CHAPTER TWO

Solutions for Section 2.1

Exercises

1. For t between 2 and 5, we have

$$\text{Average velocity} = \frac{\Delta s}{\Delta t} = \frac{400 - 135}{5 - 2} = \frac{265}{3} \text{ km/hr.}$$

The average velocity on this part of the trip was 265/3 km/hr.

2. The average velocity over a time period is the change in position divided by the change in time. Since the function $x(t)$ gives the position of the particle, we find the values of $x(0) = -2$ and $x(4) = -6$. Using these values, we find

$$\text{Average velocity} = \frac{\Delta x(t)}{\Delta t} = \frac{x(4) - x(0)}{4 - 0} = \frac{-6 - (-2)}{4} = -1 \text{ meters/sec.}$$

3. The average velocity over a time period is the change in position divided by the change in time. Since the function $x(t)$ gives the position of the particle, we find the values of $x(2) = 14$ and $x(8) = -4$. Using these values, we find

$$\text{Average velocity} = \frac{\Delta x(t)}{\Delta t} = \frac{x(8) - x(2)}{8 - 2} = \frac{-4 - 14}{6} = -3 \text{ angstroms/sec.}$$

4. The average velocity over a time period is the change in position divided by the change in time. Since the function $s(t)$ gives the distance of the particle from a point, we read off the graph that $s(0) = 1$ and $s(3) = 4$. Thus,

$$\text{Average velocity} = \frac{\Delta s(t)}{\Delta t} = \frac{s(3) - s(0)}{3 - 0} = \frac{4 - 1}{3} = 1 \text{ meter/sec.}$$

5. The average velocity over a time period is the change in position divided by the change in time. Since the function s(t) gives the distance of the particle from a point, we read off the graph that s(1) = 2 and s(3) = 6. Thus,

$$\text{Average velocity} = \frac{\Delta s(t)}{\Delta t} = \frac{s(3) - s(1)}{3 - 1} = \frac{6 - 2}{2} = 2 \text{ meters/sec.}$$

6. The average velocity over a time period is the change in position divided by the change in time. Since the function s(t) gives the distance of the particle from a point, we find the values of $s(2) = e^2 - 1 = 6.389$ and $s(4) = e^4 - 1 = 53.598$. Using these values, we find

$$\text{Average velocity} = \frac{\Delta s(t)}{\Delta t} = \frac{s(4) - s(2)}{4 - 2} = \frac{53.598 - 6.389}{2} = 23.605 \ \mu\text{m/sec.}$$

7. The average velocity over a time period is the change in the distance divided by the change in time. Since the function $s(t)$ gives the distance of the particle from a point, we find the values of $s(\pi/3) = 4 + 3\sqrt{3}/2$ and $s(7\pi/3) = 4 + 3\sqrt{3}/2$. Using these values, we find

$$\text{Average velocity} = \frac{\Delta s(t)}{\Delta t} = \frac{s(7\pi/3) - s(\pi/3)}{7\pi/3 - \pi/3} = \frac{4 + 3\sqrt{3}/2 - (4 + 3\sqrt{3}/2)}{2\pi} = 0 \text{ cm/sec.}$$

Though the particle moves, its average velocity is zero, since it is at the same position at $t = \pi/3$ and $t = 7\pi/3$.

8. (a) Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 1$ and $t = 1.1$. We have

$$\text{Average velocity} = \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{3.63 - 3}{0.1} = 6.3 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{3.0603 - 3}{0.01} = 6.03 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(1.001) - f(1)}{1.001 - 1} = \frac{3.006003 - 3}{0.001} = 6.003 \text{ m/sec.}$$

(b) We see in part (a) that as we choose a smaller and smaller interval around $t = 1$ the average velocity appears to be getting closer and closer to 6, so we estimate the instantaneous velocity at $t = 1$ to be 6 m/sec.

9. (a) Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 0$ and $t = 0.1$. We have

$$\text{Average velocity} = \frac{f(0.1) - f(0)}{0.1 - 0} = \frac{0.004 - 0}{0.1} = 0.04 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(0.01) - f(0)}{0.01 - 0} = \frac{0.000004}{0.01} = 0.0004 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(0.001) - f(0)}{1.001 - 1} = \frac{4 \times 10^{-9} - 0}{0.001} = 4 \times 10^{-6} \text{ m/sec.}$$

(b) We see in part (a) that as we choose a smaller and smaller interval around $t = 0$ the average velocity appears to be getting closer and closer to 0, so we estimate the instantaneous velocity at $t = 0$ to be 0 m/sec.

Looking at a graph of $s = f(t)$ we see that a line tangent to the graph at $t = 0$ is horizontal, confirming our result.

10. (a) Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 1$ and $t = 1.1$. We have

$$\text{Average velocity} = \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{0.808496 - 0.909297}{0.1} = -1.00801 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{0.900793 - 0.909297}{0.01} = -0.8504 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(1.001) - f(1)}{1.001 - 1} = \frac{0.908463 - 0.909297}{0.001} = -0.834 \text{ m/sec.}$$

(b) We see in part (a) that as we choose a smaller and smaller interval around $t = 1$ the average velocity appears to be getting closer and closer to -0.83, so we estimate the instantaneous velocity at $t = 1$ to be -0.83 m/sec. In this case, more estimates with smaller values of h would be very helpful in making a better estimate.

11. See Figure 2.1.

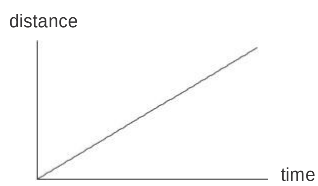


Figure 2.1

12. See Figure 2.2.

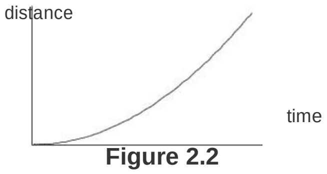


Figure 2.2

13. See Figure 2.3.

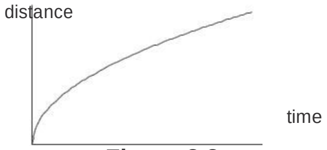


Figure 2.3

Problems

14. Using h = 0.1, 0.01, 0.001, we see

$$\frac{(3+0.1)^3-27}{0.1} = 27.91$$

$$\frac{(3+0.01)^3-27}{0.01} = 27.09$$

$$\frac{(3+0.001)^3-27}{0.001} = 27.009.$$

These calculations suggest that $\lim_{h\to 0} \frac{(3+h)^3-27}{h} = 27.$

15. Using radians,

h	(cos h − 1)/h
0.01	−0.005
0.001	−0.0005
0.0001	−0.00005

These values suggest that $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$

16. Using h = 0.1, 0.01, 0.001, we see

This suggests that $\lim \frac{7 \quad 1}{h} \approx 1.9.$

$$= 2.148$$

$$0.1$$

$$\frac{7^{0.01} - 1}{0.01} = 1.965$$

$$\frac{7^{0.001} - 1}{0.001} = 1.948$$

$$\frac{7^{0.0001} - 1}{0.0001} = 1.946.$$

7
0
.
1
−
1

h→0 h

17. Using h = 0.1, 0.01, 0.001, we see

h	$(e^{1+h} - e)/h$
0.01	2.7319
0.001	2.7196
0.0001	2.7184

These values suggest that $\lim_{h\to 0} \frac{e^{1+h} - e}{h} = 2.7$. In fact, this limit is e.

18.

Slope	−3	−1	0	1/2	1	2
Point	F	C	E	A	B	D

19. The slope is positive at A and D; negative at C and F . The slope is most positive at A; most negative at F .

20. 0 < slope at C < slope at B < slope of AB < 1 < slope at A. (Note that the line y = x, has slope 1.)

21. Since f (t) is concave down between t = 1 and t = 3, the average velocity between the two times should be less than the instantaneous velocity at t = 1 but greater than the instantaneous velocity at time t = 3, so D < A < C . For analogous reasons, F < B < E. Finally, note that f is decreasing at t = 5 so E < 0, but increasing at t = 0, so D > 0. Therefore, the ordering from smallest to greatest of the given quantities is

$$F < B < E < 0 < D < A < C.$$

22.

$$\left(\begin{array}{c}\text{Average velocity}\\ 0 < t < 0.2\end{array}\right) = \frac{s(0.2) - s(0)}{0.2 - 0} = \frac{0.5}{0.2} = 2.5 \text{ ft/sec.}$$

$$\left(\begin{array}{c}\text{Average velocity}\\ 0.2 < t < 0.4\end{array}\right) = \frac{s(0.4) - s(0.2)}{0.4 - 0.2} = \frac{1.3}{0.2} = 6.5 \text{ ft/sec.}$$

A reasonable estimate of the velocity at t = 0.2 is the average: $\frac{1}{2}(6.5 + 2.5) = 4.5$ ft/sec.

23. One possibility is shown in Figure 2.4.

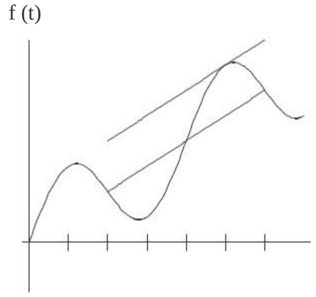


Figure 2.4

24. (a) When t = 0, the ball is on the bridge and its height is f (0) = 36, so the bridge is 36 feet above the ground.

(b) After 1 second, the ball's height is f (1) = −16 + 50 + 36 = 70 feet, so it traveled 70 − 36 = 34 feet in 1 second, and its average velocity was 34 ft/sec.

(c) At t = 1.001, the ball's height is f (1.001) = 70.017984 feet, and its velocity about $\frac{70.017984-70}{1.001-1} = 17.984$ 18 ft/sec.

(d) We complete the square:

$$
\begin{aligned}
f(t) &= -16t^2 + 50t + 36 \\
&= -16\left(t^2 - \frac{25}{8}t\right) + 36 \\
&= -16\left(t^2 - \frac{25}{8}t + \frac{625}{256}\right) + 36 + 16\,\frac{625}{256} \\
&= -16\left(t - \frac{25}{16}\right)^2 + \frac{1201}{16}
\end{aligned}
$$

so the graph of f is a downward parabola with vertex at the point (25/16, 1201/16) = (1.6, 75.1). We see from Figure 2.5 that the ball reaches a maximum height of about 75 feet. The velocity of the ball is zero when it is at the peak, since the tangent is horizontal there.

(e) The ball reaches its maximum height when $t = \frac{25}{16} = 1.6$.

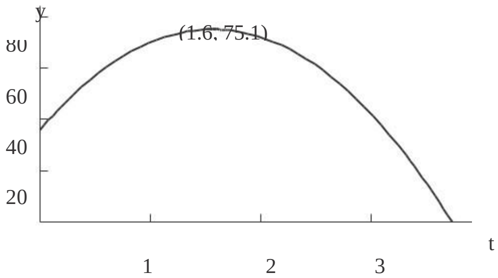


Figure 2.5

25. $\lim_{h\to 0} \frac{(2+h)^2 - 4}{h} = \lim_{h\to 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h\to 0} (4 + h) = 4$

26. $\lim_{h\to 0} \frac{(1+h)^3 - 1}{h} = \lim_{h\to 0} \frac{1 + 3h + 3h^2 + h^3 - 1}{h} = \lim_{h\to 0} \frac{h(3 + 3h + h^2)}{h} = \lim_{h\to 0} 3 + 3h + h^2 = 3.$

27. $\lim_{h\to 0} \frac{3(2+h)^2 - 12}{h} = \lim_{h\to 0} \frac{12 + 12h + 3h^2 - 12}{h} = \lim_{h\to 0} \frac{h(12 + 3h)}{h} = \lim_{h\to 0} 12 + 3h = 12.$

28. $\lim_{h\to 0} \frac{(3+h)^2 - (3-h)^2}{2h} = \lim_{h\to 0} \frac{9 + 6h + h^2 - (9 - 6h + h^2)}{2h} = \lim_{h\to 0} \frac{12h}{2h} = \lim_{h\to 0} 6 = 6.$

Strengthen Your Understanding

29. Speed is the magnitude of velocity, so it is always positive or zero; velocity has both magnitude and direction.

30. We expand and simplify first

$$\lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h\to 0} \frac{(4 + 4h + h^2) - 4}{h} = \lim_{h\to 0} \frac{4h + h^2}{h} = \lim_{h\to 0} (4 + h) = 4.$$

31. Since the tangent line to the curve at t = 4 is almost horizontal, the instantaneous velocity is almost zero. At t = 2 the slope of the tangent line, and hence the instantaneous velocity, is relatively large and positive.

32. $f(t) = t^2$. The slope of the graph of $y = f(t)$ is negative for $t < 0$ and positive for $t > 0$.

Many other answers are possible.

33. One possibility is the position function $s(t) = t^2$. Any function that is symmetric about the line $t = 0$ works.

For $s(t) = t^2$, the slope of a tangent line (representing the velocity) is negative at $t = -1$ and positive at $t = 1$, and that the magnitude of the slopes (the speeds) are the same.

34. False. For example, the car could slow down or even stop at one minute after 2 pm, and then speed back up to 60 mph at one minute before 3 pm. In this case the car would travel only a few miles during the hour, much less than 50 miles.

35. False. Its average velocity for the time between 2 pm and 4 pm is 40 mph, but the car could change its speed a lot during that time period. For example, the car might be motionless for an hour then go 80 mph for the second hour. In that case the velocity at 2 pm would be 0 mph.

36. True. During a short enough time interval the car can not change its velocity very much, and so it velocity will be nearly constant. It will be nearly equal to the average velocity over the interval.

37. True. The instantaneous velocity is a limit of the average velocities. The limit of a constant equals that constant.

38. True. By definition, Average velocity = Distance traveled/Time.

39. False. Instantaneous velocity equals a *limit* of difference quotients.

Solutions for Section 2.2

Exercises

1. The derivative, $f'(2)$, is the rate of change of x^3 at $x = 2$. Notice that each time x changes by 0.001 in the table, the value of x^3 changes by 0.012. Therefore, we estimate

$$f'(2) = \begin{matrix}\text{Rate of change} \\ \text{of } f \text{ at } x = 2\end{matrix} \approx \frac{0.012}{0.001} = 12.$$

The function values in the table look exactly linear because they have been rounded. For example, the exact value of x^3 when $x = 2.001$ is 8.012006001, not 8.012. Thus, the table can tell us only that the derivative is approximately 12. Example 5 on page 95 shows how to compute the derivative of $f(x)$ exactly.

2. With $h = 0.01$ and $h = -0.01$, we have the difference quotients

$$\frac{f(1.01) - f(1)}{0.01} = 3.0301 \quad \text{and} \quad \frac{f(0.99) - f(1)}{-0.01} = 2.9701.$$

With $h = 0.001$ and $h = -0.001$,

$$\frac{f(1.001) - f(1)}{0.001} = 3.003001 \quad \text{and} \quad \frac{f(0.999) - f(1)}{-0.001} = 2.997001.$$

The values of these difference quotients suggest that the limit is about 3.0. We say

$$f'(1) = \begin{matrix}\text{Instantaneous rate of change of } f(x) = x^3 \\ \text{with respect to } x \text{ at } x = 1\end{matrix} \approx 3.0.$$

3. (a) Using the formula for the average rate of change gives

$$\begin{matrix}\text{Average rate of change} \\ \text{of revenue for } 1 \le q \le 2\end{matrix} = \frac{R(2) - R(1)}{1} = \frac{160-90}{1} = 70 \text{ dollars/kg.}$$

$$\begin{matrix}\text{Average rate of change} \\ \text{of revenue for } 2 \le q \le 3\end{matrix} = \frac{R(3) - R(2)}{1} = \frac{210 - 160}{1} = 50 \text{ dollars/kg.}$$

So we see that the average rate decreases as the quantity sold in kilograms increases.

(b) With $h = 0.01$ and $h = -0.01$, we have the difference quotients

$$\frac{R(2.01) - R(2)}{0.01} = 59.9 \text{ dollars/kg} \quad \text{and} \quad \frac{R(1.99) - R(2)}{-0.01} = 60.1 \text{ dollars/kg.}$$

With $h = 0.001$ and $h = -0.001$,

$$\frac{R(2.001) - R(2)}{0.001} = 59.99 \text{ dollars/kg} \quad \text{and} \quad \frac{R(1.999) - R(2)}{-0.001} = 60.01 \text{ dollars/kg.}$$

The values of these difference quotients suggest that the instantaneous rate of change is about 60 dollars/kg. To confirm that the value is exactly 60, that is, that $R'(2) = 60$, we would need to take the limit as $h \to 0$.

4. (a) Using a calculator we obtain the values found in the table below:

x	1	1.5	2	2.5	3
e^x	2.72	4.48	7.39	12.18	20.09

(b) The average rate of change of $f(x) = e^x$ between $x = 1$ and $x = 3$ is

$$\text{Average rate of change} = \frac{f(3) - f(1)}{3-1} = \frac{e^3 - e}{3-1} \approx \frac{20.09 - 2.72}{2} = 8.69.$$

(c) First we find the average rates of change of $f(x) = e^x$ between $x = 1.5$ and $x = 2$, and between $x = 2$ and $x = 2.5$:

$$\text{Average rate of change} = \frac{f(2) - f(1.5)}{2-1.5} = \frac{e^2 - e^{1.5}}{2-1.5} \approx \frac{7.39 - 4.48}{0.5} = 5.82$$

$$\text{Average rate of change} = \frac{f(2.5) - f(2)}{2.5-2} = \frac{e^{2.5} - e^2}{2.5-2} \approx \frac{12.18 - 7.39}{0.5} = 9.58.$$

Now we approximate the instantaneous rate of change at $x = 2$ by averaging these two rates:

$$\text{Instantaneous rate of change} \approx \frac{5.82 + 9.58}{2} = 7.7.$$

5. (a)

Table 2.1

x	1	1.5	2	2.5	3
log x	0	0.18	0.30	0.40	0.48

(b) The average rate of change of $f(x) = \log x$ between $x = 1$ and $x = 3$ is

$$\frac{f(3) - f(1)}{3-1} = \frac{\log 3 - \log 1}{3-1} \approx \frac{0.48 - 0}{2} = 0.24$$

(c) First we find the average rates of change of $f(x) = \log x$ between $x = 1.5$ and $x = 2$, and between $x = 2$ and $x = 2.5$.

$$\frac{\log 2 - \log 1.5}{2-1.5} = \frac{0.30 - 0.18}{0.5} \approx 0.24$$

$$\frac{\log 2.5 - \log 2}{2.5-2} = \frac{0.40 - 0.30}{0.5} \approx 0.20$$

Now we approximate the instantaneous rate of change at $x = 2$ by finding the average of the above rates, i.e.

$$\begin{array}{c}\text{the instantaneous rate of change} \\ \text{of } f(x) = \log x \text{ at } x = 2\end{array} \approx \frac{0.24 + 0.20}{2} = 0.22.$$

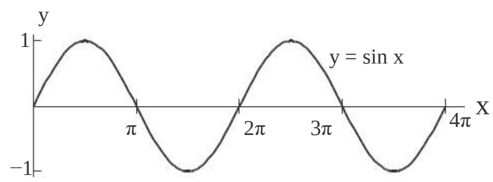
6. In Table 2.2, each x increase of 0.001 leads to an increase in $f(x)$ by about 0.031, so

$$f'(3) \approx \frac{0.031}{0.001} = 31.$$

Table 2.2

	[illegible] [illegible] [illegible] [illegible] [illegible]

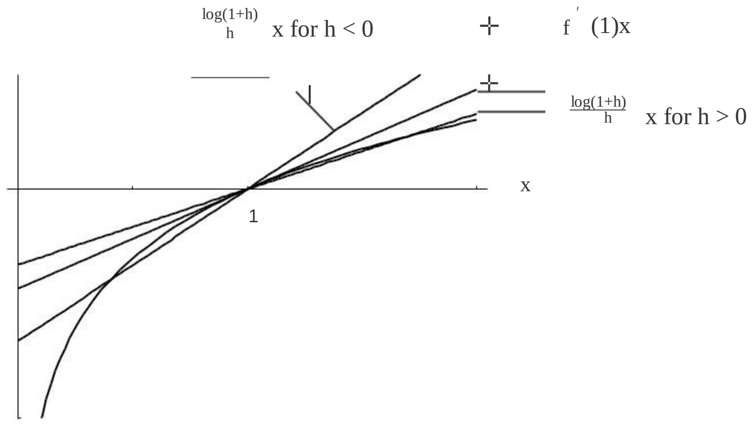
7.



Since sin x is decreasing for values near $x = 3\pi$, its derivative at $x = 3\pi$ is negative.

8. $f'(1) = \lim_{h\to 0} \frac{\log(1+h) - \log 1}{h} = \lim_{h\to 0} \frac{\log(1+h)}{h}$

Evaluating $\frac{\log(1+h)}{h}$ for h = 0.01, 0.001, and 0.0001, we get 0.43214, 0.43408, 0.43427, so $f'(1) \approx 0.43427$. The corresponding secant lines are getting steeper, because the graph of log x is concave down. We thus expect the limit to be more than 0.43427 . If we consider negative values of h, the estimates are too large. We can also see this from the graph below:



9. We estimate $f'(2)$ using the average rate of change formula on a small interval around 2. We use the interval x = 2 to x = 2.001. (Any small interval around 2 gives a reasonable answer.) We have

$$f'(2) \approx \frac{f(2.001) - f(2)}{2.001 - 2} = \frac{3^{2.001} - 3^2}{2.001 - 2} = \frac{9.00989 - 9}{0.001} = 9.89.$$

10. (a) The average rate of change from x = a to x = b is the slope of the line between the points on the curve with x = a and x = b. Since the curve is concave down, the line from x = 1 to x = 3 has a greater slope than the line from x = 3 to x = 5, and so the average rate of change between x = 1 and x = 3 is greater than that between x = 3 and x = 5.

(b) Since f is increasing, f (5) is the greater.

(c) As in part (a), f is concave down and f' is decreasing throughout so $f'(1)$ is the greater.

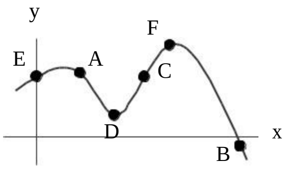
11. Since $f'(x) = 0$ where the graph is horizontal, $f'(x) = 0$ at x = d. The derivative is positive at points b and c, but the graph is steeper at x = c. Thus $f'(x) = 0.5$ at x = b and $f'(x) = 2$ at x = c. Finally, the derivative is negative at points a and e but the graph is steeper at x = e. Thus, $f'(x) = -0.5$ at x = a and $f'(x) = -2$ at x = e. See Table 2.3.

Thus, we have $f'(d) = 0$, $f'(b) = 0.5$, $f'(c) = 2$, $f'(a) = -0.5$, $f'(e) = -2$.

Table 2.3

x	$f'(x)$
d	0
b	0.5
c	2
a	−0.5
e	−2

12. One possible choice of points is shown below.



Problems

13. The statements $f(100) = 35$ and $f'(100) = 3$ tell us that at $x = 100$, the value of the function is 35 and the function is increasing at a rate of 3 units for a unit increase in x. Since we increase x by 2 units in going from 100 to 102, the value of the function goes up by approximately $2 \cdot 3 = 6$ units, so

$$f(102) \approx 35 + 2 \cdot 3 = 35 + 6 = 41.$$

14. The answers to parts (a)–(d) are shown in Figure 2.6.

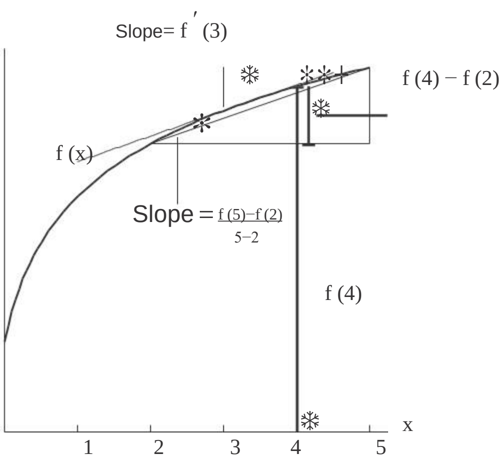


Figure 2.6

15. (a) Since f is increasing, $f(4) > f(3)$.

(b) From Figure 2.7, it appears that $f(2) - f(1) > f(3) - f(2)$.

(c) The quantity $\frac{f(2) - f(1)}{2 - 1}$ represents the slope of the secant line connecting the points on the graph at $x = 1$ and $x = 2$. This is greater than the slope of the secant line connecting the points at $x = 1$ and $x = 3$ which is $\frac{f(3) - f(1)}{3 - 1}$.

(d) The function is steeper at $x = 1$ than at $x = 4$ so $f'(1) > f'(4)$.

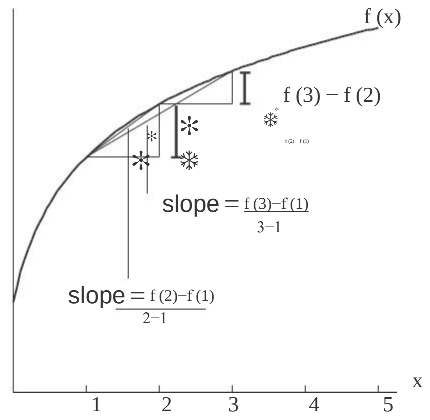


Figure 2.7

16. Figure 2.8 shows the quantities in which we are interested.

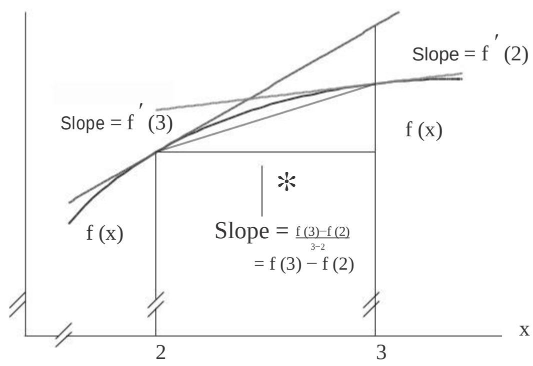


Figure 2.8

The quantities $f'(2)$, $f'(3)$ and $f(3) - f(2)$ have the following interpretations:

- $f'(2)$ = slope of the tangent line at $x = 2$
- $f'(3)$ = slope of the tangent line at $x = 3$
- $f(3) - f(2) = \frac{f(3)-f(2)}{3-2}$ = slope of the secant line from $f(2)$ to $f(3)$.

From Figure 2.8, it is clear that $0 < f(3) - f(2) < f'(2)$. By extending the secant line past the point $(3, f(3))$, we can see that it lies above the tangent line at $x = 3$.

Thus

$$0 < f'(3) < f(3) - f(2) < f'(2).$$

17. The coordinates of A are (4, 25). See Figure 2.9. The coordinates of B and C are obtained using the slope of the tangent line. Since $f'(4) = 1.5$, the slope is 1.5

From A to B, $\Delta x = 0.2$, so $\Delta y = 1.5(0.2) = 0.3$. Thus, at C we have $y = 25 + 0.3 = 25.3$. The coordinates of B are (4.2, 25.3).

From A to C , $\Delta x = -0.1$, so $\Delta y = 1.5(-0.1) = -0.15$. Thus, at C we have $y = 25 - 0.15 = 24.85$. The coordinates of C are (3.9, 24.85).

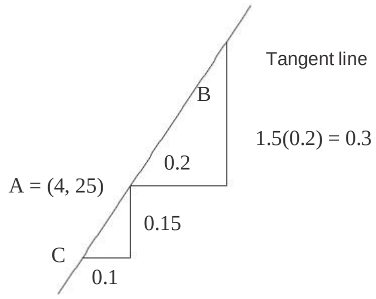


Figure 2.9

18. (a) Since the point B = (2, 5) is on the graph of g, we have g(2) = 5.

(b) The slope of the tangent line touching the graph at x = 2 is given by

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{5 - 5.02}{2 - 1.95} = \frac{-0.02}{0.05} = -0.4.$$

Thus, $g'(2) = -0.4$.

19. See Figure 2.10.

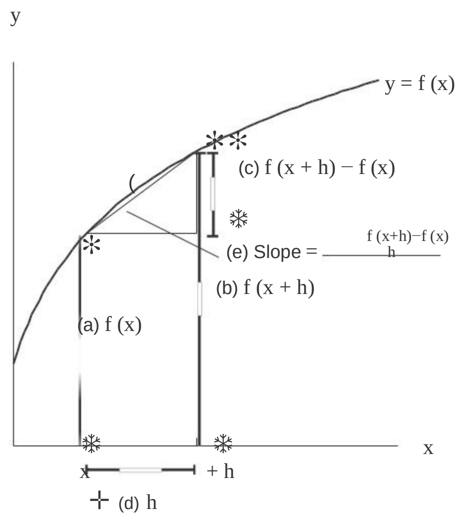


Figure 2.10

20. See Figure 2.11.

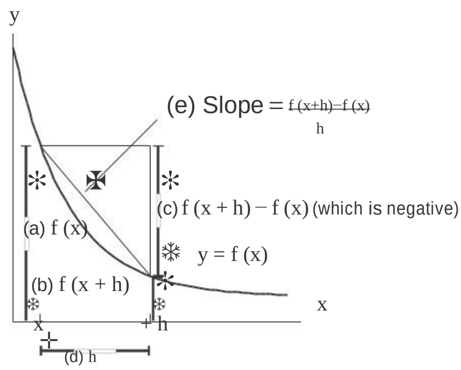


Figure 2.11

21. (a) For the line from A to B,

$$\text{Slope} = \frac{f(b) - f(a)}{b - a}.$$

(b) The tangent line at point C appears to be parallel to the line from A to B. Assuming this to be the case, the lines have the same slope.

(c) There is only one other point, labeled D in Figure 2.12, at which the tangent line is parallel to the line joining A and B.

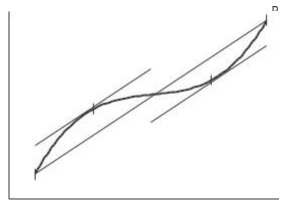


Figure 2.12

22. (a) Figure 2.13 shows the graph of an even function. We see that since f is symmetric about the y-axis, the tangent line at $x = -10$ is just the tangent line at $x = 10$ flipped about the y-axis, so the slope of one tangent is the negative of that of the other. Therefore, $f'(-10) = -f'(10) = -6$.

(b) From part (a) we can see that if f is even, then for any x, we have $f'(-x) = -f'(x)$. Thus $f'(-0) = -f'(0)$, so $f'(0) = 0$.

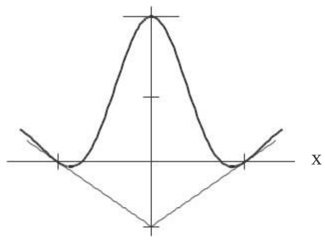


Figure 2.13

23. Figure 2.14 shows the graph of an odd function. We see that since g is symmetric about the origin, its tangent line at $x = -4$ is just the tangent line at $x = 4$ flipped about the origin, so they have the same slope. Thus, $g'(-4) = 5$.

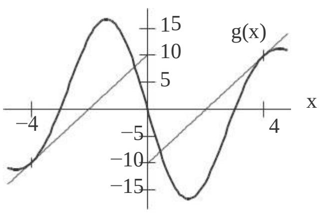


Figure 2.14

24. (a)

$$f'(0) = \lim_{h\to 0} \frac{\sin \overbrace{h}^{\text{h in degrees}} - \sin \overbrace{0}^{0}}{h} = \frac{\sin h}{h}.$$

To four decimal places,

$$\frac{\sin 0.2}{0.2} \approx \frac{\sin 0.1}{0.1} \approx \frac{\sin 0.01}{0.01} \approx \frac{\sin 0.001}{0.001} \approx 0.01745$$

so $f'(0) \approx 0.01745$.

(b) Consider the ratio $\frac{\sin h}{h}$. As we approach 0, the numerator, sin h, will be much smaller in magnitude if h is in degrees than it would be if h were in radians. For example, if $h = 1^\circ$ radian, $\sin h = 0.8415$, but if $h = 1$ degree, $\sin h = 0.01745$. Thus, since the numerator is smaller for h measured in degrees while the denominator is the same, we expect the ratio $\frac{\sin h}{h}$ to be smaller.

25. We find the derivative using a difference quotient:

$$f'(3) = \lim_{h\to 0}\frac{f(3+h) - f(3)}{h} = \lim_{h\to 0}\frac{(3+h)^2 + 3 + h - (3^2+3)}{h}$$

$$= \lim_{h\to 0}\frac{9 + 6h + h^2 + 3 + h - 9 - 3}{h} = \lim_{h\to 0}\frac{7h + h^2}{h} = \lim_{h\to 0}(7+h) = 7.$$

Thus at x = 3, the slope of the tangent line is 7. Since $f(3) = 3^2 + 3 = 12$, the line goes through the point (3, 12), and therefore its equation is

$$y - 12 = 7(x - 3) \quad \text{or} \quad y = 7x - 9.$$

The graph is in Figure 2.15.

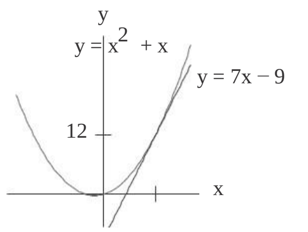


Figure 2.15

26. Using a difference quotient with h = 0.001, say, we find

$$f'(1) \approx \frac{1.001\ln(1.001) - 1\ln(1)}{1.001 - 1} = 1.0005$$

$$f'(2) \approx \frac{2.001\ln(2.001) - 2\ln(2)}{2.001 - 2} = 1.6934$$

The fact that f' is larger at x = 2 than at x = 1 suggests that f is concave up between x = 1 and x = 2.

27. We want $f'(2)$. The exact answer is

$$f'(2) = \lim_{h\to 0}\frac{f(2+h) - f(2)}{h} = \lim_{h\to 0}\frac{(2+h)^{2+h} - 4}{h},$$

but we can approximate this. If h = 0.001, then

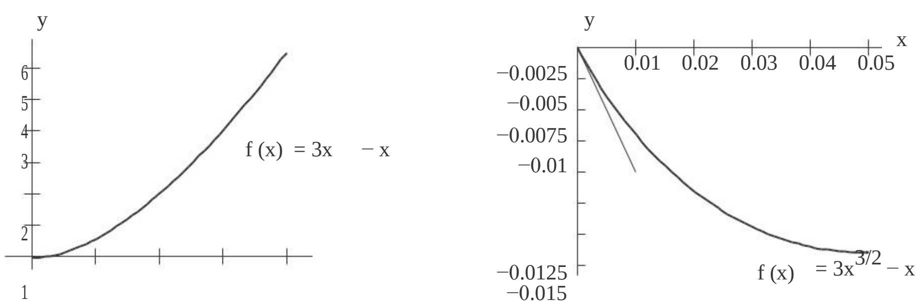
$$\frac{(2.001)^{2.001} - 4}{0.001} \approx 6.779$$

and if h = 0.0001 then

$$\frac{(2.0001)^{2.0001} - 4}{0.0001} \approx 6.773,$$

so $f'(2) \approx 6.77$.

28. Notice that we can't get all the information we want just from the graph of f for $0 \le x \le 2$, shown on the left in Figure 2.16. Looking at this graph, it looks as if the slope at x = 0 is 0. But if we zoom in on the graph near x = 0, we get the graph of f for $0 \le x \le 0.05$, shown on the right in Figure 2.16. We see that f does dip down quite a bit between x = 0 and $x \approx 0.11$. In fact, it now looks like $f'(0)$ is around −1. Note that since f(x) is undefined for x < 0, this derivative only makes sense as we approach zero from the right.



0.5 1 1.5 2

Figure 2.16

We zoom in on the graph of f near $x = 1$ to get a more accurate picture from which to estimate $f'(1)$. A graph of f for $0.7 \le x \le 1.3$ is shown in Figure 2.17. [Keep in mind that the axes shown in this graph don't cross at the origin!] Here we see that $f'(1) \approx 3.5$.

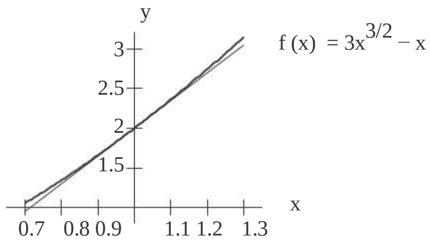


Figure 2.17

29.

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\ln(\cos(1+h)) - \ln(\cos 1)}{h}$$

For $h = 0.001$, the difference quotient $= -1.55912$; for $h = 0.0001$, the difference quotient $= -1.55758$. The instantaneous rate of change of f therefore appears to be about -1.558 at $x = 1$.

At $x = \frac{\pi}{4}$, if we try $h = 0.0001$, then

$$\text{difference quotient} = \frac{\ln[\cos(\frac{\pi}{4} + 0.0001)] - \ln(\cos\frac{\pi}{4})}{0.0001} \approx -1.0001.$$

The instantaneous rate of change of f appears to be about -1 at $x = \frac{\pi}{4}$.

30. The quantity $f(0)$ represents the population on October 17, 2006, so $f(0) = 300$ million.

The quantity $f'(0)$ represents the rate of change of the population (in millions per year). Since

$$\frac{1 \text{ person}}{11 \text{ seconds}} = \frac{1/10^6 \text{ million people}}{11/(60 \cdot 60 \cdot 24 \cdot 365) \text{ years}} = 2.867 \text{ million people/year},$$

so we have $f'(0) = 2.867$.

31. We want to approximate $P'(0)$ and $P'(7)$. Since for small h

$$P'(0) \approx \frac{P(h) - P(0)}{h},$$

if we take $h = 0.01$, we get

$$P'(0) \approx \frac{1.267(1.007)^{0.01} - 1.267}{0.01} = 0.00884 \text{ billion/year}$$
$$= 8.84 \text{ million people/year in 2000},$$

$$P'(7) \approx \frac{1.267(1.007)^{7.01} - 1.267(1.007)^{7}}{0.01} = 0.00928 \text{ billion/year}$$
$$= 9.28 \text{ million people/year in 2007}$$

32. (a) From Figure 2.18, it appears that the slopes of the tangent lines to the two graphs are the same at each x. For $x = 0$, the slopes of the tangents to the graphs of $f(x)$ and $g(x)$ at 0 are

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - 0}{h} = \lim_{h\to 0} \frac{\frac{1}{2}h^2}{h} = \lim_{h\to 0} \frac{1}{2}h = 0,$$

$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{\frac{1}{2}h^2 + 3 - 3}{h} = \lim_{h\to 0} \frac{\frac{1}{2}h^2}{h} = \lim_{h\to 0} \frac{1}{2}h = 0.$$

For $x = 2$, the slopes of the tangents to the graphs of $f(x)$ and $g(x)$ are

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{2}(2+h)^2 - \frac{1}{2}(2)^2}{h} = \lim_{h\to 0} \frac{\frac{1}{2}(4+4h+h^2) - 2}{h} = \lim_{h\to 0} \frac{2+2h+\frac{1}{2}h^2 - 2}{h} = \lim_{h\to 0} \frac{2h+\frac{1}{2}h^2}{h} = \lim_{h\to 0} \left(2 + \frac{1}{2}h\right) = 2,$$

$$g'(2) = \lim_{h\to 0} \frac{g(2+h) - g(2)}{h} = \lim_{h\to 0} \frac{\left(\frac{1}{2}(2+h)^2 + 3\right) - \left(\frac{1}{2}(2)^2 + 3\right)}{h} = \lim_{h\to 0} \frac{\frac{1}{2}(2+h)^2 - \frac{1}{2}(2)^2}{h} = \lim_{h\to 0} \frac{\frac{1}{2}(4+4h+h^2) - 2}{h} = \lim_{h\to 0} \frac{2+2h+\frac{1}{2}(h^2) - 2}{h} = \lim_{h\to 0} \frac{2h+\frac{1}{2}(h^2)}{h} = \lim_{h\to 0} \left(2 + \frac{1}{2}h\right) = 2.$$

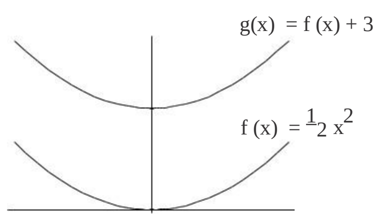


Figure 2.18

For $x = x_0$, the slopes of the tangents to the graphs of f (x) and g(x) are

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0+h)^2 - \frac{1}{2}x_0^2}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0^2 + 2x_0h + h^2) - \frac{1}{2}x_0^2}{h} \\ &= \lim_{h\to 0} \frac{x_0h + \frac{1}{2}h^2}{h} \\ &= \lim_{h\to 0} \left(x_0 + \frac{1}{2}h\right) \\ &= x_0, \end{aligned}$$

$$\begin{aligned} g'(x_0) &= \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0+h)^2 + 3 - (\frac{1}{2}(x_0)^2 + 3)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0+h)^2 - \frac{1}{2}(x_0)^2}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2}(x_0^2 + 2x_0h + h^2) - \frac{1}{2}x_0^2}{h} \\ &= \lim_{h\to 0} \frac{x_0h + \frac{1}{2}h^2}{h} \\ &= \lim_{h\to 0} \left(x_0 + \frac{1}{2}h\right) \\ &= x_0. \end{aligned}$$

(b)

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) + C - (f(x) + C)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= f'(x). \end{aligned}$$

33. As h gets smaller, round-off error becomes important. When $h = 10^{-12}$, the quantity $2^h - 1$ is so close to 0 that the calculator rounds off the difference to 0, making the difference quotient 0. The same thing will happen when $h = 10^{-20}$.

34. (a) Table 2.4 shows that near $x = 1$, every time the value of x increases by 0.001, the value of x^2 increases by approximately 0.002. This suggests that

$$f'(1) \approx \frac{0.002}{0.001} = 2.$$

Table 2.4 *Values of* $f(x) = x^2$ *near* $x = 1$

x	x^2	Difference in successive x^2 values
0.998	0.996004	
		0.001997
0.999	0.998001	
		0.001999
1.000	1.000000	
		0.002001
1.001	1.002001	
		0.002003
1.002	1.004004	
↑ x increments of 0.001		↑ All approximately 0.002

(b) The derivative is the limit of the difference quotient, so we look at

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}.$$

Using the formula for f, we have

$$f'(1) = \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h\to 0} \frac{(1+2h+h^2) - 1}{h} = \lim_{h\to 0} \frac{2h+h^2}{h}.$$

Since the limit only examines values of h close to, but not equal to zero, we can cancel h in the expression $(2h + h^2)/h$. We get

$$f'(1) = \lim_{h\to 0} \frac{h(2 + h)}{h} = \lim_{h\to 0} (2 + h).$$

This limit is 2, so $f'(1) = 2$. At $x = 1$ the rate of change of x^2 is 2.

(c) Since the derivative is the rate of change, $f'(1) = 2$ means that for small changes in x near $x = 1$, the change in $f(x) = x^2$ is about twice as big as the change in x. As an example, if x changes from 1 to 1.1, a net change of 0.1, then $f(x)$ changes by about 0.2. Figure 2.19 shows this geometrically. Near $x = 1$ the function is approximately linear with slope of 2.

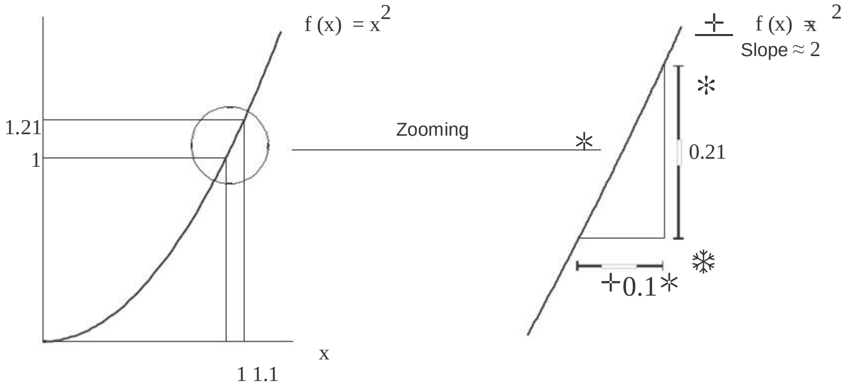


Figure 2.19: Graph of $f(x) = x^2$ near $x = 1$ has slope ≈ 2

35. $\lim_{h\to 0} \frac{(-3 + h)^2 - 9}{h} = \lim_{h\to 0} \frac{9 - 6h + h^2 - 9}{h} = \lim_{h\to 0} \frac{h(-6 + h)}{h} = \lim_{h\to 0} -6 + h = -6.$

36. $\lim_{h\to 0} \frac{(2 - h)^3 - 8}{h} = \lim_{h\to 0} \frac{8 - 12h + 6h^2 - h^3 - 8}{h} = \lim_{h\to 0} \frac{h(-12 + 6h - h^2)}{h} = \lim_{h\to 0} -12 + 6h - h^2 = -12.$

37. $\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{1 + h} - 1\right) = \lim_{h\to 0} \frac{1 - (1 + h)}{(1 + h)h} = \lim_{h\to 0} \frac{-1}{1 + h} = -1$

38. $\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(1 + h)^2} - 1\right) = \lim_{h\to 0} \frac{1 - (1 + 2h + h^2)}{h(1 + h)^2} = \lim_{h\to 0} \frac{-2 - h}{(1 + h)^2} = -2$

39. $\frac{\sqrt{4 + h} - 2}{h} = \frac{(\sqrt{4 + h} - 2)(\sqrt{4 + h} + 2)}{h(\sqrt{4 + h} + 2)} = \frac{4 + h - 4}{h(\sqrt{4 + h} + 2)} = \frac{1}{\sqrt{4 + h} + 2}.$

Therefore $\lim_{h\to 0} \frac{\sqrt{4 + h} - 2}{h} = \lim_{h\to 0} \frac{1}{\sqrt{4 + h} + 2} = \frac{1}{4}$

40. $\frac{1}{\sqrt{4 + h}} - \frac{1}{2} = \frac{2 - \sqrt{4 + h}}{2\sqrt{4 + h}} = \frac{(2 - \sqrt{4 + h})(2 + \sqrt{4 + h})}{2\sqrt{4 + h}(2 + \sqrt{4 + h})} = \frac{4 - (4 + h)}{2\sqrt{4 + h}(2 + \sqrt{4 + h})}.$

Therefore $\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{4 + h}} - \frac{1}{2}\right) = \lim_{h\to 0} \frac{-1}{2\sqrt{4 + h}(2 + \sqrt{4 + h})} = -\frac{1}{16}$

41. Using the definition of the derivative, we have

$$f'(10) = \lim_{h \to 0} \frac{f(10+h) - f(10)}{h}$$

$$= \lim_{h \to 0} \frac{5(10+h)^2 - 5(10)^2}{h}$$

$$= \lim \frac{500 + 100h + 5h^2 - 500}{}$$

$$= \lim_{h\to 0} \frac{100h + 5h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(100 + 5h)}{h}$$

$$= \lim_{h\to 0} 100 + 5h$$

$$= 100.$$

42. Using the definition of the derivative, we have

$$
\begin{aligned}
f'(-2) &= \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} \\
&= \lim_{h\to 0} \frac{(-2+h)^3 - (-2)^3}{h} \\
&= \lim_{h\to 0} \frac{(-8 + 12h - 6h^2 + h^3) - (-8)}{h} \\
&= \lim_{h\to 0} \frac{12h - 6h^2 + h^3}{h} \\
&= \lim_{h\to 0} \frac{h(12 - 6h + h^2)}{h} \\
&= \lim_{h\to 0} (12 - 6h + h^2),
\end{aligned}
$$

which goes to 12 as $h \to 0$. So $f'(-2) = 12$.

43. Using the definition of the derivative

$$
\begin{aligned}
g'(-1) &= \lim_{h\to 0} \frac{g(-1+h) - g(-1)}{h} \\
&= \lim_{h\to 0} \frac{((-1+h)^2 + (-1+h)) - ((-1)^2 + (-1))}{h} \\
&= \lim_{h\to 0} \frac{(1 - 2h + h^2 - 1 + h) - (0)}{h} \\
&= \lim_{h\to 0} \frac{-h + h^2}{h} = \lim_{h\to 0} (-1 + h) = -1.
\end{aligned}
$$

44.

$$
\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{((1+h)^3 + 5) - (1^3 + 5)}{h} \\
&= \lim_{h\to 0} \frac{1 + 3h + 3h^2 + h^3 + 5 - 1 - 5}{h} = \lim_{h\to 0} \frac{3h + 3h^2 + h^3}{h} \\
&= \lim_{h\to 0} (3 + 3h + h^2) = 3.
\end{aligned}
$$

45.

$$
\begin{aligned}
g'(2) &= \lim_{h\to 0} \frac{g(2+h) - g(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} \\
&= \lim_{h\to 0} \frac{2 - (2+h)}{h(2+h)2} = \lim_{h\to 0} \frac{-h}{h(2+h)2} \\
&= \lim_{h\to 0} \frac{1}{(2+h)2} = -\frac{1}{4}
\end{aligned}
$$

46.

$$
\begin{aligned}
g'(2) &= \lim_{h\to 0} \frac{g(2+h) - g(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{(2+h)^2} - \frac{1}{2^2}}{h} \\
&= \lim_{h\to 0} \frac{2^2 - (2+h)^2}{2^2(2+h)^2 h} = \lim_{h\to 0} \frac{4 - 4 - 4h - h^2}{4h(2+h)^2} \\
&= \lim \frac{-4h - h^2}{} = \lim \frac{-4 - h}{}
\end{aligned}
$$

$$\lim_{h\to 0} 4h(2+h)^2 \lim_{h\to 0} 4(2+h)^2$$

$$= \frac{-4}{4(2)^2} = -\frac{1}{4}.$$

47. As we saw in the answer to Problem 41, the slope of the tangent line to $f(x) = 5x^2$ at $x = 10$ is 100. When $x = 10$, $f(x) = 500$ so (10, 500) is a point on the tangent line. Thus $y = 100(x - 10) + 500 = 100x - 500$.

48. As we saw in the answer to Problem 42, the slope of the tangent line to $f(x) = x^3$ at $x = -2$ is 12. When $x = -2$, $f(x) = -8$ so we know the point $(-2, -8)$ is on the tangent line. Thus the equation of the tangent line is $y = 12(x + 2) - 8 = 12x + 16$.

49. We know that the slope of the tangent line to $f(x) = x$ when $x = 20$ is 1. When $x = 20$, $f(x) = 20$ so (20, 20) is on the tangent line. Thus the equation of the tangent line is $y = 1(x - 20) + 20 = x$.

50. First find the derivative of $f(x) = 1/x^2$ at $x = 1$.

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - \frac{1}{1^2}}{h}$$

$$= \lim_{h\to 0} \frac{1 - (1+h)^2}{h(1+h)^2} = \lim_{h\to 0} \frac{1 - (1 + 2h + h^2)}{h(1+h)^2}$$

$$= \lim_{h\to 0} \frac{-2h - h^2}{h(1+h)^2} = \lim_{h\to 0} \frac{-2 - h}{(1+h)^2} = -2$$

Thus the tangent line has a slope of -2 and goes through the point (1, 1), and so its equation is

$$y - 1 = -2(x - 1) \quad \text{or} \quad y = -2x + 3.$$

Strengthen Your Understanding

51. The graph of $f(x) = \log x$ is increasing, so $f'(0.5) > 0$.

52. The derivative of a function at a point is the slope of the tangent line, not the tangent line itself.

53. $f(x) = e^x$.
Many other answers are possible.

54. A linear function is of the form $f(x) = ax + b$. The derivative of this function is the slope of the line $y = ax + b$, so $f'(x) = a$, so $a = 2$. One such function is $f(x) = 2x + 1$.

55. True. The derivative of a function is the limit of difference quotients. A few difference quotients can be computed from the table, but the limit can not be computed from the table.

56. True. The derivative $f'(10)$ is the slope of the tangent line to the graph of $y = f(x)$ at the point where $x = 10$. When you zoom in on $y = f(x)$ close enough it is not possible to see the difference between the tangent line and the graph of f on the calculator screen. The line you see on the calculator is a little piece of the tangent line, so its slope is the derivative $f'(10)$.

57. True. This is seen graphically. The derivative $f'(a)$ is the slope of the line tangent to the graph of f at the point P where $x = a$. The difference quotient $(f(b) - f(a))/(b - a)$ is the slope of the secant line with endpoints on the graph of f at the points where $x = a$ and $x = b$. The tangent and secant lines cross at the point P. The secant line goes above the tangent line for $x > a$ because f is concave up, and so the secant line has higher slope.

58. (a). This is best observed graphically.

Solutions for Section 2.3

Exercises

1. (a) We use the interval to the right of $x = 2$ to estimate the derivative. (Alternately, we could use the interval to the left of 2, or we could use both and average the results.) We have

$$f'(2) \approx \frac{f(4) - f(2)}{4 - 2} = \frac{24 - 18}{4 - 2} = \frac{6}{2} = 3.$$

We estimate $f'(2) \approx 3$.

(b) We know that $f'(x)$ is positive when $f(x)$ is increasing and negative when $f(x)$ is decreasing, so it appears that

$f'(x)$ is positive for $0 < x < 4$ and is negative for $4 < x < 12$.

2. For x = 0, 5, 10, and 15, we use the interval to the right to estimate the derivative. For x = 20, we use the interval to the left. For x = 0, we have

$$f'(0) \approx \frac{f(5) - f(0)}{5 - 0} = \frac{70 - 100}{5 - 0} = \frac{-30}{5} = -6.$$

Similarly, we find the other estimates in Table 2.5.

Table 2.5

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

3. The graph is that of the line y = −2x + 2. The slope, and hence the derivative, is −2. See Figure 2.20.

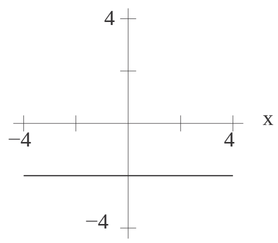


Figure 2.20

4. See Figure 2.21.

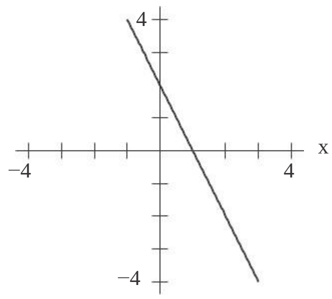


Figure 2.21

5. See Figure 2.22.

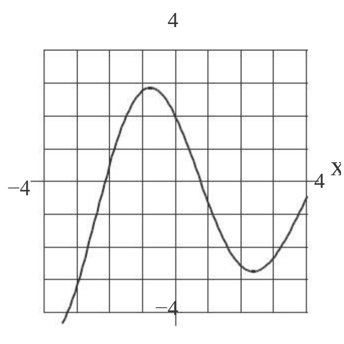


Figure 2.22

6. See Figure 2.23.

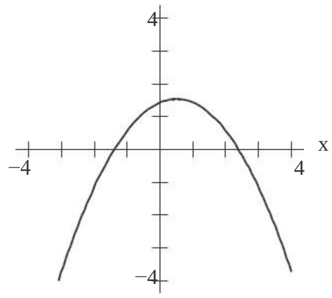


Figure 2.23

7. The slope of this curve is approximately −1 at x = −4 and at x = 4, approximately 0 at x = −2.5 and x = 1.5, and approximately 1 at x = 0. See Figure 2.24.

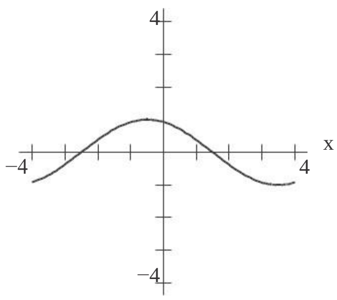


Figure 2.24

8. See Figure 2.25.

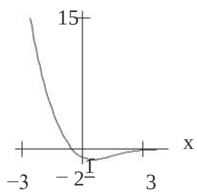


Figure 2.25

9. See Figure 2.26.

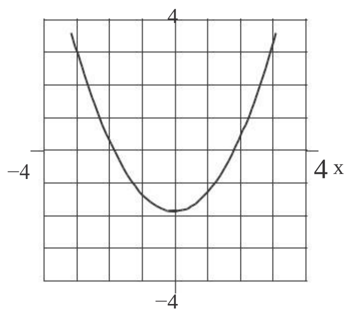


Figure 2.26

10. See Figure 2.27.

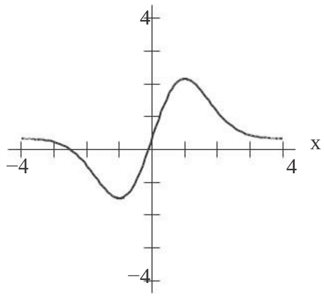


Figure 2.27

11. See Figure 2.28.

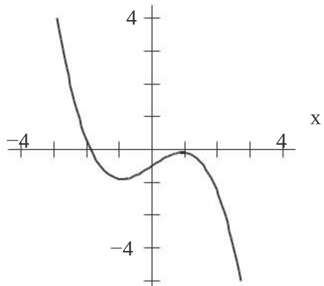


Figure 2.28

12. See Figure 2.29.

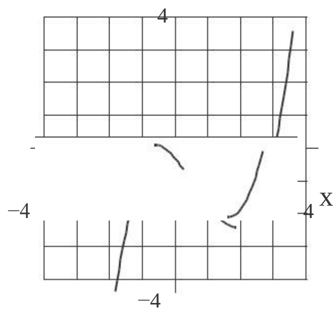


Figure 2.29

13. See Figures 2.30 and 2.31.

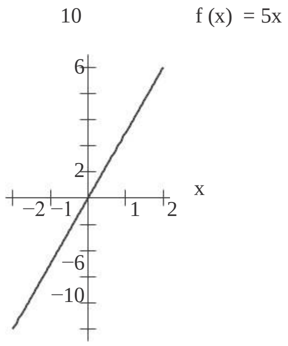


Figure 2.30

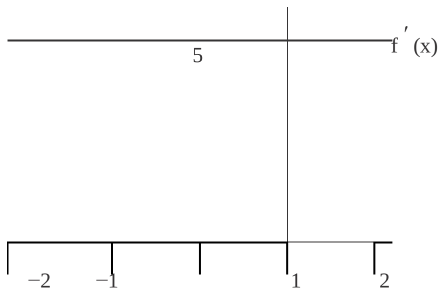


Figure 2.31

14. See Figures 2.32 and 2.33.

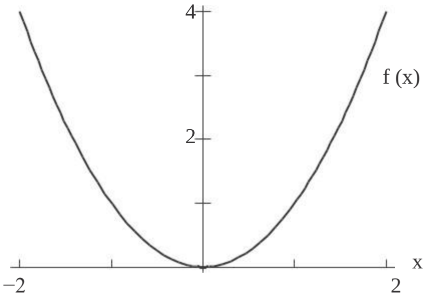


Figure 2.32

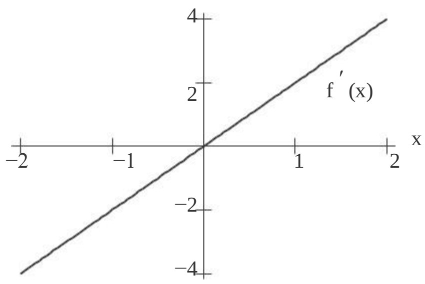


Figure 2.33

15. See Figures 2.34 and 2.35.

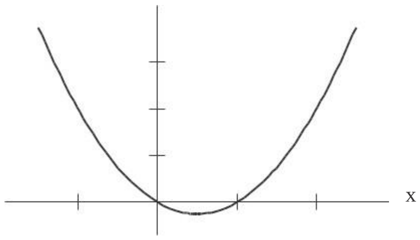


Figure 2.34

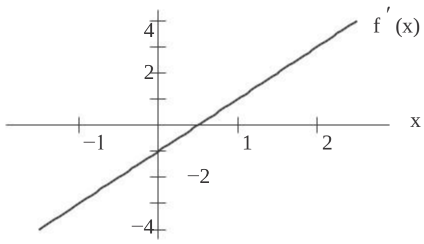


Figure 2.35

16. The graph of f (x) and its derivative look the same, as in Figures 2.36 and 2.37.

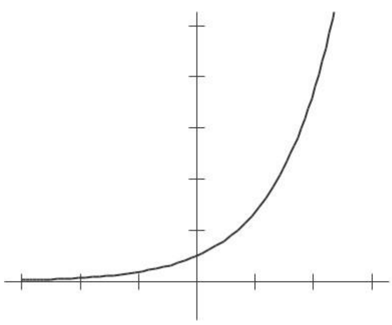


Figure 2.36

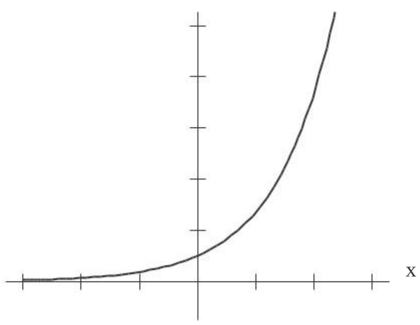


Figure 2.37

17. See Figures 2.38 and 2.39.

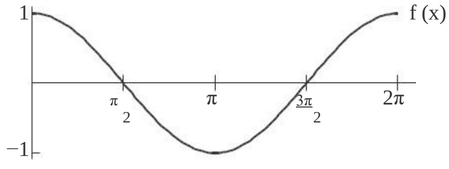


Figure 2.38

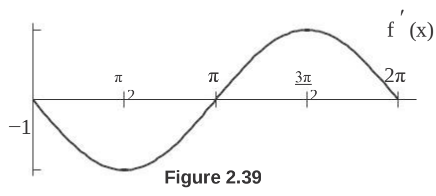


Figure 2.39

18. See Figures 2.40 and 2.41.

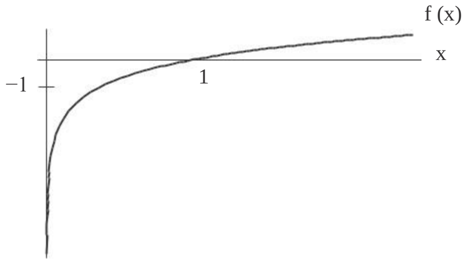


Figure 2.40

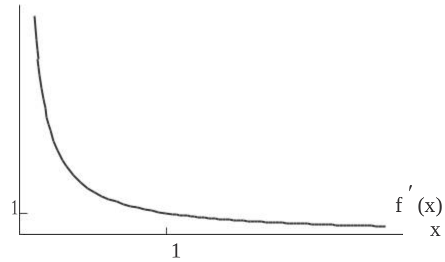


Figure 2.41

19. Since $1/x = x^{-1}$, using the power rule gives

$$k'(x) = (-1)x^{-2} = -\frac{1}{x^2}.$$

Using the definition of the derivative, we have

$$\begin{aligned} k'(x) &= \lim_{h\to 0}\frac{k(x+h)-k(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0}\frac{x-(x+h)}{h(x+h)x} \\ &= \lim_{h\to 0}\frac{-h}{h(x+h)x} = \lim_{h\to 0}\frac{-1}{(x+h)x} = -\frac{1}{x^2}. \end{aligned}$$

20. Since $1/x^2 = x^{-2}$, using the power rule gives

$$l'(x) = -2x^{-3} = -\frac{2}{x^3}.$$

Using the definition of the derivative, we have

$$\begin{aligned} l'(x) &= \lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} = \lim_{h\to 0}\frac{x^2-(x+h)^2}{h(x+h)^2x^2} \\ &= \lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{h(x+h)^2x^2} = \lim_{h\to 0}\frac{-2xh-h^2}{h(x+h)^2x^2} \\ &= \lim_{h\to 0}\frac{-2x-h}{(x+h)^2x^2} = \frac{-2x}{x^2x^2} = -\frac{2}{x^3}. \end{aligned}$$

21. Using the definition of the derivative,

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{2(x+h)^2-3-(2x^2-3)}{h} \\ &= \lim_{h\to 0}\frac{2(x^2+2xh+h^2)-3-2x^2+3}{h} = \lim_{h\to 0}\frac{4xh+2h^2}{h} \\ &= \lim_{h\to 0}(4x+2h) = 4x. \end{aligned}$$

22. Using the definition of the derivative, we have

$$m'(x) = \lim_{h\to 0}\frac{m(x+h)-m(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{x+h+1}-\frac{1}{x+1}\right)$$

$$= \lim \frac{1}{} \frac{x + 1 - x - h - 1}{} = \lim \frac{-h}{}$$

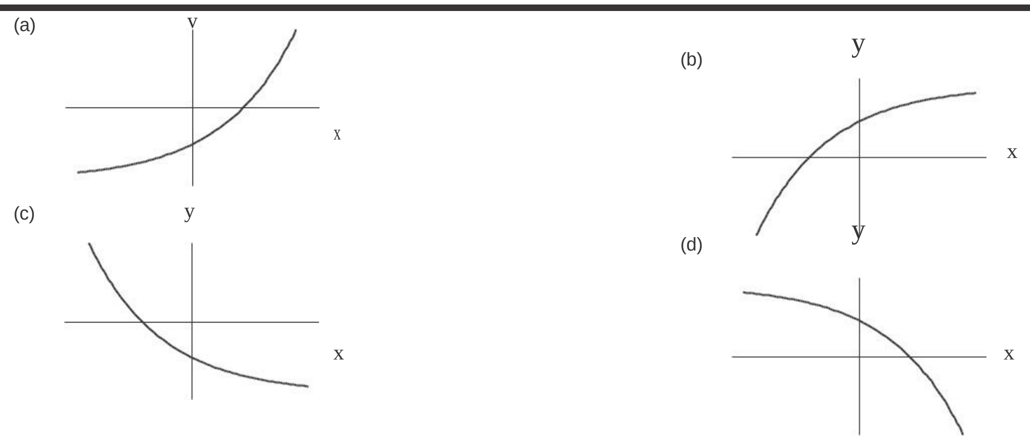
h→0 h (x + 1)(x + h + 1) h→0 h(x + 1)(x + h + 1)

$$= \lim_{h\to 0} \frac{-1}{(x+1)(x+h+1)}$$

$$= \frac{1}{(x+1)^2}.$$

Problems

23.



24. Since $f'(x) > 0$ for $x < -1$, $f(x)$ is increasing on this interval.
Since $f'(x) < 0$ for $x > -1$, $f(x)$ is decreasing on this interval.
Since $f'(x) = 0$ at $x = -1$, the tangent to $f(x)$ is horizontal at $x = -1$.
One possible shape for $y = f(x)$ is shown in Figure 2.42.

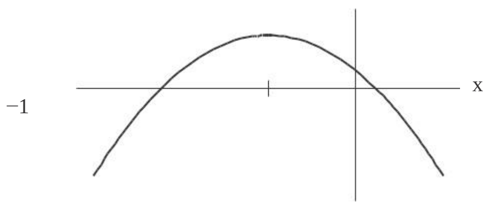


Figure 2.42

25.

x	ln x
0.998	−0.0020
0.999	−0.0010
1.000	0.0000
1.001	0.0010
1.002	0.0020

x	ln x
1.998	0.6921
1.999	0.6926
2.000	0.6931
2.001	0.6936
2.002	0.6941

x	ln x
4.998	1.6090
4.999	1.6092
5.000	1.6094
5.001	1.6096
5.002	1.6098

x	ln x
9.998	2.3024
9.999	2.3025
10.000	2.3026
10.001	2.3027
10.002	2.3028

At $x = 1$, the values of ln x are increasing by 0.001 for each increase in x of 0.001, so the derivative appears to be 1. At $x = 2$, the increase is 0.0005 for each increase of 0.001, so the derivative appears to be 0.5. At $x = 5$, ln x increases by 0.0002 for each increase of 0.001 in x, so the derivative appears to be 0.2. And at $x = 10$, the increase is 0.0001 over intervals of 0.001, so the derivative appears to be 0.1. These values suggest an inverse relationship between x and $f'(x)$, namely $f'(x) = \frac{1}{x}$.

26. We know that $f'(x) \approx \frac{f(x+h) - f(x)}{h}$. For this problem, we'll take the average of the values obtained for $h = 1$ and $h = -1$; that's the average of $f(x+1) - f(x)$ and $f(x) - f(x-1)$ which equals $\frac{f(x+1) - f(x-1)}{2}$. Thus,
$f'(0) \approx f(1) - f(0) = 13 - 18 = -5$.
$f'(1) \approx (f(2) - f(0))/2 = (10 - 18)/2 = -4$.
$f'(2) \approx (f(3) - f(1))/2 = (9 - 13)/2 = -2$.
$f'(3) \approx (f(4) - f(2))/2 = (9 - 10)/2 = -0.5$.
$f'(4) \approx (f(5) - f(3))/2 = (11 - 9)/2 = 1$.
$f'(5) \approx (f(6) - f(4))/2 = (15 - 9)/2 = 3$.
$f'(6) \approx (f(7) - f(5))/2 = (21 - 11)/2 = 5$.

$f'(7) \approx (f(8) - f(6))/2 = (30 - 15)/2 = 7.5.$
$f'(8) \approx f(8) - f(7) = 30 - 21 = 9.$
The rate of change of f (x) is positive for $4 \le x \le 8$, negative for $0 \le x \le 3$. The rate of change is greatest at about x = 8.

27. The value of g(x) is increasing at a decreasing rate for $2.7 < x < 4.2$ and increasing at an increasing rate for $x > 4.2$.

$$\frac{\Delta y}{\Delta x} = \frac{7.4 - 6.0}{5.2 - 4.7} = 2.8 \quad \text{between } x = 4.7 \text{ and } x = 5.2$$

$$\frac{\Delta y}{\Delta x} = \frac{9.0 - 7.4}{5.7 - 5.2} = 3.2 \quad \text{between } x = 5.2 \text{ and } x = 5.7$$

Thus $g'(x)$ should be close to 3 near x = 5.2.

28. (a) x_3 (b) x_4 (c) x_5 (d) x_3

29. This is a line with slope 1, so the derivative is the constant function $f'(x) = 1$. The graph is the horizontal line y = 1. See Figure 2.43.

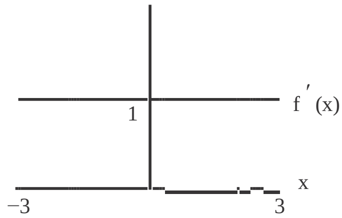


Figure 2.43

30. This is a line with slope −2, so the derivative is the constant function $f'(x) = -2$. The graph is a horizontal line at y = −2. See Figure 2.44.

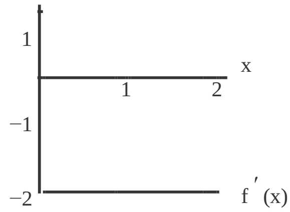


Figure 2.44

31. See Figure 2.45.

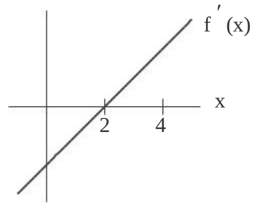


Figure 2.45

32. See Figure 2.46.

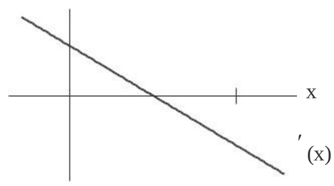
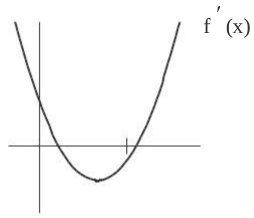


Figure 2.46

33. See Figure 2.47.



4

Figure 2.47

34. See Figure 2.48.

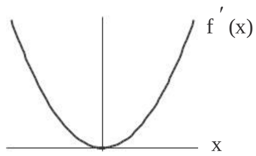


Figure 2.48

35. See Figure 2.49.

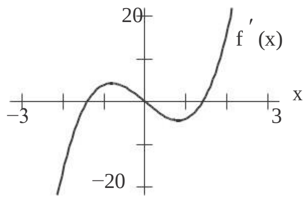


Figure 2.49

36. One possible graph is shown in Figure 2.50. Notice that as x gets large, the graph of $f(x)$ gets more and more horizontal. Thus, as x gets large, $f'(x)$ gets closer and closer to 0.

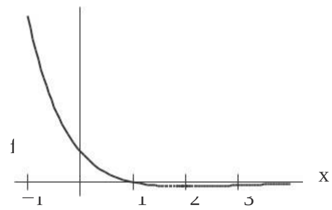


Figure 2.50

37. See Figure 2.51.

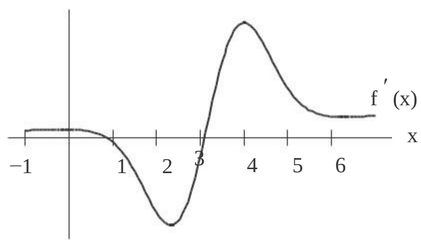


Figure 2.51

38. See Figure 2.52.

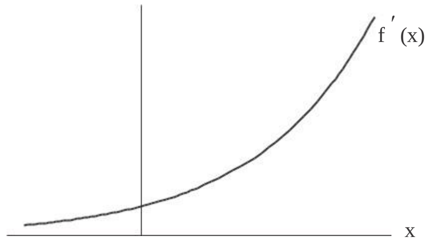


Figure 2.52

39. See Figure 2.53.

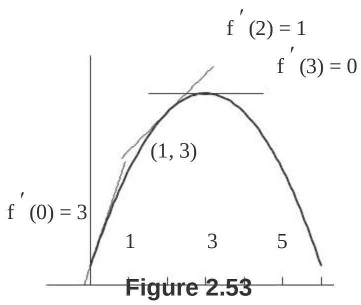
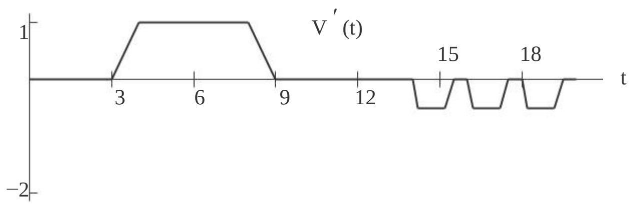


Figure 2.53

40. (a) Graph II
(b) Graph I
(c) Graph III

41. (a) $t = 3$
(b) $t = 9$
(c) $t = 14$
(d)



42. The derivative is zero whenever the graph of the original function is horizontal. Since the current is proportional to the derivative of the voltage, segments where the current is zero alternate with positive segments where the voltage is increasing and negative segments where the voltage is decreasing. See Figure 2.54. Note that the derivative does not exist where the graph has a corner.

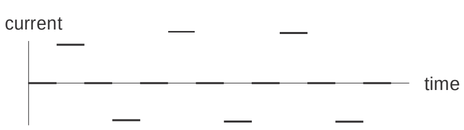


Figure 2.54

43. (a) The function f is increasing where f' is positive, so for $x_1 < x < x_3$.

(b) The function f is decreasing where f' is negative, so for $0 < x < x_1$ or $x_3 < x < x_5$.

44. On intervals where $f' = 0$, f is not changing at all, and is therefore constant. On the small interval where $f' > 0$, f is increasing; at the point where f' hits the top of its spike, f is increasing quite sharply. So f should be constant for a while, have a sudden increase, and then be constant again. A possible graph for f is shown in Figure 2.55.

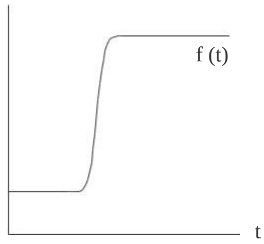
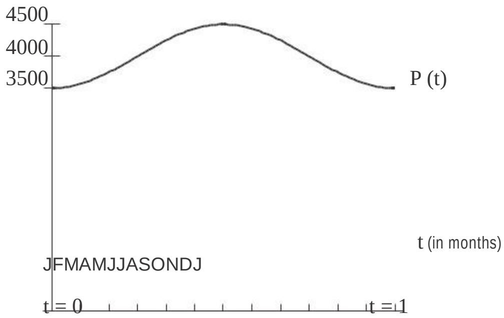


Figure 2.55: Step function

45. (a) The population varies periodically with a period of 1 year. See below.



(b) The population is at a maximum on July 1st. At this time $\sin(2\pi t - \frac{\pi}{2}) = 1$, so the actual maximum population is $4000 + 500(1) = 4500$. Similarly, the population is at a minimum on January 1st. At this time, $\sin(2\pi t - \frac{\pi}{2}) = -1$, so the minimum population is $4000 + 500(-1) = 3500$.

(c) The rate of change is most positive about April 1st and most negative around October 1st.

(d) Since the population is at its maximum around July 1st, its rate of change is about 0 then.

46. The derivative of the accumulated federal debt with respect to time is shown in Figure 2.56. The derivative represents the rate of change of the federal debt with respect to time and is measured in trillions of dollars per year.

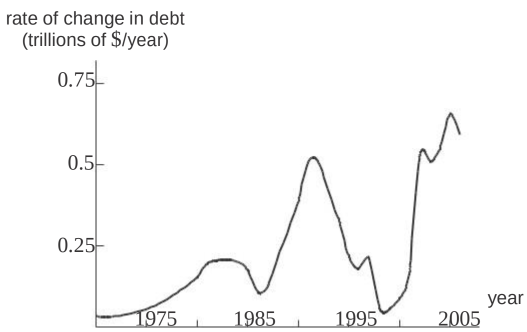


Figure 2.56

47. From the given information we know that f is increasing for values of x less than −2, is decreasing between x = −2 and x = 2, and is constant for x > 2. Figure 2.57 shows a possible graph—yours may be different.

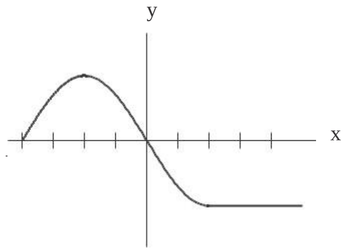


Figure 2.57

48. Since $f'(x) > 0$ for $1 < x < 3$, we see that f (x) is increasing on this interval.
Since $f'(x) < 0$ for $x < 1$ and for $x > 3$, we see that f (x) is decreasing on these intervals.
Since $f'(x) = 0$ for x = 1 and x = 3, the tangent to f (x) will be horizontal at these x's. One of many possible shapes of y = f (x) is shown in Figure 2.58.

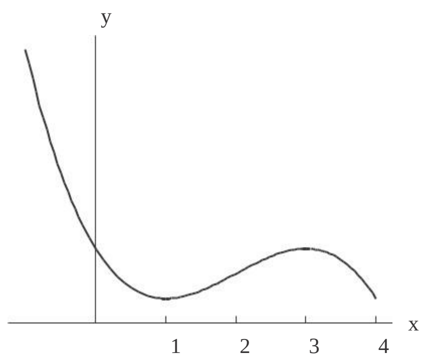


Figure 2.58

49. If $\lim_{x\to\infty} f(x) = 50$ and $f'(x)$ is positive for all x, then f (x) increases to 50, but never rises above it. A possible graph of f (x) is shown in Figure 2.59. If $\lim_{x\to\infty} f'(x)$ exists, it must be zero, since f looks more and more like a horizontal line.
If $f'(x)$ approached another positive value c, then f would look more and more like a line with positive slope c, which would eventually go above y = 50.

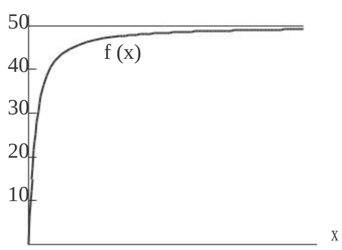
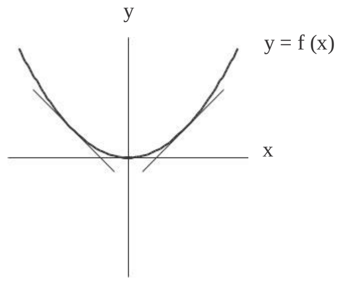
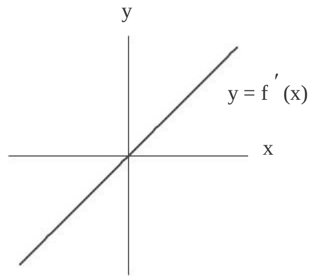


Figure 2.59

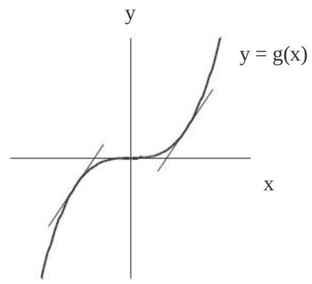
50. If f (x) is even, its graph is symmetric about the y-axis. So the tangent line to f at $x = x_0$ is the same as that at $x = -x_0$ reflected about the y-axis.

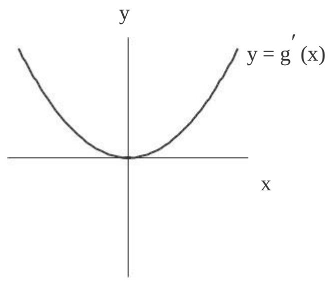




So the slopes of these two tangent lines are opposite in sign, so $f'(x_0) = -f'(-x_0)$, and f' is odd.

51. If g(x) is odd, its graph remains the same if you rotate it 180° about the origin. So the tangent line to g at $x = x_0$ is the tangent line to g at $x = -x_0$, rotated 180°.





But the slope of a line stays constant if you rotate it 180°. So $g'(x_0) = g'(-x_0)$; g' is even.

Strengthen Your Understanding

52. Since $f(x) = \cos x$ is decreasing on some intervals, its derivative $f'(x)$ is negative on those intervals, and the graph of $f'(x)$ is below the x-axis where cos x is decreasing.

53. In order for $f'(x)$ to be greater than zero, the slope of f (x) has to be greater than zero. For example, $f(x) = e^{-x}$ is positive for all x but since the graph is decreasing everywhere, f (x) has negative derivative for all x.

54. Two different functions can have the same rate of change. For example, $f(x) = 1$, $g(x) = 2$ both are constant, so $f'(x) = g'(x) = 0$ but $f(x) = g(x)$.

55. $f(t) = t(1 - t)$. We have $f(t) = t - t^2$, so $f'(t) = 1 - 2t$ so the velocity is positive for $0 < t < 0.5$ and negative for $0.5 < t < 1$.

Many other answers are possible.

56. Every linear function is of the form $f(x) = b + mx$ and has derivative $f'(x) = m$. One family of functions with the same derivative is $f(x) = b + 2x$.

57. True. The graph of a linear function $f(x) = mx + b$ is a straight line with the same slope m at every point. Thus $f'(x) = m$ for all x.

58. True. Shifting a graph vertically does not change the shape of the graph and so it does not change the slopes of the tangent lines to the graph.

59. False. If $f'(x)$ is increasing then f (x) is concave up. However, f (x) may be either increasing or decreasing. For example, the exponential decay function $f(x) = e^{-x}$ is decreasing but $f'(x)$ is increasing because the graph of f is concave up.

60. False. A counterexample is given by $f(x) = 5$ and $g(x) = 10$, two different functions with the same derivatives: $f'(x) = g'(x) = 0$.

Solutions for Section 2.4

Exercises

1. (a) The statement $f(200) = 1300$ means that it costs $1300 to produce 200 gallons of the chemical.