air ( ) in the lungs can be modeled by a sine function with amplitude $\frac{2500 - 2000}{2} = 250$ mL, average volume $\frac{2500 + 2000}{2} = 2250$ mL, and period 4 seconds. Thus, $(\ ) = 250 \sin \frac{2}{4} + 2250 = 250 \sin \frac{}{2} + 2250$, where is in mL and is in seconds.

(a) To obtain $= (|\ |)$, the portion of the graph of $= (\ )$ to the right of the -axis is reflected about the -axis.

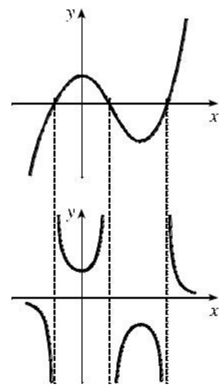
(b) $= \sin |\ |$



(c) $= \sqrt{|\ |}$



30. The most important features of the given graph are the -intercepts and the maximum and minimum points. The graph of = 1 ( ) has vertical asymptotes at the -values where there are -intercepts on the graph of = ( ). The maximum of 1 on the graph of = ( ) corresponds to a minimum of 1 1 = 1 on = 1 ( ). Similarly, the minimum on the graph of = ( ) corresponds to a maximum on the graph of = 1 ( ). As the values of get large (positively or negatively) on the graph of = ( ), the values of get close to zero on the graph of = 1 ( ).



( ) = ^{3} + 2 ^{2}; ( ) = 3 ^{2} − 1.= R for both and .

(+)( ) = (^{3} + 2 ^{2}) + (3 ^{2} − 1) = ^{3} + 5 ^{2} − 1, = (−∞ ∞), or R.

(−)( )=(^{3}+2 ^{2})−(3 ^{2}−1)= ^{3}− ^{2}+1, =R.

( )( )=(^{3}+2 ^{2})(3 ^{2}−1)=3 ^{5}+6 ^{4}− ^{3}−2 ^{2}, =R.

(d) ( ) = 3 2−1 / 3+2 2 , = | 6=±√3 since 3 ^{2} − 1 6=.0

( )= √3− , =(−∞ 3]; ( )= √ 2−1, =(−∞ −1] ∪ [1 ∞).

(+)( ) = √3 − + √ 2 − 1, = (−∞ −1] ∪ [1 3], which is the intersection of the domains of and .

(−)( )= √3− − √ 2−1, =(−∞ −1] ∪ [1 3].

( )( )= √3− · √ 2−1, =(−∞ −1] ∪ [1 3].

(d) ( ) = √ 3 − / √ 2 − 1 , = (−∞ −1) ∪ (1 3]. We must exclude = ±1 since these values would make undefined.

( ) = 3 + 5; ( ) = ^{2} + .= R for both and , and hence for their composites.

(◦)( )= (( ))= (^{2}+)=3(^{2}+)+5=3 ^{2}+3 +5, =R.

(◦)( )= (( ))= (3 +5)=(3 +5)^2+(3 +5)

9 ^{2}+30 +25+3 +5=9 ^{2}+33 +30, =R

(◦)= (( ))= (3 +5)=3(3 +5)+5=9 +15+5=9 +20, =R.

(◦)()= (())= (^{2}+)=(^{2}+)^2+(^{2}+)

^{4}+2 ^{3}+ ^{2}+ ^{2}+ = ^{4}+2 ^{3}+2 ^{2}+ , =R

( ) = ^{3} − 2; ( ) = 1 − 4 . = R for both and , and hence for their composites.

(◦)( )= (( ))= (1−4)=(1−4)^3−2

(1)^3−3(1)^2(4)+3(1)(4)^2−(4)^3−2=1−12 +48 ^{2}−64 ^{3}−2

−1−12 +48 ^{2}−64 ^{3} =R

(◦)( )= (( ))= (^{3}−2)=1−4(^{3}−2)=1−4 ^{3}+8=9−4 ^{3}, =R.

(◦)( )= (( ))= (^{3}−2)=(^{3}−2)^3−2

(^{3})^3−3(^{3})^2(2)+3(^{3})(2)^2−(2)^3−2= ^{9}−6 ^{6}+12 ^{3}−10=R

(◦)( )= (( ))= (1−4)=1−4(1−4)=1−4+16 =−3+16 , =R.

( )= √ +1, ={ | ≥−1}; ( )=4 −3, =R.

(◦)( )= (( ))= (4 −3)= (4 −3)+1= √4 −2

The domain of ◦ is { | 4 − 3 ≥ −1} = { | 4 ≥ 2} = | ≥ $\frac{1}{2}$ = $\frac{1}{2}$ ∞ .

(◦)( )= (( ))= (√ +1)=4√ +1−3

The domain of ◦ is { | is in the domain of and ( ) is in the domain of }. This is the domain of , that is,

{ | +1≥0}={ | ≥−1}=[−1 ∞).

(◦)( )= (( ))= (√ +1)= √√ +1+1

For the domain, we need + 1 ≥ 0, which is equivalent to ≥ −1, and √ + 1 ≥ −1, which is true for all real values of .

Thus, the domain of ◦ is [−1 ∞).

(◦)( )= (( ))= (4 −3)=4(4 −3)−3=16 −12−3=16 −15 =R.

( ) = sin ; ( ) = ^{2} + 1.= R for both and , and hence for their composites.

(◦)( ) = (( )) = (^{2} + 1) = sin(^{2} + 1), = R.

(◦) = (( )) = (sin) = (sin)^2 + 1 = sin^2 + 1, = R.

(◦)( ) = (( )) = (sin) = sin(sin), = R.

(◦)( )= (( ))= (^{2}+1)=(^{2}+1)^2+1= ^{4}+2 ^{2}+1+1= ^{4}+2 ^{2}+2, =R.

37. ()= + $\frac{1}{\ }$, ={ | 6=}0; ( )= $\frac{+1}{+2}$, ={ | 6=−2}

(a) (◦)( ) = (( )) = $\frac{+2}{+1}$ = $\frac{+2}{+1}$ $\frac{+\frac{+1}{+2}}{1}$ = $\frac{+2}{+1}$ $\frac{++1}{+2}$

+ 2

$\frac{(+1)(+1)+(+2)(+2)}{(+2)(+1)(+2)(+1)(+2)(+1)}$ = ^{2}+2 +1 + ^{2}+4 +4 = 2 ^{2}+6 +5

Since ( ) is not defined for = −2 and (( )) is not defined for = −2 and = −1, the domain of

(◦)( ) is = { | 6=−2 −1}.

(b) (◦)( ) = (( )) =+ $\frac{1}{\ }$ = $\frac{+\frac{1}{\ }+1}{+\frac{1}{\ }+2}$ = $\frac{^2+1+}{^2+1+2}$ = $\frac{^2+\ +1}{^2+2\ +1}$ = $\frac{^2+\ +1}{(+1)^2}$

Since ( ) is not defined for = 0 and (( )) is not defined for = −1, the

domain of (◦)( ) is = { | 6=−1 0}.

(c) (∘)()= (())= + $\frac{1}{}$ = + $\frac{1}{}$ + $\frac{1}{\frac{+}{1}}$ = + $\frac{1}{} + \frac{1}{2+1}$ = + $\frac{1}{\frac{+1}{}}$ + $\frac{}{2}$

= $\frac{()^2+1+1^2}{(2+1)}$ $\frac{+1+()}{}$ = $\frac{4+2+2+1+2}{(2+1)}$

= $\frac{4+3\,2+1}{(2+1)}$ ={ | 6=}0

(d) (∘)( ) = (( )) = $\frac{+1}{+2}$ = $\frac{\frac{+1}{+2}+1}{\frac{+1}{+2}+2}$ = $\frac{\frac{+1+1(+2)}{+2}}{\frac{+1+2(+2)}{+2}}$ = $\frac{+1+\ +2}{+1+2+4}$ = $\frac{2+3}{3+5}$

Since ( ) is not defined for = −2 and (( )) is not defined for = −$\frac{5}{3}$,

the domain of (∘)( ) is = | 6=−2 −$\frac{5}{3}$.

38. ( )= $\frac{}{1+}$, = { | 6=−1}; ( ) = sin 2 , = R.

(a) (∘)( ) = (( )) = (sin 2) = $\frac{\sin 2}{1+\sin 2}$

Domain: 1 + sin 2 6= 0 ⇒ sin 2 6=−1 ⇒26=2 $\frac{3}{}$ + 2 ⇒ 6=4 $\frac{3}{}$ + [an integer].

(b) (∘)( ) = (( )) = $\frac{}{1+}$ = sin $\frac{2}{1+}$.

Domain: { | 6=−1}

(c) (∘)( ) = (( )) = $\frac{\frac{}{1+}}{1+\frac{}{1+}}$ = $\frac{\frac{}{1+}\cdot(1+)}{\left(1+\frac{}{1+}\right)\cdot(1+)}$ = $\frac{}{1++}$ = $\frac{}{2+1}$

Since ( ) is not defined for = −1, and (( )) is not defined for = −$\frac{1}{2}$, the

domain of (∘)( ) is = { | 6=−1 −$\frac{1}{2}$ }.

(∘)( ) = (( )) = (sin 2) = sin(2 sin 2). Domain:

R

(∘ ∘)( ) = ((( ))) = ((^{2})) = (sin(^{2})) = 3 sin(^{2}) − 2

(∘ ∘)()= ((()))= (($\sqrt{}$))= (2$^{\sqrt{}}$)=2$^{\sqrt{}}$ −4

(∘ ∘)( )= ((( )))= ((^{3}+2))= [(^{3}+2)^2]

(^{6}+4 ^{3}+4)= (6+4 3+4)−3= $\sqrt{6+4\ 3+1}$

42.(∘ ∘)()= ((()))= (($\sqrt[3]{}$))= $\frac{\sqrt[3]{}}{\sqrt[3]{}-1}$ = tan $\frac{\sqrt[3]{}}{\sqrt[3]{}-1}$

Let ( ) = 2 + ^{2} and ( ) = ^{4}. Then (∘)( ) = (( )) = (2 + ^{2}) = (2 + ^{2})^4 = ( ).

Let $(\ ) = \cos$ and $(\ ) = {}^2$. Then $(\ \circ\ )(\ ) = (\ (\ )) = (\ \cos\ ) = (\cos\ )^2 = \cos^2\ = (\ )$.

45. Let ( ) = √3 and ( ) = 1 + . Then (◦)( ) = (( )) = (√3) = √3 / (1+ √3) = ().

46. Let ( ) = 1 + and ( ) = √ . Then (◦)( ) = (( )) = 3 1 + = 3 1+ = ().

Let ( ) = ^{2} and ( ) = sec tan . Then (◦)( ) = (( )) = (^{2}) = sec(^{2}) tan(^{2}) = ( ).

48. Let ( ) = tan and ( ) = 1 + . Then (◦)( ) = (( )) = (tan) = tan / (1 + tan) = ().

Let ( ) = √ , ( ) = − 1, and ( ) = √ . Then

(◦ ◦)()= ((()))= ((√))= (√ −1)= √ −1= ().

50. Let ( ) = | |, ( ) = 2 + , and ( ) = √8 . Then

(◦ ◦)()= ((()))= ((||))= (2+||)=8 2+||= ().

Let ( ) = cos , ( ) = sin , and ( ) = ^{2}. Then

(◦ ◦)( ) = ((( ))) = ((cos)) = (sin(cos)) = [sin (cos)]^2 = sin^2(cos) = ( ).

52. (a) ((1)) = (6) = 5 (b) ((1)) = (3) = 2

(c) ((1)) = (3) = 4 (d) ((1)) = (6) = 3

(e) (◦)(3) = ((3)) = (4) = 1 (f) (◦)(6) = ((6)) = (3) = 4

(a) (2) = 5, because the point (2 5) is on the graph of . Thus, ((2)) = (5) = 4, because the point (5 4) is on the graph of .

((0))= (0)=3

(◦)(0)= ((0))= (3)=0

(◦)(6) = ((6)) = (6). This value is not defined, because there is no point on the graph of that has -coordinate 6.

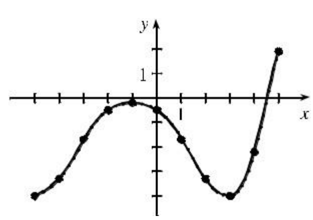
(◦)(−2) = ((−2)) = (1) = 4

(f) (◦)(4) = ((4)) = (2) = −2

To find a particular value of (( )), say for = 0, we note from the graph that (0) ≈ 2 8 and (2 8) ≈ −0 5. Thus, ((0)) ≈ (2 8) ≈ −0 5. The other values listed in the table were obtained in a similar fashion.

	( )	(())
−5	−0 2	−4
−4	1 2	−3 3
−3	2 2	−1 7
−2	2 8	−0 5
−1	3	−0 2

	( )	(())
0	2 8	−0 5
1	2 2	−1 7
2	1 2	−3 3
3	−0 2	−4
4	−1 9	−2 2
5	−4 1	1 9



(a) Using the relationship *distance* = *rate* · *time* with the radius as the distance, we have ( ) = 60 .

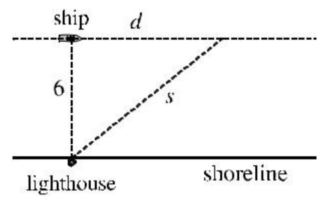
(b) = ^{2} ⇒ (◦)( ) = (( )) = (60)^2 = 3600 ^{2}. This formula gives us the extent of the rippled area (in cm^2) at any time .

(a) The radius of the balloon is increasing at a rate of 2 cm s, so ( ) = (2 cm s)(s) = 2 (in cm).

Using = $\frac{4}{3}$ ^{3}, we get (◦)( ) = (( )) = (2) = $\frac{4}{3}$ (2)^3 = $\frac{32}{3}$ ^{3}.

The result, = $\frac{32}{3}$ ^{3}, gives the volume of the balloon (in cm^3) as a function of time (in s).

57. (a) From the figure, we have a right triangle with legs 6 and , and hypotenuse . √



(b) Using = , we get = (30 km h)(hours) = 30 (in km). Thus, = ()=30.

(c) (◦)( ) = (( )) = (30) = (30)$_2$ + 36 = √900 $_2$ + 36. This function represents the distance between the lighthouse and the ship as a function of the time elapsed since noon.

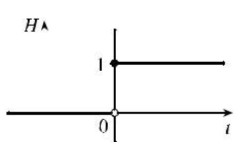
58. (a) = ⇒ ( ) = 350

There is a Pythagorean relationship involving the legs with lengths and 1 and the hypotenuse with length :

^{2} + 1^2 = ^{2}. Thus, ( ) = √ $_2$ + 1.

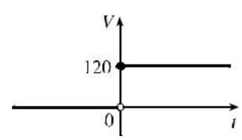
(◦)()= (())= (350)= (350)$_2$+1

59. (a)



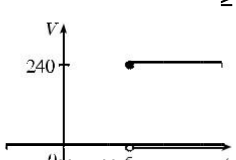
( ) = 0 if 0
1 if 0 ≥

(b)



()= 120 if 0 so ( ) = 120 ( ).
0 if0 ≥

(c)

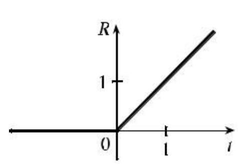


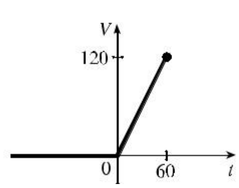
Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of = 0, we replace with − 5. Thus, the formula is ( ) = 240 (− 5).

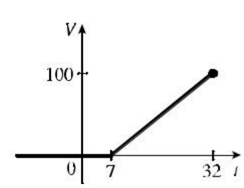
60. (a) ( ) =( )
= if0
0 if0 ≥

(b) ( ) = 2 if 0 ≤ ≤ 60
0 if0

(c) ( ) = 4(− 7) if 7 ≤ ≤ 32
0 if7







If ( ) = $_1$ + $_1$ and ( ) = $_2$ + $_2$, then

(∘)()=(())=($_2$+$_2$)= $_1$($_2$+$_2$)+$_1$= $_1$ $_2$+ $_1$$_2$+$_1$. So ∘ is a linear function with slope $_1$ $_2$.

If ( ) = 1 04 , then

(∘)( )= (( ))= (104)=104(104)=(104)^2 ,

(∘ ∘)( ) = ((∘)( )) = ((1 04)^2) = 1 04(1 04)^2 = (1 04)^3 , and

(∘ ∘ ∘)( )= ((∘ ∘)( ))= ((104)^3)=104(104)^3 =(104)^4 . These compositions represent the amount of the investment after 2, 3, and 4 years.

Based on this pattern, when we compose copies of , we get the formula (∘ ∘ · · · ∘)( ) = (1 04) .

’s

(a) By examining the variable terms in and , we deduce that we must square to get the terms 4 ^{2} and 4 in . If we let

( ) = ^{2} + , then (∘)( ) = (( )) = (2 + 1) = (2 + 1)^2 + = 4 ^{2} + 4 + (1 +). Since

( ) = 4 ^{2} + 4 + 7, we must have 1 + = 7. So = 6 and ( ) = ^{2} + 6.

We need a function so that (( )) = 3(( )) + 5 = ( ). But

( ) = 3 ^{2} + 3 + 2 = 3(^{2} +) + 2 = 3(^{2} + − 1) + 5, so we see that ( ) = ^{2} + − 1.

We need a function so that (( )) = (+ 4) = ( ) = 4 − 1 = 4(+ 4) − 17. So we see that the function must be

( )=4 −17.

We need to examine (−).

(−) = (∘)(−) = ((−)) = (( )) [because is even] = ( )

Because (−) = ( ), is an even function.

(−) = ((−)) = (− ( )). At this point, we can’t simplify the expression, so we might try to find a counterexample to show that is not an odd function. Let ( ) = , an odd function, and ( ) = ^{2} + . Then ( ) = ^{2} + which is neither even nor odd.

Now suppose is an odd function. Then (− ( )) = − (( )) = − ( ). Hence, (−) = − ( ), and so is odd if both and are odd.

Now suppose is an even function. Then (− ( )) = (( )) = ( ). Hence, (−) = ( ), and so is even if is odd and is even.

1.4 The Tangent and Velocity Problems

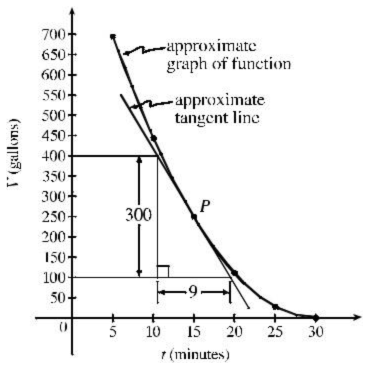
(a) Using (15 250), we construct the following table:

		slope =
5	(5 694)	$\frac{694-250}{5-15} = -\frac{444}{10} = -444$
10	(10 444)	$\frac{444-250}{10-15} = -\frac{194}{5} = -388$
20	(20 111)	$\frac{111-250}{20-15} = -\frac{139}{5} = -278$
25	(25 28)	$\frac{28-250}{25-15} = -\frac{222}{10} = -222$
30	(30 0)	$\frac{0-250}{30-15} = -\frac{250}{15} = -166$

Using the values of that correspond to the points

$\frac{-38\ 8 + (-27\ 8)}{2} = -33\ 3$

From the graph, we can estimate the slope of the tangent line at to be $\frac{-300}{9} = -33\ 3.$ −



(a) Slope = $\frac{2948-2530}{42-36} = \frac{418}{6} \approx 69\ 67$

Slope = $\frac{2948-2806}{42-40} = \frac{142}{2} = 71$

(b) Slope = $\frac{2948-2661}{42-38} = \frac{287}{4} = 71\ 75$

(d) Slope = $\frac{3080-2948}{44-42} = \frac{132}{2} = 66$

From the data, we see that the patient's heart rate is decreasing from 71 to 66 heartbeats minute after 42 minutes. After being stable for a while, the patient's heart rate is dropping.

3. (a) $= \frac{1}{1-}$, (2 −1)

		(1(1−))	
(i)	1 5	(1 5 −2)	2
(ii)	1 9	(1 9 −1 111 111)	1 111 111
(iii)	1 99	(1 99 −1 010 101)	1 010 101
(iv)	1 999	(1 999 −1 001 001)	1 001 001
(v)	2 5	(2 5 −0 666 667)	0 666 667
(vi)	2 1	(2 1 −0 909 091)	0 909 091
(vii)	2 01	(2 01 −0 990 099)	0 990 099
(viii)	2 001	(2 001 −0 999 001)	0 999 001

The slope appears to be 1.

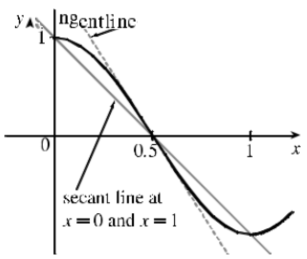
Using = 1, an equation of the tangent line to the curve at (2 −1) is − (−1) = 1(− 2), or = −3.

4. (a) = cos , (0 5 0)

(i)	0	(0 1)	−2
(ii)	0 4	(0 4 0 309017)	−3 090170
(iii)	0 49	(0 49 0 031411)	−3 141076
(iv)	0 499	(0 499 0 003142)	−3 141587
(v)	1	(1 −1)	−2
(vi)	0 6	(0 6 −0 309017)	−3 090170
(vii)	0 51	(0 51 −0 031411)	−3 141076
(viii)	0 501	(0 501 −0 003142)	−3 141587

The slope appears to be − .

− 0 = − (− 0 5) or = − + $\frac{1}{2}$.



(a) = ( ) = 40 − 16 ^{2}. At = 2, = 40(2) − 16(2)^2 = 16. The average velocity between times 2 and 2 + is

ave = $\frac{(2+\ ) - (2)}{}$ = $\frac{40(2+\ ) - 16(2+\ )^2 - 16}{}$ = $\frac{24 - 16\ ^2}{}$ = 24 16 , if = .0

(i) [2 2 5]: = 0 5, ave = −32 ft s (ii) [2 2 1]: = 0 1, ave = −25 6 ft s

(iii) [2 2 05]: = 0 05, ave = −24 8 ft s (iv) [2 2 01]: = 0 01, ave = −24 16 ft s

The instantaneous velocity when = 2 (approaches 0) is −24 ft s.

(a) = ( ) = 10 − 1 86 ^{2}. At = 1, = 10(1) − 1 86(1)^2 = 8 14. The average velocity between times 1 and 1 + is

ave = $\frac{(1+\ ) - (1)}{}$ = $\frac{10(1+\ ) - 186(1+\ )^2 - 814}{}$ = $\frac{628 - 186\ ^2}{}$ =628 1 86 , if = .0

(i) [1 2]: = 1, ave = 4 42 m s (ii) [1 1 5]: = 0 5, ave = 5 35 m s

(iii) [1 1 1]: = 0 1, ave = 6 094 m s (iv) [1 1 01]: = 0 01, ave = 6 2614 m s

[1 1 001]: = 0 001, ave = 6 27814 m s

The instantaneous velocity when = 1 (approaches 0) is 6 28 m s.

(a) (i) On the interval [2 4] , ave = $\frac{(4) - (2)}{4-2}$ = $\frac{79\ 2 - 20\ 6}{2}$ = 29 3 ft s.

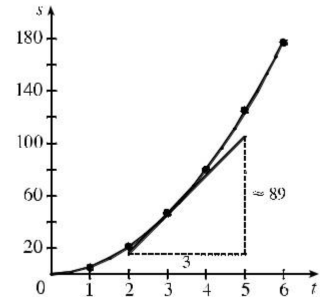
(ii) On the interval [3 4], ave = $\frac{(4) - (3)}{4-3}$ = $\frac{792-465}{1}$ = 32 7 ft s.

(iii) On the interval [4 5], ave = $\frac{(5) - (4)}{5-4}$ = $\frac{1248-792}{1}$ = 45 6 ft s.

(iv) On the interval [4 6], ave = $\frac{(6) - (4)}{6-4}$ = $\frac{1767-792}{2}$ = 48 75 ft s.

(b) Using the points $(2\ 16)$ and $(5\ 105)$ from the approximate tangent line, the instantaneous velocity at $= 3$ is about

$$\frac{105 - 16}{5 - 2} = \frac{89}{3} \approx 29\ 7 \text{ ft s.}$$



8. (a) (i) $= (\ ) = 2 \sin\ + 3 \cos$. On the interval $[1\ 2]$, $_{\text{ave}} = \dfrac{(2) - (1)}{2 - 1} = \dfrac{3 - (-3)}{1} = 6$ cm s.

(ii) On the interval $[1\ 1\ 1]$, $_{\text{ave}} = \dfrac{(1\ 1) - (1)}{1\ 1 - 1} \approx \dfrac{-3\ 471 - (-3)}{0\ 1} = 4\ 71$ cm s.

(iii) On the interval $[1\ 1\ 01]$, $_{\text{ave}} = \dfrac{(1\ 01) - (1)}{1\ 01 - 1} \approx \dfrac{-3\ 0613 - (-3)}{0\ 01} = 6\ 13$ cm s.

(iv) On the interval $[1\ 1\ 001]$, $_{\text{ave}} = \dfrac{(1\ 001) - (1)}{1\ 001 - 1} \approx \dfrac{-3\ 00627 - (-3)}{0\ 001} = 6\ 27$ cm s.

The instantaneous velocity of the particle when $= 1$ appears to be about $-6\ 3$ cm s.

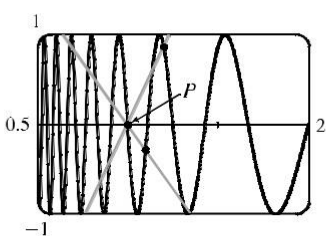
(a) For the curve $= \sin(10\ \ )$ and the point $(1\ 0)$:

2	(2 0)	0
1 5	(1 5 0 8660)	1 7321
1 4	(1 4 −0 4339)	−1 0847
1 3	(1 3 −0 8230)	−2 7433
1 2	(1 2 0 8660)	4 3301
1 1	(1 1 −0 2817)	−2 8173

0 5	(05 0)	0
0 6	(0 6 0 8660)	−2 1651
0 7	(0 7 0 7818)	−2 6061
0 8	(08 1)	−5
0 9	(0 9 −0 3420)	3 4202

As approaches 1, the slopes do not appear to be approaching any particular value.

(b)



We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at that we need to take -values much closer to 1 in order to get accurate estimates of its slope.

If we choose $= 1\ 001$, then the point is $(1\ 001\ -0\ 0314)$ and $\approx -31\ 3794$. If $= 0\ 999$, then is $(0\ 999\ 0\ 0314)$ and $= -31\ 4422$. The average of these slopes is $-31\ 4108$. So we estimate that the slope of the tangent line at is about $-31\ 4$.

1.5 The Limit of a Function

As approaches 2, ( ) approaches 5. [Or, the values of ( ) can be made as close to 5 as we like by taking sufficiently close to 2 (but 6=)2.] Yes, the graph could have a hole at (2 5) and be defined such that (2) = 3.

As approaches 1 from the left, ( ) approaches 3; and as approaches 1 from the right, ( ) approaches 7. No, the limit does not exist because the left- and right-hand limits are different.

3. (a) $\lim_{\to -3}$ ( ) = ∞ means that the values of ( ) can be made arbitrarily large (as large as we please) by taking sufficiently close to −3 (but not equal to −3).

(b) $\lim_{\to 4^+}$ ( ) = −∞ means that the values of ( ) can be made arbitrarily large negative by taking sufficiently close to 4 through values larger than 4.

(a) As approaches 2 from the left, the values of ( ) approach 3, so $\lim_{\to 2^-}$ ( ) = 3.

(b) As approaches 2 from the right, the values of ( ) approach 1, so $\lim_{\to 2^+}$ ( ) = 1.

(c) $\lim_{\to 2}$ ( ) does not exist since the left-hand limit does not equal the right-hand limit.

(d) When = 2, = 3, so (2) = 3.

(e) As approaches 4, the values of ( ) approach 4, so $\lim_{\to 4}$ ( ) = 4.

(f) There is no value of ( ) when = 4, so (4) does not exist.

5. (a) As approaches 1, the values of ( ) approach 2, so $\lim_{\to 1}$ ( ) = 2.

(b) As approaches 3 from the left, the values of ( ) approach 1, so $\lim_{\to 3^-}$ ( ) = 1.

(c) As approaches 3 from the right, the values of ( ) approach 4, so $\lim_{\to 3^+}$ ( ) = 4.

(d) $\lim_{\to 3}$ ( ) does not exist since the left-hand limit does not equal the right-hand limit.

(e) When = 3, = 3, so (3) = 3.

6. (a) ( ) approaches 4 as approaches −3 from the left, so $\lim_{\to -3^-}$ ( ) = 4.

(b) ( ) approaches 4 as approaches −3 from the right, so $\lim_{\to -3^+}$ ( ) = 4.

(c) $\lim_{\to -3}$ ( ) = 4 because the limits in part (a) and part (b) are equal.

(−3) is not defined, so it doesn't exist.

( ) approaches 1 as approaches 0 from the left, so $\lim_{\to 0^-}$ ( ) = 1.

(f) ( ) approaches −1 as approaches 0 from the right, so $\lim_{\to 0^+}$ ( ) = −1.

(g) $\lim_{\to 0}$ ( ) does not exist because the limits in part (e) and part (f) are not equal.

(h) (0) = 1 since the point (0 1) is on the graph of .

Since $\lim_{\to 2^+} (\ ) = 2$ and $\lim_{\to 2} (\ ) = 2$, we have $\lim_{\to 2} (\ ) = 2$.

(2) is not defined, so it doesn't exist.

$(\ )$ approaches 3 as approaches 5 from the right, so $\lim_{\to 5^+} (\ ) = 3$.

(l) $(\ )$ does not approach any one number as approaches 5 from the left, so $\lim_{\to 5^-} (\ )$ does not exist.

7. (a) $\lim_{\to 0^-} (\ ) = 1$ (b) $\lim_{\to 0^+} (\ ) = 2$ $-$ $-$

(c) $\lim_{\to 0} (\ )$ does not exist because the limits in part (a) and part (b) are not equal.

(d) $\lim_{\to 2^-} (\ ) = 2$ (e) $\lim_{\to 2^+} (\ ) = 0$

(f) $\lim_{\to 2} (\ )$ does not exist because the limits in part (d) and part (e) are not equal.

(g) $(2) = 1$ (h) $\lim_{\to 4} (\ ) = 3$

lim () lim () =

8. (a) $\lim_{\to -3} (\ ) = \infty$ (b) $_{\to 2}$ does not exist. (c) $_{\to 2^-}$ $-\infty$

(d) $\lim_{\to 2^+} (\ ) = \infty$ (e) $\lim_{\to -1} (\ ) = -\infty$

(f) The equations of the vertical asymptotes are $= -3$, $= -1$ and $= 2$.

9. (a) $\lim_{\to -7} (\ ) = -\infty$ (b) $\lim_{\to -3} (\ ) = \infty$ (c) $\lim_{\to 0} (\ ) = \infty$

(d) $\lim_{\to 6^-} (\ ) = -\infty$ (e) $\lim_{\to 6^+} (\ ) = \infty$

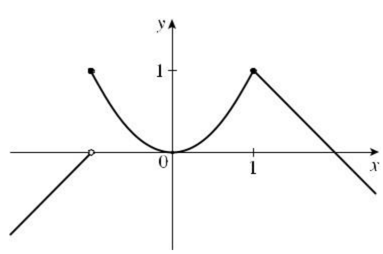
(f) The equations of the vertical asymptotes are $= -7$, $= -3$, $= 0$, and $= 6$.

10. $\lim_{\to 12^-} (\ ) = 150$ mg and $\lim_{\to 12^+} (\ ) = 300$ mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $= 12$ h. The left-hand limit represents the amount of the drug just before the fourth injection. The right-hand limit represents the amount of the drug just after the fourth injection.

11. From the graph of $(\ ) = $ 2 if 1 $-$ 1 ,

$1 +$ if 1

$2 -$ if $\geq$ 1

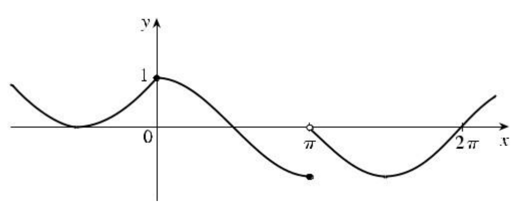


we see that $\lim_{\to} (\ )$ exists for all except $= -1$. Notice that the right and left limits are different at $= -1$.

12. From the graph of $(\ ) =$ cos if 0 ,

$1 + \sin$ if 0 $\leq$

sin if

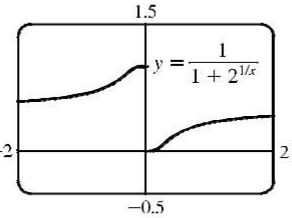


we see that $\lim_{\to} (\ )$ exists for all except $=$. Notice that the right and left limits are different at $=$.

(a) lim− ( ) = 1→0

lim+ ( ) = 0→0

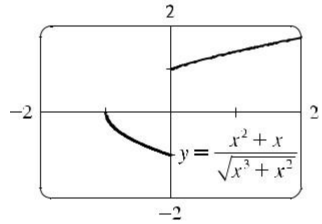
(c) lim ( ) does not exist because the limits in
→0
part (a) and part (b) are not equal.



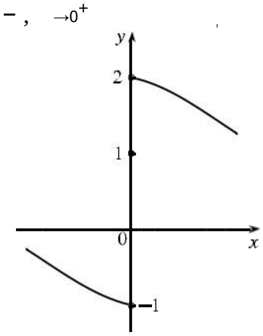
(a) lim− ( ) = −1→0

lim+ ( ) = 1→0

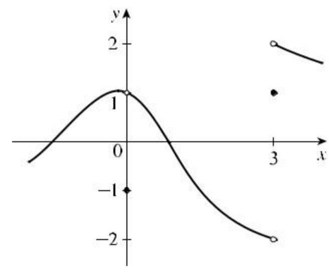
(c) lim ( ) does not exist because the limits
→0
in part (a) and part (b) are not equal.



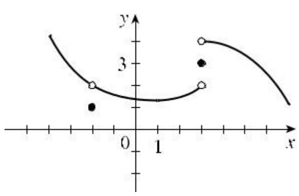
15. lim ( ) = 1 − , lim ( ) = 2 (0)=1
→0− →0+



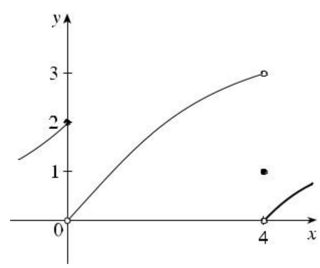
16. lim ( ) = 1, lim ( ) = −2, lim ( ) = 2,
→0 →3− →3+
(0) = −1, (3) = 1



lim ( ) = 4, lim ( ) = 2, lim ( ) = 2,
→3−→−2→3
(3) = 3, (−2) = 1



lim ( ) = 2, lim ( ) = 0, lim ( ) = 3,
→0+→4−→0
lim ( ) = 0, (0) = 2, (4) = 1
→4+



For ( ) = $\frac{2 - 3}{2 - 9}$:

	( )
3 1	0 508 197
3 05	0 504 132
3 01	0 500 832
3 001	0 500 083
3 0001	0 500 008

	( )
2 9	0 491 525
2 95	0 495 798
2 99	0 499 165
2 999	0 499 917
2 9999	0 499 992

It appears that lim $\frac{2 - 3}{2 - 9} = \frac{1}{2}$.
→3

For $( ) = \frac{2 - 3}{2 - 9}$:

	( )
−2 5	−5
−2 9	−29
−2 95	−59
−2 99	−299
−2 999	−2999
−2 9999	−29,999

	( )
−3 5	7
−3 1	31
−3 05	61
−3 01	301
−3 001	3001
−3 0001	30,001

It appears that $\lim_{\to -3^+} ( ) = -\infty$ and that $\lim_{\to -3^-} ( ) = \infty$, so $\lim_{\to -3} \frac{2 - 3}{2 - 9}$ does not exist.

21. For $( ) = \frac{\sin}{+ \tan}$:

	( )
±1	0 329 033
±0 5	0 458 209
±0 2	0 493 331
±0 1	0 498 333
±0 05	0 499 583
±0 01	0 499 983

It appears that $\lim_{\to 0} \frac{\sin}{+ \tan} = 0 5 = \frac{1}{2}$.

23. For $( ) = \frac{\sin 3}{\tan 2}$:

For $( ) = \frac{(2 +)^5 - 32}{}$:

	( )
0 5	131 312 500

	( )
−0 5	48 812 500

ars that lim $\frac{(2+}{}$ $)^5 - 32$ $= 80$.

0 0001	80 008 000

0 1	88 410 100
0 01	80 804 010
0 001	80 080 040

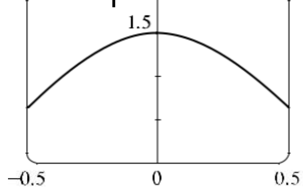
−0 0001	79 992 000

−0 1	72 390 100
−0 01	79 203 990
−0 00	79 920 040

It appears that $\lim_{\to 0} \frac{\sin 3}{\tan 2} = 1\ 5$.

The graph confirms that result.

2

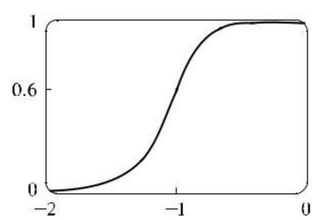


	( )
±0 1	1 457 847
±0 01	1 499 575
±0 001	1 499 996
±0 0001	1 500 000

For ( ) = $\frac{1 + 9}{1 + 15}$

	( )
−1 1	0 427 397
−1 01	0 582 008
−1 001	0 598 200
−1 0001	0 599 820

	( )
−0 9	0 771 405
−0 99	0 617 992
−0 999	0 601 800
−0 9999	0 600 180



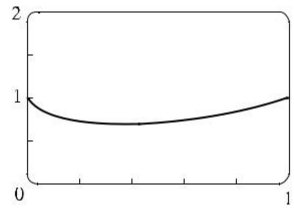
It appears that $\lim_{\to -1}$ ( ) = 0 6. The graph confirms that result.

For ( ) = :

	( )
0 1	0 794 328
0 01	0 954 993
0 001	0 993 116
0 0001	0 999 079

It appears that $\lim_{\to 0^+}$ ( ) = 1.

The graph confirms that result.



For ( ) = 5 − 1 :

	( )
0 1	1 746 189

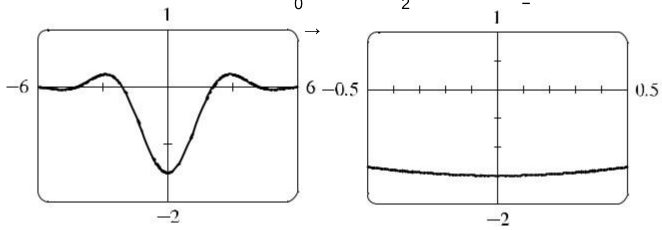
	( )
−0 1	1 486 601

0 01	1 622 459	−0 01	1 596 556
0 001	1 610 734	−0 001	1 608 143
0 0001	1 609 567	−0 0001	1 609 308

lim ( ) 1 6094

It appears that $\lim_{\to 0} \approx$. The graph confirms that result.

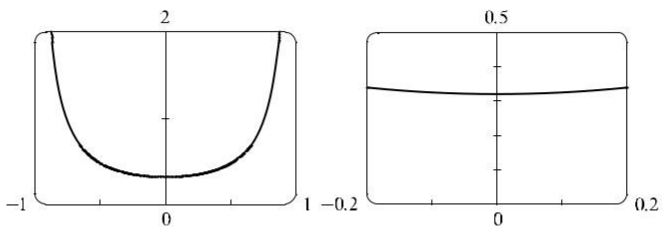
27. (a) From the graphs, it seems that $\lim_{\to 0} \frac{\cos 2 - \cos}{2} = -1\ 5$.



(b)

	( )
±0 1	−1 493 759
±0 01	−1 499 938
±0 001	−1 499 999
±0 0001	−1 500 000

28. (a) From the graphs, it seems that $\lim_{\to 0} \frac{\sin}{\sin} = 0\ 32$.



(b)

	( )
±0 1	0 323 068
±0 01	0 318 357
±0 001	0 318 310
±0 0001	0 318 310

Later we will be able to show that the exact value is $\frac{1}{}$.

29. $\lim_{\to 5^+} \frac{+1}{-5} = \infty$ since the numerator is positive and the denominator approaches 0 from the positive side as $\to 5^+$.

30. $\lim_{\to 5^-} \frac{+1}{-5} = -\infty$ since the numerator is positive and the denominator approaches 0 from the negative side as $\to 5^-$.

31. $\lim_{\to 1} \frac{2 -}{(-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $\to 1$.

32. $\lim_{\to 3^-} \frac{}{(-3)^5} = -\infty$ since the numerator is positive and the denominator approaches 0 from the negative side as $\to 3^-$.

33. $\lim_{\to -2^+} \frac{-1}{2(+2)} = -\infty$ since $(\ +2) \to 0$ as $\to -2^+$ and $\frac{-1}{2(+2)}$ 0 for 20.

34. $\lim_{\to 0} \frac{-1}{2(+2)} = -\infty$ since $^2 \to 0$ as $\to 0$ and $\frac{-1}{2(+2)}$ 0 for 01 and for 20.

35. $\lim_{\to} \frac{1}{}$ $\frac{1}{}$ $^+$

$\lim \cot = \lim \frac{\cos}{} = $ 0

36. $\lim_{\to^-} \ \lim_{\to^-} \sin$ $-\infty$ since the numerator is negative and the denominator approaches through positive values as $\to ^-$.

$\lim \csc = \lim \frac{}{} = $ 0

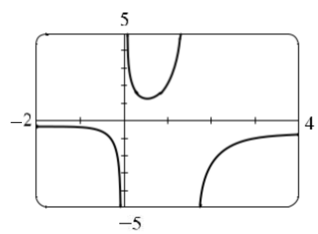
37. $\lim_{\to 2^-} \ \lim_{\to 2^-} \sin$ $-\infty$ since the numerator is positive and the denominator approaches through negative values as $\to 2^-$.

38. $\lim_{\to 2^-} \frac{^2 - 2}{^2 - 4 + 4} = \lim_{\to 2^-} \frac{(\ -2)}{(\ -2)^2} = \lim_{\to 2^-} \frac{}{-2} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $\to 2^-$.

39. $\lim_{\to 2^+} \frac{^2 - 2 - 8}{^2 - 5 + 6} = \lim_{\to 2^+} \frac{(\ -4)(\ +2)}{(\ -3)(\ -2)} = \infty$ since the numerator is negative and the denominator approaches 0 through negative values as $\to 2^+$.

40. (a) The denominator of $= \frac{^2 + 1}{3 - 2} = \frac{^2 + 1}{(3 - 2\ )}$ is equal to zero when $= 0$ and $= \frac{3}{2}$ (and the numerator is not), so $= 0$ and $= 1\ 5$ are vertical asymptotes of the function.

(b)



41. (a) $(\ ) = \frac{}{^3 - 1}$

	( )
0 5	−1 14
0 9	−3 69
0 99	−33 7
0 999	−333 7
0 9999	−3333 7
0 99999	−33,333 7

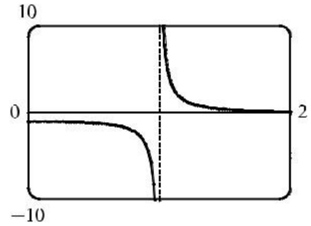
	( )
1 5	0 42
1 1	3 02
1 01	33 0
1 001	333 0
1 0001	3333 0
1 00001	33,333 3

From these calculations, it seems that $\lim_{\to 1^-} (\ ) = -\infty$ and $\lim_{\to 1^+} (\ ) = \infty$.

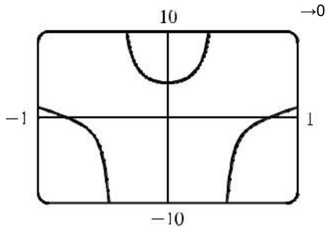
(b) If is slightly smaller than 1, then $^3 - 1$ will be a negative number close to 0, and the reciprocal of $^3 - 1$, that is, ( ), will be a negative number with large absolute value. So $\lim_{\to 1^-} (\ ) = -\infty$.

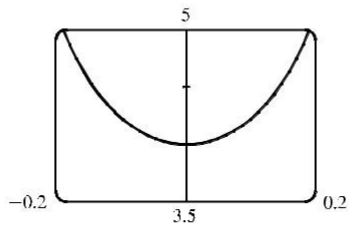
If is slightly larger than 1, then $^3 - 1$ will be a small positive number, and its reciprocal, ( ), will be a large positive number. So $\lim_{\to 1^+} (\ ) = \infty$.

(c) It appears from the graph of that $\lim_{\to 1^-} (\,) = -\infty$ and $\lim_{\to 1^+} (\,) = \infty$.



42. (a) From the graphs, it seems that $\lim_{\to 0} \frac{\tan 4}{} = 4$.





(b)

	( )
±0 1	4 227 932
±0 01	4 002 135
±0 001	4 000 021
±0 0001	4 000 000

For ( ) = ^{2} − (2 1000):

(a)

	( )
1	0 998 000
0 8	0 638 259
0 6	0 358 484
0 4	0 158 680
0 2	0 038 851
0 1	0 008 928
0 05	0 001 465

It appears that $\lim_{\to 0} (\,) = 0$.

(b)

	( )
0 04	0 000 572
0 02	−0 000 614
0 01	−0 000 907
0 005	−0 000 978
0 003	−0 000 993
0 001	−0 001 000

It appears that $\lim_{\to 0} (\,) = -0\,001$.

For ( ) = $\frac{\tan -}{3}$:

(a)

	( )
1 0	0 557 407 73
0 5	0 370 419 92
0 1	0 334 672 09
0 05	0 333 667 00
0 01	0 333 346 67
0 005	0 333 336 67

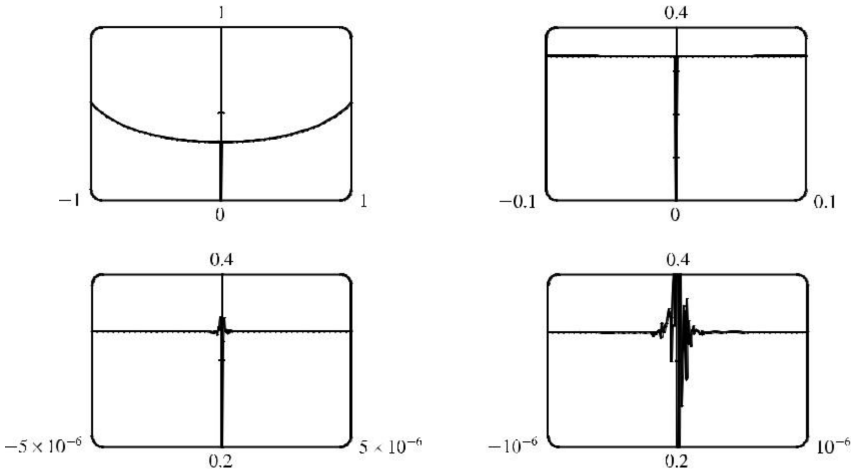
(b) It seems that $\lim_{\to 0} (\,) = \frac{1}{3}$.

(c)

	( )
0 001	0 333 333 50
0 0005	0 333 333 44
0 0001	0 333 330 00
0 00005	0 333 336 00
0 00001	0 333 000 00
0 000001	0 000 000 00

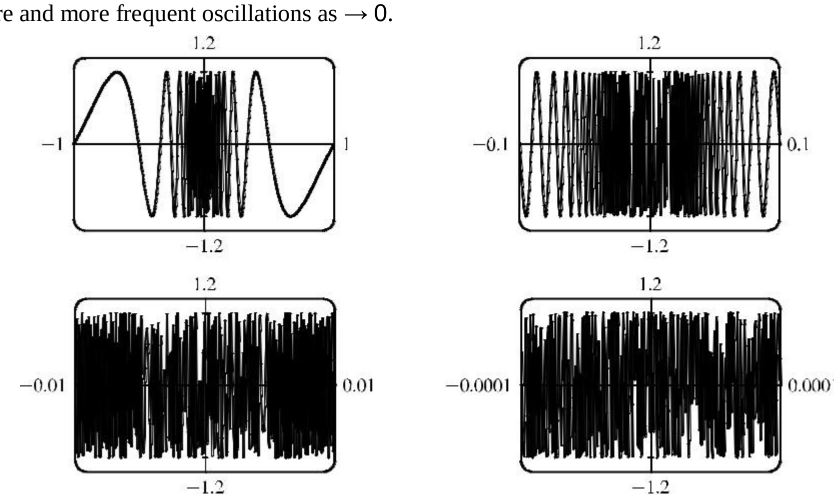
Here the values will vary from one calculator to another. Every calculator will eventually give *false values*.

(d) As in part (c), when we take a small enough viewing rectangle we get incorrect output.



No matter how many times we zoom in toward the origin, the graphs of ( ) = sin( ) appear to consist of almost-vertical lines.

This indicates more and more frequent oscillations as $\rightarrow 0$.

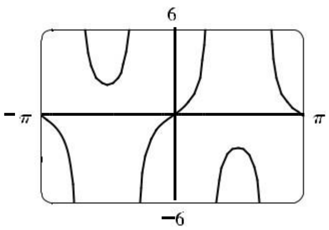


(a) For any positive integer , if = $\frac{1}{\ }$, then ( ) = tan $\frac{1}{\ }$ = tan( ) = 0. (Remember that the tangent function has period .)

(b) For any nonnegative number , if = $\frac{4}{(4 + 1)}$, then

$$( ) = \tan\frac{1}{\ } = \tan\frac{(4 \ + 1)}{4} = \tan\frac{4}{4} + \frac{}{4} = \tan + \frac{}{4} = \tan\frac{}{4} = 1$$

(c) From part (a), ( ) = 0 infinitely often as $\rightarrow 0$. From part (b), ( ) = 1 infinitely often as $\rightarrow 0$. Thus, $\lim_{\rightarrow 0} \tan\frac{1}{\ }$ does not exist since ( ) does not get close to a fixed number as $\rightarrow 0$.

47. 

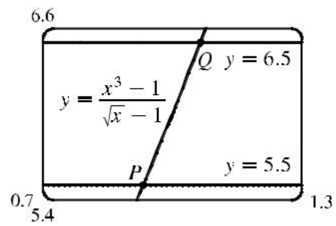
There appear to be vertical asymptotes of the curve $= \tan(2 \sin\ )$ at $\approx \pm 0\ 90$ and $\approx \pm 2\ 24$. To find the exact equations of these asymptotes, we note that the graph of the tangent function has vertical asymptotes at $= \frac{}{2} +$. Thus, we must have $2 \sin = \frac{}{2} +$, or equivalently, $\sin = \frac{}{4} + \frac{}{2}$. Since $-1 \le \sin \le 1$, we must have $\sin = \pm \frac{}{4}$ and so $= \pm \sin^{-1} \frac{}{4}$ (corresponding to $\approx \pm 0\ 90$). Just as 150° is the reference angle for 30°, $- \sin^{-1} \frac{}{4}$ is the reference angle for $\sin^{-1} \frac{}{4}$. So $= \pm \ - \sin^{-1} \frac{}{4}$ are also equations of vertical asymptotes (corresponding to $\approx \pm 2\ 24$).

48. $\to^-$ $\to^-$ $\frac{0}{}$. As $\to^-$, $\frac{}{2\ 2} \to 0^+$, and $\to \infty$.

49. (a) Let $= \frac{^3 - 1}{\sqrt{} - 1}$.

From the table and the graph, we guess that the limit of as approaches 1 is 6.

0 99	5 925 31
0 999	5 992 50
0 9999	5 999 25
1 01	6 075 31
1 001	6 007 50
1 0001	6 000 75



(b) We need to have $5\ 5 \quad \frac{^3 - 1}{\sqrt{} - 1} \quad 6\ 5$. From the graph we obtain the approximate points of intersection $(0\ 9314\ 5\ 5)$ and $(1\ 0649\ 6\ 5)$. Now $1 - 0\ 9314 = 0\ 0686$ and $1\ 0649 - 1 = 0\ 0649$, so by requiring that be within $0\ 0649$ of 1, we ensure that is within $0\ 5$ of 6.

1.6 Calculating Limits Using the Limit Laws

1. (a) $\lim_{\to 2} [\ (\ ) + 5\ (\ )] = \lim_{\to 2}\ (\ ) + \lim_{\to 2} [5\ (\ )]$ [Limit Law 1]

$= \lim_{\to 2}\ (\ ) + 5 \lim_{\to 2}\ (\ )$ [Limit Law 3]

$= 4 + 5(-2) = -6$

(b) $\lim_{\to 2} [\ (\ )]^3 = \left[\lim_{\to 2}\ (\ )\right]^3$ [Limit Law 6]

$= (-2)^3 = -8$

(c) $\lim_{\to 2} \sqrt{(\ )} = \sqrt{\lim_{\to 2}\ (\ )}$ [Limit Law 11]

$= \sqrt{4} = 2$

(d) $\lim_{\to 2} \frac{3\ (\ )}{(\ )} = \frac{\lim_{\to 2} [3\ (\ )]}{\lim_{\to 2}\ (\ )}$ [Limit Law 5]

$= \frac{3 \lim_{\to 2}\ (\ )}{\lim_{\to 2}\ (\ )}$

$= \frac{3(4)}{-2} = -6$

Because the limit of the denominator is 0, we can't use Limit Law 5. The given limit, $\lim_{\to 2} \frac{(\ )}{(\ )}$, does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

(f) $\lim_{\to 2} \frac{(\ )(\ )}{(\ )} = \frac{\lim_{\to 2} [(\ )\ (\ )]}{\lim_{\to 2} (\ )}$ [Limit Law 5]

$= \frac{\lim_{\to 2} (\ ) \cdot \lim_{\to 2} (\ )}{\lim_{\to 2} (\ )}$ [Limit Law 4]

$= \frac{-2 \cdot 0}{4} = 0$

(a) $\lim_{\to 2} [(\ ) + (\ )] = \lim_{\to 2} (\ ) + \lim_{\to 2} (\ )$ [Limit Law 1]

$= -1 + 2$

$= 1$

(b) $\lim_{\to 0} (\ )$ exists, but $\lim_{\to 0} (\ )$ does not exist, so we cannot apply Limit Law 2 to $\lim_{\to 0} [(\ ) - (\ )]$. The limit does not exist.

(c) $\lim_{\to 1^-} [(\ )\ (\ )] = \lim_{\to 1^-} (\ ) \cdot \lim_{\to 1^-} (\ )$ [Limit Law 4]

$= 1 \cdot 2$

$= 2$

(d) $\lim_{\to 3} (\ ) = 1$, but $\lim_{\to 3} (\ ) = 0$, so we cannot apply Limit Law 5 to $\lim_{\to 3} \frac{(\ )}{(\ )}$. The limit does not exist.

$\lim \frac{(\ )}{(\ )}$ + 3^- $\lim \frac{(\ )}{(\ )}$ – 3^+

Note: $\lim_{\to 3^-} (\ ) = \infty$ since $(\ ) \to 0$ as $\to$ and $\lim_{\to 3^+} (\ ) = -\infty$ since $(\ ) \to 0$ as $\to$.

Therefore, the limit does not exist, even as an infinite limit.

(e) $\lim_{\to 2} {}^2 (\ ) = \lim_{\to 2} {}^2 \cdot \lim_{\to 2} (\ )$ [Limit Law 4]

$= 2^2 \cdot (-1)$

$= -4$

(f) $(-1) + \lim_{\to 1^-} (\ )$ is undefined since (-1) is not defined.

3. $\lim_{\to 3} (5^3 - 3^2 + - 6) = \lim_{\to 3} (5^3) - \lim_{\to 3} (3^2) + \lim_{\to 3} - \lim_{\to 3} 6$ [Limit Laws 2 and 1]

$= 5 \lim_{\to 3} {}^3 - 3 \lim_{\to 3} {}^2 + \lim_{\to 3} - \lim_{\to 3} 6$ [3]

$= 5(3^3) - 3(3^2) + 3 - 6$ [9, 8, and 7]

$= 105$

4. $\lim_{\to-1}(\ ^4 - 3\ )(\ ^2 + 5\ + 3) = \lim_{\to-1}(\ ^4 - 3\ )\lim_{\to-1}(\ ^2 + 5\ + 3)$ [Limit Law 4]

$= \left(\lim_{\to-1}\ ^4 - \lim_{\to-1} 3\ \right)\left(\lim_{\to-1}\ ^2 + \lim_{\to-1} 5\ + \lim_{\to-1} 3\right)$ [2, 1]

$= \left(\lim_{\to-1}\ ^4 - 3\lim_{\to-1}\ \right)\left(\lim_{\to-1}\ ^2 + 5\lim_{\to-1}\ + \lim_{\to-1} 3\right)$ [3]

$= (1 + 3)(1 - 5 + 3)$ [9, 8, and 7]

$= 4(-1) = -4$

5. $\lim_{\to-2} \dfrac{\ ^4 - 2}{2\ ^2 - 3\ + 2} = \dfrac{\lim_{\to-2}(\ ^4 - 2)}{\lim_{\to-2}(2\ ^2 - 3\ + 2)}$ [Limit Law 5]

$= \dfrac{\lim_{\to-2}\ ^4 - \lim_{\to-2} 2}{2\lim_{\to-2}\ ^2 - 3\lim_{\to-2}\ + \lim_{\to-2} 2}$ [1, 2, and 3]

$= \dfrac{16 - 2}{2(4) - 3(-2) + 2}$ [9, 7, and 8]

$= \dfrac{14}{16} = \dfrac{7}{8}$

6. $\lim_{\to-2}\sqrt{\ ^4 + 3\ + 6} = \sqrt{\lim_{\to-2}(\ ^4 + 3\ + 6)}$ [11]

$= \sqrt{\lim_{\to-2}\ ^4 + 3\lim_{\to-2}\ + \lim_{\to-2} 6}$ [1, 2, and 3]

$= \sqrt{(-2)^4 + 3(-2) + 6}$ [9, 8, and 7]

$= \sqrt{16 - 6 + 6} = \sqrt{16} = 4$

7. $\lim_{\to 8}(1 + \sqrt[3]{\ })(2 - 6\ ^2 + \ ^3) = \lim_{\to 8}(1 + \sqrt[3]{\ }) \cdot \lim_{\to 8}(2 - 6\ ^2 + \ ^3)$ [Limit Law 4]

$= \left(\lim_{\to 8} 1 + \lim_{\to 8}\sqrt[3]{\ }\right) \cdot \left(\lim_{\to 8} 2 - 6\lim_{\to 8}\ ^2 + \lim_{\to 8}\ ^3\right)$ [1, 2, and 3]

$= \left(1 + \sqrt[3]{8}\right) \cdot \left(2 - 6 \cdot 8^2 + 8^3\right)$ [7, 10, 9]

$= (3)(130) = 390$

8. $\lim_{\to 2}\left(\dfrac{\ ^2 - 2}{\ ^3 - 3\ + 5}\right)^2 = \left(\lim_{\to 2}\dfrac{\ ^2 - 2}{\ ^3 - 3\ + 5}\right)^2$ [Limit Law 6]

$= \left(\dfrac{\lim_{\to 2}(\ ^2 - 2)}{\lim_{\to 2}(\ ^3 - 3\ + 5)}\right)^2$ [5]

$= \left(\dfrac{\lim_{\to 2}\ ^2 - \lim_{\to 2} 2}{\lim_{\to 2}\ ^3 - 3\lim_{\to 2}\ + \lim_{\to 2} 5}\right)^2$ [1, 2, and 3]

$\to 2$ $\quad 8-3(2)+5$ $\quad -$ $\quad \to 2$ $\quad \to 2$

2

$= \quad \dfrac{}{4-2}$ [9, 7, and 8]

$2^2 = 4$

749

$$\lim_{\to 2} \sqrt{\frac{2^2+1}{3-2}} = \sqrt{\lim_{\to 2} \frac{2^2+1}{3-2}}$$ [Limit Law 11]

$$= \sqrt{\frac{\lim_{\to 2}(2^2+1)}{\lim_{\to 2}(3-2)}}$$ [5]

$$= \sqrt{\frac{2\lim_{\to 2}{}^2 + \lim_{\to 2} 1}{3\lim_{\to 2} - \lim_{\to 2} 2}}$$ [1, 2, and 3]

$$= \sqrt{\frac{2(2)^2+1}{3(2)-2}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$$ [9, 8, and 7]

(a) The left-hand side of the equation is not defined for $= 2$, but the right-hand side is.

Since the equation holds for all $\neq 2$, it follows that both sides of the equation approach the same limit as $\to 2$, just as in Example 3. Remember that in finding $\lim_{\to}(\ )$, we never consider $=$.

11. $\lim_{\to 5} \frac{^2-6+5}{-5} = \lim_{\to 5} \frac{(-5)(-1)}{-5} = \lim_{\to 5}(-1) = 5-1 = 4$

12. $\lim_{\to -3} \frac{^2+3}{^2--12} = \lim_{\to -3} \frac{(+3)}{(-4)(+3)} = \lim_{\to -3} \frac{}{-4} = \frac{-3}{-3-4} = \frac{3}{7}$

13. $\lim_{\to 5} \frac{^2-5+6}{-5}$ does not exist since $-5 \to 0$, but $^2-5+6 \to 6$ as $\to 5$.

14. $\lim_{\to 4} \frac{^2+3}{^2--12} = \lim_{\to 4} \frac{(+3)}{(-4)(+3)} = \lim_{\to 4} \frac{}{-4}$. The last limit does not exist since $\lim_{\to 4^-} \frac{}{-4} = -\infty$ and $\lim_{\to 4^+} \frac{}{-4} = \infty$.

15. $\lim_{\to -3} \frac{^2-9}{2^2+7+3} = \lim_{\to -3} \frac{(+3)(-3)}{(2+1)(+3)} = \lim_{\to -3} \frac{-3}{2+1} = \frac{-3-3}{2(-3)+1} = \frac{-6}{-5} = \frac{6}{5}$

$$\lim_{\to -1} \frac{2^2+3+1}{^2-2-3} = \lim_{\to -1} \frac{(2+1)(+1)}{(-3)(+1)} = \lim_{\to -1} \frac{2+1}{-3} = \frac{2(-1)+1}{-1-3} = \frac{-1}{-4} = \frac{1}{4}$$

17. $\lim_{\to 0} \frac{(-5+)^2-25}{} = \lim_{\to 0} \frac{(25-10+^2)-25}{} = \lim_{\to 0} \frac{-10+^2}{} = \lim_{\to 0} \frac{(-10+)}{} = \lim_{\to 0}(-10+) = -10$

18. $\lim_{\to 0} \frac{(2+)^3-8}{} = \lim_{\to 0} \frac{8+12+6^2+^3-8}{} = \lim_{\to 0} \frac{12+6^2+^3}{}$
$= \lim_{\to 0}(12+6+^2) = 12+0+0 = 12$

By the formula for the sum of cubes, we have

$$\lim_{\to -2} \frac{+2}{^3+8} = \lim_{\to -2} \frac{+2}{(+2)(^2-2+4)} = \lim_{\to -2} \frac{1}{^2-2+4} = \frac{1}{4+4+4} = \frac{1}{12}.$$

We use the difference of squares in the numerator and the difference of cubes in the denominator.

lim 4 − 1 / 3 − 1 = lim (2−1)(2+1) / (−1)(2+ +1) = lim (−1)(+1)(2+1) / (−1)(2+ +1) = lim (+1)(2+1) / 2+ +1 = 2(2) / 3 = 4 / 3

→1 →1 →1 →1

21. lim √ 9 + − 3 = lim √ 9 + − 3 · √ 9 + + 3 / √ 9 + + 3 = lim (√ 9 +)2 − 32 / (√ 9+ +3) = lim (9 +) − 9 / (√ 9+ +3)

0 0 0 0

= lim / (√ 9+ +3) = lim 1 / √ 9+ +3 = 1 / 3 + 3 = 1 / 6

→0 →0

22. lim √ 4 + 1 − 3 / − 2 = lim √ 4 + 1 − 3 / − 2 · √ 4 + 1 + 3 / √ 4 + 1 + 3 = lim (√ 4 + 1)2 − 32 / (− 2)(√ 4 +1+3)

→2 →2 →2

= lim 4 + 1 − 9 / (− 2)(√ 4 +1+3) = lim 4(− 2) / (− 2)(√ 4 +1+3)

→2 →2

= lim 4 / √ 4 +1+3 = 4 / 6 = 2/3

→2

23. lim (1/ − 1/3) / (− 3) = lim (1/ − 1/3) / (− 3) · 3 / 3 = lim 3 − / 3(− 3) = lim −1 / 3 = − 1/9

→3 →3 →3 →3

24. lim (3 +)−1 − 3−1 / = lim (1/(3 +) − 1/3) / = lim (3−(3+) / (3+)3) / = lim − / (3+)3

→0 →0 →0 →0

= lim − 1 / 3(3+) = −1 / lim [3(3 +)] = −1 / 3(3 + 0) = − 1/9

→0 →0

25. lim √ 1 + − √ 1 − / = lim (√ 1 + − √ 1 −) / · (√ 1 + + √ 1 −) / (√ 1 + + √ 1 −) = lim (√ 1 +)2 − (√ 1 −)2 / (√ 1 + + √ 1 −)

→0 →0 →0

= lim (1 +) − (1 −) / (√ 1+ + √ 1−) = lim 2 / (√ 1+ + √ 1−) = lim 2 / (√ 1+ + √ 1−)

→0 →0 →0

= 2 / (√ 1 + √ 1) = 2/2 = 1

26. lim (1/ − 1/(2 +)) = lim (1/ − 1/(+1)) = lim +1−1 / (+1) = lim 1 / + 1 = 1 / 0 + 1 = 1

→0 →0 →0 →0

27. $\lim\limits_{\to 16} \dfrac{4 - \sqrt{\ }}{16 - {}^2} = \lim\limits_{\to 16} \dfrac{(4 - \sqrt{\ })(4 + \sqrt{\ })}{(16 - {}^2)(4 + \sqrt{\ })} = \lim\limits_{\to 16} \dfrac{16 - }{(16 -)(4 + \sqrt{\ })}$

$= \lim\limits_{\to 16} \dfrac{1}{(4 + \sqrt{\ })} = \dfrac{1}{16\ 4 + \sqrt{16}} = \dfrac{1}{16(8)} = \dfrac{1}{128}$

28. $\lim\limits_{\to 2} \dfrac{{}^2 - 4 + 4}{{}^4 - 3\,{}^2 - 4} = \lim\limits_{\to 2} \dfrac{(\ -2)^2}{(\ {}^2 - 4)(\ {}^2 + 1)} = \lim\limits_{\to 2} \dfrac{(\ -2)^2}{(\ +2)(\ -2)(\ {}^2 + 1)}$

$= \lim\limits_{\to 2} \dfrac{\ - 2}{(\ +2)(\ {}^2 + 1)} = \dfrac{0}{4 \cdot 5} = 0$

29. lim $\frac{1}{\quad}$ $\frac{1}{}$ = lim $\frac{1- \sqrt{1+}}{}$ = lim $\frac{1- \sqrt{1+}}{}$ $\frac{1+ \sqrt{1+}}{}$ = lim –

$\to 0$ $\frac{1}{\sqrt{1+}}$ 1 – $\frac{1}{2}$

= lim = = = =

30. $\lim_{\to -4} \frac{\sqrt{^2+9}-5}{+4} = \lim_{\to -4} \frac{\sqrt{^2+9}-5}{+4} \cdot \frac{\sqrt{^2+9}+5}{\sqrt{^2+9}+5} = \lim_{\to -4} \frac{(^2+9)-25}{(+4)\sqrt{^2+9}+5}$

$= \lim_{\to -4} \frac{^2-16}{(+4)\sqrt{^2+9}+5} = \lim_{\to -4} \frac{(+4)(-4)}{(+4)\sqrt{^2+9}+5}$

$= \lim_{\to -4} \frac{-4}{\sqrt{^2+9}+5} = \frac{-4-4}{\sqrt{16+9}+5} = \frac{-8}{5+5} = -\frac{4}{5}$

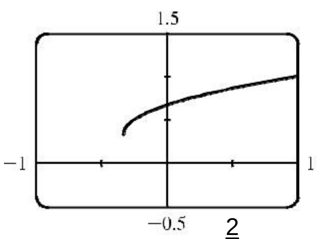
31. $\lim_{\to 0} \frac{(+)^3-^3}{} = \lim_{\to 0} \frac{(^3+3^2+3\,^2+^3)-^3}{} = \lim_{\to 0} \frac{3^2+3\,^2+^3}{}$

$= \lim_{\to 0} \frac{(3^2+3+^2)}{} = \lim_{\to 0} (3^2+3+^2) = 3^2$

32. $\lim_{\to 0} \frac{\frac{1}{(+)^2} - \frac{1}{^2}}{} = \lim_{\to 0} \frac{^2-(+)^2}{(+)^2\,^2} = \lim_{\to 0} \frac{^2-(^2+2+^2)}{^2(+)^2} = \lim_{\to 0} \frac{-(2+)}{^2(+)^2}$

$= \lim_{\to 0} \frac{-(2+)}{^2(+)^2} = \frac{-2}{^2 \cdot ^2} = -\frac{2}{^3}$

33. (a)



lim $\frac{}{} = \frac{2}{3}$

(b)

	( )
−0 001	0 666 166 3
−0 000 1	0 666 616 7
−0 000 01	0 666 661 7
−0 000 001	0 666 666 2
0 000 001	0 666 667 2
0 000 01	0 666 671 7
0 0001	0 666 716 7
0 001	0 667 166 3

The limit appears to be $\frac{2}{3}$.

(c) $\lim_{\to 0} \frac{}{\sqrt{1+3}-1} \cdot \frac{\sqrt{1+3}+1}{\sqrt{1+3}+1} = \lim_{\to 0} \frac{(\sqrt{1+3}+1)}{(1+3)-1} = \lim_{\to 0} \frac{\sqrt{1+3}+1}{3}$

$= \frac{1}{3} \lim_{\to 0} \left(\sqrt{1+3}+1\right)$ [Limit Law 3]

$= \frac{1}{3}\left[\sqrt{\lim_{\to 0}(1+3)} + \lim_{\to 0} 1\right]$ [1 and 11]

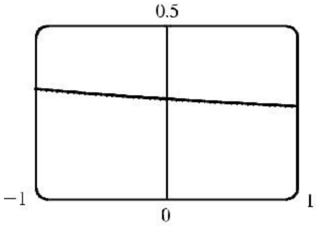
$= \frac{1}{3}\left(\sqrt{\lim_{\to 0} 1 + 3 \lim_{\to 0}} + 1\right)$ [1, 3, and 7]

$\frac{1}{}$ $\sqrt{}$

$= 3^{1+3\cdot0+1}$ [7 and 8]

$= \frac{1}{3}(1+1) = \frac{2}{3}$

34. (a)



$\displaystyle\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}\approx 0.29$

(b)

x	$f(x)$
-0.001	$0.288\,699\,2$
$-0.000\,1$	$0.288\,677\,5$
$-0.000\,01$	$0.288\,675\,4$
$-0.000\,001$	$0.288\,675\,2$
$0.000\,001$	$0.288\,675\,1$
$0.000\,01$	$0.288\,674\,9$
0.0001	$0.288\,672\,7$
0.001	$0.288\,651\,1$

The limit appears to be approximately 0.2887.

(c) $\displaystyle\lim_{x\to 0}\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\right)=\lim_{x\to 0}\frac{(3+x)-3}{x\left(\sqrt{3+x}+\sqrt{3}\right)}=\lim_{x\to 0}\frac{1}{\sqrt{3+x}+\sqrt{3}}$

$\displaystyle=\frac{\lim_{x\to 0}1}{\lim_{x\to 0}\sqrt{3+x}+\lim_{x\to 0}\sqrt{3}}$ [Limit Laws 5 and 1]

$\displaystyle=\frac{1}{\sqrt{\lim_{x\to 0}(3+x)}+\sqrt{3}}$ [7 and 11]

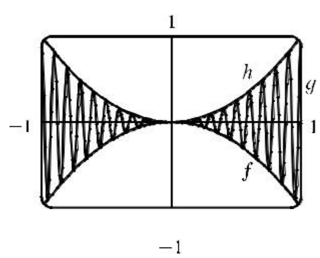
$\displaystyle=\frac{1}{\sqrt{3+0}+\sqrt{3}}$ [1, 7, and 8]

$\displaystyle=\frac{1}{2\sqrt{3}}$

35. Let $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$ and $h(x)=x^2$. Then

$-1\le\cos 20\pi x\le 1\;\Rightarrow\;-x^2\le x^2\cos 20\pi x\le x^2\;\Rightarrow\;f(x)\le g(x)\le h(x)$.

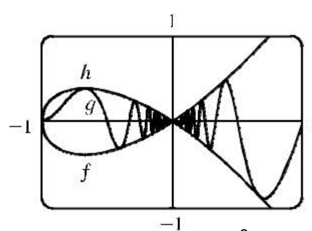
So since $\lim_{x\to 0}f(x)=\lim_{x\to 0}h(x)=0$, by the Squeeze Theorem we have $\lim_{x\to 0}g(x)=0$.



36. Let $f(x)=-\sqrt{x^3+x^2}$, $g(x)=\sqrt{x^3+x^2}\sin(\pi/x)$, and $h(x)=\sqrt{x^3+x^2}$. Then

$-1\le\sin(\pi/x)\le 1\;\Rightarrow\;-\sqrt{x^3+x^2}\le\sqrt{x^3+x^2}\sin(\pi/x)\le\sqrt{x^3+x^2}\;\Rightarrow$

$f(x)\le g(x)\le h(x)$. So since $\lim_{x\to 0}f(x)=\lim_{x\to 0}h(x)=0$, by the Squeeze Theorem we have $\lim_{x\to 0}g(x)=0$.



37. We have $\lim_{x\to 4}(4x-9)=4(4)-9=7$ and $\lim_{x\to 4}\left(x^2-4x+7\right)=4^2-4(4)+7=7$. Since $4x-9\le f(x)\le x^2-4x+7$ for $x\ge 0$, $\lim_{x\to 4}f(x)=7$ by the Squeeze Theorem.

38. We have $\lim_{x\to 1}(2x)=2(1)=2$ and $\lim_{x\to 1}\left(x^4-x^2+2\right)=1^4-1^2+2=2$. Since $2x\le g(x)\le x^4-x^2+2$ for all x, $\lim_{x\to 1}g(x)=2$ by the Squeeze Theorem.

39. $-1 \le 4 \cos(2\ ) \le 1 \Rightarrow -\ ^4 \le\ ^4 \cos(2\ ) \le\ ^4$. Since $\lim_{\to 0} -\ _4 = 0$ and $\lim_{\to 0}\ ^4 = 0$, we have

$\lim_{\to 0} \cos(2\ ) = 0$ by the Squeeze Theorem.

40. $-1 \le \sin(2) \le 1 \Rightarrow 0 \le \sin^2(2\ ) \le 1 \Rightarrow 1 \le 1 + \sin^2(2\ ) \le 2 \Rightarrow$

$\sqrt{-}\quad \sqrt{-}\quad 2 \quad \sqrt{-}\quad \lim \quad \sqrt{\ } = 0 \quad \lim 2 \sqrt{\ } = 0$

$\le\ 1 + \sin(2\ )\ \le 2$. Since $\to 0+$ and $\to 0^+$, we have

$\lim\limits_{\to 0^+} \sqrt{\ }(1 + \sin^2(2\ )) = 0$ by the Squeeze Theorem.

41. $|-3| =$ (3) if $-3\ 0 =$ 3 if 3

-3 if $3 \ge 0$ -3 if ≥ 3

Thus, $\lim\limits_{\to 3^+}(2 + |-|\ 3) = \lim\limits_{\to 3^+}(2 + -3) = \lim\limits_{\to 3^+}(3 - 3) = 3(3) - 3 = 6$ and

$\lim\limits_{\to 3^-}(2 + |-|\ 3) = \lim(2 + 3 -) = \lim(+3) = 3 + 3 = 6$. Since the left and right limits are equal, $\lim\limits_{3}(2 + |-|\ 3) = 6$

42. $|+6| = -(+6)$ if $+6\ 0$ $= -(+6)$ if -6

$+6$ if $+6 \ge 0$ $+6$ if ≥ -6

We'll look at the one-sided limits.

$\lim\limits_{\to -6^+} \dfrac{2 + 12}{|+6|} = \lim\limits_{\to -6^+} \dfrac{2(+6)}{+6} = 2$ and $\lim\limits_{\to -6^-} \dfrac{2 + 12}{|+6|} = \lim\limits_{\to -6^-} \dfrac{2(+6)}{-(+6)} = -2$

The left and right limits are different, so $\lim\limits_{\to -6} \dfrac{2 + 12}{|+6|}$ does not exist.

$|2^3 - ^2| = |^2(2 - 1)| = |^2| \cdot |2 - 1| = ^2|2 - 1|$

$|2 - 1| =$ $(2\ 1)$ if $2\ 1\ 0$ $(2\ 1)$ if $0\ 5$

$2 - 1$ if $21 \ge 0$ $2 - 1$ if $\ge 0\ 5$

So $|2^3 - ^2| = ^2[-(2 - 1)]$ for $0\ 5$.

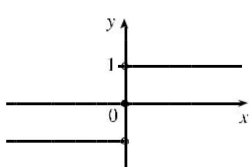
Thus, $\lim\limits_{\to 05^-} \dfrac{2 - 1}{|2^3 - ^2|} = \lim\limits_{\to 05^-} \dfrac{2 - 1}{^2[-(2 - 1)]} = \lim\limits_{\to 0\,5^-} \dfrac{-1}{^2} = \dfrac{-1}{(0\ 5)^2} = \dfrac{-1}{0\ 25} = -4.$

44. Since $|| = -$ for 0, we have $\lim\limits_{\to -} \dfrac{2 - ||}{2 + } = \lim\limits_{\to -} \dfrac{2 - (-)}{2 + } = \lim\limits_{\to -} \dfrac{2 +}{2 +} = \lim\limits_{\to -} 1 = 1.$

45. Since $|\ | = -$ for 0, we have $\lim\limits_{\to 0^-}\left(\dfrac{1}{} - \dfrac{1}{||}\right) = \lim\limits_{\to 0^-}\left(\dfrac{1}{} - \dfrac{1}{-}\right) = \lim\limits_{\to 0^-} \dfrac{2}{}$, which does not exist since the denominator approaches 0 and the numerator does not.

46. Since $|\ | =$ for 0, we have $\lim\limits_{\to 0+}\left(\dfrac{1}{} - \dfrac{1}{||}\right) = \lim\limits_{\to 0+}\left(\dfrac{1}{} - \dfrac{1}{}\right) = \lim\limits_{\to 0^+} 0 = 0$.

47. (a)



(b) (i) Since $\operatorname{sgn} = 1$ for 0, $\lim\limits_{\to 0^+} \operatorname{sgn} = \lim\limits_{\to 0^+} 1 = 1.$

(ii) Since $\operatorname{sgn} = -1$ for 0, $\lim\limits_{\to 0^-} \operatorname{sgn} = \lim\limits_{\to 0^-} -1 = -1.$

(iii) Since $\lim\limits_{\to 0^-} \operatorname{sgn} \ne \lim\limits_{\to 0^+} \operatorname{sgn}$, $\lim\limits_{\to 0} \operatorname{sgn}$ does not exist.

$\operatorname{sgn} = 1$

(iv) Since | | = 0lim lim 1 = 1
for 6 , →0|sgn | = →0 .

−1 if sin0

48. (a) ( ) = sgn(sin) =0 if sin = 0

(i) lim ( ) = lim sgn(sin) = 1 since sin is positive for small positive values of .
$\to 0^+$ $\to 0^+$ sgn(sin) = 1 sin

(ii) $\to 0^-$ $\to 0^-$ − since is negative for small negative values of .

(iii) $\to 0$ does not exist since $\to 0^+$ 6 $\to 0^-$.

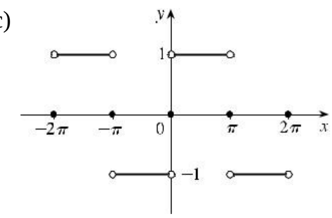
(iv) $\to ^+$ $\to ^+$ − since is negative for values of slightly greater than .

(v) lim ( ) = lim sgn(sin) = 1 since sin is positive for values of slightly less than .
$\to ^-$ $\to ^-$ lim ( ) = lim ( )

(vi) $\to$ does not exist since $\to +$ 6 $\to ^-$.

(b) The sine function changes sign at every integer multiple of , so the signum function equals 1 on one side and −1 on the other side of , an integer. Thus, lim ( ) does not exist $\to$ for = , an integer.

(c)



(a) (i) lim ( ) = lim $\dfrac{^2 + -6}{|-2|}$ lim $\dfrac{(+3)(-2)}{|-2|}$
$\to 2^+$ $\to 2$ $\to 2$

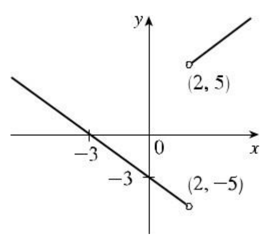
$= \lim\limits_{\to 2^+} \dfrac{(+3)(-2)}{-2}$ [since − 2 0 if $\to 2^+$]

$= \lim\limits_{\to 2^+} (+3) = 5$

(ii) The solution is similar to the solution in part (i), but now | − 2| = 2 − since − 2 0 if $\to 2^-$.
Thus, $\lim\limits_{\to 2^-}$ ( ) = $\lim\limits_{\to 2^-}$ −(+ 3) = −5.

(b) Since the right-hand and left-hand limits of at = 2 are not equal, $\lim\limits_{\to 2}$ ( ) does not exist.

(c)

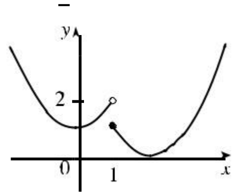


50. (a) ( ) = (− 2)2 if ≥ 1
2 + 1 if 1

$\lim\limits_{\to 1^-}$ ( ) = $\lim\limits_{\to 1^-}$ (^{2} +1)=1^2+1=2 , $\lim\limits_{\to 1^+}$ ( ) = $\lim\limits_{\to 1^+}$ (− 2)^2 = (− 1)^2 = 1

(b) Since the right-hand and left-hand limits of at = 1 are not equal, $\lim\limits_{\to 1}$ ( ) does not exist.

(c)



lim ( ) = 2 lim ( ) = lim 4 1 = 4 1 = 3

51. For the →2 √ to exist, the one-sided limits at must be equal. →2− →2− − 2 − and

lim ( ) = lim + = √ 2 + 3 = √ 2 + 9=2+ = 7
→2+ →2+ . Now ⇒ ⇔ .

(a) (i) lim ( ) = lim = 1

→1−→1

lim ( ) = lim (2 2)=2 1² = 1 lim ( ) = 1 lim ( ) = 1 lim ( ) = 1
(ii) →1+ →1+ − − . Since →1− and →1+ , we have →1

Note that the fact (1) = 3 does not affect the value of the limit.

When = 1, ( ) = 3, so (1) = 3.

lim ( ) = lim (2 − ²) = 2 − 2² = 2 − 4 = −2

→2−→2

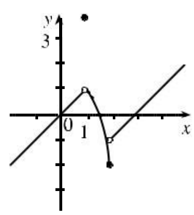
lim ( ) = lim (− 3) = 2 − 3 = −1

lim ( ) lim ( ) = lim ( ) →2+→2

(vi) →2 does not exist since →2− 6 →2+ .

(b) ( ) = 3 if 1 if = 1

− if 2 ≤



53. (a) (i) [[ ]] = −2 for −2 ≤−1, so lim2+[[ ]] = lim2+ (−2) = −2

(ii) [[ ]] = −3 for −3 ≤−2, so lim →−2− [[ ]]= lim →−2− (− 3) = − 3 .

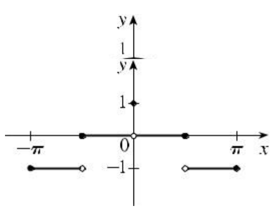
The right and left limits are different, so lim →−2 [[ ]] does not exist.

(iii) [[ ]] = −3 for −3 ≤−2, so lim →−2 4 [[ ]]= lim →−2 4 (− 3) = − 3 .

(b) (i) [[ ]]= − 1 for − ≤ 1 , so lim →- [[ ]] = lim →- (− 1) = − 1 .

(ii) [[ ]] = for ≤+ 1, so lim → + [[ ]] = lim → + = .

(c) → [[ ]] exists ⇔ is not an integer.



54. (a) See the graph of = cos .

Since −1 ≤ cos 0 on [− − 2), we have = ( ) = [[cos]] = −1 on [− − 2).

Since 0 ≤ cos 1 on [− 2 0) ∪ (0 2], we have ( ) = 0 on [− 2 0) ∪ (0 2].

Since $-1 \le \cos$ 0 on (2], we have ( ) $= -1$ on (2].

Note that $(0) = 1$.

(b) (i) $\lim_{\to 0^-}$ ( ) = 0 and $\lim_{\to 0^+}$ ( ) = 0, so $\lim_{\to 0}$ ( ) = 0.

(ii) As → (2) − , ( ) → 0 $\lim_{\to}$ ( ) = 0 .

(iii) As → (2) + , ( ) → 1 $\lim_{\to}$ ( ) = 1

(iv) Since the answers in parts (ii) and (iii) are not equal, lim ( ) does not exist.

lim ( ) ( ) → 2 2 = 2

(c) → exists for all in the open interval − except − and .

The graph of ( ) = [[ ]] + [[−]] is the same as the graph of ( ) = −1 with holes at each integer, since ( ) = 0 for any integer . Thus, $\lim_{\to 2^-}$ ( ) = 1 − and $\lim_{\to 2^+}$ ( ) = 1 − , so $\lim_{\to 2}$ ( ) = 1 − . However,

(2) = [[2]] + [[2]] = 2 + (2) = 0 lim ( ) = (2)

− , so →2 6 .

56. $\lim_{\to -0}$ 1 − 2 = 0 − . As the velocity approaches the speed of light, the length approaches .

A left-hand limit is necessary since is not defined for .

Since ( ) is a polynomial, ( ) = $_0$ + $_1$ + $_2$ ^{2} + · · · + . Thus, by the Limit Laws,

$\lim_{\to}$ ( ) = $\lim_{\to}$ $_0$ $_1$ $_2$ + + ^{2} + · · · + = $_0$ + $_1$ lim + $_2$ lim ^{2} + · · · + lim

$_0$ + $_1$ + $_2$ ^{2} + · · · + = ( )

Thus, for any polynomial , $\lim_{\to}$ ( ) = ( ).

Let ( ) = ( ) / ( ) where ( ) and ( ) are any polynomials, and suppose that ( ) ≠ 0. Then

$\lim_{\to}$ ( ) = $\lim_{\to}$ ( ) / ( ) = $\lim_{\to}$ ( ) / $\lim_{\to}$ ( ) [Limit Law 5] = ( ) / ( ) [Exercise 57] = ( ).

59. $\lim_{\to 1}$ [( ) − 8] = $\lim_{\to 1}$ (( ) − 8) / (− 1) · (− 1) = $\lim_{\to 1}$ (( ) − 8) / (− 1) · $\lim_{\to 1}$ (− 1) = 10 · 0 = 0.

Thus, $\lim_{\to 1}$ ( ) = $\lim_{\to 1}$ {[( ) − 8] + 8} = $\lim_{\to 1}$ [( ) − 8] + $\lim_{\to 1}$ 8 = 0 + 8 = 8.

Note: The value of $\lim_{\to 1}$ (( ) − 8) / (− 1) does not affect the answer since it's multiplied by 0. What's important is that $\lim_{\to 1}$ (( ) − 8) / (− 1) exists.

60. (a) $\lim_{\to 0}$ ( ) = $\lim_{\to 0}$ ( ) / ^{2} · ^{2} = $\lim_{\to 0}$ ( ) / ^{2} · $\lim_{\to 0}$ ^{2} = 5 · 0 = 0

(b) $\lim_{\to 0}$ ( ) / = $\lim_{\to 0}$ ( ) / ^{2} · = $\lim_{\to 0}$ ( ) / ^{2} · $\lim_{\to 0}$ = 5 · 0 = 0

61. Observe that ≤ ≤ for all , and →0 →0 . So, by the Squeeze Theorem, →0 .

62. Let $( ) = [[ ]]$ and $( ) = -[[ ]]$. Then $\lim_{\to 3} ( )$ and $\lim_{\to 3} ( )$ do not exist [Example 10]

but $\lim_{\to 3} [( ) + ( )] = \lim_{\to 3} ([[ ]] - [[ ]]) = \lim_{\to 3} 0 = 0$.

Let $( ) = ( )$ and $( ) = 1 - ( )$, where is the Heaviside function defined in Exercise 1.3.59.

Thus, either or is 0 for any value of . Then $\lim_{\to 0} ( )$ and $\lim_{\to 0} ( )$ do not exist, but $\lim_{\to 0} [( ) ( )] = \lim_{\to 0} 0 = 0$.

64. $\lim_{\to 2} \dfrac{\sqrt{6} - -2}{\sqrt{3} - -1} = \lim_{\to 2} \dfrac{\sqrt{6}}{2} \cdot \dfrac{\sqrt{6} + 2}{\sqrt{3} + 1}$ $- 1 \cdot \sqrt{6} + 2$

$$\lim \frac{(2-)\sqrt{3} + 1}{(2 -)\sqrt{6} + 2} = \lim_{\to 2} \frac{\sqrt{3} + 1}{\sqrt{6} + 2} = \frac{1}{2}$$

Since the denominator approaches 0 as $\to -2$, the limit will exist only if the numerator also approaches 0 as $\to -2$. In order for this to happen, we need $\lim_{\to -2} (3^2 + + 3) = 0 \Leftrightarrow$

$3(-2)^2 + (-2) + + 3 = 0 \Leftrightarrow 12 - 2 + + 3 = 0 \Leftrightarrow = 15$. With $= 15$, the limit becomes

$$\lim_{\to -2} \frac{3^2 + 15 + 18}{^2 + - 2} = \lim_{\to -2} \frac{3(+ 2)(+ 3)}{(- 1)(+ 2)} = \lim_{\to -2} \frac{3(+ 3)}{- 1} = \frac{3(-2 + 3)}{-2 - 1} = \frac{3}{-3} = -1.$$

Solution 1: First, we find the coordinates of and as functions of . Then we can find the equation of the line determined by these two points, and thus find the -intercept (the point), and take the limit as $\to 0$. The coordinates of are $(0\ )$.

The point is the point of intersection of the two circles $^2 + ^2 = ^2$ and $(- 1)^2 + ^2 = 1$. Eliminating from these equations, we get $^2 - ^2 = 1 - (- 1)^2 \Leftrightarrow ^2 = 1 + 2 - 1 \Leftrightarrow = \frac{1}{2}{}^2$. Substituting back into the equation of the shrinking circle to find the -coordinate, we get $\left(\frac{1}{2}{}^2\right)^2 + ^2 = ^2 \Leftrightarrow ^2 = ^2\left(1 - \frac{1}{4}{}^2\right) \Leftrightarrow = \sqrt{1 - \frac{1}{4}{}^2}$ (the positive -value). So the coordinates of are $\left(\frac{1}{2}{}^2\ \sqrt{1 - \frac{1}{4}{}^2}\right)$. The equation of the line joining and is thus

$$- = \frac{\sqrt{1 - \frac{1}{4}{}^2} - }{\frac{1}{2}{}^2 - 0}(- 0).$$ We set $= 0$ in order to find the -intercept, and get

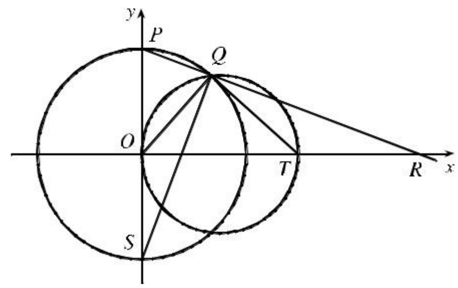
$$= \frac{\frac{1}{2}{}^2}{\sqrt{1 - \frac{1}{4}{}^2} - 1} = \frac{\frac{1}{2}{}^2\left(\sqrt{1 - \frac{1}{4}{}^2} + 1\right)}{1 - \frac{1}{4}{}^2 - 1} = 2\left(\sqrt{1 - \frac{1}{4}{}^2} + 1\right)$$

Now we take the limit as $\to 0^+$: $\lim_{\to 0^+} = \lim 2\left(\sqrt{1 - \frac{1}{4}{}^2} + 1\right) = \lim 2\left(\sqrt{1} + 1\right) = 4$

So the limiting position of   is the point (4 0).

[continued]

Solution 2: We add a few lines to the diagram, as shown. Note that $\angle = 90^\circ$ (subtended by diameter). So $\angle = 90^\circ = \angle$ (subtended by diameter). It follows that $\angle = \angle$. Also $\angle = 90^\circ - \angle = \angle$. Since 4 is isosceles, so is 4 , implying that = . As the circle $_2$ shrinks, the point plainly approaches the origin, so the point must approach a point twice as far from the origin as , that is, the point (4 0), as above.



1.7 The Precise Definition of a Limit

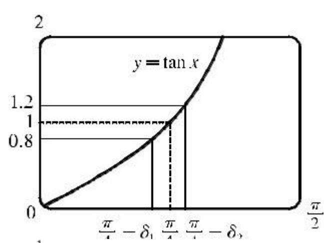
1. If | ( ) − 1| 0 2, then −0 2( ) − 1 0 2 ⇒ 0 8 ( ) 1 2. From the graph, we see that the last inequality is true if 0 71 1, so we can choose = min {1 − 0 7 1 1 − 1} = min {0 3 0 1} = 0 1 (or any smaller positive number).

2. If | ( ) − 2| 0 5, then −0 5( ) − 2 0 5 ⇒ 1 5 ( ) 2 5. From the graph, we see that the last inequality is true if 2 6 3 8, so we can take = min {3 − 2 6 3 8 − 3} = min {0 4 0 8} = 0 4 (or any smaller positive number). Note that 6= .3

The leftmost question mark is the solution of $\sqrt{\ } = 1\ 6$ and the rightmost, $\sqrt{\ } = 2\ 4$. So the values are $1\ 6^2 = 2\ 56$ and $2\ 4^2 = 5\ 76$. On the left side, we need | − 4| |2 56 − 4| = 1 44. On the right side, we need | − 4| |5 76 − 4| = 1 76. To satisfy both conditions, we need the more restrictive condition to hold — namely, | − 4| 1 44. Thus, we can choose = 1 44, or any smaller positive number.

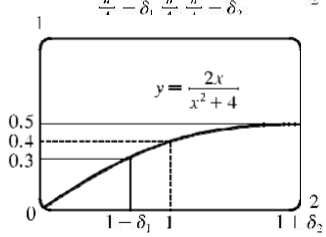
4. The leftmost question mark is the positive solution of $^2 = \frac{1}{2}$, that is, $= \frac{1}{\sqrt{2}}$, and the rightmost question mark is the positive solution of $^2 = \frac{3}{2}$, that is, $= \sqrt{\frac{3}{2}}$. On the left side, we need $| - 1| \ \frac{1}{\sqrt{2}} - 1 \approx 0\ 292$ (rounding down to be safe). On the right side, we need $| - 1| \ \sqrt{\frac{3}{2}} - 1 \approx 0\ 224$. The more restrictive of these two conditions must apply, so we choose = 0 224 (or any smaller positive number).

5.

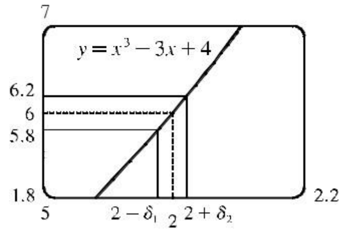


From the graph, we find that = tan = 0 8 when ≈ 0 675, so $\frac{}{4} - _1 \approx 0\ 675 \Rightarrow _1 \approx \frac{}{4} - 0\ 675 \approx 0\ 1106$. Also, = tan = 1 2 when ≈ 0 876, so $\frac{}{4} + _2 \approx 0\ 876 \Rightarrow _2 = 0\ 876 - \frac{}{4} \approx 0\ 0906$. Thus, we choose = 0 0906 (or any smaller positive number) since this is the smaller of $_1$ and $_2$.

6.



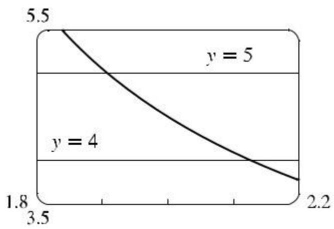
From the graph, we find that $= 2 \ (^2 + 4) = 0\ 3$ when $= \frac{2}{3}$, so $1 - _1 = \frac{2}{3} \Rightarrow _1 = \frac{1}{3}$. Also, $= 2 \ (^2 + 4) = 0\ 5$ when = 2, so $1 + _2 = 2 \Rightarrow _2 = 1$. Thus, we choose $= \frac{1}{3}$ (or any smaller positive number) since this is the smaller of $_1$ and $_2$.

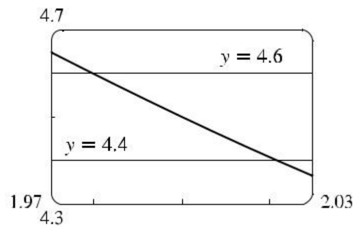
7. 

From the graph with $= 0\ 2$, we find that $= {}^3 - 3 + 4 = 5\ 8$ when $\approx 1\ 9774$, so $2 - {}_1 \approx 1\ 9774 \Rightarrow {}_1 \approx 0\ 0226$. Also, $= {}^3 - 3 + 4 = 6\ 2$ when $\approx 2\ 022$, so $2 + {}_2 \approx 2\ 0219 \Rightarrow {}_2 \approx 0\ 0219$. Thus, we choose $= 0\ 0219$ (or any smaller positive number) since this is the smaller of ${}_1$ and ${}_2$.

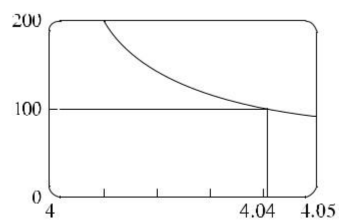
For $= 0\ 1$, we get ${}_1 \approx 0\ 0112$ and ${}_2 \approx 0\ 0110$, so we choose $= 0\ 011$ (or any smaller positive number).

For $= (4 + 1)\ (3 - 4)$ and $= 0\ 5$, we need $1\ 91 \leq \ \leq 2\ 125$. So since $|2 - 1\ 91| = 0\ 09$ and $|2 - 2\ 125| = 0\ 125$, we can take $0 \leq 0\ 09$. For $= 0\ 1$, we need $1\ 980 \leq 2\ 021$. So since $|2 - 1\ 980| = 0\ 02$ and $|2 - 2\ 021| = 0\ 021$, we can take $= 0\ 02$ (or any smaller positive number).

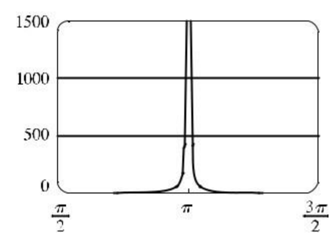




9. (a) The graph of $= \dfrac{{}^2 + 4}{\sqrt{\ - 4}}$ shows that $= 100$ when $\approx 4\ 04$ (more accurately, $4\ 04134$). Thus, we choose $= 0\ 04$ (or any smaller positive number).



(b) From part (a), we see that as gets closer to 4 from the right, increases without bound. In symbols, $\lim\limits_{\to 4^+} \dfrac{{}^2 + 4}{\sqrt{\ - 4}} = \infty$.

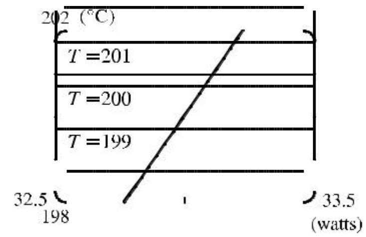


We graph $= \csc^2$ and $= 500$. The graphs intersect at $\approx 3\ 186$, so we choose $= 3\ 186 - \approx 0\ 044$. Thus, if $0\ |\ -\ |\ 0\ 044$, then $\csc^2\ 500$. Similarly, for $= 1000$, we get $= 3\ 173 - \approx 0\ 031$.

11. (a) $= {}^2$ and $= 1000\ \text{cm}^2 \Rightarrow {}^2 = 1000 \Rightarrow {}^2 = \dfrac{1000}{} \Rightarrow = \sqrt{\dfrac{1000}{}}\ (0) \approx 17\ 8412$ cm.

(b) $|\ - 1000| \leq 5 \Rightarrow -5 \leq {}^2 - 1000 \leq 5 \Rightarrow 1000 - 5 \leq {}^2 \leq 1000 + 5 \Rightarrow \sqrt{\dfrac{995}{}} \leq \ \leq \sqrt{\dfrac{1005}{}} \Rightarrow 17\ 7966 \leq \ \leq 17\ 8858$. $\sqrt{\dfrac{1000}{}} - \sqrt{\dfrac{995}{}} \approx 0\ 04466$ and $\sqrt{\dfrac{1005}{}} - \sqrt{\dfrac{1000}{}} \approx 0\ 04455$. So if the machinist gets the radius within $0\ 0445$ cm of $17\ 8412$, the area will be within $5\ \text{cm}^2$ of 1000.

is the radius, $(\ )$ is the area, is the target radius given in part (a), is the target area ($1000\ \text{cm}^2$), is the magnitude of the error tolerance in the area ($5\ \text{cm}^2$), and is the tolerance in the radius given in part (b).



12. (a) = 0 1 ^{2} + 2 155 + 20 and = 200 ⇒
01 ^{2}+2155 +20=200 ⇒ [by the quadratic formula or
from the graph]≈ 33 0 watts (0)

(b) From the graph, 199 ≤ ≤ 201 ⇒ 32 8933 11.

(c) is the input power, ( ) is the temperature, is the target input power given in part (a), is the target temperature (200),

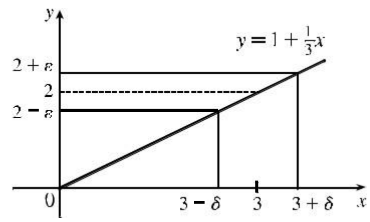
is the tolerance in the temperature (1), and is the tolerance in the power input in watts indicated in part (b) (0 11 watts).
0

1 0 1
13. (a) |4 − 8| = 4 | − 2| 0 1 ⇔|−2| 4 , so = 4 = 0 025.

0 01 0 01
(b) |4 − 8| = 4 | − 2| 0 01 ⇔|−2| 4 , so = 4 = 0 0025.

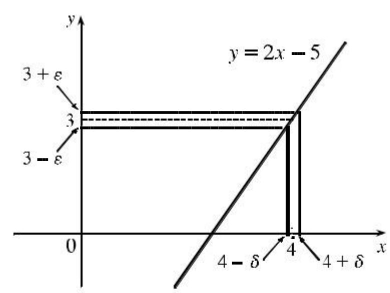
14. |(5 − 7) − 3| = |5 − 10| = |5(− 2)| = 5 | − 2|. We must have | ( ) − | , so 5 | − 2|⇔

| − 2| 5. Thus, choose = 5. For = 0 1, = 0 02; for = 0 05, = 0 01; for = 0 01, = 0 002.



15. Given0, we need0 such that if 0 | − 3| , then
(1 + 1) 2 (1 + 1) 2 1 1
3 . But 3 − ⇔ 3 − ⇔
1
3 3 3 3 . So if we choose = 3 , then
0 | − | 3 ⇔ | − | (1 + 1 3) 2 . Thus, lim (1 + 1 3) = 2 by
| −| ⇒ - →3

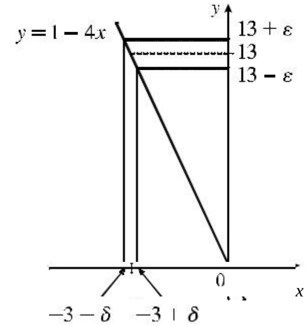
the definition of a limit.



16. Given0, we need0 such that if 0 | − 4| , then

|(2 − 5) − 3| . But |(2 − 5) − 3|⇔ |2 − 8| ⇔

|2|| −4| ⇔ | − 4| 2. So if we choose = 2, then
0 4 (2 5) 3 lim (2 5)=3
| − | ⇒ | − − | . Thus, →4 − by the
definition of a limit.



17. Given0, we need0 such that if 0 | − (−3)| , then
|(1 − 4) − 13| . But |(1 − 4) − 13| ⇔

|−4 − 12|⇔ |−4| | + 3| ⇔ | − (−3)| 4. So if

we choose = 4, then 0 | − (−3)| ⇒ |(1−4)−13| .
Thus, lim3(1 − 4) = 13 by the definition of a limit.
→−

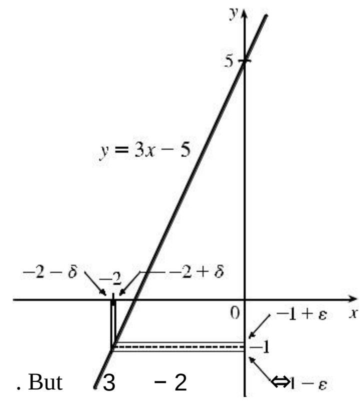
18. Given 0, we need 0 such that if 0 | − (−2)| , then |(3 + 5) − (−1)| .

But |(3 + 5) − (−1)| ⇔

|3 + 6| ⇔ |3| | + 2| ⇔ | + 2| 3. So if we choose = 3, then 0 | + 2| ⇒ |(3

+ 5) − (−1)| . Thus,

lim (3 + 5) = −1 by the definition of a limit.
→−2



19. Given 0, we need 0 such that if 0 | − 1| , then 2 3 − 2 . But 3 − 2
+

4 2 + 4

4 3− 4 ⇔ 4 3 . So if we choose = 3 4 , then 0 | − 1| ⇒
2

+

4 lim 2 + 4

= 2 by the definition of a limit.

20. Given 4 0, we need 5 4 0 such that if 0 | − 10| , then 3 − 4 5 (5) . But 3 − 4 5 (5) ⇔ 8 − 4 5 ⇔ − 4 5 | − 10| ⇔ | − 10| 4 . So if we choose = 5 4 , then 0 | − 10| ⇒ . Thus,

3 (5) lim (3) = 5 by the definition of a limit.

21. Given 0, we need 0 such that if 0 | − 4| , then 2 − 2 − 8 − 4 − 6 ⇔

(4)(+ 2) − 4 − 6 ⇔ | + 2 − 6| [6= 4] ⇔ | − 4| . So choose = . Then

0 | − 4| ⇒ | − 4| ⇒ | + 2 − 6| ⇒ (4)(+ 2) − 4 − 6 [6= 4] ⇒

2 − 2 − 8 − 2 − 2 − 8

22. Given 0, we need 0 such that if 0 | + 1 5| , then 9 − 4 2 3 + 2 − 6 ⇔ (3 + 2)(3 − 2) 3 + 2 − 6

+ 1 52. So choose = 2. Then 0 + 1 5 + 1 5 2 + 1 5

| ⇒ | − − | ⇒ || ⇒ | | 6 − ⇒ |−|| |− ⇒

|− 2 3| 3 2 6 (3+2)(3−2) 3 + 2 − 6 [=15] ⇒ 9−4 2 3 + 2 6 .

By the definition of a limit, $\lim_{x \to -1.5} \frac{9 - 4x^2}{3 + 2x} = 6.$

Given0, we need0 such that if 0 | − | , then | − | . So = will work.

Given 0, we need 0 such that if 0 | − | , then | − | . But | − | = 0, so this will be true no matter what we pick.

25. 0 0 such that if 0 0, then 2 0 2 √ −. Take = √

Given | , we need ⇒ 2 − | − | 2 − ⇔ ⇔ | |

Then 0 0 0 . Thus, lim = 0 by the definition of a limit.

26. 0 0 such that if 0 0, then 3 0 3 √ 3 − . Take = √ 3

Given | , we need ⇒ 3 − 3 | − | →0 3 − ⇔ | | ⇔ | |

Then 0 0 0 = . Thus, lim = 0 by the definition of a limit.

27. Given0, we need0 such that if 0 | − 0| , then | |−0 . But | | = | |. So this is true if we pick = .

Thus, lim = 0 by the definition of a limit.
→0 | |

8 √ 8 − − 8 − 8 (6) − ⇔

28. 6 + 6 + (6). So if we choose = , then 0

Given0, we need0 such that if 0 (6) , then √ 6 + 0. But √ 6 + 0

⇔ ⇔ − − − − ⇒

√8 lim √8 6+ =0

6 + − 0. Thus, →−6+ by the definition of a right-hand limit.

0 0 02 2 4 + 5 1 2 4 + 4

29. (2) . So take = − √. Then 0 2 , then 2 √ (2) . Thus, ⇔

Given 2 , we need such that if | − | − − ⇔2 −

→2 − 2 | − | ⇔|−| ⇔ −

lim 4 + 5 = 1 by the definition of a limit.

30. Given0, we need0 such that if 0 2, then (2+2 7) 1. But (2+2 7) 1

− ⇔ | ||−| 2 1. Then 1 | − | 13 | |

+ 2 8 + 4 2. Thus our goal is to make 2 small enough so that its product with + 4

| − | − − ⇒ ⇒ ⇒

| + 4| 7, and this gives us 7 | − 2| ⇒ | − 2| 7. Choose = min {1 7}. Then if 0 | − 2| , we

have | − | 27 and 2 | | + 4 7, so (2 + 2 − 7) − 1 = (+4)(− 2) = + 4 | | − | 2 7(7) = , as

desired. Thus, lim (+ 2 − 7) = 1 by the definition of a limit.
→2

Given0, we need0 such that if 0 | − (−2)| , then 2 − 1 − 3or upon simplifying we need

2 − 4 whenever 0 | + 2| . Notice that if | + 2| 1, then −1 + 2 1 ⇒−5 −2−3 ⇒

2 5. So take = min 5 1 . Then 0 + 2 2 5 and + 25, so

| − | { } | | ⇒ | − | | | − .

2 − 1 − 3 = |(+ 2)(− 2)| = | + 2| | − 2| (5)(5) = . Thus, by the definition of a limit, →−2 lim (2 1)=3

Given 0, we need 0 such that if 0 | − 2| , then 3 − 8 . Now 3 − 8 = (− 2) 2 + 2 + 4 . If | − 2| 1, that is, 1 3, then 2 + 2 +

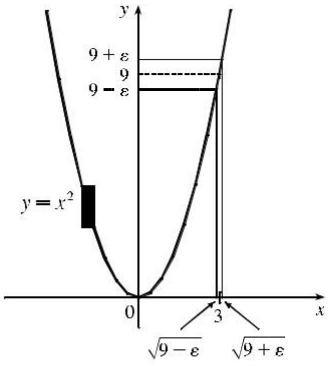
4 32 + 2(3) + 4 = 19 and so

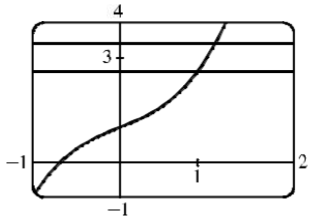
3 $-8 = |-2|\ 2\ +2\ +4\ \ |-|$. So if we take 19, then 3 $|\ -|$
3 $-\ \ |-|\ \ -19\ \ 2\ \ = \min\ 1\ \ -\ \ 0_{\to 2}\ \ 2 \Rightarrow$
19

$8 = 2 + 2 + 4$ 8. If $|\ -|$, $2\ |-|$ $19 =$. Thus, by the definition of a limit, $\lim_{\Rightarrow -} = 8$. $\Rightarrow$

33. Given 0, we let $= \min\ 2$ $4+38$ $-\ 0$ 3 then $\Rightarrow |+3|\ 8$. Also $|-3|\ 8$, so -9 $3\ 2$ $=|+3||-3|\ 8\cdot$ $2\ \ 3\ 2$ $8 =$. Thus, $\lim_{\to 3}\ ^2 = 9$.

34. From the figure, our choices for are $_1 = 3 - \sqrt{9 -}$ and $_2 = \sqrt{9 +} - 3$. The *largest* possible choice for is the minimum value of $\{_1\ _2\}$; that is, $= \min\{_1\ _2\} = {_2} = \sqrt{9 +} - 3$.





35. (a) The points of intersection in the graph are $(_1\ 2\ 6)$ and $(_2\ 3\ 4)$ with $_1 \approx 0\ 891$ and $_2 \approx 1\ 093$. Thus, we can take to be the smaller of $1 - {_1}$ and $_2 - 1$. So $= {_2} - 1 \approx 0\ 093$.

(b) Solving $^3 + \ + 1 = 3 +$ gives us two nonreal complex roots and one real root, which is
$() = \dfrac{6\ 216 + 108 + 12\sqrt{336 + 324 + 81\ 2}}{}$ ... $\dfrac{216 + 108 + 12\sqrt{336+324+81\ 2}}{}$ $2\,3$ 12 . Thus, $= () - 1$.

(c) If $= 0\ 4$, then $() \approx 1\ 093\ 272\ 342$ and $= () - 1 \approx 0\ 093$, which agrees with our answer in part (a).

1 **36.** 1. Guessing a value for ... Let be given. We have to find a number such that $-\ \dfrac{1}{}$ whenever
1

$|-|$ $\dfrac{1}{} - \dfrac{1}{}$ $\dfrac{2}{}$ $\dfrac{2}{}$ $|\ 2$ $\dfrac{1}{2}$ $\Rightarrow$
0 2. But $= - = -|$. We find a positive constant such that
2
$|\ _2 - |$ $|\ -|$ 2 and we can make $|-|$ 2 by taking $|\ -|$ 2 = . We restrict to lie in the interval
$||$
$|-2|\ 1 \Rightarrow 13$ so 1 $\dfrac{1}{3} \Rightarrow \dfrac{1}{6}\ \dfrac{1}{2}\ \dfrac{1}{2} \Rightarrow \dfrac{1}{|2|}\ \dfrac{1}{2}$. So $= \dfrac{1}{2}$ is suitable. Thus, we should choose $= \min\{1\ 2\}$.

2. Showing that works Given 0 we let $= \min\{1\ 2\}$. If $0\ |-2|$, then $|-2|\ 1 \Rightarrow 13 \Rightarrow$
$\dfrac{1}{}\ \dfrac{1}{}$ (as in part 1). Also $2\ 2$, so $\dfrac{1}{}\ \dfrac{1}{} = \dfrac{-2|}{}\ \dfrac{1}{}\ 2 =$. This shows that $\lim_{\to 2}(1\ ) = \dfrac{1}{}$.

1. Guessing a value for Given 0, we must find 0 such that $|\sqrt{\ } - \sqrt{\ }|$ whenever 0 $|-|$. But

$| - | \quad \sqrt{\ } + \sqrt{\ }$

$\sqrt{\ } \quad \sqrt{\ } = | - |$ (from the hint). Now if we can find a positive constant such that $\sqrt{\ } + \sqrt{\ }$ then

$| - | \quad \dfrac{| - |}{\ }$, and we take . We can find this number by restricting to lie in some interval

centered at . If $| - |$ $\frac{1}{2}$, then $-$ $\frac{1}{2}$ $\frac{1}{2}$ $-$ $\frac{1}{2}$ $\Rightarrow$ $\frac{1}{2}$ $\frac{3}{2}$ $\Rightarrow \sqrt{\ } + \sqrt{\ }$ $-$ $-$ $\frac{1}{2} + \sqrt{\ }$, and so

$= \frac{1}{2} + \sqrt{\ }$ is a suitable choice for the constant. So $| - |$ $\frac{1}{2} + \sqrt{\ }$. This suggests that we let

$= \min \frac{1}{2} \quad \frac{1}{2} + \sqrt{\ }$.

2. Showing that works Given 0, we let $= \min \frac{1}{2} \quad \frac{1}{2} + \sqrt{\ }$. If 0 $| - |$, then

$| - | \quad \frac{1}{2} \Rightarrow \sqrt{\ } + \sqrt{\ } \quad \frac{1}{2} + \sqrt{\ }$ (as in part 1). Also $| - |$ $\frac{1}{2} + \sqrt{\ }$, so

$| - | \quad \sqrt{\ } + \sqrt{\ } \quad \dfrac{\frac{1}{2} + \sqrt{\ }}{\ } = \quad \lim$ ■ 1

38. Suppose that $\lim_{\to 0} ( ) =$. Given $= \frac{1}{2}$, there exists 0 such that 0 $| |$ $\Rightarrow | ( ) - | \frac{1}{2} \Leftrightarrow$

$- \frac{1}{2} ( ) + \frac{1}{2}$. For 0, $( ) = 1$, so $1 + \frac{1}{2} \Rightarrow \frac{1}{2}$. For -0, $( ) = 0$,

so $- \frac{1}{2} \quad 0 \Rightarrow \frac{1}{2}$. This contradicts $\frac{1}{2}$. Therefore, $\lim_{\to 0} ( )$ does not exist.

39. Suppose that $\lim_{\to 0} ( ) =$. Given $= \frac{1}{2}$, there exists 0 such that 0 $| |$ $\Rightarrow | ( ) - | \frac{1}{2}$. Take any rational

number with 0 $| |$. Then $( ) = 0$, so $|0 - | \frac{1}{2}$, so $\le | | \frac{1}{2}$. Now take any irrational number with

0 $| |$. Then $( ) = 1$, so $|1 - | \frac{1}{2}$. Hence, $1 - \frac{1}{2}$, so $\frac{1}{2}$. This contradicts $\frac{1}{2}$, so $\lim_{\to 0} ( )$ does not exist.

40. First suppose that $\lim_{\to} ( ) =$. Then, given 0 there exists 0 so that 0 $| - |$ $\Rightarrow | ( ) - |$.

Then $-$ $\Rightarrow 0 \quad | - |$ so $| ( ) - |$. Thus, $\lim_{\to^-} ( ) =$. Also $+$ $\Rightarrow$

0 $| - |$ so $| ( ) - |$. Hence, $\lim_{\to^+} ( ) =$.

Now suppose $\lim_{\to^-} ( ) = = \lim_{\to^+} ( )$. Let 0 be given. Since $\lim_{\to^-} ( ) =$, there exists $_1$ 0 so that

$- _1$ $\Rightarrow | ( ) - |$. Since $\lim_{\to^+} ( ) =$, there exists $_2$ 0 so that $+ _2$ $\Rightarrow$

$| ( ) - |$. Let be the smaller of $_1$ and $_2$. Then 0 $| - |$ $\Rightarrow - _1$ or $+ _2$ so

$( )$ $\lim ( ) =$ $\lim ( ) =$

$|\quad - |\quad$. Hence, $\to$. So we have proved that $\to$ $\Leftrightarrow$ $\lim_{\to^-} (\,) = = \lim_{\to^+} (\,)$.

41. $\dfrac{1}{(+3)^4}\quad 10{,}000 \quad\Leftrightarrow (+3)^4 \dfrac{1}{10{,}000} \quad\Leftrightarrow |+3| \dfrac{1}{\sqrt[4]{10{,}000}} \quad\Leftrightarrow\quad | - (-3)| \quad \dfrac{1}{10}$

42. Given 0, we need 0 such that 0 | + 3| ⇒ 1 (+ 3)^4. Now 1/(+3)4 ⇔

(+3)^4 1 ⇔|+3| 1/√4 . So take = 1/√4 . Then 0 | + 3|= 1/√4 ⇒ 1/(+3)4 , so

lim3 1/(+3)4 = ∞.
→−

43. Let 0 be given. Then, for −1, we have (+ 1)3 ⇔(+1)^3 ⇔ + 1 .

Let − . Then − − − ⇒ 5 5 3 5 ⇒ (, so →−1- (+ 1)3 −∞.
+1)3

= 3 5 1 1 3 5 + 1 0 5 lim 5 =

lim ( ) = 0 ()+1

44. (a) Let be given. Since → ∞, there exists 1 such that 0 | − |1 ⇒ − . Since lim
→ ( ) = , there exists 2 0 such that 0 | − | 2 ⇒ |()−| 1 ⇒ ( )− 1. Let be the smaller of 1 and 2. Then 0 | − | ⇒ ( ) + ( ) (+ 1 −) + (− 1) = . Thus,
lim
→ [( )+ ( )]=∞.

lim ()=0 0 0

(b) Let 0 be given. Since → , there exists 1 such that | − | 1 ⇒
|( ) 2 ⇒ ( )2 lim ( ) = ∞, there exists 0 such that 0 − ⇒
− | . Since → 2 | | 2
2 = lim

( ) 2 . Let = min { 1 2}. Then 0 | − | ⇒ ()() 2 , so → ()()=∞.

lim ( ) =0 0 0

(c) Let 0 be given. Since → , there exists 1 such that | − | 1 ⇒

|( ) 2 ⇒ ( ) 2. Since lim ( ) = ∞, there exists 0 such that 0 ⇒
− | − → 2 | − | 2

( ) 2 . (Note that 0 and 0 ⇒ 20.) Let = min { 1 2}. Then 0 | − | ⇒
( ) 2 ⇒ ()() 2 lim ( ) ( ) = −∞.
· 2 = , so →

1.8 Continuity

1. From Definition 1, lim
→4 ( ) = (4).

The graph of has no hole, jump, or vertical asymptote.

(a) is discontinuous at −4 since (−4) is not defined and at −2, 2, and 4 since the limit does not exist (the left and right limits are not the same).

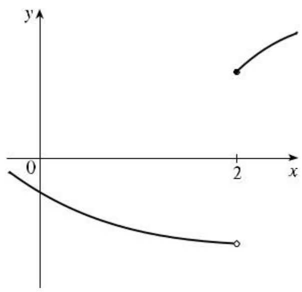
lim

(b) is continuous from the left at −2 since →-2$^-$ ( ) = (−2). is continuous from the right at 2 and 4 since

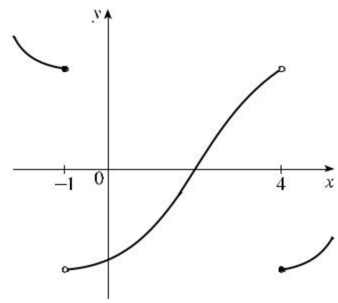
lim ( ) = (2) lim ( )= (4) 4 (4)

$\to 2^+$ and $\to 4^+$. It is continuous from neither side at $-$ since $-$ is undefined. From the graph of , we see that is continuous on the intervals $[-3\ -2)$, $(-2\ -1)$, $(-1\ 0]$, $(0\ 1)$, and $(1\ 3]$.

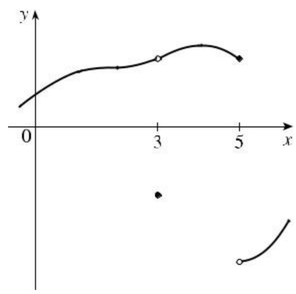
The graph of = ( ) must have a discontinuity at = 2



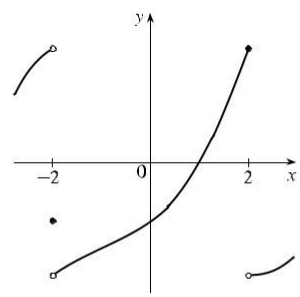
The graph of = ( ) must have discontinuities at = −1 and = 4. It must show that $\lim_{\to -1^-}$ ()= (1) − and $\lim_{\to 4^+}$ ( )= (4) .



The graph of = ( ) must have a removable discontinuity (a hole) at = 3 and a jump discontinuity at = 5.

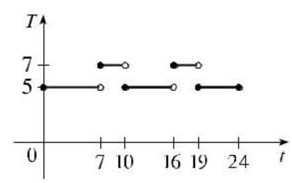


The graph of = ( ) must have a discontinuity at = −2 with $\lim_{\to -2^-}$ ( ) 6= (−2) and $\lim_{\to -2^+}$ ( ) 6= (−2). It must also show that $\lim_{\to 2^-}$ ( ) = (2) and $\lim_{\to 2^+}$ ( ) 6= (2).



9. (a) The toll is $7 between 7:00 AM and 10:00 AM and between 4:00 PM and 7:00 PM.

(b) The function has jump discontinuities at = 7, 10, 16, and 19. Their significance to someone who uses the road is that, because of the sudden jumps in the toll, they may want to avoid the higher rates between = 7 and = 10 and between = 16 and = 19 if feasible.



(a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.

Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values — at a cliff, for example.

Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.

(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

11. $\lim_{x\to-1} f(x) = \lim_{x\to-1}\left(x+2x^3\right)^4 = \left(\lim_{x\to-1} x + 2\lim_{x\to-1} x^3\right)^4 = \left[-1+2(-1)^3\right]^4 = (-3)^4 = 81 = f(-1)$.

By the definition of continuity, f is continuous at $a=-1$.

12. $\lim_{t\to2} g(t) = \lim_{t\to2}\dfrac{t^2+5t}{2t+1} = \dfrac{\lim_{t\to2}(t^2+5t)}{\lim_{t\to2}(2t+1)} = \dfrac{\lim_{t\to2} t^2 + 5\lim_{t\to2} t}{2\lim_{t\to2} t + \lim_{t\to2} 1} = \dfrac{2^2+5(2)}{2(2)+1} = \dfrac{14}{5} = g(2)$.

By the definition of continuity, g is continuous at $a=2$.

13. $\lim_{v\to1} p(v) = \lim_{v\to1} 2\sqrt{3v^2+1} = 2\lim_{v\to1}\sqrt{3v^2+1} = 2\sqrt{\lim_{v\to1}(3v^2+1)} = 2\sqrt{3\lim_{v\to1} v^2 + \lim_{v\to1} 1}$
$= 2\sqrt{3(1)^2+1} = 2\sqrt{4} = 4 = p(1)$

By the definition of continuity, p is continuous at $a=1$.

14. $\lim_{x\to2} f(x) = \lim_{x\to2}\left(3x^4 - 5x + \sqrt[3]{x^2+4}\right) = 3\lim_{x\to2} x^4 - 5\lim_{x\to2} x + \sqrt[3]{\lim_{x\to2}(x^2+4)}$

$= 3(2)^4 - 5(2) + \sqrt[3]{2^2+4} = 48 - 10 + 2 = 40 = f(2)$

By the definition of continuity, f is continuous at $a=2$.

15. For $a>4$, we have

$\lim_{x\to a} f(x) = \lim_{x\to a}\left(x+\sqrt{x-4}\right) = \lim_{x\to a} x + \lim_{x\to a}\sqrt{x-4}$ [Limit Law 1]

$= a + \sqrt{\lim_{x\to a} x - \lim_{x\to a} 4}$ [8, 11, and 2]

$= a + \sqrt{a-4}$ [8 and 7]

$= f(a)$

So f is continuous at $x=a$ for every a in $(4,\infty)$. Also, $\lim_{x\to4^+} f(x) = 4 = f(4)$, so f is continuous from the right at 4. Thus, f is continuous on $[4,\infty)$.

16. For $a<-2$, we have

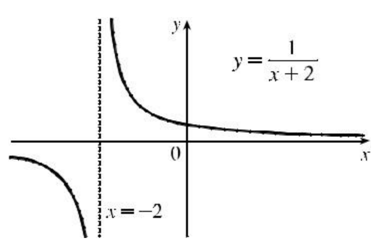
$\lim_{x\to a} g(x) = \lim_{x\to a}\dfrac{x-1}{3x+6} = \dfrac{\lim_{x\to a}(x-1)}{\lim_{x\to a}(3x+6)}$ [Limit Law 5]

$= \dfrac{\lim_{x\to a} x - \lim_{x\to a} 1}{3\lim_{x\to a} x + \lim_{x\to a} 6}$ [2, 1, and 3]

$= \dfrac{a-1}{3a+6}$ [8 and 7]

Thus, g is continuous at $x=a$ for every a in $(-\infty,-2)$; that is, g is continuous on $(-\infty,-2)$.

$f(x) = \dfrac{1}{x+2}$ is discontinuous at $a=-2$ because $f(-2)$ is undefined.



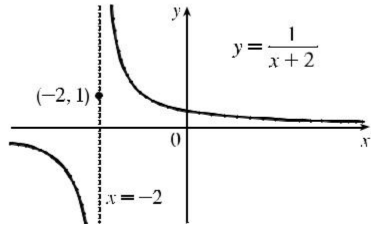
18. ( )= $\frac{1}{}$ + 2 if 6=−2

1 if = 2

(2)=1 , but lim ( ) = lim ( ) =

Here − →−2⁻ −∞ and →−2⁺ ∞,

so lim ( ) does not exist and is discontinuous at −2.
→−2



19. ( ) = 1 if 1

1 − 2 if1

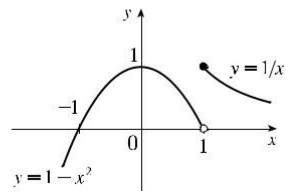
≥

The left-hand limit of at = 1 is

lim ( ) = lim (1 2)=0 = 1
→1⁻ →1⁻ − The right-hand limit of at is

lim ( ) = lim (1) = 1 Since these limits are not equal, lim ( )
→ →

does not exist and is discontinuous at 1.

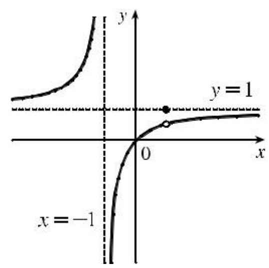


20. ()=2 − 1 6

2
1 − if = 1

lim ( ) = lim 2 − = lim (−1) = lim = 1,
→1 →1 2 − 1 →1 (+1)(−1) →1 +1 2

but (1) = 1, so is discontinous at 1

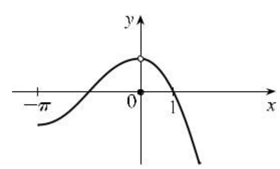


21. ( )=0 if = 0

cos if 0

1 − if 0

lim ( ) = 1, but (0) = 0 6= ,1so is discontinuous at 0.
→0

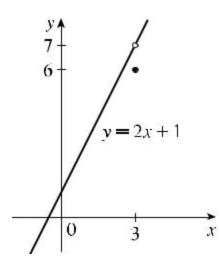


22. ( )= − 3 6

2 2
6 −5 −3 if = 3

lim ( ) = lim 2 2−5 −3 = lim (2 +1)(−3) = lim (2 +1)=7,
→3 →3 − 3 →3 − 3 →3

but (3) = 6, so is discontinuous at 3.



( ) = 2 − − 2 = (− 2)(+ 1) = + 1 for 6= .2Since lim ( ) = 2 + 1 = 3, define (2) = 3. Then is

−2−2→2

continuous at 2.

( ) = 3 − 8 = (− 2)(2 + 2 + 4) = 2 + 2 + 4 for 6= .2Since lim ( ) = 4 + 4 + 4 = 3, define (2) = 3.

2−4(−2)(+2)+2→22+2

Then is continuous at 2.

$(\ ) = \dfrac{2\ {}_2 - - 1}{{}_2 + 1}$ is a rational function, so it is continuous on its domain, $(-\infty\ \infty)$, by Theorem 5(b).

26. $(\ ) = \dfrac{{}^2 + 1}{2{}_2 - -1} = \dfrac{{}^2 + 1}{(2\ +1)(\ -1)}$ is a rational function, so it is continuous on its domain,

$-\infty\ -\ \frac{1}{2} \cup -\ \frac{1}{2}1 \cup (1\ \infty)$, by Theorem 5(b).

27. ${}^3\ -\ 2 = 0 \Rightarrow\ {}^3\ = 2 \Rightarrow\ = \sqrt[3]{2}$, so $(\ ) = \dfrac{\sqrt{3} - 2}{3 - 2}$ has domain $-\infty\ \sqrt[3]{2} \cup \sqrt[3]{2}\ \infty$. Now ${}^3\ -\ 2$ is

continuous everywhere by Theorem 5(a) and $\sqrt{3 - 2}$ is continuous everywhere by Theorems 5(a), 7, and 9. Thus, is continuous on its domain by part 5 of Theorem 4.

By Theorem 7, the trigonometric function sin and the polynomial function + 1 are continuous on R.
By part 5 of Theorem 4, $(\ ) = \dfrac{\sin}{+ 1}$ is continuous on its domain, { | 6=−1}.

By Theorem 5, the polynomial $1 - {}^2$ is continuous on $(-\infty\ \infty)$. By Theorem 7, cos is continuous on its domain, R. By Theorem 9, $\cos 1 - {}^2$ is continuous on its domain, which is R.

By Theorem 7, the trigonometric function tan is continuous on its domain, | 6=2 + . By Theorems 5(a), 7, and 9, the composite function $\sqrt{4 - {}_2}$ is continuous on its domain [−2 2]. By part 5 of Theorem 4, $(\ ) = \dfrac{\tan}{\sqrt{4 - 2}}$ is continuous on its

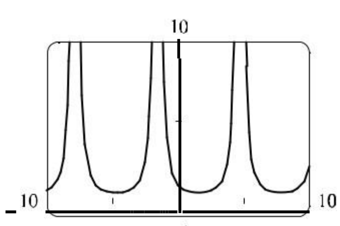
domain, (−2 − 2) ∪ (− 2 2) ∪ (2 2).

31. $(\ ) = \dfrac{1}{}1 + = \dfrac{}{+ 1}$ is defined when $\dfrac{}{+ 1} \geq 0 \Rightarrow + 1 \geq 0$ and 0 or $+ 1 \leq 0$ and $0 \Rightarrow 0$

or ≤ −1, so has domain (−∞ −1] ∪ (0 ∞). is the composite of a root function and a rational function, so it is continuous at every number in its domain by Theorems 7 and 9.

The sine and cosine functions are continuous everywhere by Theorem 7, so ( ) = sin(cos(sin)), which is the composite of sine, cosine, and (once again) sine, is continuous everywhere by Theorem 9.

33.

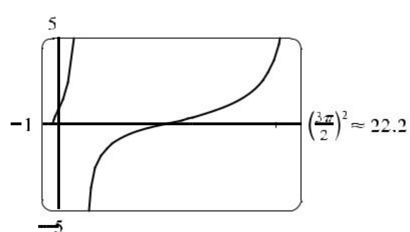


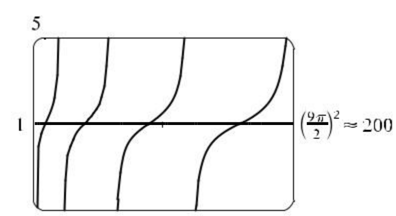
$= \dfrac{1}{1 + \sin}$ is undefined and hence discontinuous when

$1 + \sin\ = 0 \Leftrightarrow \sin\ = -1 \Leftrightarrow\ = -\ \frac{}{2} + 2$, an

integer. The figure shows discontinuities for = −1, 0, and 1; that is, $-\frac{5}{2} \approx -785, -\frac{}{2} \approx -1\ 57$, and $\frac{3}{2} \approx 471$.

34.





The function = ( ) = tan

is continuous throughout its domain because it is the composite of a trigonometric function

–

and a root function. The square root function has domain [0 ∞) and the tangent function has domain | 6 = $\frac{}{2}$ +.

So is discontinuous when 0 and when √ = $\frac{}{2}$ + ⇒ = $\frac{}{2}$ + ^{2}, where is a nonnegative integer. Note

that as increases, the distance between discontinuities increases.

Because is continuous on R and $\sqrt{20 - {}^2}$ is continuous on its domain, $-\sqrt{20} \le \le \sqrt{20}$, the product

( ) = $\sqrt{20 - {}^2}$ is continuous on $-\sqrt{20} \le \le \sqrt{20}$. The number 2 is in that domain, so is continuous at 2, and

$\lim\limits_{\to 2}$ ( ) = (2) = 2 $\sqrt{16}$ = 8.

Because is continuous on R, sin is continuous on R, and + sin is continuous on R, the composite function ( ) = sin(+ sin) is continuous on R, so $\lim\limits_{\to}$ ( ) = ( ) = sin(+ sin) = sin = 0.

The function ( ) = ^{2} tan is continuous throughout its domain because it is the product of a polynomial and a trigonometric function. The domain of is the set of all real numbers that are not odd multiples of 2 ; that is, domain = { | 6= 2, an odd integer}. Thus, 4 is in the domain of and

$$\lim_{\to 4} {}^2 \tan = = {}^2 \tan = {}^2 \quad 1 = {}^2 \quad 16$$

38. The function ( ) = $\frac{3}{\sqrt{\;\;+\;\;-2}}$ is continuous throughout its domain because it is the quotient of a polynomial and the square root of a polynomial. The domain of is | +−2 0={|(3 + 2)(1) 0 = } { | − or } −∞− ∪ ∞

$$\lim \frac{2}{} = (2) = \frac{2}{\sqrt{}} = 8 = 4$$

Thus, 2 is in the domain of and →2 √ 2 + − 2 2 2 2+2−2 √4 2 1 = (2) (1) .

39. ( )= $\sqrt{-1}$ if 1

1 − ^{2} if ≤ 1

By Theorem 5, since ( ) equals the polynomial 1 − ^{2} on (−∞ 1], is continuous on (−∞ 1]. By Theorem 7, since ( ) equals the root function $\sqrt{\;\;-1}$ on (1 ∞), is continuous on (1 ∞). At = 1, $\lim\limits_{\to 1^-}$ ( ) = $\lim\limits_{\to 1^-}$ (1 − ^{2}) = 1 − 1^2 = 0 and $\lim\limits_{\to 1^+}$ ( ) = $\lim\limits_{\to 1^+}$ $\sqrt{\;\;-1}$ = $\sqrt{1-1}$ = 0. Thus, $\lim\limits_{\to 1}$ ( ) exists and equals 0. Also, (1) = 1 − 1^2 = 0. Therefore, is continuous at = 1. We conclude that is continuous on (−∞ ∞).

40. ( ) = sin if 4

cos if ≥ 4

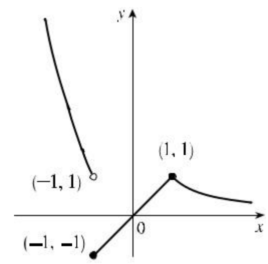
By Theorem 7, the trigonometric functions are continuous. Since ( ) = sin on (−∞ 4) and ( ) = cos on (4 ∞), is continuous on (−∞ 4) ∪ (4 ∞) $\lim\limits_{\to}$ ( ) = lim sin = sin 4 = 1 $\sqrt{2}$ since the sine function is continuous at 4 Similarly, lim ( ) = $\lim\limits_{\to}$ cos = 1 $\sqrt{2}$ by continuity of the cosine function

$\rightarrow(\ 4)^{+}$ $\rightarrow(\ 4)^{+}$

at 4. Thus, $\lim\limits_{\rightarrow(\ 4)}$ ( ) exists and equals $1\ \sqrt{\ \ 2}$, which agrees with the value (4). Therefore, is continuous at 4,

so is continuous on $(-\infty\ \infty)$.

41. ( )= if 1− 1
2 if 1
1 if ≥ 1

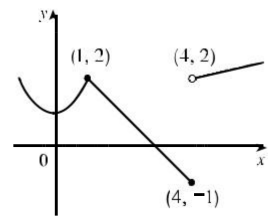


is continuous on (−∞ −1), (−1 1), and (1 ∞), where it is a polynomial, a polynomial, and a rational function, respectively.

Now lim ( ) = lim ^{2} = 1 and lim ( ) = lim = 1
→−1- →−1$^-$ →−1$^+$ →−1$^+$ − ,

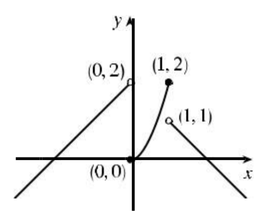
so is discontinuous at − . Since − − , is continuous from the right at − . Also, →1$^-$ →1$^-$ and
lim ( ) = lim 1 = 1 = (1), so is continuous at 1.
→ →

42. ( )=3 if 1 4
2 + 1 if ≤ 1 ≤
√ if 4
−



is continuous on (−∞ 1), (1 4), and (4 ∞), where it is a polynomial, a polynomial, and a root function, respectively. Now
lim lim (^{2} + 1) = 2 lim ( ) = lim (3) = 2
→1$^-$ ( ) = →1$^-$ and →1$^+$ →1$^+$ − . √ = 2
−
Since , we have continuity at . Also, →4$^-$ →4$^-$ − − and →4+ →4$^+$,
so is discontinuous at 4, but it is continuous from the left at 4.

43. ( )=2 ^{2} if 0 1
+ 2 if 0 ≤
2 − if 1



is since on each of
−∞ ∞
these intervals it is a polynomial. Now lim ( ) = lim (+ 2) = 2 and
→0$^-$ →0$^-$
lim ( ) = lim 2 ^{2} = 0, so is discontinuous at 0. Since (0) = 0, is continuous from the right at 0. Also
→0$^+$ →0$^+$
→1$^-$ ( )= →1- and →1$^+$ →1$^+$ − , so is discontinuous at . Since ,

is continuous from the left at 1.

By Theorem 5, each piece of is continuous on its domain. We need to check for continuity at = .
lim ( ) = lim 3 = 2 and lim ( ) = lim 2 = 2 , so lim ( ) = 2 . Since ( ) = 2 ,
→ $^-$ → $^-$ → $^+$ → $^+$ →
is continuous at . Therefore, is a continuous function of .

45. ( )= 3 − if ≥ 2
2 + 2 if2

lim ( ) = lim ^{2} + 2 =4 +4
is continuous on (−∞ 2) and (2 ∞). Now →2$^-$ →2$^-$ and

$\lim_{\to 2^+} (\,) = \lim_{\to 2^+} (\;^3 - \; - \;) = 8 - 2 - 2$. So is continuous $\Leftrightarrow$ $4 + 4 = 8 - 2 - 2 \Leftrightarrow 6 = 4 \Leftrightarrow \; = \frac{2}{3}$. Thus, for to be continuous on $(-\infty, \infty)$, $= \frac{2}{3}$.

46. $(\,) = \begin{cases} \dfrac{^2 - 4}{- 2} & \text{if } \; 2 \\ \;^2 - \; + 3 & \text{if } 2 \le \; 3 \\ 2 - \; + & \text{if } \ge 3 \end{cases}$

At $= 2$: $\lim_{\to 2^-} (\,) = \lim_{\to 2^-} \dfrac{^2 - 4}{- 2} = \lim_{\to 2^-} \dfrac{(\;+2)(\;-2)}{- 2} = \lim_{\to 2^-} (\; + 2) = 2 + 2 = 4$

$\lim_{\to 2^+} (\,) = \lim_{\to 2^+} (\;^2 - \; + 3) = 4 - 2 + 3$

We must have $4 - 2 + 3 = 4$, or $4 - 2 = 1$ **(1)**.

At $= 3$: $\lim_{\to 3^-} (\,) = \lim_{\to 3^-} (\;^2 - \; + 3) = 9 - 3 + 3$

$\lim_{\to 3^+} (\,) = \lim_{\to 3^+} (2 - \; + \;) = 6 - \; +$

We must have $9 - 3 + 3 = 6 - \; +$, or $10 - 4 = 3$ **(2)**.

Now solve the system of equations by adding -2 times equation **(1)** to equation **(2)**.

$$\begin{aligned} -8 + 4 &= -2 \\ 10 - 4 &= 3 \\ \hline 2 &= 1 \end{aligned}$$

So $= \frac{1}{2}$. Substituting $\frac{1}{2}$ for in **(1)** gives us $-2 = -1$, so $= \frac{1}{2}$ as well. Thus, for to be continuous on $(-\infty, \infty)$, $= = \frac{1}{2}$.

47. If and are continuous and $(2) = 6$, then $\lim_{\to 2} [3\,(\,) + (\,)\,(\,)] = 36 \Rightarrow$

$3 \lim_{\to 2} (\,) + \lim_{\to 2} (\,) \cdot \lim_{\to 2} (\,) = 36 \Rightarrow 3\,(2) + (2) \cdot 6 = 36 \Rightarrow 9\,(2) = 36 \Rightarrow (2) = 4$.

(a) $(\,) = \frac{1}{}$ and $(\,) = \frac{1}{^2}$, so $(\circ)(\,) = (\,(\,)) = (1/^2) = 1/(1/^2) = \;^2$.

The domain of $\circ$ is the set of numbers in the domain of (all nonzero reals) such that $(\,)$ is in the domain of (also all nonzero reals). Thus, the domain is $\{\; \neq 0 \text{ and } \frac{1}{} \neq 0\} = \{\;|\; \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$. Since $\circ$ is

the composite of two rational functions, it is continuous throughout its domain; that is, everywhere except $= 0$.

49. (a) $(\,) = \dfrac{^4 - 1}{- 1} = \dfrac{(\;^2+1)(\;^2-1)}{- 1} = \dfrac{(\;^2+1)(\;+1)(\;-1)}{- 1} = (\;^2 + 1)(\; + 1)$ [or $^3 + \;^2 + \; + 1$]

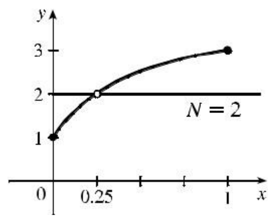
for $\neq 1$. The discontinuity is removable and $(\,) = \;^3 + \;^2 + \; + 1$ agrees with for $\neq 1$ and is continuous on $\mathbb{R}$.

(b) $(\,) = \dfrac{^3 - \;^2 - 2}{- 2} = \dfrac{(\;^2 - \; - 2)}{- 2} = \dfrac{(\;-2)(\;+1)}{- 2} = (\; + 1)$ [or $^2 +$] for $\neq 2$. The discontinuity

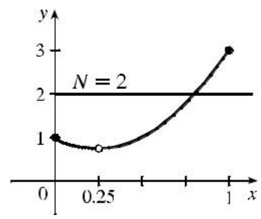
is removable and $(\,) = \;^2 +$ agrees with for $\neq 2$ and is continuous on $\mathbb{R}$.

(c) $\lim_{\to^-} (\,) = \lim_{\to^-} [[\sin\;]] = \lim_{\to^-} 0 = 0$ and $\lim_{\to^+} (\,) = \lim_{\to^+} [[\sin\;]] = \lim_{\to^+} (-1) = -1$, so $\lim_{\to} (\,)$ does not exist. The discontinuity at $=$ is a jump discontinuity.

50.



does not satisfy the conclusion of the Intermediate Value Theorem.



does satisfy the conclusion of the Intermediate Value Theorem.

( ) = ^{2} + 10 sin is continuous on the interval [31 32], (31) ≈ 957, and (32) ≈ 1030. Since 957 1000 1030, there is a number c in (31 32) such that ( ) = 1000 by the Intermediate Value Theorem. *Note:* There is also a number c in (−32 −31) such that ( ) = 1000

Suppose that (3) 6. By the Intermediate Value Theorem applied to the continuous function on the closed interval [2 3], the fact that (2) = 8 6 and (3) 6 implies that there is a number in (2 3) such that ( ) = 6. This contradicts the fact that the only solutions of the equation ( ) = 6 are = 1 and = 4. Hence, our supposition that (3) 6 was incorrect. It follows that (3) ≥ 6. But (3) 6= 6because the only solutions of ( ) = 6 are = 1 and = 4. Therefore, (3) 6.

( ) = ^{4} + − 3 is continuous on the interval [1 2] (1) = −1, and (2) = 15. Since −1 0 15, there is a number in (1 2) such that ( ) = 0 by the Intermediate Value Theorem. Thus, there is a root of the equation ^{4} + − 3 = 0 in the interval (1 2)

The equation 2 = − √ is equivalent to the equation 2 − + √ = 0. ( ) = 2 − + √ is continuous on the interval [2 3], (2) = 1 − 2 √ + 2 ≈ 0 414, and (3) = $\frac{2}{3}$ − 3 + 3 ≈ −0 601. Since (2) 0 (3), there is a number in (2 3) such that ( ) = 0 by the Intermediate Value Theorem. Thus, there is a root of the equation 2 − + √ = 0, or 2 = − √, in the interval (2 3).

( ) = cos − is continuous on the interval [0 1], (0) = 1, and (1) = cos 1 − 1 ≈ −0 46. Since −0 46 0 1, there is a number in (0 1) such that ( ) = 0 by the Intermediate Value Theorem. Thus, there is a root of the equation cos − = 0, or cos = , in the interval (0 1).

The equation sin = ^{2} − is equivalent to the equation sin − ^{2} + = 0. ( ) = sin − ^{2} + is continuous on the interval [1 2] (1) = sin 1 ≈ 0 84, and (2) = sin 2 − 2 ≈ −1 09. Since sin 1 0 sin 2 − 2, there is a number in (1 2) such that ( ) = 0 by the Intermediate Value Theorem. Thus, there is a root of the equation sin − ^{2} + = 0, or sin = ^{2} − , in the interval (1 2).

(a) ( ) = cos − ^{3} is continuous on the interval [0 1], (0) = 1 0, and (1) = cos 1 − 1 ≈ −0 46 0. Since 1 0 −0 46, there is a number in (0 1) such that ( ) = 0 by the Intermediate Value Theorem. Thus, there is a root of the equation cos − ^{3} = 0, or cos = ^{3}, in the interval (0 1).

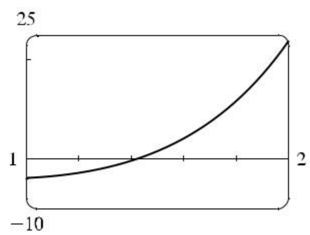
(0 86) ≈ 0 016 0 and (0 87) ≈ −0 014 0, so there is a root between 0 86 and 0 87, that is, in the interval (0 86 0 87).

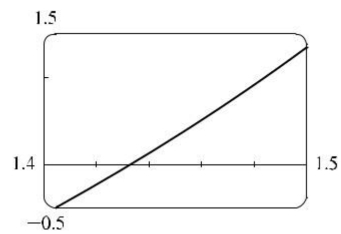
(a) ( ) = ^{5} − ^{2} + 2 + 3 is continuous on [−1 0], (−1) = −1 0, and (0) = 3 0. Since −1 0 3, there is a number in (−1 0) such that ( ) = 0 by the Intermediate Value Theorem. Thus, there is a root of the equation ^{5} − ^{2} + 2 + 3 = 0 in the interval (−1 0).

(−0 88) ≈ −0 062 0 and (−0 87) ≈ 0 0047 0, so there is a root between −0 88 and −0 87.

(a) Let ( ) = ^{5} − ^{2} − 4. Then (1) = $1^5 - 1^2$ − 4 = −4 0 and (2) = $2^5 - 2^2$ − 4 = 24 0. So by the Intermediate Value Theorem, there is a number in (1 2) such that ( ) = ^{5} − ^{2} − 4 = 0.

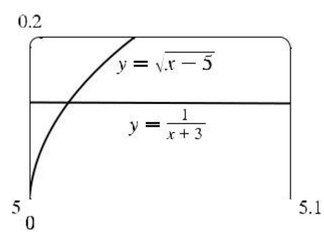
We can see from the graphs that, correct to three decimal places, the root is ≈ 1 434.





(a) Let ( ) = √ − 5 − + 1 3 . Then (5) = −$\frac{1}{8}$ 0 and (6) = $\frac{8}{9}$ 0, and is continuous on [5 ∞). So by the Intermediate Value Theorem, there is a number in (5 6) such that ( ) = 0. This implies that $\frac{1}{+3}$ = √ − 5.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is = 5 016, correct to three decimal places.



Let ( ) = sin ^{3}. Then is continuous on [1 2] since is the composite of the sine function and the cubing function, both of which are continuous on R. The zeros of the sine are at , so we note that 0 1 $\frac{3}{2}$ 2 8 3 , and that the pertinent cube roots are related by 1 $\sqrt[3]{\frac{3}{2}}$ [call this value] 2. [By observation, we might notice that = √$_3$ and = $\sqrt[3]{2}$ are zeros of .]

Now (1) = sin 1 0, ( ) = sin $\frac{3}{2}$ = −1 0, and (2) = sin 8 0. Applying the Intermediate Value Theorem on [1] and then on [2], we see there are numbers and in (1) and (2) such that ( ) = ( ) = 0. Thus, has at least two -intercepts in (1 2).

Let ( ) = ^{2} − 3 + 1 . Then is continuous on (0 2] since is a rational function whose domain is (0 ∞). By inspection, we see that $\frac{1}{4}$ = $\frac{17}{16}$ 0, (1) = −1 0, and (2) = $\frac{3}{2}$ 0. Appling the Intermediate Value Theorem on

$\frac{1}{4}$ 1 and then on [1 2], we see there are numbers and in $\frac{1}{4}$ 1 and (1 2) such that ( ) = ( ) = 0. Thus, has at least two -

intercepts in (0 2).

(⇒) If is continuous at , then by Theorem 8 with ( ) = + , we have

$\lim_{\to 0} (+) = \lim_{\to 0} (+) = ().$

(⇐) Let 0. Since $\to 0$, there exists such that | | ⇒

| (+) − ( )| . So if 0 | − | , then | ( ) − ( )| = | (+ (−)) − ( )| .
Thus, $\lim_{\to}$ ( ) = ( ) and so is continuous at .

64. $\lim_{\to 0} \sin(+) = \lim_{\to 0} (\sin \cos + \cos \sin) = \lim_{\to 0} (\sin \cos) + \lim_{\to 0} (\cos \sin)$

$= \lim_{\to 0} \sin \lim_{\to 0} \cos + \lim_{\to 0} \cos \lim_{\to 0} \sin = (\sin)(1) + (\cos)(0) = \sin$

65. As in the previous exercise, we must show that $\lim_{\to 0} \cos(+) = \cos$ to prove that the cosine function is continuous.

$\lim_{\to 0} \cos(+) = \lim_{\to 0} (\cos \cos - \sin \sin) = \lim_{\to 0} (\cos \cos) - \lim_{\to 0} (\sin \sin)$

$= \lim_{\to 0} \cos \lim_{\to 0} \cos - \lim_{\to 0} \sin \lim_{\to 0} \sin = (\cos)(1) - (\sin)(0) = \cos$

66. (a) Since is continuous at , $\lim_{\to}$ ( ) = ( ). Thus, using the Constant Multiple Law of Limits, we have

$\lim_{\to} ( )( ) = \lim_{\to} ( ) = \lim_{\to} ( ) = ( ) = ( )( )$. Therefore, is continuous at .

(b) Since and are continuous at , $\lim_{\to}$ ( ) = ( ) and $\lim_{\to}$ ( ) = ( ). Since ( ) 6= 0, we can use the Quotient Law of Limits: $\lim_{\to} ( ) = \lim_{\to} \frac{( )}{( )} = \frac{\lim_{\to} ( )}{\lim_{\to} ( )} = \frac{( )}{( )} = ( )$. Thus, is continuous at .

67. $( ) = \begin{cases} 0 & \text{if is rational} \\ 1 & \text{if is irrational} \end{cases}$ is continuous nowhere. For, given any number and any 0, the interval (− +)

contains both infinitely many rational and infinitely many irrational numbers. Since ( ) = 0 or 1, there are infinitely many numbers with 0 | − | and | ( ) − ( )| = 1. Thus, $\lim_{\to} ( ) \neq ( )$. [In fact, $\lim_{\to} ( )$ does not even exist.]

68. $( ) = \begin{cases} 0 & \text{if is rational} \\ & \text{if is irrational} \end{cases}$ is continuous at 0. To see why, note that −| | ≤ ( ) ≤ | |, so by the Squeeze Theorem

$\lim_{\to 0} ( ) = 0 = (0)$. But is continuous nowhere else. For if 6= 0 and 0, the interval (− +) contains both infinitely many rational and infinitely many irrational numbers. Since ( ) = 0 or , there are infinitely many numbers with 0 | − | and | ( ) − ( )| | | 2. Thus, $\lim_{\to}$ ( ) 6= ( ).

69. If there is such a number, it satisfies the equation $^3 + 1 = ⇔ ^3 − + 1 = 0$. Let the left-hand side of this equation be called ( ). Now (−2) = −5 0, and (−1) = 1 0. Note also that ( ) is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number between −2 and −1 such that ( ) = 0, so that $= ^3 + 1$.

70. $\frac{}{3+2\,2-1}$ + $\frac{}{3+\;-2}$ = 0 ⇒ (^{3} + − 2) + (^{3} + 2 ^{2} − 1) = 0. Let ( ) denote the left side of the last equation. Since is continuous on [−1 1], (−1) = −4 0, and (1) = 2 0, there exists a in (−1 1) such that ( ) = 0 by the Intermediate Value Theorem. Note that the only root of either denominator that is in (−1 1) is

(−1 + $\sqrt{5}$) 2 = , but ( ) = (3 $\sqrt{5}$ − 9) 2 6= .0Thus, is not a root of either denominator, so ( ) = 0 ⇒ = is a root of the given equation.

( ) = ^{4} sin(1) is continuous on (−∞ 0) ∪ (0 ∞) since it is the product of a polynomial and a composite of a trigonometric function and a rational function. Now since −1 ≤ sin(1) ≤ 1, we have − ^{4} ≤ ^{4} sin(1) ≤ ^{4}. Because lim (− ^{4}) = 0 and lim ^{4} = 0, the Squeeze Theorem gives us lim (^{4} sin(1)) = 0, which equals (0). Thus, is continuous at 0 and, hence, on (−∞ ∞).

72. (a) lim ( ) = 0 and lim ( ) = 0, so lim ( ) = 0, which is (0), and hence is continuous at = if = 0. For

, → → . For ,→ → − − . Thus, is continuous at = ; that is, continuous everywhere.

(b) Assume that is continuous on the interval . Then for ∈ , lim ( ) = lim ( ) = ( ) by Theorem 8. (If is an endpoint of , use the appropriate one-sided limit.) So | | is continuous on .

(c) No, the converse is false. For example, the function ( ) = 1 if ≥ 0, −1 if 0 is not continuous at = 0, but | ( )| = 1 is continuous on R.

Define ( ) to be the monk's distance from the monastery, as a function of time (in hours), on the first day, and define ( ) to be his distance from the monastery, as a function of time, on the second day. Let be the distance from the monastery to the top of the mountain. From the given information we know that (0) = 0, (12) = , (0) = and (12) = 0. Now consider the function − , which is clearly continuous. We calculate that (−)(0) = − and (−)(12) = .

So by the Intermediate Value Theorem, there must be some time $_0$ between 0 and 12 such that (−)($_0$) = 0 ⇔ ($_0$) = ($_0$). So at time $_0$ after 7:00 AM, the monk will be at the same place on both days.

1 Review

TRUE-FALSE QUIZ

1. False. Let ( ) = ^{2} , = −1, and = 1. Then (+) = $(-1 + 1)^2 = 0^2 = 0$, but ( )+ ()=$(-1)^2+1^2$=26=0=(+).

2. False. Let ( ) = ^{2} . Then (−2) = 4 = (2), but −2 6= .2

3. False. Let ( ) = ^{2}. Then (3) = $(3\ )^2$ = 9 ^{2} and 3 ( ) = 3 ^{2}. So (3) 6= 3( ).

True. If 12 and is a decreasing function, then the -values get smaller as we move from left to right. Thus, $(_1)(_2)$.

True. See the Vertical Line Test.

False. For example, if $= -3$, then $\sqrt{(-3)^2} = \sqrt{9} = 3$, not -3.

False. Limit Law 2 applies only if the individual limits exist (these don't).

False. Limit Law 5 cannot be applied if the limit of the denominator is 0 (it is).

True. Limit Law 5 applies.

False. $\frac{^2 - 9}{- 3}$ is not defined when $= 3$, but $+ 3$ is.

True. $\lim_{\to 3} \frac{^2 - 9}{- 3} = \lim_{\to 3} \frac{(+3)(-3)}{(-3)} = \lim_{\to 3} (+3)$

True. The limit doesn't exist since $()\ ()$ doesn't approach any real number as approaches 5. (The denominator approaches 0 and the numerator doesn't.)

False. Consider $\lim_{\to 5} \frac{(-5)}{- 5}$ or $\lim_{\to 5} \frac{\sin(-5)}{- 5}$. The first limit exists and is equal to 5. By Example 1.5.3, we know that the latter limit exists (and it is equal to 1).

False. If $() = 1$, $() = -1$, and $= 0$, then $\lim_{\to 0} ()$ does not exist, $\lim_{\to 0} ()$ does not exist, but $\lim_{\to 0} [() + ()] = \lim_{\to 0} 0 = 0$ exists.

True. Suppose that $\lim_{\to} [() + ()]$ exists. Now $\lim_{\to} ()$ exists and $\lim_{\to} ()$ does not exist, but $\lim_{\to} () = \lim_{\to} \{[() + ()] - ()\} = \lim_{\to} [() + ()] - \lim_{\to} ()$ [by Limit Law 2], which exists, and we have a contradiction. Thus, $\lim_{\to} [() + ()]$ does not exist.

False. Consider $\lim_{\to 6} [()\,()] = \lim_{\to 6} \left[(-6)\frac{1}{-6}\right]$. It exists (its value is 1) but $(6)=0$ and (6) does not exist, so $(6)\,(6)\ 6= 1$.

True. A polynomial is continuous everywhere, so $\lim_{\to} ()$ exists and is equal to $()$.

False. Consider $\lim_{\to 0} [() - ()] = \lim_{\to 0} \left(\frac{1}{^2} - \frac{1}{^4}\right)$. This limit is $-\infty$ (not 0), but each of the individual functions approaches ∞.

False. Consider $() = \begin{cases} 1(-1) & \text{if } 6= 1 \\ 2 & \text{if } = 1 \end{cases}$

20. False. The function must be *continuous* in order to use the Intermediate Value Theorem. For example, let

$$( )= \begin{cases} 1 & \text{if } 0 \le 3 \\ -1 & \text{if } = 3 \end{cases}$$

There is no number $\in [0\ 3]$ with $( ) = 0$.

21. True. Use Theorem 1.8.8 with $= 2$, $= 5$, and $( ) = 4^{2} - 11$. Note that $(4) = 3$ is not needed.

22. True. Use the Intermediate Value Theorem with $= -1$, $= 1$, and $=$, since 34.

True, by the definition of a limit with $= 1$.

24. False. For example, let $( ) = \begin{cases} ^2 + 1 & \text{if } 6= 0 \\ 2 & \text{if } = 0 \end{cases}$

Then $( )$ 1 for all , but $\lim_{\to 0} ( ) = \lim_{\to 0} {}^2 + 1 = 1$.

True. $( ) = {}^{10} - 10^{2} + 5$ is continuous on the interval $[0\ 2]$, $(0) = 5$, $(1) = -4$, and $(2) = 989$. Since $-4\ 0\ 5$, there is a number in $(0\ 1)$ such that $( ) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation ${}^{10} - 10^{2} + 5 = 0$ in the interval $(0\ 1)$. Similarly, there is a root in $(1\ 2)$.

26. True. See Exercise 1.8.72(b).

27. False See Exercise 1.8.72(c).

EXERCISES

1. (a) When $= 2$, $\approx 2\ 7$. Thus, $(2) \approx 2\ 7$.

(b) $( ) = 3 \Rightarrow \approx 2\ 3, 5\ 6$

(c) The domain of is $-6 \le \le 6$, or $[-6\ 6]$.

(d) The range of is $-4 \le \le 4$, or $[-4\ 4]$.

(e) is increasing on $[-4\ 4]$, that is, on $-4 \le \le 4$.

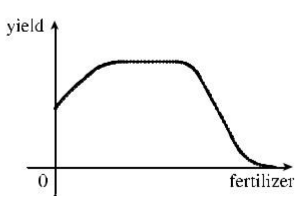
(f) is odd since its graph is symmetric about the origin.

(a) This curve *is not* the graph of a function of since it *fails* the Vertical Line Test.

This curve *is* the graph of a function of since it *passes* the Vertical Line Test. The domain is $[-3\ 3]$ and the range is $[-2\ 3]$.

$( ) = {}^2 - 2 + 3$, so $(+) = (+)^2 - 2(+) + 3 = {}^2 + 2 + {}^2 - 2 - 2 + 3$, and $\underline{(+) - ( )} = \underline{({}^2 + 2 + {}^2 - 2 - 2 + 3) - ({}^2 - 2 + 3)} = \underline{(2 + - 2)} = 2 + - 2$.

4. There will be some yield with no fertilizer, increasing yields with increasing fertilizer use, a leveling-off of yields at some point, and disaster with too much fertilizer use.



5. ( )=2 (3 −1). Domain: 3 −16=0⇒ 3 6=1⇒ 6=3$\frac{1}{}$. = −∞ $\frac{1}{3}$ ∪ $\frac{1}{3}$ ∞

Range: all reals except 0 (= 0 is the horizontal asymptote for .)

=(−∞ 0) ∪ (0 ∞)

6. ( )= √ 16− 4 Domain: 16− ^{4}≥0 ⇒ ^{4}≤16 ⇒ | |≤ √4 16 ⇒ | |≤2. = [−2 2]

Range: ≥ 0 and ≤ √ 16 ⇒ 0≤ ≤4.
=[0 4]

7. = 1 + sin . Domain: R.

Range: −1 ≤ sin ≤ 1 ⇒ 0 ≤ 1 + sin ≤ 2 ⇒ 0≤ ≤2. =[0 2]

8. = ( ) = 3 + cos 2 . Domain: R.= (−∞ ∞)

Range: −1 ≤ cos 2 ≤ 1 ⇒ 2 ≤ 3 + cos 2 ≤ 4 ⇒ 2 ≤ ≤ 4.
=[2 4]

(a) To obtain the graph of = ( ) + 8, we shift the graph of = ( ) up 8 units.

To obtain the graph of = (+ 8), we shift the graph of = ( ) left 8 units.

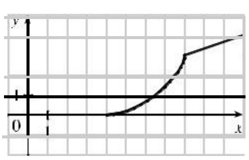
To obtain the graph of = 1 + 2 ( ), we stretch the graph of = ( ) vertically by a factor of 2, and then shift the resulting graph 1 unit upward.

To obtain the graph of = (− 2) − 2, we shift the graph of = ( ) right 2 units (for the "−2" inside the parentheses), and then shift the resulting graph 2 units downward.

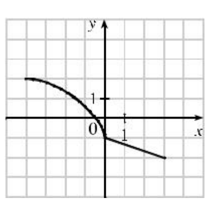
To obtain the graph of = − ( ), we reflect the graph of = ( ) about the -axis.

To obtain the graph of = 3 − ( ), we reflect the graph of = ( ) about the -axis, and then shift the resulting graph 3 units upward.

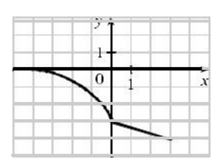
10. (a) To obtain the graph of = (− 8), we shift the graph of = ( ) right 8 units.



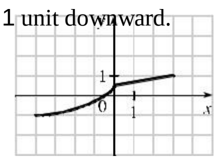
To obtain the graph of = 2 − ( ), we reflect the graph of = ( ) about the -axis, and then shift the resulting graph 2 units upward.



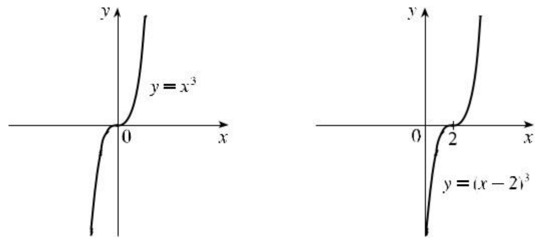
(b) To obtain the graph of = − ( ), we reflect the graph of = ( ) about the -axis.



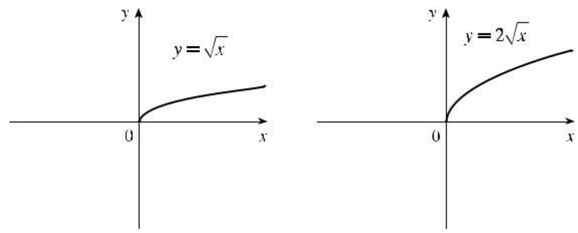
To obtain the graph of = $\frac{1}{2}$ ( ) − 1, we shrink the graph of = ( ) by a factor of 2, and then shift the resulting graph 1 unit downward.



11. $= (\; - 2)^3$: Start with the graph of $= \; ^3$ and shift 2 units to the right.

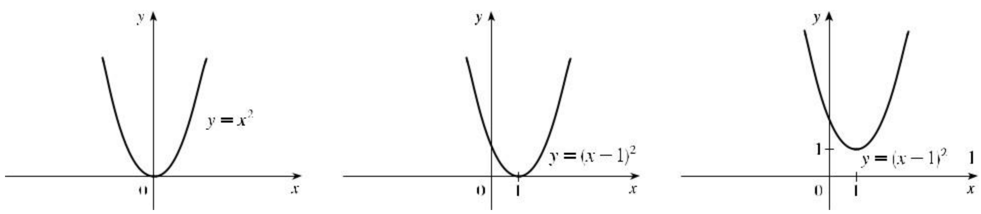


√ √

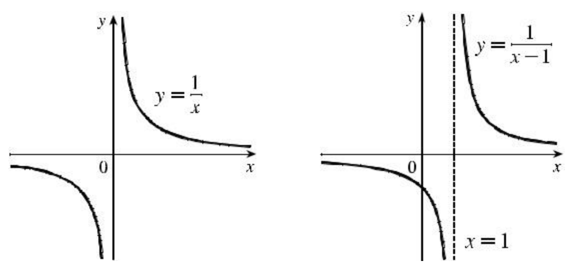


12. $= 2$: Start with the graph of $=$ and stretch

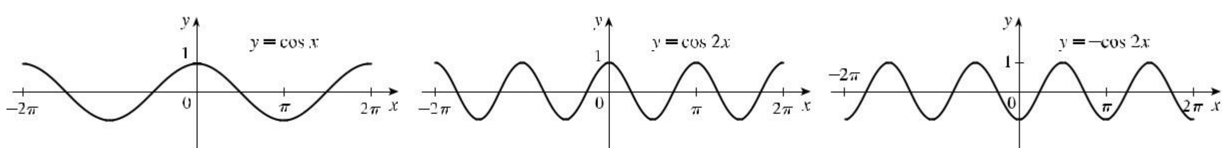
$= {}^2 - 2 + 2 = ({}^2 - 2 + 1) + 1 = (- 1)^2 + 1$: Start with the graph of $= {}^2$, shift 1 unit to the right, and shift 1 unit upward.



14. $= \dfrac{1}{- 1}$: Start with the graph of $= \dfrac{1}{}$ and shift 1 unit to

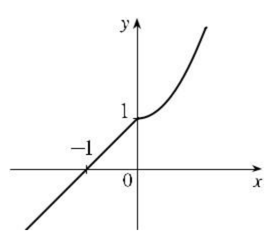


$( ) = -\cos 2$: Start with the graph of $= \cos$, shrink horizontally by a factor of 2, and reflect about the -axis.



16. $( ) =$ $1 + {}^2$ if 0
$1 +$ if0

≥



On $(-\infty\ 0)$, graph $= 1 +$ (the line with slope 1 and -intercept 1) with open endpoint (0 1).

On $[0\ \infty)$, graph $= 1 \quad + {}^2$ (the rightmost half of the parabola $= {}^2$ shifted 1 unit upward) with closed endpoint (0 1).

(a) The terms of are a mixture of odd and even powers of , so is neither even nor odd.

The terms of are all odd powers of , so is odd.

$(-\ ) = \cos(-\ )^2 = \cos(\ ^2) = (\ )$, so is even.

$(-\ ) = 1 + \sin(-\ ) = 1 - \sin$. Now $(-\ ) \neq (\ )$ and $(-\ ) \neq -(\ )$, so is neither even nor odd.

18. For the line segment from (2 2) to (1 0), the slope is $\frac{0-2}{-1+2} = -2$, and an equation is $-0 = -2(\ +1)$ or, equivalently, $= -2 - 2$. The circle has equation $^2 + ^2 = 1$; the top half has equation $= \sqrt{1 - ^2}$ (we have solved for positive). Thus, $(\ ) = \sqrt{1 - ^2}$ if $-1 \le \ \le 1$

$-2\ -2$ if $-2 \le \ \le -1$

19. $(\ ) = \sqrt{\ }$, $= [0\ \infty)$; $(\ ) = \sin$, $= \mathbb{R}$.

(a) $(\ \circ\ )(\ ) = (\ (\ )) = (\sin\ ) = \sqrt{\sin}$. For $\sqrt{\sin}$ to be defined, we must have $\sin \ge 0 \Leftrightarrow$

$\in [0\ ], [2\ 3\ ], [-2\ -\ ], [4\ 5\ ], [-4\ -3\ ],$, so $= \{\ \mid\ \in [2 + 2\ ]$, where is an integer}.

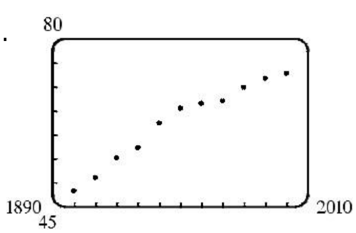
$(\ \circ\ )(\ ) = (\ (\ )) = (\sqrt{\ }) = \sin\sqrt{\ }$. must be greater than or equal to 0 for $\sqrt{\ }$ to be defined, so $= [0\ \infty)$.

$(\ \circ\ )(\ ) = (\ (\ )) = (\sqrt{\ }) = \sqrt{\sqrt{\ }} = \sqrt[4]{\ }$. $= [0\ \infty)$.

$(\ \circ\ )(\ ) = (\ (\ )) = (\sin\ ) = \sin(\sin\ )$. $= \mathbb{R}$.

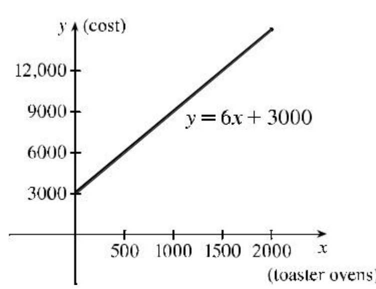
Let $(\ ) = +\sqrt{\ }$, $(\ ) = \sqrt{\ }$, and $(\ ) = 1$. Then $(\ \circ\ \circ\ )(\ ) = \frac{1}{+\sqrt{\ }} = (\ )$.

21.



Many models appear to be plausible. Your choice depends on whether you think medical advances will keep increasing life expectancy, or if there is bound to be a natural leveling-off of life expectancy. A linear model, $= 0\ 2493 - 423\ 4818$, gives us an estimate of 77 6 years for the year 2010.

22. (a) Let denote the number of toaster ovens produced in one week and the associated cost. Using the points (1000 9000) and (1500 12,000), we get an equation of a line:



$$-9000 = \frac{12{,}000 - 9000}{1500 - 1000}(\ -1000) \Rightarrow$$

$$= 6(\ -1000) + 9000 \Rightarrow \ = 6\ + 3000.$$

The slope of 6 means that each additional toaster oven produced adds \$6 to the weekly production cost.

The -intercept of 3000 represents the overhead cost — the cost incurred without producing anything.

23. (a) (i) $\lim_{\to 2^+} (\ ) = 3$ (ii) $\lim_{\to -3^+} (\ ) = 0$

(iii) $\lim_{\to -3} (\ )$ does not exist since the left and right limits are not equal. (The left limit is -2.)

(iv) $\lim_{\to 4} (\ ) = 2$

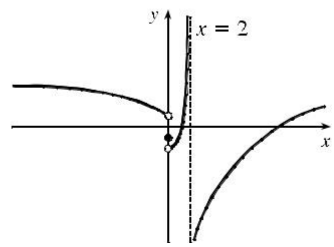
(v) $\lim_{\to 0} (\ ) = \infty$ (vi) $\lim_{\to 2^-} (\ ) = -\infty$

The equations of the vertical asymptotes are $= 0$ and $= 2$.

is discontinuous at $= -3, 0, 2$, and 4. The discontinuities are jump, infinite, infinite, and removable, respectively.

24. $\lim_{\to 0^+} (\ ) = -2$, $\lim_{\to -}$ $\lim_{\to} (\ ) = 1$ $(0) = 1$

$\lim_{\to 2^-} (\ ) = \infty$, $\lim_{\to 2^+} (\ ) = -\infty$



25. $\lim_{\to 0} \cos(\ + \sin\ ) = \cos \lim_{\to 0} (\ + \sin\ )$ [by Theorem 1.8.8] $= \cos 0 = 1$

26. Since rational functions are continuous, $\lim_{\to 3} \dfrac{^2 - 9}{^2 + 2 - 3} = \dfrac{3^2 - 9}{3^2 + 2(3) - 3} = \dfrac{0}{12} = 0$.

27. $\lim_{\to -3} \dfrac{^2 - 9}{^2 + 2 - 3} = \lim_{\to -3} \dfrac{(+3)(-3)}{(+3)(-1)} = \lim_{\to -3} \dfrac{-3}{-1} = \dfrac{-3-3}{-3-1} = \dfrac{-6}{-4} = \dfrac{3}{2}$

28. $\lim_{\to 1^+} \dfrac{^2 - 9}{^2 + 2 - 3} = -\infty$ since $^2 + 2 - 3 \to 0^+$ as $\to 1^+$ and $\dfrac{^2 - 9}{^2 + 2 - 3} \to 0$ for $1 < < 3$.

29. $\lim_{\to 0} \dfrac{(-1)^3 + 1}{} = \lim_{\to 0} \dfrac{^3 - 3^2 + 3 - 1 + 1}{} = \lim_{\to 0} \dfrac{^3 - 3^2 + 3}{} = \lim_{\to 0} (^2 - 3 + 3) = 3$

Another solution: Factor the numerator as a sum of two cubes and then simplify.

$\lim_{\to 0} \dfrac{(-1)^3 + 1}{} = \lim_{\to 0} \dfrac{(-1)^3 + 1^3}{} = \lim_{\to 0} \dfrac{[(-1) + 1][(-1)^2 - 1(-1) + 1^2]}{}$

$= \lim_{\to 0} [(-1)^2 - + 2] = 1 - 0 + 2 = 3$

30. $\lim_{\to 2} \dfrac{^2 - 4}{^3 - 8} = \lim_{\to 2} \dfrac{(+2)(-2)}{(-2)(^2 + 2 + 4)} = \lim_{\to 2} \dfrac{+2}{^2 + 2 + 4} = \dfrac{2+2}{4+4+4} = \dfrac{4}{12} = \dfrac{1}{3}$

31. $\lim_{\to 9} \dfrac{\sqrt{\ }}{(-9)^4} = \infty$ since $(-9)^4 \to 0^+$ as $\to 9$ and $\dfrac{\sqrt{\ }}{(-9)^4} > 0$ for $\neq 9$.

32. $\lim_{\to 4^+} \dfrac{4 -}{|4 - |} = \lim_{\to 4^+} \dfrac{4 -}{-(4 -)} = \lim_{\to 4^+} \dfrac{1}{-1} = -1$

33. $\lim_{\to 1} \frac{\;^4 - 1}{\;^3+5\;^2-6} = \lim_{\to 1} \frac{(\;^2+1)(\;^2-1)}{(\;^2+5-6)} = \lim_{\to 1} \frac{(\;^2+1)(+1)(-1)}{(+6)(-1)} = \lim_{\to 1} \frac{(\;^2+1)(+1)}{(+6)} = \frac{2(2)}{1(7)} = \frac{4}{7}$

34. lim √ + 6 − = lim √ + 6 − √+ 6 + = lim (√+6)^2 − ^{2}

→3 3 3 2 →3 2(3) √+6+ →3 2(3) √+6+

= lim + 6 − ^{2} = lim −(− − 6) = lim −(−3)(+ 2)

2 3) √+6+ 3 2 √ 3 2 √

= lim → (−(+2) 5 → = (5−3) + 6 + → (−3) + 6 +

→3 2 +6+ 9(3 + 3) 54

35. lim 4 − = lim 4 − = lim −1 = − 1 = 1

→16 − 16 →16 (√ + 4)(√ − 4) →16 √ + 4 √16+4 − 8

36. lim ^{2}+2 −8 = lim (+4)(−2) = lim + 4 = 2 + 4 = 3

→2 4−16 →2 (+2)(−2)(2+4) →2 (+2)(2+4) (2 + 2)(22 + 4) 16

37. lim 1 − 1 − 1 + √ 1 − 2 = 1−(1 − 2) = lim − 2 = lim − = 0

lim

→0 1 + 1− 2 →0 1+ 1− 2 →0 1 + 1− 2 →01+ 1− 2

→1 − 1 2−3 +2 →1 − 1 (−1)(−2) →1 (−1)(−2) (−1)(−2)

38. lim 1 + 1 = lim 1 + 1 = lim − 2 + 1

→1 (−1)(−2) →1 − 2 1 − 2 −

= lim − 1 = lim 1 = 1 = 1

2 lim (2 1) = 1 = lim 2 lim ( ) = 1

39. Since 2 − 1 ≤ ( ) ≤ for 0 3 and →1 − →1 , we have →1 by the Squeeze Theorem.

Let ( ) = − 2, ( ) = 2 cos 1 2 and ( ) = 2. Then since cos 1 2 ≤ 1 for 6= ,0 we have

( ) ( ) ( ) = 0 lim ( ) = lim ( ) = 0 lim ( ) = 0

⇒

≤ ≤ for 6 , and so →0 →0 →0 by the Squeeze Theorem.

41. Given 0, we need 0 such that if 0 | − 2| , then |(14 − 5) − 4| . But |(14 − 5) − 4| ⇔

|−5 + 10| ⇔ |−5|| − 2| ⇔ | − 2| 5. So if we choose = 5, then 0 | − 2| ⇒

lim

|(14 − 5) − 4| . Thus, →2 (14 − 5) = 4 by the definition of a limit.

42. Given 0 we must find 0 so that if 0 | − 0| , then | √3 − 0| . Now | √3 − 0| = | √3 | ⇒

= 3 3 3 3 0 = 3 √ 3 0 = 3 = 3 √3 3 =

| | | | . So take = . Then 0 3 − | | | ⇒ | − | |√ | | | .

Therefore, by the definition of a limit, lim = 0.

→0

Given 0, we need 0 so that if 0 | − 2| , then ^{2} − 3 − (−2). First, note that if | − 2| 1, then

−1 − 2 1, so 0 − 1 2 ⇒ | − 1| 2. Now let = min { 2 1}. Then 0 | − 2| ⇒

^{2}−3 −(2−2) =|(−2)(−1)|=| −2|| −1| (2)(2)= . →

Thus, lim (− 3) = 2 by the definition of a limit.

44. Given 0, we need 0 such that if 0 − 4 , then 2

$\sqrt{} - 4$. This is true $\Leftrightarrow$ $\sqrt{}$ $- 4 \quad 2 \quad \Leftrightarrow - 4 \quad 4 \quad ^2$.

So if we choose

$= 4 \quad ^2$, then $0 \quad - 4 \quad \Rightarrow 2 \sqrt{} - 4$. So by the definition of a limit,

$\lim_{\to 4^+} 2 \sqrt{ - 4} = \infty$.

(a) $(\,) = \sqrt{-}$ if 0, $(\,) = 3 -$ if $0 \le 3$, $(\,) = (\;-3)^2$ if 3.

(i) $\lim_{\to 0^+} (\,) = \lim_{\to 0^+} (3 \; - \;) = 3$

(ii) $\lim_{\to 0^-} (\,) = \lim_{\to 0^-} \sqrt{-} = 0$

(iii) Because of (i) and (ii), $\lim_{\to 0} (\,)$ does not exist.

$\lim (\,) = \lim (3 - \,) = 0$

$\to 3^- \to 3$

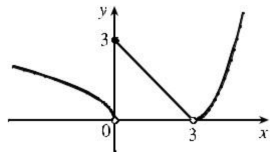
(v) $\lim_{\to 3^+} (\,) = \lim_{\to 3^+} (\; - 3)^2 = 0$

(vi) Because of (iv) and (v), $\lim_{\to 3} (\,) = 0$.

(b) is discontinuous at 0 since $\lim_{\to 0} (\,)$ does not exist.

is discontinuous at 3 since (3) does not exist.

(c)



(a) $(\,) = 2 \; - ^2$ if $0 \le \; \le 2$, $(\,) = 2 -$ if $2 \le 3$, $(\,) = \; - 4$ if $3 4$, $(\,) =$ if ≥ 4.

Therefore, $\lim_{\to 2^-} (\,) = \lim_{\to 2^-} 2 \; - {}^2 = 0$ and $\lim_{\to 2^+} (\,) = \lim_{\to 2^+} (2 \; - \;) = 0$. Thus, $\lim_{\to 2} (\,) = 0 = (2)$,

2 lim ( ) = lim(2) = 1 lim ( ) = lim (4) = 1

so is continuous at . $\to 3^-$ $\to 3^-$ $-$ $-$ and $\to 3^+$ $\to 3^+$ $-$ $-$. Thus,

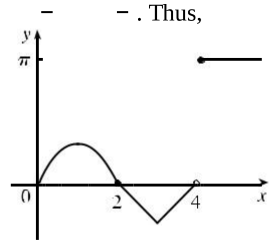
$\lim_{\to 3} (\,) = -1 = (3)$, so is continuous at 3.

$\lim_{\to 4^-} (\,) = \lim_{\to 4^-} (\; - 4) = 0$ and $\lim_{\to 4^+} (\,) = \lim_{\to 4^+} = \,$.

Thus, $\lim_{\to 4} (\,)$ does not exist, so is discontinuous at 4. But

$\lim_{\to 4^+} (\,) = \; = (4)$, so is continuous from the right at 4.

(b)



^{3} is continuous on $\mathbb{R}$ since it is a polynomial and $\cos$ is also continuous on $\mathbb{R}$, so the product $^3 \cos$ is continuous on $\mathbb{R}$. The root function $\sqrt[4]{\;}$ is continuous on its domain, $[0\ \infty)$, and so the sum, $(\,) = \sqrt[4]{\;} + {}^3 \cos$, is continuous on its domain, $[0$

$\infty)$.

$^2 - 9$ is continuous on $\mathbb{R}$ since it is a polynomial and $\sqrt{\;}$ is continuous on $[0\ \infty)$ by Theorem 1.8.7, so the composition $\sqrt{{}^2 - 9}$ is continuous on $|\ {}^2 - 9 \ge 0 = (-\infty\ -3] \cup [3\ \infty)$ by Theorem 1.8.9. Note that $^2 - 2\ 6 = 0$ on this set and so the quotient function $(\,) = \dfrac{\sqrt{{}^2 - 9}}{{}^2 - 2}$ is continuous on its domain, $(-\infty\ -3] \cup [3\ \infty)$ by Theorem 1.8.4.

$(\,) = {}^5 - {}^3 + 3 - 5$ is continuous on the interval $[1\ 2]$, $(1) = -2$, and $(2) = 25$. Since $-2\ 0\ 25$, there is a number in $(1\ 2)$ such that $(\,) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation

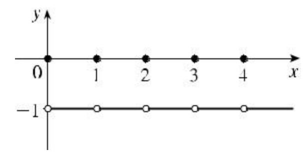
$^5 - {}^3 + 3 \; - 5 = 0$ in the interval $(1\ 2)$.

Let $(\,) = 2 \sin - 3 + 2$. Now is continuous on $[0\ 1]$ and $(0) = -3\ 0$ and $(1) = 2 \sin 1 - 1 \approx 0\ 68\ 0$. So by the Intermediate Value Theorem there is a number in $(0\ 1)$ such that $(\,) = 0$, that is, the equation $2 \sin = 3 - 2$ has a root in $(0\ 1)$.

( ) ( ) ( ) ( ) ( ) lim ( ) = 0 = lim ( )
51. | | ≤ ⇔ − ≤ ≤ and → → − .

Thus, by the Squeeze Theorem, lim → ( ) = 0.

52. (a) Note that is an even function since ( ) = (−). Now for any integer , [[ ]] + [[−]] = − = 0, and for any real number which is not an integer, [[ ]] + [[−]] = [[ ]] + (−[[ ]] − 1) = −1. So lim → ( ) exists (and is equal to −1) for all values of .



(b) is discontinuous at all integers.

PRINCIPLES OF PROBLEM SOLVING

1. $|2 - 1| = 1 \;\; 2$ if $\quad$ 1and $\quad$ $| +5| = \;\; 5$ if5

$\frac{1}{2}$

$2 \;\; 1$ if $\geq$ $\frac{2}{2}$ $\quad$ $+ 5$ $\quad$ if ≥ -5

$-$ $\quad$ $- -$ $\quad$ $-$

Therefore, we consider the three cases-5, $-5 \leq$ $\frac{1}{2}$, and $\geq \frac{1}{2}$.

If-5, we must have $1 - 2 - (- - 5) = 3 \Leftrightarrow = 3$, which is false, since we are considering -5.

If $-5 \leq$ $\frac{1}{2}$, we must have $1 - 2 - (+ 5) = 3 \Leftrightarrow = -$ $\frac{7}{3}$.

If $\geq$ $\frac{1}{2}$, we must have $2 - 1 - (+ 5) = 3 \Leftrightarrow = 9$.

So the two solutions of the equation are $= -$ $\frac{7}{3}$ and $= 9$.

2. $| - 1| = 1$ $\quad$ if1 $\quad$ and $\quad$ $| -3| = 3$ $\quad$ if $\quad$ 3

1 if ≥ 1 $\quad$ 3 if ≥ 3

$-$ $\quad$ $-$

Therefore, we consider the three cases1, $1 \leq 3$, and ≥ 3.

If1, we must have $1 - - (3 -) \geq 5 \Leftrightarrow 0 \geq 7$, which is false.

If $1 \leq 3$, we must have $- 1 - (3 -) \geq 5 \Leftrightarrow \geq$ $\frac{9}{2}$, which is false because3.

If ≥ 3, we must have $- 1 - (- 3) \geq 5 \Leftrightarrow 2 \geq 5$, which is false.

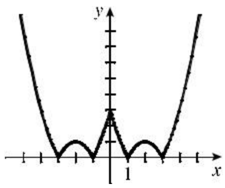
All three cases lead to falsehoods, so the inequality has no solution.

$( ) = {}^2 - 4 | | + 3$. If ≥ 0, then $( ) = {}^2 - 4 + 3 = |(- 1)(- 3)|$.

Case (i): If $0 \quad \leq 1$, then $( ) = {}^2 - 4 + 3$.

Case (ii): If $1 \quad \leq 3$, then $( ) = -({}^2 - 4 + 3) = - {}^2 + 4 - 3$.

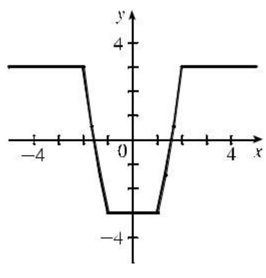
Case (iii): If $\quad 3$, then $( ) = {}^2 - 4 + 3$.



This enables us to sketch the graph for ≥ 0. Then we use the fact that is an even function to reflect this part of the graph about the -axis to obtain the entire graph. Or, we could consider also the cases -3, $-3 \leq -1$, and $-1 \leq 0$.

4. $()= \frac{}{2} \; {}_2 - 12 - {}_2 \;\; - 4$. $\quad$ if $\;$ 1 and $\;$ 2 $\quad$ 2 $\quad$ 4 if2

$-$ $\quad$ $| \; |^{\geq}$ $\quad$ $-$ $\quad$ $| \; |^{\geq}$



So for $0 \leq | | \;\; 1$, $( ) = 1 - {}^2 - (4 - {}^2) = -3$, for

$1 \leq | | \; 2$, $() = {}^2 - 1 - (4 - {}^2) = 2 {}^2 - 5$, and for

$| \ | \geq 2, \ (\ ) = {}^2 - 1 - ({}^2 - 4) = 3$

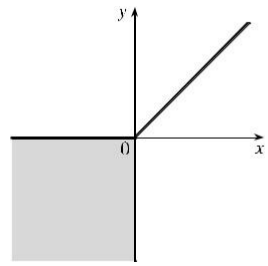
Remember that | | = if ≥ 0 and that | | = − if0. Thus,

+| |= 0 if0 and+ | | = 0 if0

2 if ≥ 0 2 if ≥ 0

We will consider the equation + | | = + | | in four cases.

(1) ≥0 ≥0	(2) ≥0,0	(3)0, ≥0	(4)00
2 = 2	2 = 0	0 = 2	0 = 0
=	= 0	0 =	



Case 1 gives us the line = with nonnegative and .

Case 2 gives us the portion of the -axis with negative.

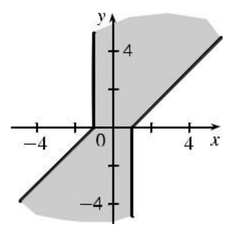
Case 3 gives us the portion of the -axis with negative.

Case 4 gives us the entire third quadrant.

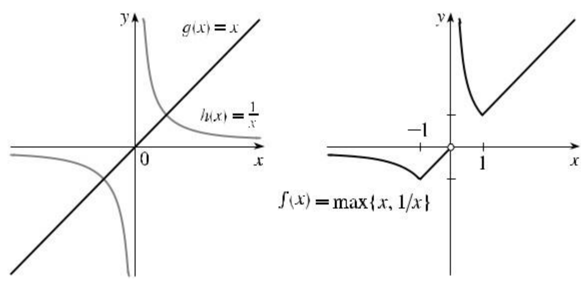
6. | − |+| |−| |≤2 [call this inequality ( )]

Case (i):	≥ ≥ 0.	Then ( )	⇔	−+−≤2	⇔	− ≤ 1	⇔ ≥ −1.
Case (ii):	≥ ≥0.	Then ( )	⇔	−+−≤2	⇔	0 ≤ 2 (true).	
Case (iii):	≥ 0 and ≤ 0.	Then ( )	⇔	−++≤2	⇔	2 ≤ 2	⇔ ≤ 1.
Case (iv):	≤ 0 and ≥ 0.	Then ( )	⇔	−−−≤2	⇔	−2 ≤2	⇔ ≥ −1.
Case (v):	≤ ≤0.	Then ( )	⇔	−−+≤2	⇔	0 ≤ 2 (true).	
Case (vi):	≤ ≤0.	Then ( )	⇔	−−+≤2	⇔	− ≤ 1	⇔ ≤ +1.

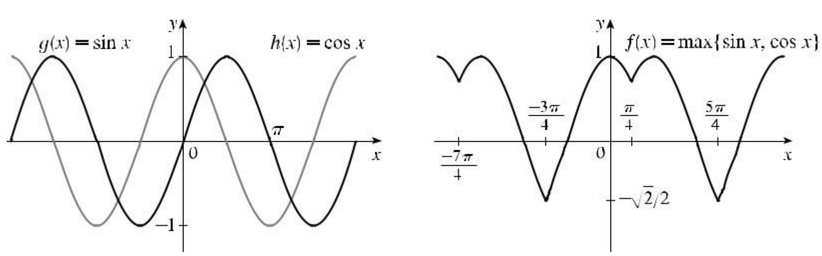
Note: Instead of considering cases (iv), (v), and (vi), we could have noted that the region is unchanged if and are replaced by − and − , so the region is symmetric about the origin. Therefore, we need only draw cases (i), (ii), and (iii), and rotate through 180° about the origin.



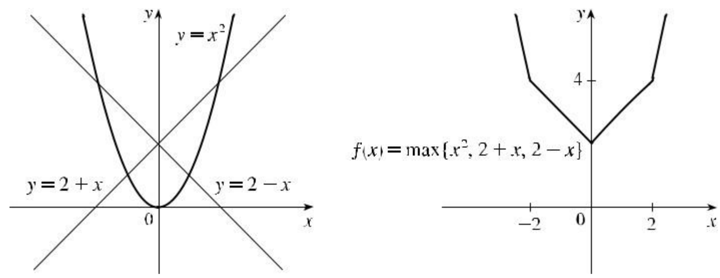
7. (a) To sketch the graph of ( ) = max { 1 }, we first graph ( ) = and ( ) = 1 on the same coordinate axes. Then create the graph of by plotting the largest -value of and for every value of .



(b)



(c)



On the TI-84 Plus, max is found under LIST, then under MATH. To graph $( ) = \max$ ^{2} $2 +$ $2 -$, use

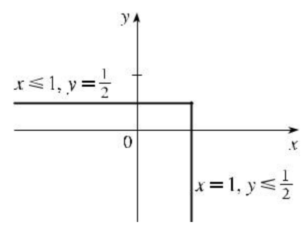
$= \max($ ^{2} $\max(2 +$ $2 -))$.

8. (a) If $\max\{$ $2\} = 1$, then either $= 1$ *and* $2 \leq 1$

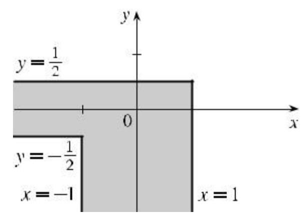
≤ 1 *and* $2 = 1$. Thus, we obtain the set of points such that $= 1$ *and* $\leq \frac{1}{2}$ [a vertical line

2 2

a horizontal line with rightmost point $(1\ \frac{1}{2})$.



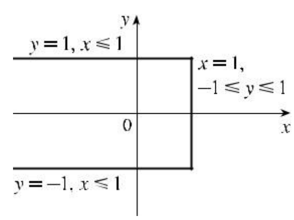
(b) The graph of $\max\{$ $2\} = 1$ is shown in part (a), and the graph of $\max\{$ $2\} = -1$ can be found in a similar manner. The inequalities in $-1 \leq \max\{$ $2\} \leq 1$ give us all the points on or inside the boundaries.



(c) $\max\{$ $^2\} = 1 \iff$

$= 1$ *and* $^2 \leq 1$ $[-1 \leq \ \leq 1]$

≤ 1 *and* $^2 = 1$ $[\ = \pm 1]$.



Let be the distance traveled on each half of the trip. Let $_1$ and $_2$ be the times taken for the first and second halves of the trip. For the first half of the trip we have $_1 = 30$ and for the second half we have $_2 = 60$. Thus, the average speed for the entire trip is $\dfrac{\text{total distance}}{\text{total time}} = \dfrac{2}{_1 + _2} = \dfrac{2}{\frac{}{30} + \frac{}{60}} \cdot \dfrac{60}{60} = \dfrac{120}{2 +} = \dfrac{120}{3} = 40$. The average speed for the entire trip is 40 mi h.

Let $( ) = \sin$, $( ) =$, and $( ) =$. Then the left-hand side of the equation is

$[\ \circ (\ +\ )](\ ) = \sin(\ +\ ) = \sin 2\ = 2 \sin\ \cos$; and the right-hand side is

$(\ \circ\ )(\ ) + (\ \circ\ )(\ ) = \sin\ + \sin\ = 2 \sin$. The two sides are not equal, so the given statement is false.

Let be the statement that $7 - 1$ is divisible by 6

$_1$ is true because $7^1 - 1 = 6$ is divisible by 6.

Assume is true, that is, $7 - 1$ is divisible by 6. In other words, $7 - 1 = 6$ for some positive integer . Then $7^{+1} - 1$ $= 7 \cdot 7 - 1 = (6 + 1) \cdot 7 - 1 = 42 + 6 = 6(7 + 1)$, which is divisible by 6, so $_{+1}$ is true.

Therefore, by mathematical induction, $7 - 1$ is divisible by 6 for every positive integer .

Let be the statement that $1 + 3 + 5 + \cdots + (2 - 1) = {}^2$.

$_1$ is true because $[2(1) - 1] = 1 = 1^2$.

Assume is true, that is, $1 + 3 + 5 + \cdots + (2 - 1) = {}^2$. Then

$1+3+5+\cdots+(2-1)+[2(+1)-1]=1+3+5+\cdots+(2-1)+(2+1)={}^2+(2+1)=(+1)^2$ which shows that $_{+1}$ is true.

Therefore, by mathematical induction, $1 + 3 + 5 + \cdots + (2 - 1) = {}^2$ for every positive integer .

$_0(\,) = {}^2$ and $_{+1}(\,) = {}_0(\,(\,))$ for $= 0\ 1\ 2$.

$_1(\,) = {}_0({}_0(\,)) = {}_0\,{}^2 = {}^{2\,2} = {}^4$, $_2(\,) = {}_0({}_1(\,)) = {}_0({}^4) = ({}^4)^2 = {}^8$,

$_3(\,) = {}_0({}_2(\,)) = {}_0({}^8) = ({}^8)^2 = {}^{16}$, . Thus, a general formula is $(\,) = {}^{2^{+1}}$.

(a) $_0(\,) = 1\,(2 -)$ and $_{+1} = {}_0 \circ$ for $= 0\ 1\ 2$.

$$_1()={}_0\frac{2-}{1} = \frac{2-\;2\;1}{1} = \frac{2(2-)-1}{2-} = \frac{3-2}{2-},$$

$$_2()={}_0\frac{3-2}{2-} = \frac{2\quad 2-}{1 - 3-2} = \frac{2(3-2)-(2-)}{3-2} = \frac{4-3}{3-2},$$

$$_3()={}_0\frac{4-3}{3-2} = \frac{2\quad 3-2}{1 - 4-3} = \frac{2(4-3)-(3-2)}{4-3} = \frac{5-4}{4-3}$$

Thus, we conjecture that the general formula is $(\,) = \dfrac{+1-}{+2-(+1)}$.

To prove this, we use the Principle of Mathematical Induction. We have already verified that is true for $= 1$. Assume that the formula is true for $=$; that is, $(\,) = \dfrac{+1-}{+2-(+1)}$. Then

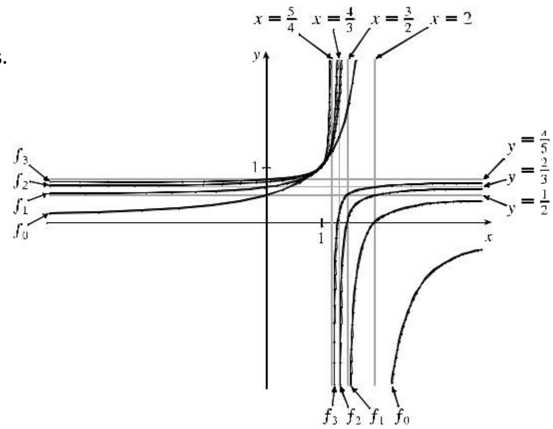
$$_{+1}(\,) = ({}_0 \circ)(\,) = {}_0(\,(\,)) = {}_0\frac{+1-}{+2-(+1)} = \frac{2\;\;\frac{+1-}{+2-(+1)}}{1-\frac{+1-}{+2-(+1)}}$$

$$= \frac{+2-(+1)}{2[+2-(+1)]-(+1-)} = \frac{+2-(+1)}{+3-(+2)}$$

This shows that the formula for is true for $= + 1$. Therefore, by mathematical induction, the formula is true for all positive integers .

(b) From the graph, we can make several observations:

- The values at each fixed = keep increasing as increases.
- The vertical asymptote gets closer to = 1 as increases.
- The horizontal asymptote gets closer to = 1 as increases.
- The -intercept for +1 is the value of the vertical asymptote for .
- The -intercept for is the value of the horizontal asymptote for +1.



15. Let $= \sqrt[6]{\ }$, so $= {}^6$. Then $\to 1$ as $\to 1$, so

$$\lim_{\to 1} \frac{\sqrt[3]{\ } - 1}{\sqrt{\ } - 1} = \lim_{\to 1} \frac{^2 - 1}{^3 - 1} = \lim_{\to 1} \frac{(-1)(+1)}{(-1)(^2 + +1)} = \lim_{\to 1} \frac{+1}{^2 + +1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}.$$

Another method: Multiply both the numerator and the denominator by $(\sqrt{\ } + 1)\left(\sqrt[3]{^2} + \sqrt[3]{\ } + 1\right)$.

16. First rationalize the numerator: $\lim_{\to 0} \frac{\sqrt{+} - 2}{} \cdot \frac{\sqrt{+} + 2}{\sqrt{+} + 2} = \lim_{\to 0} \frac{+ - 4}{\left(\sqrt{+} + 2\right)}$. Now since the denominator

approaches 0 as $\to 0$, the limit will exist only if the numerator also approaches 0 as $\to 0$. So we require that

$(0) + - 4 = 0 \Rightarrow = 4$. So the equation becomes $\lim_{\to 0} \frac{}{\sqrt{+4} + 2} = 1 \Rightarrow \frac{}{\sqrt{4}+2} = 1 \Rightarrow = 4$.
Therefore, $=$ $= 4$.

For $-\frac{1}{2} < < \frac{1}{2}$, we have $2 - 1 < 0$ and $2 + 1 > 0$, so $|2 - 1| = -(2 - 1)$ and $|2 + 1| = 2 + 1$.

Therefore, $\lim_{\to 0} \frac{|2-1| - |2+1|}{} = \lim_{\to 0} \frac{-(2-1) - (2+1)}{} = \lim_{\to 0} \frac{-4}{} = \lim_{\to 0} (-4) = -4$.

Let be the midpoint of , so the coordinates of are $\left(\frac{1}{2}, \frac{1}{2}{}^2\right)$ since the coordinates of are ^{2}. Let $= (0,)$.

Since the slope $= \frac{^2}{} = $, $= -\frac{1}{}$ (negative reciprocal). But $= \frac{\frac{1}{2}{}^2 - }{\frac{1}{2} - 0} = \frac{^2 - 2}{}$, so we conclude that

$-1 = {}^2 - 2 \Rightarrow 2 = {}^2 + 1 \Rightarrow = \frac{1}{2}{}^2 + \frac{1}{2}$. As $\to 0$, $\to \frac{1}{2}$ and the limiting position of is $\left(0, \frac{1}{2}\right)$.

19. (a) For $0 < < 1$, $[[\ ]] = 0$, so $\lim_{\to 0^-} \frac{[[\ ]]}{} = 0$, and $\lim_{\to 0^+} \frac{[[\ ]]}{} = 0$. For $-1 < < 0$, $[[\ ]] = -1$, so $\frac{[[\ ]]}{} = \frac{-1}{}$, and

$\lim_{\to 0^-} \frac{[[\ ]]}{} = \lim_{\to 0^-} \frac{-1}{} = \infty$. Since the one-sided limits are not equal, $\lim_{\to 0} \frac{[[\ ]]}{}$ does not exist.

(b) For 0, 1 − 1 ≤ [[1]] ≤ 1 ⇒ (1 −1) ≤ [[1]] ≤ (1) ⇒ 1 − ≤ [[1]] ≤ 1.

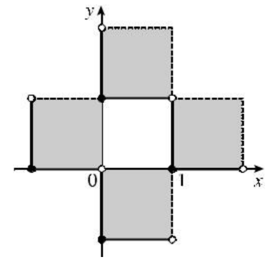
As → 0 , 1 − → 1, so by the Squeeze Theorem, $\to 0^{+}$.

For 0, 1 − 1 ≤ [[1]] ≤ 1 ⇒ (1 −1) ≥ [[1]] ≥ (1) ⇒ 1 − ≥ [[1]] ≥ 1.

As → 0 , 1 − → 1, so by the Squeeze Theorem, $\to 0^{-}$.

Since the one-sided limits are equal, $\lim_{\to 0}$ [[1]] = 1.

20. (a) [[ ]]^2 + [[ ]]^2 = 1. Since [[ ]]^2 and [[ ]]^2 are positive integers or 0, there are only 4 cases:



Case (i): [[ ]] = 1, [[ ]] = 0 ⇒1 ≤2 and 0 ≤1

Case (ii): [[ ]] = −1, [[ ]] = 0 ⇒−1 ≤0 and 0 ≤1

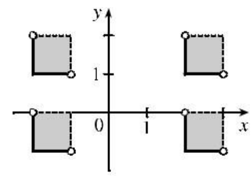
Case (iii):[[ ]] = 0, [[ ]] = 1 ⇒0 ≤1 and 1 ≤2

Case (iv): [[ ]] = 0, [[ ]] = −1 ⇒0 ≤1 and −1 ≤0

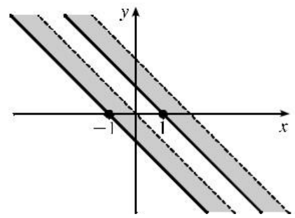
(b) [[ ]]^2 − [[ ]]^2 = 3. The only integral solution of ^{2} − ^{2} = 3 is = ±2 and = ±1. So the graph is

{( )|[[ ]]=±2, [[ ]]=±1}= ( ) 1 ≤ 2 or −1 ≤0 .

2 ≤ 3 or 2 1

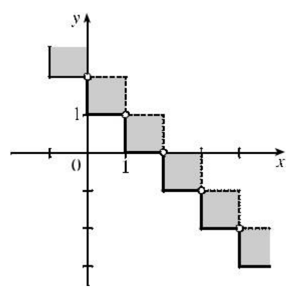


(c) [[+]] ^{2} = 1 ⇒ [[+]] = ±1 ⇒ 1 ≤ +2 or −1 ≤ + 0



(d) For ≤ + 1, [[ ]] = . Then [[ ]] + [[ ]] = 1 ⇒ [[ ]] = 1 − ⇒

1 − ≤ 2 − . Choosing integer values for produces the graph.

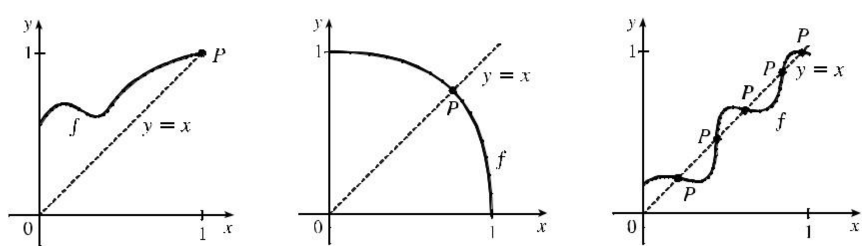


is continuous on (−∞) and (∞). To make continuous on R, we must have continuity at . Thus,

lim$_{\to^+}$ ( ) = lim$_{\to^-}$ ( ) ⇒ lim$_{\to^+}$ ^{2} = lim$_{\to^-}$ (+ 1) ⇒ ^{2} = + 1 ⇒ ^{2} − − 1 = 0 ⇒

[by the quadratic formula] = 1± √5 2 ≈ 1 618 or −0 618.

(a) Here are a few possibilities:



The “obstacle” is the line = (see diagram). Any intersection of the graph of with the line = constitutes a fixed point, and if the graph of the function does not cross the line somewhere in (0 1), then it must either start at (0 0) (in which case 0 is a fixed point) or finish at (1 1) (in which case 1 is a fixed point).

Consider the function ( ) = ( ) − , where is any continuous function with domain [0 1] and range in [0 1]. We shall prove that has a fixed point. Now if (0) = 0 then we are done: has a fixed point (the number 0), which is what we are trying to prove. So assume (0) 6= .0For the same reason we can assume that (1) 6= .1Then (0) = (0) 0 and (1) = (1) − 1 0. So by the Intermediate Value Theorem, there exists some number in the interval (0 1) such that ( ) = ( ) − = 0. So ( ) = , and therefore has a fixed point.

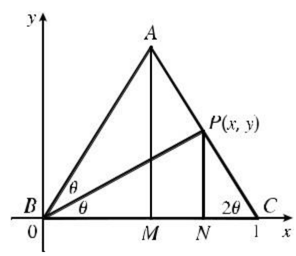
23. lim [( ) + ( )] = 2 → ⇒ lim ( ) + lim ( ) = 2 → → **(1)**

lim [( )−( )]=1 → lim → ( ) − lim ( ) = 1 **(2)**

Adding equations **(1)** and **(2)** gives us 2 lim ( ) = 3 → ⇒ lim ( ) = 3/2 → . From equation **(1)**, lim ( ) = 1/2 → . Thus,

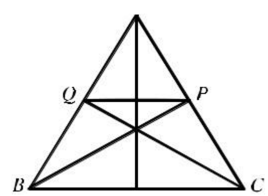
lim →[( )( )] = lim ( ) → · lim ( ) → = 3/2 · 1/2 = 3/4 .

24. (a) *Solution 1:* We introduce a coordinate system and drop a perpendicular from , as shown. We see from ∠ that tan 2 = 1 − , and from ∠ that tan = . Using the double-angle formula for tangents, we get 1 − = tan 2 = 2 tan / (1 − tan) = 2( ) / (1−( )) . After a bit of simplification, this becomes 1/(1 −) = 2/(2 − 2) ⇔ 2= (3 −2).

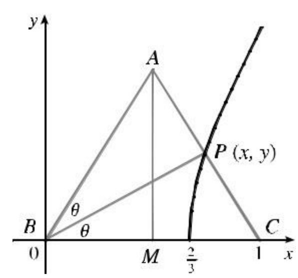


As the altitude decreases in length, the point will approach the -axis, that is, → 0, so the limiting location of must be one of the roots of the equation (3 − 2) = 0. Obviously it is not = 0 (the point can never be to the left of the altitude , which it would have to be in order to approach 0) so it must be 3 − 2 = 0, that is, = 2/3 .

Solution 2: We add a few lines to the original diagram, as shown. Now note that ∠ = ∠ (alternate angles; k by symmetry) and similarly ∠ = ∠ . So ∆ and ∆ are isosceles, and the line segments , and are all of equal length. As | | → 0, and approach points on the base, and the point is seen to approach a position two-thirds of the way between and , as above.



(b) The equation 2 = (3 − 2) calculated in part (a) is the equation of the curve traced out by . Now as | | → ∞, 2 → 2 , → 4 , → 1, and since tan = , → 1. Thus, only traces out the part of the curve with 0 ≤ 1.



(a) Consider $( ) = (+ 180^{\circ}) - ( )$. Fix any number . If $( ) = 0$, we are done: Temperature at = Temperature at $+ 180^{\circ}$. If $( )$ 0, then $(+ 180^{\circ}) = (+ 360^{\circ}) - (+ 180^{\circ}) = ( ) - (+ 180^{\circ}) = - ( )$ 0. Also, is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, has a zero on the interval $[+ 180^{\circ}]$. If $( )$ 0, then a similar argument applies.

Yes. The same argument applies.

The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.