CHAPTER 2

Differentiation

CHAPTER 2 Differentiation

Section 2.1 The Derivative and the Tangent Line Problem

The problem of finding the tangent line at a point P is essentially finding the slope of the tangent line at point P. To do so for a function f, if f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through the point $P(c, f(c))$ with slope m is the tangent line to the graph of f at the point P.

Some alternative notations for $f'(x)$ are

$$\frac{dy}{dx},\ y',\ \frac{d}{dx}\left[f(x)\right],\ \text{and}\ D_x[y].$$

The limit used to define the slope of a tangent line is also used to define differentiation. The key is to rewrite the difference quotient so that Δx does not occur as a factor of the denominator.

If a function f is differentiable at a point $x = c$, then f is continuous at $x = c$. The converse is not true. That is, a function could be continuous at a point, but not differentiable there. For example, the function $y = |x|$ is continuous at $x = 0$, but is not differentiable there.

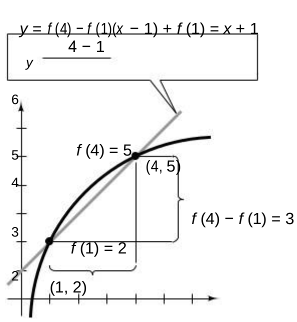
5. At (x_1, y_1), slope $= 0$.

At (x_2, y_2), slope $= \frac{5}{2}$.

6. At (x_1, y_1), slope $= \frac{2}{3}$.

At (x_2, y_2), slope $= -\frac{2}{5}$.

7. (a)–(c)



8. (a) $\frac{f(4) - f(1)}{4 - 1} = \frac{5 - 2}{3} = 1$

$\frac{f(4) - f(3)}{4 - 3} \approx \frac{5 - 4.75}{1} = 0.25$

So, $\frac{f(4) - f(1)}{4 - 1} > \frac{f(4) - f(3)}{4 - 3}$.

(b) The slope of the tangent line at $(1, 2)$ equals $f'(1)$. This slope is steeper than the slope of the line through $(1, 2)$ and $(4, 5)$. So, $\frac{f(4) - f(1)}{4 - 1} < f'(1)$.

9. $f(x) = 3 - 5x$ is a line. Slope $= -5$

10. $g(x) = \frac{3}{2}x + 1$ is a line. Slope $= \frac{3}{2}$

11. Slope at $(2, 5)$

$$= \lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2(2 + \Delta x)^2 - 3 - \left[2(2)^2 - 3\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\left[4 + 4\Delta x + (\Delta x)^2\right] - 3 - (5)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{8\Delta x + 2(\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (8 + 2\Delta x) = 8$$

12. Slope at $(3, -4)$

$$= \lim_{\Delta x \to 0} \frac{f(3 + \Delta x) - f(3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{5 - (3 + \Delta x)^2 - (-4)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{5 - 9 - 6\Delta x - (\Delta x)^2 + 4}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-6\Delta x - (\Delta x)^2}{\Delta x}$$

1 2 3 4 5 6 x

$$= \lim_{\Delta x \to 0} (-6 - \Delta x) = -6$$

(d) $y = \dfrac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$

$$= \frac{3}{3}(x - 1) + 2$$

$$= 1(x - 1) + 2$$

$$= x + 1$$

13. Slope at $(0, 0) = \lim_{t \to 0} \dfrac{f(0 + t) - f(0)}{t}$

$$= \lim_{t \to 0} \frac{3t - t^2 - 0}{t}$$

$$= \lim_{t \to 0} (3 - t) = 3$$

14. Slope at $(1, 5) = \lim_{\Delta t \to 0} \frac{h(1 + \Delta t) - h(1)}{\Delta t}$

$= \lim_{\Delta t \to 0} \frac{(1 + \Delta t)^2 + 4(1 + \Delta t) - 5}{\Delta t}$

$= \lim_{\Delta t \to 0} \frac{1 + 2(\Delta t) + (\Delta t)^2 + 4 + 4(\Delta t) - 5}{\Delta t}$

$= \lim_{\Delta t \to 0} \frac{6(\Delta t) + (\Delta t)^2}{\Delta t}$

$= \lim_{\Delta t \to 0} (6 + \Delta t) = 6$

15. $f(x) = 7$

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{7 - 7}{\Delta x}$

$= \lim_{\Delta x \to 0} 0 = 0$

16. $g(x) = -3$

$g'(x) = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{-3 - (-3)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$

17. $f(x) = -5x$

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{-5(x + \Delta x) - (-5x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{-5x - 5\Delta x + 5x}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{-5\Delta x}{\Delta x}$

$= \lim_{\Delta x \to 0} -5 = -5$

18. $f(x) = 7x - 3$

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{7(x + \Delta x) - 3 - (7x - 3)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{7x + 7\Delta x - 3 - 7x + 3}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{7(\Delta x)}{\Delta x}$

$= \lim_{\Delta x \to 0} 7 = 7$

19. $h(s) = 3 + \frac{2}{3}s$

$h'(s) = \lim_{\Delta s \to 0} \frac{h(s + \Delta s) - h(s)}{\Delta s}$

$= \lim_{\Delta s \to 0} \frac{3 + \frac{2}{3}(s + \Delta s) - \left(3 + \frac{2}{3}s\right)}{\Delta s}$

$= \lim_{\Delta s \to 0} \frac{3 + \frac{2}{3}s + \frac{2}{3}\Delta s - 3 - \frac{2}{3}s}{\Delta s}$

$= \lim_{\Delta s \to 0} \frac{\frac{2}{3}\Delta s}{\Delta s} = \frac{2}{3}$

20. $f(x) = 5 - \frac{2}{3}x$

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{5 - \frac{2}{3}(x + \Delta x) - \left(5 - \frac{2}{3}x\right)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{5 - \frac{2}{3}x - \frac{2}{3}\Delta x - 5 + \frac{2}{3}x}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{-\frac{2}{3}(\Delta x)}{\Delta x}$

$= \lim$

$x \to$ $_0$)

$x \to 0$ x

$$= \lim \left(-\frac{2}{3} \right) = -\frac{2}{3}$$

$($

$x \to 0$

$$f(x) = x^2 + x - 3$$

$$f'(x) = \lim_{x \to 0} \frac{f(x + x) - f(x)}{x}$$

$$= \lim_{x \to 0} \frac{(x + x)^2 + (x + x) - 3 - (x^2 + x - 3)}{x}$$

$$= \lim_{x \to 0} \frac{x^2 + 2x(x) + (x)^2 + x + x - 3 - x^2 - x + 3}{x}$$

$$= \lim_{x \to 0} \frac{2x(x) + (x)^2 + x}{x}$$

$$= \lim_{x \to 0} (2x + x + 1) = 2x + 1$$

$$f(x) = x^2 - 5$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 5 - (x^2 - 5)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - 5 - x^2 + 5}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x(\Delta x) + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

$$f(x) = x^3 - 12x$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^3 - 12(x + \Delta x)\right] - \left[x^3 - 12x\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12x - 12\Delta x - x^3 + 12x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(3x^2 + 3x\Delta x + (\Delta x)^2 - 12\right) = 3x^2 - 12$$

24. $g(t) = t^3 + 4t$

$$g'(t) = \lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left[(t + \Delta t)^3 + 4(t + \Delta t)\right] - \left[t^3 + 4t\right]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{t^3 + 3t^2\Delta t + 3t(\Delta t)^2 + (\Delta t)^3 + 4t + 4\Delta t - t^3 - 4t}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{3t^2\Delta t + 3t(\Delta t)^2 + (\Delta t)^3 + 4\Delta t}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left(3t^2 + 3t\Delta t + (\Delta t)^2 + 4\right)$$

$$= 3t^2 + 4$$

25. $f(x) = \dfrac{1}{x-1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{x+\Delta x-1} - \dfrac{1}{x-1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x-1) - (x+\Delta x-1)}{\Delta x(x+\Delta x-1)(x-1)}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x(x+\Delta x-1)(x-1)}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x-1)(x-1)}$$

$$= -\frac{1}{(x-1)^2}$$

26. $f(x) = \dfrac{1}{x^2}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{(x+\Delta x)^2} - \dfrac{1}{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - (x+\Delta x)^2}{\Delta x(x+\Delta x)^2 x^2}$$

$$= \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x(x+\Delta x)^2 x^2}$$

$$= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{(x+\Delta x)^2 x^2}$$

$$= \frac{-2x}{x^4}$$

$$= -\frac{2}{x^3}$$

27. $f(x) = \sqrt{x+4}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+4} - \sqrt{x+4}}{\Delta x} \cdot \left(\frac{\sqrt{x+\Delta x+4} + \sqrt{x+4}}{\sqrt{x+\Delta x+4} + \sqrt{x+4}}\right)$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x+4) - (x+4)}{\Delta x\left[\sqrt{x+\Delta x+4} + \sqrt{x+4}\right]}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x+4} + \sqrt{x+4}} = \frac{1}{\sqrt{x+4} + \sqrt{x+4}} = \frac{1}{2\sqrt{x+4}}$$

28. $h(s) = -2\sqrt{s}$

$$h'(s) = \lim_{\Delta s \to 0} \frac{h(s+\Delta s) - h(s)}{\Delta s}$$

$$= \lim_{\Delta s \to 0} \frac{-2\sqrt{s+\Delta s} - \left(-2\sqrt{s}\right)}{\Delta s}$$

$$= \lim_{\Delta s \to 0} \frac{-2\left(\sqrt{s+\Delta s} - \sqrt{s}\right)}{\Delta s} \cdot \frac{\sqrt{s+\Delta s} + \sqrt{s}}{\sqrt{s+\Delta s} + \sqrt{s}}$$

$$= \lim_{\Delta s \to 0} \frac{-2\left(s+\Delta s - s\right)}{\Delta s\left(\sqrt{s+\Delta s} + \sqrt{s}\right)}$$

$$= \lim_{s \to 0} -\frac{2}{\sqrt{s + \quad s} + \sqrt{s}}$$

$$= -\frac{2}{2\sqrt{s}} = -\frac{1}{\sqrt{s}}$$

29. (a) $f(x) = x^2 + 3$

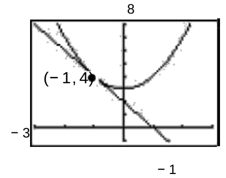
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^2 + 3\right] - \left(x^2 + 3\right)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 3 - x^2 - 3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

At $(-1, 4)$, the slope of the tangent line is $m = 2(-1) = -2$.

The equation of the tangent line is

$$y - 4 = -2(x + 1)$$
$$y - 4 = -2x - 2$$
$$y = -2x + 2$$

(b)



(c) Graphing utility confirms $\frac{dy}{dx} = -2$ at $(-1, 4)$.

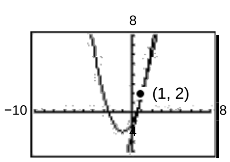
30. (a) $f(x) = x^2 + 2x - 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^2 + 2(x + \Delta x) - 1\right] - \left[x^2 + 2x - 1\right]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\left[x^2 + 2x\Delta x + (\Delta x)^2 + 2x + 2\Delta x - 1\right] - \left[x^2 + 2x - 1\right]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 + 2\Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2x + \Delta x + 2) = 2x + 2$$

At $(1, 2)$, the slope of the tangent line is $m = 2(1) + 2 = 4$.

The equation of the tangent line is

$$y - 2 = 4(x - 1)$$
$$y - 2 = 4x - 4$$
$$y = 4x - 2.$$



Graphing utility confirms $\frac{dy}{dx} = 4$ at $(1, 2)$.

31. (a) $f(x) = x^3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$$

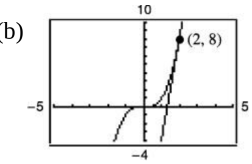
$$= \lim_{\Delta x \to 0} \frac{3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(3x^2 + 3x\Delta x + (\Delta x)^2\right) = 3x^2$$

At $(2, 8)$, the slope of the tangent is $m = 3(2)^2 = 12$.

The equation of the tangent line is

$$y - 8 = 12(x - 2)$$
$$y - 8 = 12x - 24$$
$$y = 12x - 16.$$

(b) 

(c) Graphing utility confirms $\frac{dy}{dx} = 12$ at $(2, 8)$.

32. (a) $f(x) = x^3 + 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^3 + 1\right] - (x^3 + 1)}{\Delta x}$$

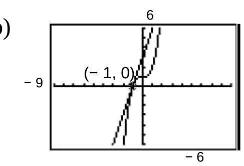
$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + 1 - x^3 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[3x^2 + 3x(\Delta x) + (\Delta x)^2\right] = 3x^2$$

At $(-1, 0)$, the slope of the tangent line is $m = 3(-1)^2 = 3$.

The equation of the tangent line is

$$y - 0 = 3(x + 1)$$
$$y = 3x + 3.$$

(b) 

(c) Graphing utility confirms $\frac{dy}{dx} = 3$ at $(-1, 0)$.

33. (a) $f(x) = \sqrt{x}$

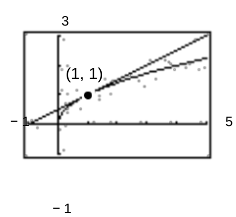
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

At $(1, 1)$, the slope of the tangent line is $m = \frac{1}{2\sqrt{1}} = \frac{1}{2}$.

(b) 

(c) Graphing utility confirms $\frac{dy}{dx} = \frac{1}{2}$ at $(1, 1)$.

The equation of the tangent line is

2 1 2

$-1 = 2(x - 1)$

$$y - 1 = \frac{1}{2}x - \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$

34. (a) $f(x) = \sqrt{x-1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x-1} - \sqrt{x-1}}{\Delta x} \cdot \left(\frac{\sqrt{x+\Delta x-1} + \sqrt{x-1}}{\sqrt{x+\Delta x-1} + \sqrt{x-1}}\right)$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x-1) - (x-1)}{\Delta x\left(\sqrt{x+\Delta x-1} + \sqrt{x-1}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x-1} + \sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

At $(5, 2)$, the slope of the tangent line is $m = \frac{1}{2\sqrt{5-1}} = \frac{1}{4}$.

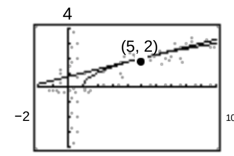
The equation of the tangent line is

$$y - 2 = \frac{1}{4}(x-5)$$

$$y - 2 = \frac{1}{4}x - \frac{5}{4}$$

$$y = \frac{1}{4}x + \frac{3}{4}.$$

(b)



Graphing utility confirms $\frac{dy}{dx} = \frac{1}{4}$ at $(5, 2)$.

35. (a) $f(x) = x + \frac{4}{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x) + \frac{4}{x+\Delta x} - \left(x + \frac{4}{x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x(x+\Delta x)(x+\Delta x) + 4x - x^2(x+\Delta x) - 4(x+\Delta x)}{x(\Delta x)(x+\Delta x)}$$

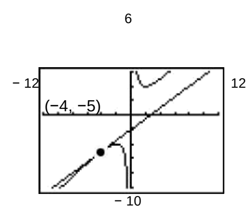
$$= \lim_{\Delta x \to 0} \frac{x^3 + 2x^2(\Delta x) + x(\Delta x)^2 - x^3 - x^2(\Delta x) - 4(\Delta x)}{x(\Delta x)(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^2(\Delta x) + x(\Delta x)^2 - 4(\Delta x)}{x(\Delta x)(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + x(\Delta x) - 4}{x(x+\Delta x)}$$

$$= \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2}$$

(b)



Graphing utility confirms $\frac{dy}{dx} = \frac{3}{4}$ at $(-4, -5)$.

At $(-4, -5)$, the slope of the tangent line is $m = 1 - \frac{4}{(-4)^2} = \frac{3}{4}$.
The equation of the tangent line is

$$+ 5 = \frac{3}{4}(x + 4)$$

$$y + 5 = \frac{3}{4}x + 3$$

$$y = \frac{3}{4}x - 2.$$

36. (a) $f(x) = x - \dfrac{1}{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(x + \Delta x - \dfrac{1}{x + \Delta x}\right) - \left(x - \dfrac{1}{x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)(x + \Delta x)x - x - x^2(x + \Delta x) + (x + \Delta x)}{x(x + \Delta x)\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 2x^2(\Delta x) + x(\Delta x)^2 - x - x^3 - x^2(\Delta x) + x + \Delta x}{x(x + \Delta x)\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x^2(\Delta x) + x(\Delta x)^2 - x^2(\Delta x) + \Delta x}{x(x + \Delta x)\Delta x}$$

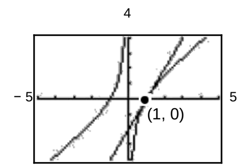
$$= \lim_{\Delta x \to 0} \frac{2x^2 + x(\Delta x) - x^2 + 1}{x(x + \Delta x)}$$

$$= \frac{x^2 + 1}{x^2} = 1 + \frac{1}{x^2}$$

At $(1, 0)$, the slope of the tangent line is $m = f'(1) = 2$. The equation of the tangent line is

$$y - 0 = 2(x - 1)$$
$$y = 2x - 2.$$

(b)



$\dfrac{dy}{dx} = 2$ confirms at $(1, 0)$.

37. Using the limit definition of a derivative, $f'(x) = -\dfrac{1}{2}x$.

Because the slope of the given line is -1, you have

$$-\frac{1}{2}x = -1$$
$$x = 2.$$

At the point $(2, -1)$, the tangent line is parallel to $x + y = 0$. The equation of this line is

$$y - (-1) = -1(x - 2)$$
$$y = -x + 1.$$

38. Using the limit definition of derivative, $f'(x) = 4x$.

Because the slope of the given line is –4, you have

$$4x = -4$$
$$x = -1.$$

At the point $(-1, 2)$ the tangent line is parallel to $4x + y + 3 = 0$. The equation of this line is

$$y - 2 = -4(x + 1)$$
$$y = -4x - 2.$$

39. From Exercise 31 we know that $f'(x) = 3x^2$.

Because the slope of the given line is 3, you have

$$3x^2 = 3$$
$$x = \pm 1.$$

Therefore, at the points $(1, 1)$ and $(-1, -1)$ the tangent lines are parallel to $3x - y + 1 = 0$.

These lines have equations

$$y - 1 = 3(x - 1) \quad \text{and} \quad y + 1 = 3(x + 1)$$
$$y = 3x - 2 \qquad\qquad y = 3x + 2.$$

40. Using the limit definition of derivative, $f'(x) = 3x^2$.

Because the slope of the given line is 3, you have

$$3x^2 = 3$$
$$x^2 = 1 \Rightarrow x = \pm 1.$$

Therefore, at the points $(1, 3)$ and $(-1, 1)$ the tangent lines are parallel to $3x - y - 4 = 0$. These lines have equations

$$y - 3 = 3(x - 1) \quad \text{and} \quad y - 1 = 3(x + 1)$$
$$y = 3x \qquad\qquad y = 3x + 4.$$

Using the limit definition of derivative,

$$f'(x) = \frac{-1}{2x\sqrt{x}}.$$

Because the slope of the given line is $-\frac{1}{2}$, you have

$$\frac{1}{x\sqrt{x}} = -\frac{1}{2}$$

$$= 1.$$

Therefore, at the point $(1, 1)$ the tangent line is parallel to $x + 2y - 6 = 0$. The equation of this line is

$$-1 = -\frac{1}{2}(x - 1)$$

$$y - 1 = -\frac{1}{2}x + \frac{1}{2}$$

$$= -\frac{1}{2}x + \frac{3}{2}.$$

Using the limit definition of derivative,

$$f'(x) = \frac{-1}{2(x-1)^{3/2}}.$$

Because the slope of the given line is $-\frac{1}{2}$, you have

$$\frac{-1}{2(x-1)^{3/2}} = -\frac{1}{2}$$

$$= (x-1)^{3/2}$$

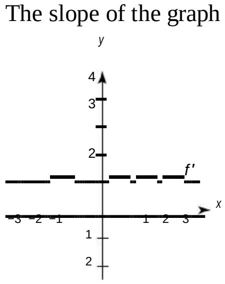
$$1 = x - 1 \Rightarrow x = 2.$$

At the point $(2, 1)$, the tangent line is parallel to $x + 2y + 7 = 0$. The equation of the tangent line is

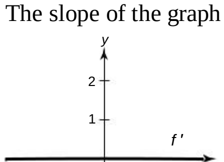
$$-1 = -\frac{1}{2}(x - 2)\, y$$

$$= -\frac{1}{2}x + 2.$$

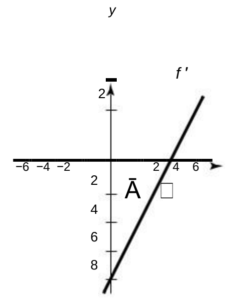
The slope of the graph of f is 1 for all x-values.



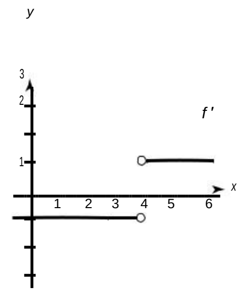
The slope of the graph of f is 0 for all x-values.



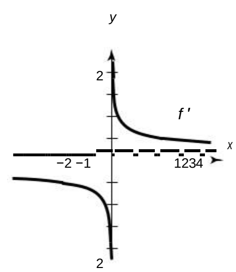
The slope of the graph of f is negative for $x < 4$, positive for $x > 4$, and 0 at $x = 4$.



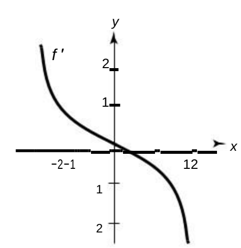
The slope of the graph of f is –1 for $x < 4$, 1 for $x > 4$, and undefined at $x = 4$.



The slope of the graph of f is negative for $x < 0$ and positive for $x > 0$. The slope is undefined at $x = 0$.



48. The slope is positive for $-2 < x < 0$ and negative for $0 < x < 2$. The slope is undefined at $x = \pm 2$, and 0 at $x = 0$.



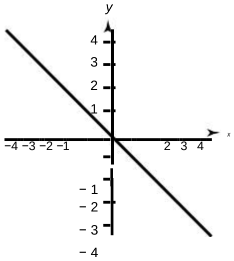
−2 −1 1 2

1

2

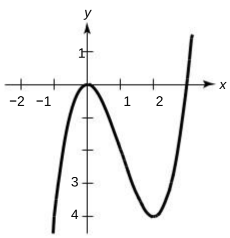
Answers will vary.

Sample answer: $y = -x$



The derivative of $y = -x$ is $y' = -1$. So, the derivative is always negative.

50. Answers will vary. *Sample answer*: $y = x^3 - 3x^2$



Note that $y' = 3x^2 - 6x = 3x(x - 2)$.
So, $y' = 0$ at $x = 0$ and $x = 2$.

51. No. For example, the domain of $f(x) = \sqrt{x}$ is $x \geq 0$, whereas the domain of $f'(x) = \frac{1}{2\sqrt{x}}$ is $x > 0$.

No. For example, $f(x) = x^3$ is symmetric with respect to the origin, but its derivative, $f'(x) = 3x^2$, is symmetric with respect to the y-axis.

$g(4) = 5$ because the tangent line passes through $(4, 5)$.

$$g'(4) = \frac{5 - 0}{4 - 7} = -\frac{5}{3}$$

$h(-1) = 4$ because the tangent line passes through $(-1, 4)$.

$$h'(-1) = \frac{6 - 4}{3 - (-1)} = \frac{2}{4} = \frac{1}{2}$$

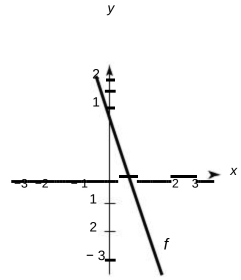
$f(x) = 5 - 3x$ and $c = 1$

$f(x) = x^3$ and $c = -2$

$f(x) = -\sqrt{x}^2$ and $c = 6$

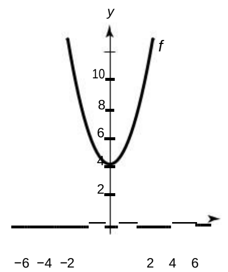
59. $f(0) = 2$ and $f'(x) = -3, -\infty < x < \infty$

$f(x) = -3x + 2$



60. $f(0) = 4, f'(0) = 0; f'(x) < 0$ for $x < 0, f'(x) > 0$ for $x > 0$

Answers will vary: *Sample answer:* $f(x) = x^2 + 4$



Let (x_0, y_0) be a point of tangency on the graph of f. By the limit definition for the derivative, $f'(x) = 4 - 2x$. The slope of the line through $(2, 5)$ and (x_0, y_0) equals the derivative of f at x_0:

$$\frac{5 - y_0}{2 - x_0} = 4 - 2x_0$$

$$5 - y_0 = (2 - x_0)(4 - 2x_0)$$

$$5 - (4x_0 - x_0^2) = 8 - 8x_0 + 2x_0^2$$

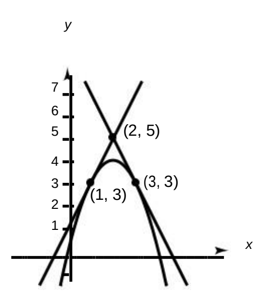
$$0 = x_0^2 - 4x_0 + 3$$

$$= (x_0 - 1)(x_0 - 3) \Rightarrow x_0 = 1, 3$$

Therefore, the points of tangency are $(1, 3)$ and $(3, 3)$, and the corresponding slopes are 2 and –2. The equations of the tangent lines are:

$y - 5 = 2(x - 2)$ $\quad$ $y - 5 = -2(x - 2)$

$y = 2x + 1$ $\quad$ $y = -2x + 9$



$f(x) = 2\ x$ and $c = 9$

−2 1 2 3 6

Let (x_0, y_0) be a point of tangency on the graph of *f*.

By the limit definition for the derivative, $f'(x) = 2x$.

The slope of the line through $(1, -3)$ and (x_0, y_0) equals the derivative of *f* at x_0:

$$\frac{-3 - y_0}{1 - x_0} = 2x_0$$

$$-3 - y_0 = (1 - x_0)2x_0$$

$$-3 - x_0^2 = 2x_0 - 2x_0^2$$

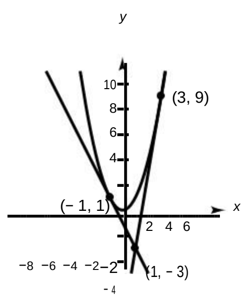
$$x_0^2 - 2x_0 - 3 = 0$$

$$(x_0 - 3)(x_0 + 1) = 0 \Rightarrow x_0 = 3, -1$$

Therefore, the points of tangency are $(3, 9)$ and $(-1, 1)$, and the corresponding slopes are 6 and –2. The equations of the tangent lines are:

$$y + 3 = 6(x - 1) \qquad y + 3 = -2(x - 1)$$

$$y = 6x - 9 \qquad y = -2x - 1$$



63. (a) $f(x) = x^2$

$$f'(x) = \lim_{x\to 0} \frac{f(x + x) - f(x)}{x}$$

$$= \lim_{x\to 0} \frac{(x + x)^2 - x^2}{x}$$

$$= \lim_{x\to 0} \frac{x^2 + 2x(x) + (x)^2 - x^2}{x}$$

$$= \lim_{x\to 0} \frac{x(2x + x)}{x}$$

$$= \lim_{x\to 0} (2x + x) = 2x$$

At $x = -1, f'(-1) = -2$ and the tangent line is

$y - 1 = -2(x + 1)$ or $y = -2x - 1.$

At $x = 0, f'(0) = 0$ and the tangent line is $y = 0$.

At $x = 1, f'(1) = 2$ and the tangent line is

$= 2x - 1.$

2

(b) $$g'(x) = \lim_{x\to 0} \frac{g(x + x) - g(x)}{x}$$

$$= \lim_{x\to 0} \frac{(x + x)^3 - x^3}{x}$$

$$= \lim_{x\to 0} \frac{x^3 + 3x^2(x) + 3x(x)^2 + (x)^3 - x^3}{x}$$

$$= \lim_{x\to 0} \frac{x\left(3x^2 + 3x(x) + (x)^2\right)}{x}$$

$$= \lim_{x\to 0} \left(3x^2 + 3x(x) + (x)^2\right) = 3x^2$$

At $x = -1, g'(-1) = 3$ and the tangent line is

$y + 1 = 3(x + 1)$ or $y = 3x + 2.$

At $x = 0, g'(0) = 0$ and the tangent line is $y = 0.$

At $x = 1, g'(1) = 3$ and the tangent line is

$y - 1 = 3(x - 1)$ or $y = 3x - 2.$

2

− 3 3

2

For this function, the slopes of the tangent lines are sometimes the same.

(a) $g'(0) = -3$

$g'(3) = 0$

(c) Because $g'(1) = -\frac{8}{3}$, *g* is decreasing (falling) at $= 1$.

Because $g'(-4) = \frac{7}{3}$, *g* is increasing (rising) at $= -4$.

Because $g'(4)$ and $g'(6)$ are both positive, $g(6)$ is greater than $g(4)$, and $g(6) - g(4) > 0$.

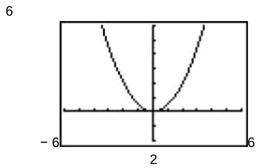
No, it is not possible. All you can say is that *g* is decreasing (falling) at $x = 2$.

− 3 3

3

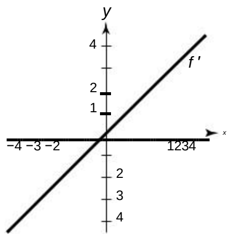
For this function, the slopes of the tangent lines are always distinct for different values of x.

$f(x) = \frac{1}{2}x^2$



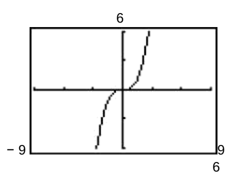
$f'(0) = 0, f'(1/2) = 1/2, f'(1) = 1, f'(2) = 2$

By symmetry: $f'(-1/2) = -1/2, f'(-1) = -1, f'(-2) = -2$



(d) $f'(x) = \lim_{x\to 0} \frac{f(x+ x) - f(x)}{x} = \lim_{x\to 0} \frac{\frac{1}{2}(x+ x)^2 - \frac{1}{2}x^2}{x} = \lim_{x\to 0} \frac{\frac{1}{2}\left(x^2 + 2x(x) + (x)^2\right) - \frac{1}{2}x^2}{x} = \lim_{x\to 0}\left(x + \frac{x}{2}\right) = x$

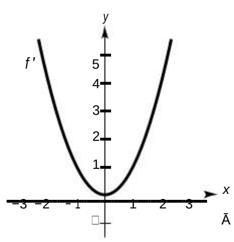
$f(x) = \frac{1}{3}x^3$



$f'(0) = 0, f'(1/2) = 1/4, f'(1) = 1, f'(2) = 4, f'(3) = 9$

(b) By symmetry: $f'(-1/2) = 1/4, f'(-1) = 1, f'(-2) = 4, f'(-3) = 9$

(c)



(d) $f'(x) = \lim_{x\to 0} \frac{f(x+ x) - f(x)}{x}$

$= \lim_{x\to 0} \frac{\frac{1}{3}(x + x)^3 - \frac{1}{3}x^3}{x}$

$= \lim_{x\to 0} \frac{\frac{1}{3}\left(x^3 + 3x^2(x) + 3x(x)^2 + (x)^3\right) - \frac{1}{3}x^3}{x}$

$= \lim_{x\to 0}\left[x^2 + x(x) + \frac{1}{3}(x)^2\right] = x^2$

67. $f(2) = 2(4-2) = 4,\ f(2.1) = 2.1(4-2.1) = 3.99$

$f'(2) \approx \dfrac{3.99-4}{2.1-2} = -0.1$ $\left[\text{Exact: } f'(2) = 0\right]$

$f(2) = \frac{1}{4}(2^3) = 2,\ f(2.1) = 2.31525$

$f'(2) \approx \dfrac{2.31525-2}{2.1-2} = 3.1525$ $\left[\text{Exact: } f'(2) = 3\right]$

69. $f(x) = x^3 + 2x^2 + 1,\ c = -2$

$$f'(-2) = \lim_{x\to-2} \frac{f(x)-f(-2)}{x+2} = \lim_{x\to-2} \frac{(x^3+2x^2+1)-1}{x+2}$$

$$= \lim_{x\to-2} \frac{x^2(x+2)}{x+2} = \lim_{x\to-2} x^2 = 4$$

$g(x) = x^2 - x,\ c = 1$

$$g'(1) = \lim_{x\to1} \frac{g(x)-g(1)}{x-1} = \lim_{x\to1} \frac{x^2-x-0}{x-1} = \lim_{x\to1} \frac{x(x-1)}{x-1} = \lim_{x\to1} x = 1$$

71. $g(x) = \sqrt{|x|},\ c = 0$

$g'(0) = \lim_{x\to0} \dfrac{g(x)-g(0)}{x-0} = \lim_{x\to0} \dfrac{\sqrt{|x|}}{x}$. Does not exist.

As $x \to 0^-$, $\dfrac{\sqrt{|x|}}{x} = \dfrac{-1}{\sqrt{|x|}} \to -\infty$.

As $x \to 0^+$, $\dfrac{\sqrt{|x|}}{x} = \dfrac{1}{\sqrt{x}} \to \infty$.

Therefore $g(x)$ is not differentiable at $x = 0$.

$f(x) = (x-6)^{2/3},\ c = 6$

$$f'(6) = \lim_{x\to6} \frac{f(x)-f(6)}{x-6} = \lim_{x\to6} \frac{(x-6)^{2/3}-0}{x-6} = \lim_{x\to6} \frac{1}{(x-6)^{1/3}}.$$

Does not exist.

Therefore $f(x)$ is not differentiable at $x = 6$.

$g(x) = (x+3)^{1/3},\ c = -3$

$$g'(-3) = \lim_{x\to-3} \frac{g(x)-g(-3)}{x-(-3)} = \lim_{x\to-3} \frac{(x+3)^{1/3}-0}{x+3} = \lim_{x\to-3} \frac{1}{(x+3)^{2/3}}.$$

Does not exist.

Therefore $g(x)$ is not differentiable at $x = -3$.

$h(x) = |x+7|,\ c = -7$

$$h'(-7) = \lim_{x\to-7} \frac{h(x)-h(-7)}{x-(-7)} = \lim_{x\to-7} \frac{|x+7|-0}{x+7} = \lim_{x\to-7} \frac{|x+7|}{x+7}$$

Does not exist.

Therefore $h(x)$ is not differentiable at $x = -7$.

$f(x) = |x-6|,\ c = 6$

$$f'(6) = \lim_{x\to6} \frac{f(x)-f(6)}{x-6} = \lim_{x\to6} \frac{|x-6|-0}{x-6} = \lim_{x\to6} \frac{|x-6|}{x-6}$$

Does not exist.

Therefore $f(x)$ is not differentiable at $x = 6$.

$f(x)$ is differentiable everywhere except at $x = -4$. (Sharp turn in the graph)

$f(x)$ is differentiable everywhere except at $x = \pm 2$. (Discontinuities)

$f(x)$ is differentiable on the interval $(-1, \infty)$. (At $x = -1$ the tangent line is vertical.)

$f(x)$ is differentiable everywhere except at $x = 0$. (Discontinuity)

$f(x) = \frac{3}{x}, c = 4$

$$f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4}$$

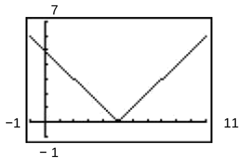
$$= \lim_{x \to 4} \frac{\frac{3}{x} - \frac{3}{4}}{x - 4}$$

$$= \lim_{x \to 4} \frac{12 - 3x}{4x(x - 4)}$$

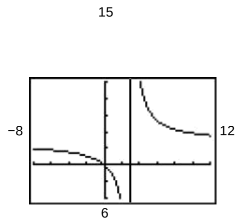
$$= \lim_{x \to 4} \frac{-3(x - 4)}{4x(x - 4)}$$

$$= \lim_{x \to 4} -\frac{3}{4x} = -\frac{3}{16}$$

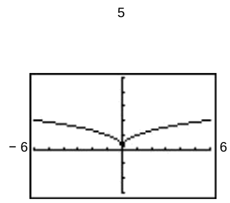
81. $f(x) = |x - 5|$ is differentiable everywhere except at $x = 5$. There is a sharp corner at $x = 5$.



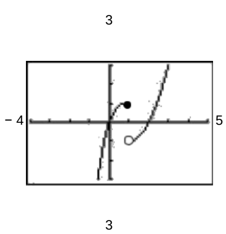
82. $f(x) = \frac{4x}{x - 3}$ is differentiable everywhere except at $x = 3$. f is not defined at $x = 3$. (Vertical asymptote)



$f(x) = x^{2/5}$ is differentiable for all $x \neq 0$. There is a sharp corner at $x = 0$.



84. f is differentiable for all $x \neq 1$. f is not continuous at $x = 1$.



$f(x) = |x - 1|$

The derivative from the left is

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{|x - 1| - 0}{x - 1} = -1.$$

The derivative from the right is

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{|x - 1| - 0}{x - 1} = 1.$$

The one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

86. $f(x) = \sqrt{1 - x^2}$

The derivative from the left does not exist because

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{\sqrt{1 - x^2} - 0}{x - 1}$$

87. $f(x) = \begin{cases} (x - 1)^3, & x \le 1 \\ (x - 1)^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{(x - 1)^3 - 0}{x - 1}$$

$$= \lim_{x \to 1^-} (x - 1)^2 = 0.$$

The derivative from the right is

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{(x - 1)^2 - 0}{x - 1}$$

$$= \lim_{x \to 1^+} (x - 1) = 0.$$

The one-sided limits are equal. Therefore, f is differentiable at $x = 1$. $\left(f'(1) = 0\right)$

$f(x) = (1 - x)^{2/3}$

The derivative from the left does not exist.

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{(1 - x)^{2/3} - 0}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{-1}{(1 - x)^{1/3}} = -\infty$$

Similarly, the derivative from the right does not exist because the limit is ∞.

Therefore, f is not differentiable at $x = 1$.

89. Note that f is continuous at $x = 2$.

$f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$

The derivative from the left is

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{(x^2 + 1) - 5}{x - 2}$$

$$= \lim_{x \to 2^-} (x + 2) = 4.$$

The derivative from the right is

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{(4x - 3) - 5}{x - 2} = \lim_{x \to 2^+} 4 = 4.$$

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. $\left(f'(2) = 4\right)$

$$= \lim_{x \to 1^-} \frac{1 - x^2}{x - 1} \cdot \frac{1 - x^2}{1 - x^2}$$

$$= \lim_{x \to 1^-} \frac{-(1 + x)}{\sqrt{1 - x^2}} = -\infty.$$

(Vertical tangent)

The limit from the right does not exist since f is undefined for $x > 1$. Therefore, f is not differentiable at $= 1$.

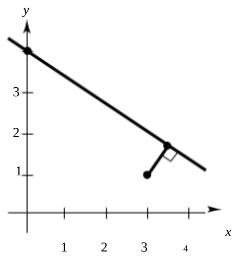
90. $f(x) = \begin{cases} \frac{1}{2}x + 2, & x < 2 \\ \sqrt{2x}, & x \geq 2 \end{cases}$

is not differentiable at $x = 2$ because it is not continuous at $x = 2$.

$$\lim_{x \to 2^-} f(x) = \frac{1}{2}(2) + 2 = 3$$

$$\lim_{x \to 2^+} f(x) = \sqrt{2(2)} = 2$$

91. (a) The distance from $(3, 1)$ to the line $mx - y + 4 = 0$ is $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2+B^2}} = \dfrac{|m(3) - 1(1) + 4|}{\sqrt{m^2+1}} = \dfrac{|3m+3|}{\sqrt{m^2+1}}$.



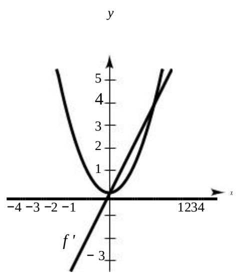
(b) 5

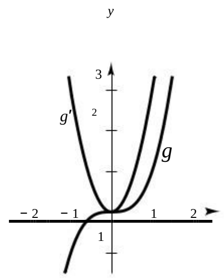
−4 4

1

The function d is not differentiable at $m = -1$. This corresponds to the line $y = -x + 4$, which passes through the point $(3, 1)$.

(a) $f(x) = x^2$ and $f'(x) = 2x$ (b) $g(x) = x^3$ and $g'(x) = 3x^2$





The derivative is a polynomial of degree 1 less than the original function. If $h(x) = x^n$, then $h'(x) = nx^{n-1}$.

If $f(x) = x^4$, then

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^4 - x^4}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 - x^4}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x\left(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3\right)}{\Delta x} = \lim_{\Delta x \to 0}\left(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3\right) = 4x^3.
\end{aligned}$$

So, if $f(x) = x^4$, then $f'(x) = 4x^3$ which is consistent with the conjecture. However, this is not a proof because you must verify the conjecture for all integer values of n, $n \geq 2$.

93. False. The slope is $\displaystyle\lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x}$.

False. $y = |x - 2|$ is continuous at $x = 2$, but is not differentiable at $x = 2$. (Sharp turn in the graph)

False. If the derivative from the left of a point does not equal the derivative from the right of a point, then the derivative does not exist at that point. For example, if $f(x) = |x|$, then the derivative from the left at $x = 0$ is –1 and the derivative from the right at $x = 0$ is 1. At $x = 0$, the derivative does not exist.

True. See Theorem 2.1.

97. $f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Using the Squeeze Theorem, you have $-|x| \leq x \sin(1/x) \leq |x|$, $x \neq 0$. So, $\lim_{x \to 0} x \sin(1/x) = 0 = f(0)$ and is continuous at $x = 0$. Using the alternative form of the derivative, you have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \to 0}\left(\sin\frac{1}{x}\right)$$

Because this limit does not exist ($\sin(1/x)$ oscillates between –1 and 1), the function is not differentiable at $x = 0$.

$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Using the Squeeze Theorem again, you have $-x^2 \leq x^2 \sin(1/x) \leq x^2$, $x \neq 0$. So, $\lim_{x \to 0} x^2 \sin(1/x) = 0 = g(0)$ and g is continuous at $x = 0$. Using the alternative form of the derivative again, you have

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} x \sin\frac{1}{x} = 0.$$

Therefore, g is differentiable at $x = 0$, $g'(0) = 0$.

98.

3

−3 3

1

As you zoom in, the graph of $y_1 = x^2 + 1$ appears to be locally the graph of a horizontal line, whereas the graph $y_2 = |x| + 1$ always has a sharp corner at $(0, 1)$. y_2 is not differentiable at $(0, 1)$.

Section 2.2 Basic Differentiation Rules and Rates of Change

The derivative of a constant function is 0. $\frac{d}{dx}[c] = 0$

To find the derivative of $f(x) = cx^n$, multiply n by c, and reduce the power of x by 1.

$f'(x) = ncx^{n-1}$

The derivative of the sine function, $f(x) = \sin x$, is the cosine function, $f'(x) = \cos x$.

The derivative of the cosine function, $g(x) = \cos x$, is the negative of the sine function, $g'(x) = -\sin x$.

The average velocity of an object is the change in distance divided by the change in time. The velocity is the instantaneous change in velocity. It is the derivative of the position function.

(a) $y = x^{1/2}$

$y' = \frac{1}{2}x^{-1/2}$

$y'(1) = \frac{1}{2}$

$y = x^3$

$y' = 3x^2$

$y'(1) = 3$

(a) $y = x^{-1/2}$

$y' = -\frac{1}{2}x^{-3/2}$

$y = x^{-1}$

$y' = -x^{-2}$

$y'(1) = -1$

$y = 12$
$y' = 0$

$f(x) = -9$
$f'(x) = 0$

$y = x^7$
$y' = 7x^6$

$y = x^{12}$
$y' = 12x^{11}$

$y = \frac{1}{x^5} = x^{-5}$

$y' = -5x^{-6} = -\frac{5}{x^6}$

$y = \frac{3}{x^7} = 3x^{-7}$

$y' = 3(-7x^{-8}) = -\frac{21}{x^8}$

13. $f(x) = \sqrt[9]{x} = x^{1/9}$

$f'(x) = \frac{1}{9}x^{-8/9} = \frac{1}{9x^{8/9}}$

14. $y = \sqrt[4]{x} = x^{1/4}$

$y' = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}}$

15. $f(x) = x + 11$
$f'(x) = 1$

$g(x) = 6x + 3$
$g'(x) = 6$

$f(t) = -3t^2 + 2t - 6$
$f'(t) = -6t + 2$

$y = t^2 - 3t + 1$
$y' = 2t - 3$

$g(x) = x^2 + 4x^3$
$g'(x) = 2x + 12x^2$

$y = 4x - 3x^3$
$y' = 4 - 9x^2$

$s(t) = t^3 + 5t^2 - 3t + 8$
$s'(t) = 3t^2 + 10t - 3$

$y = 2x^3 + 6x^2 - 1$
$y' = 6x^2 + 12x$

$y = \frac{\pi}{2}\sin\theta$ $y' = \frac{\pi}{2}\cos\theta$

$g(t) = \pi\cos t$ $g'(t) = -\pi\sin t$

$y = x^2 - \frac{1}{2}\cos x$ $y' = 2x + \frac{1}{2}\sin x$

$y = 7x^4 + 2\sin x$
$y' = 7(4x^3) + 2\cos x = 28x^3 + 2\cos x$

Function	Rewrite	Differentiate	Simplify
27. $y = \frac{2}{7x^4}$	$y = \frac{2}{7}x^{-4}$	$y' = -\frac{8}{7}x^{-5}$	$y' = -\frac{8}{7x^5}$
28. $y = \frac{8}{5x^{-5}}$	$y = \frac{8}{5}x^5$	$y' = \frac{8}{5}(5x^4)$	$y' = 8x^4$
29. $y = \frac{6}{(5x)^3}$	$y = \frac{6}{125}x^{-3}$	$y' = -\frac{18}{125}x^{-4}$	$y' = -\frac{18}{125x^4}$
30. $\frac{3}{(2x)^{-2}}$	$y =$	$\frac{3}{}$	

125

$y = 12x^2$ $y' = 12(2x)$ $y' = 24x$

31. $f(x) = \frac{8}{x^2} = 8x^{-2}, (2, 2)$

$f'(x) = -16x^{-3} = -\frac{16}{x^3}$

$f'(2) = -2$

32. $f(t) = 2 - \frac{4}{t} = 2 - 4t^{-1}, (4, 1)$

$f'(t) = 4t^{-2} = \frac{4}{t^2}$

$f'(4) = \frac{1}{4}$

33. $f(x) = -\frac{1}{2} + \frac{7}{5}x^3, \left(0, -\frac{1}{2}\right)$

$f'(x) = \frac{21}{5}x^2$

$f'(0) = 0$

$y = 2x^4 - 3, (1, -1)$

$y' = 8x^3$

$y'(1) = 8$

$y = (4x + 1)^2, (0, 1)$

$16x^2 + 8x + 1$

$y' = 32x + 8$

$y'(0) = 32(0) + 8 = 8$

$f(x) = 2(x - 4)^2, (2, 8)$

$= 2x^2 - 16x + 32$

$f'(x) = 4x - 16$

$f'(2) = 8 - 16 = -8$

37. $f(\theta) = 4\sin\theta - \theta, (0, 0)$

$f'(\theta) = 4\cos\theta - 1$

$f'(0) = 4(1) - 1 = 3$

$g(t) = -2\cos t + 5, (\pi, 7)$

$g'(t) = 2\sin t$

$g'(\pi) = 0$

$f(x) = x^2 + 5 - 3x^{-2}$

$g(t) = t^2 - \frac{4}{t^3} = t^2 - 4t^{-3}$

$g'(t) = 2t + 12t^{-4} = 2t + \frac{12}{t^4}$

42. $f(x) = 8x + \frac{3}{x^2} = 8x + 3x^{-2}$

$f'(x) = 8 - 6x^{-3} = 8 - \frac{6}{x^3}$

$f(x) = \frac{x^3 - 3x^2 + 4}{x^2} = x - 3 + 4x^{-2}$

$f'(x) = 1 - \frac{8}{x^3} = \frac{x^3 - 8}{x^3}$

$h(x) = \frac{4x^3 + 2x + 5}{x} = 4x^2 + 2 + 5x^{-1}$

$h'(x) = 8x - 5x^{-2} = 8x - \frac{5}{x^2}$

45. $g(t) = \frac{3t^2 + 4t - 8}{t^{3/2}} = 3t^{1/2} + 4t^{-1/2} - 8t^{-3/2}$

$g'(t) = \frac{3}{2}t^{-1/2} - 2t^{-3/2} + 12t^{-5/2}$

$= \frac{3t^2 - 4t + 24}{2t^{5/2}}$

$h(s) = \frac{s^5 + 2s + 6}{s^{1/3}} = s^{14/3} + 2s^{2/3} + 6s^{-1/3}$

$h'(s) = \frac{14}{3}s^{11/3} + \frac{4}{3}s^{-1/3} - 2s^{-4/3}$

$= \frac{14s^5 + 4s - 6}{3s^{4/3}}$

$y = x(x^2 + 1) = x^3 + x$

$y' = 3x^2 + 1$

$y = x^2(2x^2 - 3x) = 2x^4 - 3x^3$

$y' = 8x^3 - 9x^2 = x^2(8x - 9)$

49. $f(x) = \sqrt{x} - 6\sqrt[3]{x} = x^{1/2} - 6x^{1/3}$

$f'(x) = \frac{1}{2}x^{-1/2} - 2x^{-2/3} = \frac{1}{2\sqrt{x}} - \frac{2}{x^{2/3}}$

$f(t) = t^{2/3} - t^{1/3} + 4$

$$f'(x) = 2x + 6x^{-3} = 2x + \frac{6}{x^3}$$

$$f(x) = x^3 - 2x + 3x^{-3}$$

$$f'(x) = 3x^2 - 2 - 9x^{-4} = 3x^2 - 2 - \frac{9}{x^4}$$

$$f'(t) = \frac{2}{3}t^{-1/3} - \frac{1}{3}t^{-2/3} = \frac{2}{3t^{1/3}} - \frac{1}{3t^{2/3}}$$

51. $f(x) = 6\sqrt{x} + 5\cos x = 6x^{1/2} + 5\cos x$

$$f'(x) = 3x^{-1/2} - 5\sin x = \frac{3}{\sqrt{x}} - 5\sin x$$

52. $f(x) = \frac{2}{\sqrt[3]{x}} + 3\cos x = 2x^{-1/3} + 3\cos x$

$f'(x) = -\frac{2}{3}x^{-4/3} - 3\sin x = -\frac{2}{3x^{4/3}} - 3\sin x$

53. $y = \frac{1}{(3x)^{-2}} - 5\cos x = (3x)^2 - 5\cos x = 9x^2 - 5\cos x$

$y' = 18x + 5\sin x$

54. $y = \frac{3}{(2x)^3} + 2\sin x = \frac{3}{8}x^{-3} + 2\sin x$

$y' = -\frac{9}{8}x^{-4} + 2\cos x = -\frac{9}{8x^4} + 2\cos x$

(a) $f(x) = -2x^4 + 5x^2 - 3$

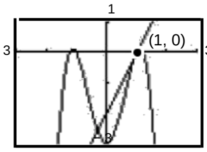
$f'(x) = -8x^3 + 10x$

At $(1, 0)$: $f'(1) = -8(1)^3 + 10(1) = 2$

Tangent line: $y - 0 = 2(x - 1)$

$= 2x - 2$

(b) and (c)



(a) $y = x^3 - 3x$ $y' = 3x^2 - 3$

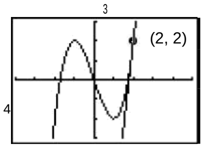
At $(2, 2)$: $y' = 3(2)^2 - 3 = 9$

Tangent line: $y - 2 = 9(x - 2)$

$y = 9x - 16$

$9x - y - 16 = 0$

(b) and (c)



57. (a) $f(x) = \frac{2}{\sqrt[4]{x^3}} = 2x^{-3/4}$

$f'(x) = -\frac{3}{2}x^{-7/4} = -\frac{3}{2x^{7/4}}$

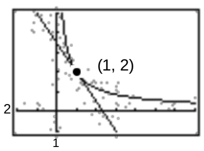
At $(1, 2)$: $f'(1) = -\frac{3}{2}$

Tangent line: $y - 2 = -\frac{3}{2}(x - 1)$

$y = -\frac{3}{2}x + \frac{7}{2}$

$3x + 2y - 7 = 0$

(b) and (c)



(a) $y = (x - 2)(x^2 + 3x) = x^3 + x^2 - 6x$ $y' = 3x^2 + 2x - 6$

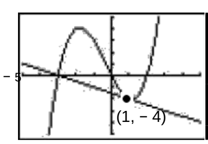
At $(1, -4)$: $y' = 3(1)^2 + 2(1) - 6 = -1$

Tangent line: $y - (-4) = -1(x - 1)$

$y = -x - 3$

$x + y + 3 = 0$

(b) and (c)



$y = x^4 - 2x^2 + 3$

$y' = 4x^3 - 4x$

$4x(x^2 - 1)$

$4x(x - 1)(x + 1)$

$y' = 0 \Rightarrow x = 0, \pm 1$

Horizontal tangents: $(0, 3), (1, 2), (-1, 2)$

$y = x^3 + x$

$y' = 3x^2 + 1 > 0$ for all x.

Therefore, there are no horizontal tangents.

$y = \frac{1}{x^2} = x^{-2}$

$y' = -2x^{-3} = -\frac{2}{x^3}$ cannot equal zero.

$y = x^2 + 9$

$y' = 2x = 0 \Rightarrow x = 0$

At $x = 0$, $y = 1$.

Horizontal tangent: $(0, 9)$

$y = x + \sin x$, $0 \le x < 2\pi$ $y' = 1 + \cos x = 0$

$\cos x = -1 \Rightarrow x = \pi$ At $x = \pi : y = \pi$ Horizontal tangent: (π, π)

$y = \sqrt{3}x + 2\cos x$, $0 \le x < 2\pi$ $y' = \sqrt{3} - 2\sin x = 0$

$\sin x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

At $x = \frac{\pi}{3}$: $y = \frac{\sqrt{3}\pi + 3}{3}$

At $x = \frac{2\pi}{3}$: $y = \frac{2\sqrt{3}\pi - 3}{3}$

Horizontal tangents: $\left(\frac{\pi}{3}, \frac{\sqrt{3}\pi + 3}{3}\right), \left(\frac{2\pi}{3}, \frac{2\sqrt{3}\pi - 3}{3}\right)$

65. $f(x) = k - x^2$, $y = -6x + 1$

$f'(x) = -2x$ and slope of tangent line is $m = -6$.

$f'(x) = -6$

$-2x = -6$

$x = 3$

$y = -6(3) + 1 = -17$

$-17 = k - 3^2$

$-8 = k$

66. $f(x) = kx^2$, $y = -2x + 3$

$f'(x) = 2kx$ and slope of tangent line is $m = -2$.

$f'(x) = -2$

$2kx = -2$

$x = -\frac{1}{k}$

$y = -2\left(-\frac{1}{k}\right) + 3 = \frac{2}{k} + 3$

$\frac{2}{k} + 3 = k\left(-\frac{1}{k}\right)^2$

$\frac{2}{k} + 3 = \frac{1}{k}$

$\frac{1}{k} = -3$

$k = -\frac{1}{3}$

67. $f(x) = \frac{k}{x}$, $y = -\frac{3}{4}x + 3$

$f'(x) = -\frac{k}{x^2}$ and slope of tangent line is $m = -\frac{3}{4}$.

$f'(x) = -\frac{3}{4}$

$-\frac{k}{x^2} = -\frac{3}{4}$

$x^2 = \frac{4k}{3}$

$x = \sqrt{\frac{4k}{3}}$

$y = -\frac{3}{4}x + 3 = -\frac{3}{4}\sqrt{\frac{4k}{3}} + 3$

$-\frac{3}{4}\sqrt{\frac{4k}{3}} + 3 = k\sqrt{\frac{3}{4k}}$

$k = 3$

$f(x) = k\sqrt{x},\ y = x + 4$

$f'(x) = \dfrac{k}{2\sqrt{x}}$ and slope of tangent line is $m = 1$.

$$f'(x) = 1$$

$$\frac{k}{2\sqrt{x}} = 1$$

$$\frac{k}{2} = \sqrt{x}$$

$$\frac{k^2}{4} = x$$

$$y = x + 4 = \frac{k^2}{4} + 4$$

$$\frac{k^2}{4} + 4 = k \cdot \frac{k}{2}$$

$$\frac{k^2}{4} - \frac{k^2}{2} = -4$$

$$-\frac{k^2}{4} = -4$$

$$k^2 = 16$$

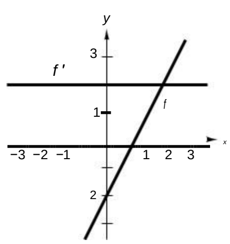
$$k = 4$$

$g(x) = f(x) + 6 \Rightarrow g'(x) = f'(x)$

$g(x) = 2f(x) \Rightarrow g'(x) = 2f'(x)$

$g(x) = -5f(x) \Rightarrow g'(x) = -5f'(x)$

$g(x) = 3f(x) - 1 \Rightarrow g'(x) = 3f'(x)$

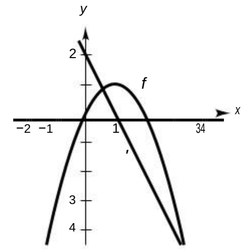


If f is linear then its derivative is a constant function.

$f(x) = ax + b$

$f'(x) = a$

74.

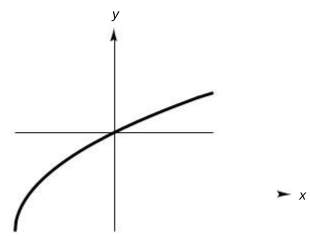


If f is quadratic, then its derivative is a linear function.

$f(x) = ax^2 + bx + c$

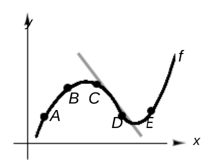
$f'(x) = 2ax + b$

The graph of a function f such that $f' > 0$ for all x and the rate of change of the function is decreasing (i.e., as x increases, f' decreases) would, in general, look like the graph below.



(a) The slope appears to be steepest between A and B.

The average rate of change between A and B is **greater** than the instantaneous rate of change at B.



Let (x_1, y_1) and (x_2, y_2) be the points of tangency on $y = x^2$ and $y = -x^2 + 6x - 5$, respectively.
The derivatives of these functions are:

$y' = 2x \Rightarrow m = 2x_1$ and $y' = -2x + 6 \Rightarrow m = -2x_2 + 6$

$= 2x_1 = -2x_2 + 6$

$x_1 = -x_2 + 3$

Because $y_1 = x_1^2$ and $y_2 = -x_2^2 + 6x_2 - 5$:

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{\left(-x_2^2 + 6x_2 - 5\right) - \left(x_1^2\right)}{x_2 - x_1} = -2x_2 + 6$$

$$\frac{\left(-x_2^2 + 6x_2 - 5\right) - \left(-x_2 + 3\right)^2}{x_2 - \left(-x_2 + 3\right)} = -2x_2 + 6$$

$$\left(-x_2^2 + 6x_2 - 5\right) - \left(x_2^2 - 6x_2 + 9\right) = \left(-2x_2 + 6\right)\left(2x_2 - 3\right)$$

$$-2x_2^2 + 12x_2 - 14 = -4x_2^2 + 18x_2 - 18$$

$$2x_2^2 - 6x_2 + 4 = 0$$

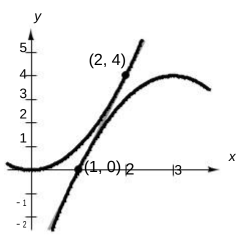
$$2\left(x_2 - 2\right)\left(x_2 - 1\right) = 0$$

$$x_2 = 1 \text{ or } 2$$

$x_2 = 1 \Rightarrow y_2 = 0, x_1 = 2$ and $y_1 = 4$

So, the tangent line through $(1, 0)$ and $(2, 4)$ is

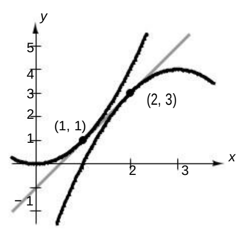
$$y - 0 = \left(\frac{4 - 0}{2 - 1}\right)(x - 1) \Rightarrow y = 4x - 4.$$



$x_2 = 2 \Rightarrow y_2 = 3, x_1 = 1$ and $y_1 = 1$

So, the tangent line through $(2, 3)$ and $(1, 1)$ is

$$y - 1 = \left(\frac{3 - 1}{2 - 1}\right)(x - 1) \Rightarrow y = 2x - 1.$$



78. m_1 is the slope of the line tangent to $y = x$. m_2 is the slope of the line tangent to $y = 1/x$. Because

$$y = x \Rightarrow y' = 1 \Rightarrow m_1 = 1 \text{ and } y = \frac{1}{x} \Rightarrow y' = -\frac{1}{x^2} \Rightarrow m_2 = -\frac{1}{x^2}.$$

The points of intersection of $y = x$ and $y = 1/x$ are

$$x = \frac{1}{x} \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

At $x = \pm 1$, $m_2 = -1$. Because $m_2 = -1/m_1$, these tangent lines are perpendicular at the points of intersection.

79. $f(x) = 3x + \sin x + 2$

$f'(x) = 3 + \cos x$

Because $|\cos x| \le 1$, $f'(x) \ne 0$ for all x and f does not have a horizontal tangent line.

80. $f(x) = x^5 + 3x^3 + 5x$

$f'(x) = 5x^4 + 9x^2 + 5$

Because $5x^4 + 9x^2 \ge 0$, $f'(x) \ge 5$. So, f does not have a tangent line with a slope of 3.

81. $f(x) = \sqrt{x},\ (-4, 0)$

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$\frac{1}{2\sqrt{x}} = \frac{0 - y}{-4 - x}$

$4 + x = 2\sqrt{x}\,y$

$4 + x = 2\sqrt{x}\sqrt{x}$

$4 + x = 2x$

$x = 4,\ y = 2$

The point $(4, 2)$ is on the graph of f.

Tangent line: $y - 2 = \frac{0 - 2}{-4 - 4}(x - 4)$

$4y - 8 = x - 4$

$0 = x - 4y + 4$

$f(x) = \frac{2}{x},\ (5, 0)$

$f'(x) = -\frac{2}{x^2}$

$-\frac{2}{x^2} = \frac{0 - y}{5 - x}$

$-10 + 2x = -x^2 y$

$-10 + 2x = -x^2\left(\frac{2}{x}\right)$

$-10 + 2x = -2x$

$4x = 10$

$x = \frac{5}{2},\ y = \frac{4}{5}$

The point $\left(\frac{5}{2}, \frac{4}{5}\right)$ is on the graph of f. The slope of the tangent line is $f'\left(\frac{5}{2}\right) = -\frac{8}{25}$.

Tangent line: $y - \frac{4}{5} = -\frac{8}{25}\left(x - \frac{5}{2}\right)$

$25y - 20 = -8x + 20$

$8x + 25y - 40 = 0$

(a) One possible secant is between $(3.9, 7.7019)$ and $(4, 8)$:

$$y - 8 = \frac{8 - 7.7019}{4 - 3.9}(x - 4)$$

$y - 8 = 2.981(x - 4)$

$y = S(x) = 2.981x - 3.924$

20

(4, 8)

−2

12

−2

(b) $f'(x) = \frac{3}{2}x^{1/2} \Rightarrow f'(4) = \frac{3}{2}(2) = 3$

$T(x) = 3(x - 4) + 8 = 3x - 4$

The slope (and equation) of the secant line approaches that of the tangent line at $(4, 8)$ as you choose points closer and closer to $(4, 8)$.

(c) As you move further away from $(4, 8)$, the accuracy of the approximation T gets worse.

20

f

T

−2

12

−2

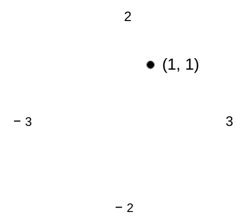
(d)

Δx	–3	–2	–1	–0.5	–0.1	0	0.1	0.5	1	2	3
$f(4 + \Delta x)$	1	2.828	5.196	6.548	7.702	8	8.302	9.546	11.180	14.697	18.520
$T(4 + \Delta x)$	–1	2	5	6.5	7.7	8	8.3	9.5	11	14	17

(a) Nearby point: $(1.0073138, 1.0221024)$

Secant line: $y - 1 = \frac{1.0221024 - 1}{1.0073138 - 1}(x - 1)$

$y = 3.022(x - 1) + 1$

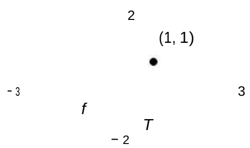


(Answers will vary.)

(b) $f'(x) = 3x^2$

$T(x) = 3(x - 1) + 1 = 3x - 2$

(c) The accuracy worsens as you move away from $(1, 1)$.



(d)

Δx	–3	–2	–1	–0.5	–0.1	0	0.1	0.5	1	2	3
$f(x)$	–8	–1	0	0.125	0.729	1	1.331	3.375	8	27	64
$T(x)$	–8	–5	–2	–0.5	0.7	1	1.3	2.5	4	7	10

The accuracy decreases more rapidly than in Exercise 85 because $y = x^3$ is less "linear" than $y = x^{3/2}$.

85. False. Let $f(x) = x$ and $g(x) = x + 1$. Then

$f'(x) = g'(x) = x$, but $f(x) \neq g(x)$.

True. If $y = x^{a+2} + bx$, then

$\frac{dy}{dx} = (a + 2)x^{(a+2)-1} + b = (a + 2)x^{a+1} + b.$

False. If $y = \pi^2$, then $dy/dx = 0$. (π^2 is a constant.)

88. True. If $f(x) = -g(x) + b$, then

$f'(x) = -g'(x) + 0 = -g'(x)$.

False. If $f(x) = 0$, then $f'(x) = 0$ by the Constant Rule.

90. False. If $f(x) = \frac{1}{x^n} = x^{-n}$, then

$f'(x) = -nx^{-n-1} = \frac{-n}{x^{n+1}}$.

91. $f(t) = 3t + 5$, $[1, 2]$

$f'(t) = 3$. So, $f'(1) = f'(2) = 3$.

92. $f(t) = t^2 - 7$, $[3, 3.1]$

$f'(t) = 2t$

Instantaneous rate of change:

At $(3, 2)$: $f'(3) = 6$

At $(3.1, 2.61)$: $f'(3.1) = 6.2$

Average rate of change:

$\frac{f(3.1) - f(3)}{3.1 - 3} = \frac{2.61 - 2}{0.1} = 6.1$

x [ ]

$f'(x) = \frac{1}{x}$

Instantaneous rate of change:

$(1, -1) \Rightarrow f'(1) = 1$

$\left(2, -\frac{1}{2}\right) \Rightarrow f'(2) = \frac{1}{4}$

Average rate of change:

$f(2)$

$-\ f1$ -12 $-\ -1$ 2

/ 1

Instantaneous rate of change is the constant 3.

$2-1$ $2-1$

Average rate of change:

$$\frac{f(2)-f(1)}{2-1}=\frac{11-8}{1}=3$$

94. $f(x) = \sin x, \left[0, \frac{\pi}{6}\right]$

$f'(x) = \cos x$

Instantaneous rate of change:

$(0, 0) \Rightarrow f'(0) = 1$

$\left(\frac{\pi}{6}, \frac{1}{2}\right) \Rightarrow f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \approx 0.866$

Average rate of change:

$\frac{f(\pi/6) - f(0)}{(\pi/6) - 0} = \frac{(1/2) - 0}{(\pi/6) - 0} = \frac{3}{\pi} \approx 0.955$

(a) $s(t) = -16t^2 + 1362$ $v(t) = -32t$

$\frac{s(2) - s(1)}{2 - 1} = 1298 - 1346 = -48$ ft/sec

$v(t) = s'(t) = -32t$

When $t = 1$: $v(1) = -32$ ft/sec

When $t = 2$: $v(2) = -64$ ft/sec

(d) $-16t^2 + 1362 = 0$

$t^2 = \frac{1362}{16} \Rightarrow t = \frac{\sqrt{1362}}{4} \approx 9.226$ sec

(e) $v\left(\frac{\sqrt{1362}}{4}\right) = -32\left(\frac{\sqrt{1362}}{4}\right)$

$-8\sqrt{1362} \approx -295.242$ ft/sec

$s(t) = -16t^2 - 22t + 220$ $v(t) = -32t - 22$

$v(3) = -118$ ft/sec

$s(t) = -16t^2 - 22t + 220$

112 (height after falling 108 ft)

$(t = 2)$

$v(2) = -32(2) - 22$

-86 ft/sec

98. (a) $s(t) = -4.9t^2 + v_0t + s_0 = -4.9t^2 + 214$

$s'(t) = v(t) = -9.8t$

Average velocity $= \frac{s(5) - s(2)}{5 - 2}$

$\frac{91.5 - 194.4}{3}$

-34.3 m/sec

$s'(2) = -9.8(2) = -19.6$ m/sec

$s'(5) = -9.8(5) = 49.0$ m/sec

$s(t) = -4.9t^2 + 214 = 0$

$4.9t^2 = 214$

$t^2 = \frac{214}{4.9}$

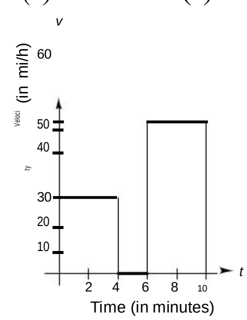
$t \approx 6.61$ sec

$v(6.61) = -9.8(6.61) \approx -64.8$ m/sec

From $(0, 0)$ to $(4, 2)$, $s(t) = \frac{1}{2}t \Rightarrow v(t) = \frac{1}{2}$ mi/min. $v(t) = \frac{1}{2}(60) = 30$ mi/h for $0 < t < 4$

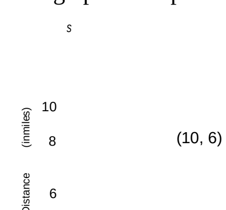
Similarly, $v(t) = 0$ for $4 < t < 6$. Finally, from $(6, 2)$ to $(10, 6)$,

$s(t) = t - 4 \Rightarrow v(t) = 1$ mi/min. $= 60$ mi/h.



(The velocity has been converted to miles per hour.)

This graph corresponds with Exercise 101.

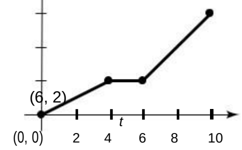


$s(t) = -4.9t^2 + v_0t + s_0$

4

(4, 2)

2



$-4.9t^2 + 120t$

$v(t) = -9.8t + 120$

$v(5) = -9.8(5) + 120 = 71$ m/sec

$v(10) = -9.8(10) + 120 = 22$ m/sec

$V = s^3, \frac{dV}{ds} = 3s^2$

When $s = 6$ cm, $\frac{dV}{ds} = 108 \text{ cm}^3$ per cm change in s.

$A = s^2, \frac{dA}{ds} = 2s$

When $s = 6$ m, $\frac{dA}{ds} = 12\text{ m}^2$ per m change in s.

(a) Using a graphing utility,

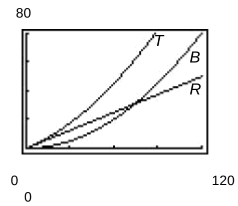
$R(v) = 0.417v - 0.02.$

(b) Using a graphing utility,

$B(v) = 0.0056v^2 + 0.001v + 0.04.$

(c) $T(v) = R(v) + B(v) = 0.0056v^2 + 0.418v + 0.02$

(d)



$\frac{dT}{dv} = 0.0112v + 0.418$

For $v = 40$, $T'(40) \approx 0.866$

For $v = 80$, $T'(80) \approx 1.314$

For $v = 100$, $T'(100) \approx 1.538$

(f) For increasing speeds, the total stopping distance increases.

$C = (\text{gallons of fuel used})(\text{cost per gallon})$

$= \left(\frac{15{,}000}{x}\right)(3.48) = \frac{52{,}200}{x}$

$\frac{dC}{dx} = -\frac{52{,}200}{x^2}$

x	10	15	20	25	30	35	40
C	5220	3480	2610	2088	1740	1491.4	1305
dC/dx	–522	–232	–130.5	–83.52	–58	–42.61	–32.63

The driver who gets 15 miles per gallon would benefit more. The rate of change at $x = 15$ is larger in absolute value than that at $x = 35$.

105. $s(t) = -\frac{1}{2}at^2 + c$ and $s'(t) = -at$

Average velocity: $\frac{s(t_0 + \Delta t) - s(t_0 - \Delta t)}{(t_0 + \Delta t) - (t_0 - \Delta t)} = \frac{\left[-\frac{1}{2}a(t_0 + \Delta t)^2 + c\right] - \left[-\frac{1}{2}a(t_0 - \Delta t)^2 + c\right]}{2\Delta t}$

$= \frac{-\frac{1}{2}a\left(t_0^2 + 2t_0\Delta t + (\Delta t)^2\right) + \frac{1}{2}a\left(t_0^2 - 2t_0\Delta t + (\Delta t)^2\right)}{2\Delta t}$

$= \frac{-2at_0\Delta t}{2\Delta t} = -at_0 = s'(t_0)$ instantaneous velocity at $t = t_0$

$C = \frac{1{,}008{,}000}{Q} + 6.3Q$

$\frac{dC}{dQ} = -\frac{1{,}008{,}000}{Q^2} + 6.3$

$C(351) - C(350) \approx 5083.095 - 5085 \approx -\$1.91$

When $Q = 350$, $\frac{dC}{dQ} \approx -\$1.93.$

$y = ax^2 + bx + c$

Because the parabola passes through $(0, 1)$ and $(1, 0)$, you have:

$(0, 1): 1 = a(0)^2 + b(0) + c \Rightarrow c = 1$

$(1, 0): 0 = a(1)^2 + b(1) + 1 \Rightarrow b = -a - 1$

So, $y = ax^2 + (-a - 1)x + 1$. From the tangent line $y = x - 1$, you know that the derivative is 1 at the point $(1, 0)$.

$y' = 2ax + (-a - 1)$

$= 2a(1) + (-a - 1)$

$= a - 1$

$a = 2$

$b = -a - 1 = -3$

Therefore, $y = 2x^2 - 3x + 1$.

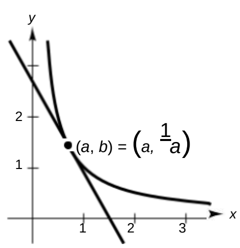
$y = \frac{1}{x}, x > 0$

$y' = -\frac{1}{x^2}$

At (a, b), the equation of the tangent line is $y - \frac{1}{a} = -\frac{1}{a^2}(x - a)$ or $y = -\frac{x}{a^2} + \frac{2}{a}$.

The x-intercept is $(2a, 0)$. The y-intercept is $\left(0, \frac{2}{a}\right)$.

The area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(2a)\left(\frac{2}{a}\right) = 2$.



$y = x^3 - 9x$

$y' = 3x^2 - 9$

Tangent lines through $(1, -9)$:

$y + 9 = (3x^2 - 9)(x - 1)$

$(x^3 - 9x) + 9 = 3x^3 - 3x^2 - 9x + 9$

$0 = 2x^3 - 3x^2 = x^2(2x - 3)$

$x = 0$ or $x = \frac{3}{2}$

The points of tangency are $(0, 0)$ and $\left(\frac{3}{2}, -\frac{81}{8}\right)$. At $(0, 0)$, the slope is $y'(0) = -9$. At $\left(\frac{3}{2}, -\frac{81}{8}\right)$, the slope is $y'\left(\frac{3}{2}\right) = -\frac{9}{4}$.

$y - 0 = -9(x - 0)$ and $y + \frac{81}{8} = -\frac{9}{4}\left(x - \frac{3}{2}\right)$

$y = -9x$ $\quad$ $y = -\frac{9}{4}x - \frac{27}{4}$

$9x + y = 0$ $\quad$ $9x + 4y + 27 = 0$

$y = x^2$ y'
$= 2x$

Tangent lines through $(0, a)$:

$- a = 2x(x -$
$0)$ $x^2 - a = 2x^2$
$- a = x^2$
$\sqrt{-a} = x$

The points of tangency are $\left(\pm\sqrt{-a}, -a\right)$. At $\left(\sqrt{-a}, -a\right)$, the slope is $y'\left(\sqrt{-a}\right) = 2\sqrt{-a}$.

At $\left(-\sqrt{-a}, -a\right)$, the slope is $y'\left(-\sqrt{-a}\right) = -2\sqrt{-a}$.

Tangent lines: $y + a = 2\sqrt{-a}\left(x - \sqrt{-a}\right)$ and $y + a = -2\sqrt{-a}\left(x + \sqrt{-a}\right)$

$y = 2\sqrt{-a}x + a$ $\quad$ $y = -2\sqrt{-a}x + a$

Restriction: a must be negative.

(b) Tangent lines through $(a, 0)$:

$- 0 = 2x(x - a)$
$x^2 = 2x^2 - 2ax$
$0 = x^2 - 2ax = x(x - 2a)$

The points of tangency are $(0, 0)$ and $(2a, 4a^2)$. At $(0, 0)$, the slope is $y'(0) = 0$. At $(2a, 4a^2)$, the slope is $y'(2a) = 4a$.

Tangent lines: $y - 0 = 0(x - 0)$ and $y - 4a^2 = 4a(x - 2a)$

$y = 0$ $\quad$ $y = 4ax - 4a^2$

Restriction: None, a can be any real number.

111. $f(x) = \begin{cases} ax^3, & x \le 2 \\ x^2 + b, & x > 2 \end{cases}$

f must be continuous at $x = 2$ to be differentiable at $x = 2$.

$$\left.\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} ax^3 = 8a \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+}\left(x^2 + b\right) = 4 + b \end{aligned}\right\} \begin{aligned} 8a &= 4 + b \\ 8a - 4 &= b \end{aligned}$$

$$f'(x) = \begin{cases} 3ax^2, & x < 2 \\ 2x, & x > 2 \end{cases}$$

For f to be differentiable at $x = 2$, the left derivative must equal the right derivative.

$3a(2)^2 = 2(2)$
$12a = 4$
$a = \frac{1}{3}$

$b = 8a - 4 = -\frac{4}{3}$

112. $f(x) = \begin{cases} \cos x, & x < 0 \\ ax + b, & x \geq 0 \end{cases}$

$f(0) = b = \cos(0) = 1 \Rightarrow b = 1$

$f'(x) = \begin{cases} -\sin x, & x < 0 \\ a, & x > 0 \end{cases}$

So, $a = 0$.

Answer: $a = 0$, $b = 1$

$f_1(x) = |\sin x|$ is differentiable for all $x \neq n\pi$, n an integer.

$f_2(x) = \sin|x|$ is differentiable for all $x \neq 0$.

You can verify this by graphing f_1 and f_2 and observing the locations of the sharp turns.

Let $f(x) = \cos x$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos x(\cos \Delta x - 1)}{\Delta x} - \lim_{\Delta x \to 0} \sin x \left(\frac{\sin \Delta x}{\Delta x}\right)$$

$$= 0 - \sin x(1) = -\sin x$$

You are given $f : R \to R$ satisfying

$$(*) \quad f'(x) = \frac{f(x + n) - f(x)}{n}$$ for all real numbers x and all positive integers n. You claim that

$f(x) = mx + b$, $m, b \in R$.

For this case, $\frac{\left[m(x + n) + b\right] - \left[mx + b\right]}{n} = m$.

Furthermore, these are the only solutions:

Note first that $f'(x + 1) = \frac{f(x + 2) - f(x + 1)}{1}$, and

$f'(x) = f(x + 1) - f(x)$. From $*$ you have

$$2f'(x) = f(x + 2) - f(x)$$

$$= \left[f(x + 2) - f(x + 1)\right] + \left[f(x + 1) - f(x)\right]$$

$$f'(x + 1) + f'(x).$$

Thus, $f'(x) = f'(x + 1)$.

Let $g(x) = f(x + 1) - f(x)$.

Let $m = g(0) = f(1) - f(0)$.

Let $b = f(0)$. Then

$g'(x) = f'(x + 1) - f'(x) = 0$

$g(x) = \text{constant} = g(0) = m$

$f'(x) = f(x + 1) - f(x) = g(x) = m$

$\Rightarrow f(x) = mx + b.$

Section 2.3 Product and Quotient Rules and Higher-Order Derivatives

To find the derivative of the product of two differentiable functions f and g, multiply the first function f by the derivative of the second function g, and then add the second function g times the derivative of the first function f.

To find the derivative of the quotient of two differentiable functions f and g, where $g(x) \neq 0$, multiply the denominator by the derivative of the numerator minus the numerator times the derivative of the denominator, all of which is divided by the square of the denominator.

$\frac{d}{dx}\tan x = \sec^2 x$ $\quad \frac{d}{dx}\cot x = -\csc^2 x$

$g(x) = (2x - 3)(1 - 5x)$

$g'(x) = (2x - 3)(-5) + (1 - 5x)(2)$

$- 10x + 15 + 2 - 10x$

$- 20x + 17$

$y = (3x - 4)(x^3 + 5)$

$y' = (3x - 4)(3x^2) + (x^3 + 5)(3)$

$9x^3 - 12x^2 + 3x^3 + 15$

$12x^3 - 12x^2 + 15$

sec

$$h(t) = \sqrt{t}(1 - t^2) = t^{1/2}(1 - t^2)$$

$$h'(t) = t^{1/2}(-2t) + (1 - t^2)\frac{1}{2}t^{-1/2}$$

$$= -2t^{3/2} + \frac{1}{2t^{1/2}} - \frac{1}{2}t^{3/2}$$

$$= -\frac{5}{2}t^{3/2} + \frac{1}{2t^{1/2}}$$

$$= \frac{1 - 5t^2}{2t^{1/2}} = \frac{1 - 5t^2}{2\sqrt{t}}$$

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

Higher-order derivatives are successive derivatives of a function.

8. $g(s) = \sqrt{s}(s^2 + 8) = s^{1/2}(s^2 + 8)$

$g'(s) = s^{1/2}(2s) + (s^2 + 8)\frac{1}{2}s^{-1/2}$

$= 2s^{3/2} + \frac{1}{2}s^{3/2} + 4s^{-1/2}$

$= \frac{5}{2}s^{3/2} + \frac{4}{s^{1/2}}$

$= \frac{5s^2 + 8}{2\sqrt{s}}$

9. $f(x) = x^3 \cos x$

$f'(x) = x^3(-\sin x) + \cos x(3x^2)$
$= 3x^2 \cos x - x^3 \sin x$

$= x^2(3\cos x - x\sin x)$

10. $g(x) = \sqrt{x}\sin x$

$g'(x) = \sqrt{x}\cos x + \sin x\left(\frac{1}{2\sqrt{x}}\right)$

$= \sqrt{x}\cos x + \frac{1}{2\sqrt{x}}\sin x$

11. $f(x) = \frac{x}{x - 5}$

$f'(x) = \frac{(x-5)(1) - x(1)}{(x-5)^2} = \frac{x - 5 - x}{(x-5)^2} = -\frac{5}{(x-5)^2}$

$g(t) = \frac{3t^2 - 1}{2t + 5}$

$g'(t) = \frac{(2t + 5)(6t) - (3t^2 - 1)(2)}{(2t + 5)^2}$

$= \frac{6t^2 + 30t + 2}{(2t + 5)^2}$

13. $h(x) = \frac{\sqrt{x}}{x^3 + 1} = \frac{x^{1/2}}{x^3 + 1}$

$h'(x) = \frac{(x^3 + 1)\frac{1}{2}x^{-1/2} - x^{1/2}(3x^2)}{(x^3 + 1)^2}$

$= \frac{x^3 + 1 - 6x^3}{2x^{1/2}(x^3 + 1)^2}$

$= \frac{1 - 5x^3}{2\sqrt{x}(x^3 + 1)^2}$

14. $f(x) = \frac{x^2}{2\sqrt{x} + 1}$

$f'(x) = \frac{(2\sqrt{x} + 1)(2x) - x^2(x^{-1/2})}{(2\sqrt{x} + 1)^2}$

$= \frac{4x^{3/2} + 2x - x^{3/2}}{(2\sqrt{x} + 1)^2}$

$= \frac{3x^{3/2} + 2x}{(2\sqrt{x} + 1)^2}$

$= \frac{x(3\sqrt{x} + 2)}{(2\sqrt{x} + 1)^2}$

15. $g(x) = \frac{\sin x}{x^2}$

$g'(x) = \frac{x^2(\cos x) - \sin x(2x)}{(x^2)^2} = \frac{x\cos x - 2\sin x}{x^3}$

$f(t) = \frac{\cos t}{t^3}$

$f'(t) = \frac{t^3(-\sin t) - \cos t(3t^2)}{(t^3)^2} = -\frac{t\sin t + 3\cos t}{t^4}$

$f(x) = (x^3 + 4x)(3x^2 + 2x - 5)$

$= (x$

$f'(x) = x^3 + 4x(6x + 2) + 3x^2 + 2x - 5\ 3x^2 + 4$ $f'(t) = (t)^2 = - t^4$

$= 6x^4 + 24x^2 + 2x^3 + 8x + 9x^4 + 6x^3 - 15x^2 + 12x^2$

$( ) = 15x^4 + 8x^3 + 21x^2 + 16x - 20$

$f'\ 0 = -20$

$8x - 20$

$$f(x) = (2x^2 - 3x)(9x + 4)$$

$$f'(x) = (2x^2 - 3x)(9) + (9x + 4)(4x - 3)$$

$$= 18x^2 - 27x + 36x^2 + 16x - 27x - 12$$

$$= 54x^2 - 38x - 12$$

$$f'(-1) = 54(-1)^2 - 38(-1) - 12 = 80$$

$$f(x) = \frac{x^2 - 4}{x - 3}$$

$$f'(x) = \frac{(x - 3)(2x) - (x^2 - 4)(1)}{(x - 3)^2}$$

$$= \frac{2x^2 - 6x - x^2 + 4}{(x - 3)^2}$$

$$= \frac{x^2 - 6x + 4}{(x - 3)^2}$$

$$f'(1) = \frac{1 - 6 + 4}{(1 - 3)^2} = -\frac{1}{4}$$

$$f(x) = \frac{x - 4}{x + 4}$$

$$f'(x) = \frac{(x + 4)(1) - (x - 4)(1)}{(x + 4)^2}$$

$$= \frac{x + 4 - x + 4}{(x + 4)^2}$$

$$= \frac{8}{(x + 4)^2}$$

$$f'(3) = \frac{8}{(3 + 4)^2} = \frac{8}{49}$$

$$f(x) = x\cos x$$

$$f'(x) = x(-\sin x) + (\cos x)(1) = \cos x - x\sin x$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{8}(4 - \pi)$$

$$f(x) = \frac{\sin x}{x}$$

$$f'(x) = \frac{(x)(\cos x) - (\sin x)(1)}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

$$f'\left(\frac{\pi}{6}\right) = \frac{(\pi/6)\left(\sqrt{3}/2\right) - (1/2)}{\pi^2/36} = \frac{3\sqrt{3}\pi - 18}{\pi^2}$$

$$= \frac{3\left(\sqrt{3}\pi - 6\right)}{\pi^2}$$

Function	Rewrite	Differentiate	Simplify
23. $y = \frac{x^3 + 6x}{3}$	$y = \frac{1}{3}x^3 + 2x$	$y' = \frac{1}{3}(3x^2) + 2$	$y' = x^2 + 2$
24. $y = \frac{5x^2 - 3}{4}$	$y = \frac{5}{4}x^2 - \frac{3}{4}$	$y' = \frac{10}{4}x$	$y' = \frac{5x}{2}$
25. $y = \frac{6}{7x^2}$	$y = \frac{6}{7}x^{-2}$	$y' = -\frac{12}{7}x^{-3}$	$y' = -\frac{12}{7x^3}$
26. $y = \frac{10}{3x^3}$	$y = \frac{10}{3}x^{-3}$	$y' = -\frac{30}{3}x^{-4}$	$y' = -\frac{10}{x^4}$
$\frac{4x^{3/2}}{x}$	$/\ 0$	$y' = 2x^{-1/2}$	$y' = \frac{2}{\sqrt{x}},\ x > 0$

27. $y = 4\sqrt{x}$ $\quad y = 4x^{1/2},\ x >$ $\quad y' = \frac{4}{3}x^{-1/3}$ $\quad y' = \frac{4}{3x^{1/3}}$

28. $y = \frac{2x}{x^{1/3}}$ $\quad y = 2x^{2/3}$

29. $f(x) = \dfrac{4-3x-x^2}{x^2-1}$

$$f'(x) = \frac{(x^2-1)(-3-2x)-(4-3x-x^2)(2x)}{(x^2-1)^2}$$

$$= \frac{-3x^2+3-2x^3+2x-8x+6x^2+2x^3}{(x^2-1)^2}$$

$$= \frac{3x^2-6x+3}{(x^2-1)^2}$$

$$= \frac{3(x^2-2x+1)}{(x^2-1)^2}$$

$$= \frac{3(x-1)^2}{(x-1)^2(x+1)^2} = \frac{3}{(x+1)^2},\ x \neq 1$$

30. $f(x) = \dfrac{x^2+5x+6}{x^2-4}$

$$f'(x) = \frac{(x^2-4)(2x+5)-(x^2+5x+6)(2x)}{(x^2-4)^2}$$

$$= \frac{2x^3+5x^2-8x-20-2x^3-10x^2-12x}{(x^2-4)^2}$$

$$= \frac{-5x^2-20x-20}{(x^2-4)^2}$$

$$= \frac{-5(x^2+4x+4)}{(x-2)^2(x+2)^2}$$

$$= \frac{-5(x+2)^2}{(x-2)^2(x+2)^2}$$

$$= -\frac{5}{(x-2)^2},\ x \neq 2, -2$$

Alternate solution:

$f(x) = \dfrac{x^2+5x+6}{\ldots}$

31. $f(x) = x\left(1-\dfrac{4}{x+3}\right) = x - \dfrac{4x}{x+3}$

$$f'(x) = 1 - \frac{(x+3)4-4x(1)}{(x+3)^2}$$

$$= \frac{(x^2+6x+9)-12}{(x+3)^2}$$

$$= \frac{x^2+6x-3}{(x+3)^2}$$

$f(x) = x^4\left[1-\dfrac{2}{x+1}\right] = x^4\left[\dfrac{x-1}{x+1}\right]$

$$f'(x) = x^4\left[\frac{(x+1)-(x-1)}{(x+1)^2}\right] + \left[\frac{x-1}{x+1}\right](4x^3)$$

$$= x^4\left[\frac{2}{(x+1)^2}\right] + \left[\frac{x-1}{x+1}\right]4x^3$$

$$= 2x^3\left[\frac{2x^2+x-2}{(x+1)^2}\right]$$

33. $f(x) = \dfrac{3x-1}{\sqrt{x}} = 3x^{1/2} - x^{-1/2}$

$$f'(x) = \frac{3}{2}x^{-1/2} + \frac{1}{2}x^{-3/2} = \frac{3x+1}{2x^{3/2}}$$

$f(x) = \dfrac{3x-1}{\sqrt{x}} = \dfrac{3x-1}{x^{1/2}}$

$$f'(x) = \frac{x^{1/2}(3)-(3x-1)\left(\frac{1}{2}x^{-1/2}\right)}{x}$$

$$= \frac{\frac{1}{2}x^{-1/2}(3x+1)}{x}$$

$$= \frac{3x+1}{2x^{3/2}}$$

$$f(x)=\frac{x^2-4}{(x+3)(x+2)}$$

$$=\frac{(x+2)(x-2)}{(x+3)(x+2)}$$

$$=\frac{x-2}{x+3},\ x\neq -2$$

$$f'(x)=\frac{(x+3)(1)-(x-2)(1)}{(x+3)^2}$$

$$=\frac{5}{(x+3)^2}$$

$$f(x)=\sqrt[3]{x}\left(\sqrt{x}+3\right)=x^{1/3}\left(x^{1/2}+3\right)$$

$$f'(x)=x^{1/3}\left(\frac{1}{2}x^{-1/2}\right)+\left(x^{1/2}+3\right)\left(\frac{1}{3}x^{-2/3}\right)$$

$$=\frac{5}{6}x^{-1/6}+x^{-2/3}$$

$$=\frac{5}{6x^{1/6}}+\frac{1}{x^{2/3}}$$

Alternate solution:

$$f(x)=\sqrt[3]{x}\left(\sqrt{x}+3\right)=x^{5/6}+3x^{1/3}$$

$$f'(x)=\frac{5}{6}x^{-1/6}+x^{-2/3}=\frac{5}{6x^{1/6}}+\frac{1}{x^{2/3}}$$

(

$\sqrt{}$ $\sqrt{}$ / /

/ /

/ /

6 $6x^{1/6}$ $x^{2/3}$

35. $f(x) = \dfrac{2 - \frac{1}{x}}{x - 3} = \dfrac{2x - 1}{x(x-3)} = \dfrac{2x-1}{x^2 - 3x}$

$$f'(x) = \frac{(x^2 - 3x)2 - (2x - 1)(2x - 3)}{(x^2 - 3x)^2}$$

$$= \frac{2x^2 - 6x - 4x^2 + 8x - 3}{(x^2 - 3x)^2}$$

$$= \frac{-2x^2 + 2x - 3}{(x^2 - 3x)^2} = -\frac{2x^2 - 2x + 3}{x^2(x - 3)^2}$$

36. $h(x) = \dfrac{\frac{1}{x^2} + 5x}{x + 1} = \dfrac{1 + 5x^3}{x^3 + x^2}$

$$h'(x) = \frac{(x^3 + x^2)(15x^2) - (1 + 5x^3)(3x^2 + 2x)}{(x^3 + x^2)^2}$$

$$= \frac{15x^5 + 15x^4 - 3x^2 - 2x - 15x^5 - 10x^4}{x^4(x+1)^2}$$

$$= \frac{5x^4 - 3x^2 - 2x}{x^4(x+1)^2}$$

$$= \frac{5x^3 - 3x - 2}{x^3(x+1)^2}$$

37. $g(s) = s^3\left(5 - \dfrac{s}{s + 2}\right) = 5s^3 - \dfrac{s^4}{s + 2}$

$$g'(s) = 15s^2 - \frac{(s + 2)4s^3 - s^4(1)}{(s + 2)^2}$$

$$= 15s^2 - \frac{3s^4 + 8s^3}{(s + 2)^2}$$

$$= \frac{15s^2(s + 2)^2 - (3s^4 + 8s^3)}{(s + 2)^2}$$

$$= \frac{15s^2(s^2 + 4s + 4) - 3s^4 - 8s^3}{(s + 2)^2}$$

$$= \frac{12s^4 + 52s^3 + 60s^2}{(s + 2)^2}$$

$$= \frac{4s^2(3s^2 + 13s + 15)}{(s + 2)^2}$$

38. $g(x) = x^2\left(\dfrac{2}{x} - \dfrac{1}{x + 1}\right) = 2x - \dfrac{x^2}{x + 1}$

$$g'(x) = 2 - \frac{(x + 1)2x - x^2(1)}{(x + 1)^2} = \frac{2(x^2 + 2x + 1) - x^2 - 2x}{(x + 1)^2} = \frac{x^2 + 2x + 2}{(x + 1)^2}$$

39. $f(x) = (2x^3 + 5x)(x - 3)(x + 2)$

$$f'(x) = (6x^2 + 5)(x - 3)(x + 2) + (2x^3 + 5x)(1)(x + 2) + (2x^3 + 5x)(x - 3)(1)$$

$$= (6x^2 + 5)(x^2 - x - 6) + (2x^3 + 5x)(x + 2) + (2x^3 + 5x)(x - 3)$$

$$= (6x^4 + 5x^2 - 6x^3 - 5x - 36x^2 - 30) + (2x^4 + 4x^3 + 5x^2 + 10x) + (2x^4 + 5x^2 - 6x^3 - 15x)$$

$$= 10x^4 - 8x^3 - 21x^2 - 10x - 30$$

Note: You could simplify first: $f(x) = (2x^3 + 5x)(x^2 - x - 6)$

40. $f(x) = (x^3 - x)(x^2 + 2)(x^2 + x - 1)$

$$f'(x) = (3x^2 - 1)(x^2 + 2)(x^2 + x - 1) + (x^3 - x)(2x)(x^2 + x - 1) + (x^3 - x)(x^2 + 2)(2x + 1)$$

$$= (3x^4 + 5x^2 - 2)(x^2 + x - 1) + (2x^4 - 2x^2)(x^2 + x - 1) + (x^5 + x^3 - 2x)(2x + 1)$$

$$= (3x^6 + 5x^4 - 2x^2 + 3x^5 + 5x^3 - 2x - 3x^4 - 5x^2 + 2)$$

$$+ (2x^6 - 2x^4 + 2x^5 - 2x^3 - 2x^4 + 2x^2)$$

$$\left(2x^6 + 2x^4 - 4x^2 + x^5 + x^3 - 2x\right)$$
$$7x^6 + 6x^5 + 4x^3 - 9x^2 - 4x + 2$$

41. $f(t) = t^2 \sin t$

$f'(t) = t^2 \cos t + 2t \sin t = t(t\cos t + 2\sin t)$

42. $f(\theta) = (\theta + 1)\cos\theta$

$f'(\theta) = (\theta + 1)(-\sin\theta) + \cos\theta(1)$

$= \cos\theta - (\theta + 1)\sin\theta$

43. $f(t) = \dfrac{\cos t}{t}$

$f'(t) = \dfrac{-t\sin t - \cos t}{t^2} = -\dfrac{t\sin t + \cos t}{t^2}$

$f(x) = \dfrac{\sin x}{x^3}$

$f'(x) = \dfrac{x^3\cos x - \sin x(3x^2)}{(x^3)^2} = \dfrac{x\cos x - 3\sin x}{x^4}$

45. $f(x) = -x + \tan x$

$f'(x) = -1 + \sec^2 x = \tan^2 x$

$y = x + \cot x$

$y' = 1 - \csc^2 x = -\cot^2 x$

47. $g(t) = \sqrt[4]{t} + 6\csc t = t^{1/4} + 6\csc t$

$g'(t) = \frac{1}{4}t^{-3/4} - 6\csc t\cot t$

$= \dfrac{1}{4t^{3/4}} - 6\csc t\cot t$

$h(x) = \dfrac{1}{x} - 12\sec x = x^{-1} - 12\sec x$

$h'(x) = -x^{-2} - 12\sec x\tan x$

$= \dfrac{-1}{x^2} - 12\sec x\tan$

49. $y = \dfrac{3(1 - \sin x)}{2\cos x} = \dfrac{3 - 3\sin x}{2\cos x}$

$y' = \dfrac{(-3\cos x)(2\cos x) - (3 - 3\sin x)(-2\sin x)}{(2\cos x)^2}$

$= \dfrac{-6\cos^2 x + 6\sin x - 6\sin^2 x}{4\cos^2 x}$

$y = \dfrac{\sec x}{x}$

$y' = \dfrac{x\sec x\tan x - \sec x}{x^2}$

$= \dfrac{\sec x(x\tan x - 1)}{x^2}$

51. $y = -\csc x - \sin x$

$y' = \csc x\cot x - \cos x$

$= \dfrac{\cos x}{\sin^2 x} - \cos x$

$= \cos x(\csc^2 x - 1)$

$= \cos x\cot^2 x$

$y = x\sin x + \cos x$

$y' = x\cos x + \sin x - \sin x = x\cos x$

$f(x) = x^2\tan x$

$f'(x) = x^2\sec^2 x + 2x\tan x = x(x\sec^2 x + 2\tan x)$

$f(x) = \sin x\cos x$

$f'(x) = \sin x(-\sin x) + \cos x(\cos x) = \cos 2x$

$y = 2x\sin x + x^2\cos x$

$y' = 2x\cos x + 2\sin x + x^2(-\sin x) + 2x\cos x$

$= 4x\cos x + (2 - x^2)\sin x$

$h(\theta) = 5\theta\sec\theta + \theta\tan\theta$

$h'(\theta) = 5\theta\sec\theta\tan\theta + 5\sec\theta + \theta\sec^2\theta + \tan\theta$

$g(x) = \left(\dfrac{x + 1}{x + 2}\right)(2x - 5)$

$g'(x) = \left(\dfrac{x + 1}{x + 2}\right)(2) + (2x - 5)\left[\dfrac{(x + 2)(1) - (x + 1)(1)}{(x + 2)^2}\right]$

$= \dfrac{2x^2 + 8x - 1}{(x + 2)^2}$

$= \frac{3}{2}(-1 + \tan x\sec x - \tan^2 x)$

$= \frac{3}{2}\sec x(\tan x - \sec x)$

$(x+2)$ (Form of answer may vary.)

58. $f(x) = \dfrac{\cos x}{1 - \sin x}$

$f'(x) = \dfrac{(1-\sin x)(-\sin x) - (\cos x)(-\cos x)}{(1-\sin x)^2}$

$= \dfrac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$

$= \dfrac{1 - \sin x}{(1-\sin x)^2}$

$= \dfrac{1}{1 - \sin x}$

(Form of answer may vary.)

$y = \dfrac{1 + \csc x}{1 - \csc x}$

$y' = \dfrac{(1-\csc x)(-\csc x \cot x) - (1+\csc x)(\csc x \cot x)}{(1-\csc x)^2} = \dfrac{-2\csc x \cot x}{(1-\csc x)^2}$

$y'\left(\dfrac{\pi}{6}\right) = \dfrac{-2(2)\sqrt{3}}{(1-2)^2} = -4\sqrt{3}$

60. $f(x) = \tan x \cot x = 1$

$f'(x) = 0$

$f'(1) = 0$

61. $h(t) = \dfrac{\sec t}{t}$

$h'(t) = \dfrac{t(\sec t \tan t) - (\sec t)(1)}{t^2} = \dfrac{\sec t\,(t \tan t - 1)}{t^2}$

$h'(\pi) = \dfrac{\sec \pi\,(\pi \tan \pi - 1)}{\pi^2} = \dfrac{1}{\pi^2}$

$f(x) = \sin x(\sin x + \cos x)$

$f'(x) = \sin x(\cos x - \sin x) + (\sin x + \cos x)\cos x$

$\sin x \cos x - \sin^2 x + \sin x \cos x + \cos^2 x$

$\sin 2x + \cos 2x$

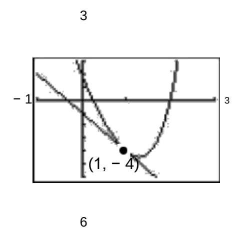
$f'\left(\dfrac{\pi}{4}\right) = \sin \dfrac{\pi}{2} + \cos \dfrac{\pi}{2} = 1$

63. (a) $f(x) = (x^3 + 4x - 1)(x - 2), (1, -4)$

$f'(x) = (x^3 + 4x - 1)(1) + (x - 2)(3x^2 + 4)$

$= x^3 + 4x - 1 + 3x^3 - 6x^2 + 4x - 8$

$= 4x^3 - 6x^2 + 8x - 9$



Graphing utility confirms $\dfrac{dy}{dx} = -3$ at $(1, -4)$.

(a) $f(x) = (x - 2)(x^2 + 4), (1, -5)$

$f'(x) = (x - 2)(2x) + (x^2 + 4)(1)$

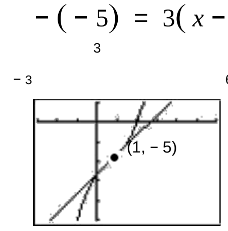
$2x^2 - 4x + x^2 + 4$

$3x^2 - 4x + 4$

$f'(1) = -3$; Slope at $(1, -5)$

Tangent line:

$-(-5) = 3(x - 1) \Rightarrow y = 3x - 8$



$f'(1) = \dfrac{dy}{}$ $= -3$; Slope at $(1, -4)$

Tangent line: $y + 4 = -3(x - 1) \Rightarrow y = -3x - 1$

Graphing utility confirms $dx = 3$ at $(1, -5)$.

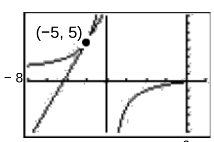
65. (a) $f(x) = \dfrac{x}{x+4}$, $(-5, 5)$

$$f'(x) = \frac{(x+4)(1) - x(1)}{(x+4)^2} = \frac{4}{(x+4)^2}$$

$$f'(-5) = \frac{4}{(-5+4)^2} = 4; \text{ Slope at } (-5, 5)$$

Tangent line: $y - 5 = 4(x+5) \Rightarrow y = 4x + 25$

(b)



Graphing utility confirms $\dfrac{dy}{dx} = 4$ at $(-5, 5)$.

66. (a) $f(x) = \dfrac{x+3}{x-3}$, $(4, 7)$

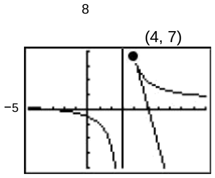
$$f'(x) = \frac{(x-3)(1) - (x+3)(1)}{(x-3)^2} = -\frac{6}{(x-3)^2}$$

$$f'(4) = \frac{-6}{1} = -6; \text{ Slope at } (4, 7)$$

Tangent line:

$y - 7 = -6(x-4) \Rightarrow y = -6x + 31$

(b)



(c) Graphing utility confirms $\dfrac{dy}{dx} = -6$ at $(4, 7)$.

67. (a) $f(x) = \tan x$, $\left(\dfrac{\pi}{4}, 1\right)$

$f'(x) = \sec^2 x$

$\left(\dfrac{\pi}{4}\right)$

68. (a) $f(x) = \sec x$, $\left(\dfrac{\pi}{3}, 2\right)$

$$f'\left(\frac{\pi}{3}\right) = 2\sqrt{3}; \text{ Slope at } \left(\frac{\pi}{3}, 2\right)$$

Tangent line:

$$y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$$

$$\sqrt{3}x - 3y + 6 - 2\sqrt{3}\pi = 0$$

6

$-\pi$ $\left(\frac{\pi}{3}, 2\right)$ π

2

Graphing utility confirms $\dfrac{dy}{dx} = 2\sqrt{3}$ at $\left(\dfrac{\pi}{3}, 2\right)$.

69. $f(x) = \dfrac{8}{x^2+4}$; $(2, 1)$

$$f'(x) = \frac{(x^2+4)(0) - 8(2x)}{(x^2+4)^2} = \frac{-16x}{(x^2+4)^2}$$

$$f'(2) = \frac{-16(2)}{(4+4)^2} = -\frac{1}{2}$$

$$y - 1 = -\frac{1}{2}(x - 2)$$

$$y = -\frac{1}{2}x + 2$$

$$2y + x - 4 = 0$$

$f(x) = \dfrac{27}{x^2+9}$; $\left(-3, \dfrac{3}{2}\right)$

$$f'(x) = \frac{(x^2+9)(0) - 27(2x)}{(x^2+9)^2} = \frac{-54x}{(x^2+9)^2}$$

$f'(-3) =$

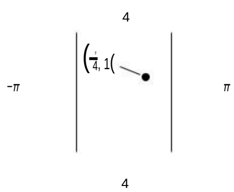
$f\left(\frac{\pi}{4}\right) = 2;$ Slope at $\left(\frac{\pi}{4}, 1\right)$

Tangent line: $y - 1 = 2\left(x - \frac{\pi}{4}\right)$

$$y - 1 = 2x - \frac{\pi}{2}$$

$$4x - 2y - \pi + 2 = 0$$

(b)



Graphing utility confirms $\frac{dy}{dx} = 2$ at $\left(\frac{\pi}{4}, 1\right)$.

$\frac{-54(-3)}{(9+9)^2} = \frac{1}{2}$

$$y - \frac{3}{2} = \frac{1}{2}(x + 3)$$

$$= \frac{1}{2}x + 3$$

$$2y - x - 6 = 0$$

71. $f(x) = \frac{16x}{x^2 + 16}; \left(-2, -\frac{8}{5}\right)$

$$f'(x) = \frac{(x^2 + 16)(16) - 16x(2x)}{(x^2 + 16)^2} = \frac{256 - 16x^2}{(x^2 + 16)^2}$$

$$f'(-2) = \frac{256 - 16(4)}{20^2} = \frac{12}{25}$$

$$y + \frac{8}{5} = \frac{12}{25}(x + 2)$$

$$= \frac{12}{25}x - \frac{16}{25}$$

$$25y - 12x + 16 = 0$$

72. $f(x) = \frac{4x}{x^2 + 6}; \left(2, \frac{4}{5}\right)$

$$f'(x) = \frac{(x^2 + 6)(4) - 4x(2x)}{(x^2 + 6)^2} = \frac{24 - 4x^2}{(x^2 + 6)^2}$$

$$f'(2) = \frac{24 - 16}{10^2} = \frac{2}{25}$$

$$y - \frac{4}{5} = \frac{2}{25}(x - 2)$$

$$y = \frac{2}{25}x + \frac{16}{25}$$

$$25y - 2x - 16 = 0$$

$$f(x) = \frac{2x - 1}{x^2} = 2x^{-1} - x^{-2}$$

$$f'(x) = -2x^{-2} + 2x^{-3} = \frac{2(-x + 1)}{x^3}$$

$f'(x) = 0$ when $x = 1$, and $f(1) = 1$.

Horizontal tangent at $(1, 1)$.

$$f(x) = \frac{x^2}{x^2 + 1}$$

75. $f(x) = \frac{x^2}{x - 1}$

$$f'(x) = \frac{(x - 1)(2x) - x^2(1)}{(x - 1)^2} = \frac{x^2 - 2x}{(x - 1)^2} = \frac{x(x - 2)}{(x - 1)^2}$$

$f'(x) = 0$ when $x = 0$ or $x = 2$.

Horizontal tangents are at $(0, 0)$ and $(2, 4)$.

$$f(x) = \frac{x - 4}{x^2 - 7}$$

$$f'(x) = \frac{(x^2 - 7)(1) - (x - 4)(2x)}{(x^2 - 7)^2}$$

$$= \frac{x^2 - 7 - 2x^2 + 8x}{(x^2 - 7)^2} = \frac{(x - 7)(x - 1)}{(x^2 - 7)^2}$$

$f'(x) = 0$ for $x = 1, 7$; $f(1) = \frac{1}{2}$, $f(7) = \frac{1}{14}$

f has horizontal tangents at $\left(1, \frac{1}{2}\right)$ and $\left(7, \frac{1}{14}\right)$.

$$f(x) = \frac{x + 1}{x - 1}$$

$$f'(x) = \frac{(x - 1) - (x + 1)}{(x - 1)^2} = \frac{-2}{(x - 1)^2}$$

$2y + x = 6 \Rightarrow y = -\frac{1}{2}x + 3$; Slope: $-\frac{1}{2}$

$$\frac{-2}{(x - 1)^2} = -\frac{1}{2}$$

$$(x - 1)^2 = 4$$

$f'(x) = \frac{(x^2 + 1)2x - (x^2 - 1)2x}{(x^2 + 1)^2}$

$= \frac{2x}{(x^2 + 1)}$

$f'(x) = 0$ when $x^2 = 0$.

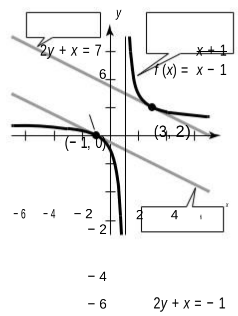
Horizontal tangent is at $(0, 0)$.

$-1 = \pm 2$

$x = -1, 3;\ f(-1) = 0,\ f(3) = 2$

$y - 0 = -\frac{1}{2}(x + 1) \Rightarrow y = -\frac{1}{2}x - \frac{1}{2}$

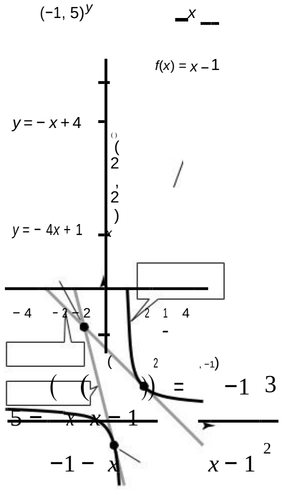
$y - 2 = -\frac{1}{2}(x - 3) \Rightarrow y = -\frac{1}{2}x + \frac{7}{2}$



78. $f(x) = \dfrac{x}{x-1}$

$f'(x) = \dfrac{(x-1) - x}{(x-1)^2} = \dfrac{-1}{(x-1)^2}$

Let $(x, y) = \left(x, \dfrac{x}{x-1}\right)$ be a point of tangency on the graph of *f*.



$\dfrac{5 - \dfrac{x}{x-1}}{-1 - x} = \dfrac{-1}{(x-1)^2}$

$\dfrac{4x-5}{(x-1)(x+1)} = \dfrac{1}{(x-1)^2}$

$(4x-5)(x-1) = x+1$

$4x^2 - 10x + 4 = 0$

$(x-2)(2x-1) = 0 \Rightarrow x = \dfrac{1}{2}, 2$

$f\left(\dfrac{1}{2}\right) = -1, f(2) = 2; f'\left(\dfrac{1}{2}\right) = -4, f'(2) = -1$

Two tangent lines:

$y + 1 = -4\left(x - \dfrac{1}{2}\right) \Rightarrow y = -4x + 1$

$y - 2 = -1(x - 2) \Rightarrow y = -x + 4$

79. $f'(x) = \dfrac{(x+2)3 - 3x(1)}{(x+2)^2} = \dfrac{6}{(x+2)^2}$

$f'(x) = \dfrac{x\cos x - (\sin x)(1)}{x^2} = \dfrac{x\cos x - \sin x}{x^2}$

$g'(x) = \dfrac{x(\cos x + 2) - (\sin x + 2x)(1)}{x^2} = \dfrac{x\cos x - \sin x}{x^2}$

$g(x) = \dfrac{\sin x + 2x}{x} = \dfrac{\sin x - 3x + 5x}{x} = f(x) + 5$

f and *g* differ by a constant.

81. (a) $p'(x) = f'(x)g(x) + f(x)g'(x)$

$p'(1) = f'(1)g(1) + f(1)g'(1) = 1(4) + 6\left(-\dfrac{1}{2}\right) = 1$

(b) $q'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

$q'(4) = \dfrac{3(-1) - 7(0)}{3^2} = -\dfrac{1}{3}$

82. (a) $p'(x) = f'(x)g(x) + f(x)g'(x)$

$p'(4) = \dfrac{1}{2}(8) + 1(0) = 4$

(b) $q'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

$q'(7) = \dfrac{4(2) - 4(-1)}{4^2} = \dfrac{12}{16} = \dfrac{3}{4}$

83. Area $= A(t) = (6t + 5)\sqrt{t} = 6t^{3/2} + 5t^{1/2}$

$A'(t) = 9t^{1/2} + \dfrac{5}{2}t^{-1/2} = \dfrac{18t + 5}{2\sqrt{t}}$ cm^2/sec

84. $V = \pi r^2 h = \pi(t + 2)\left(\dfrac{1}{2}\sqrt{t}\right) = \dfrac{1}{2}\left(t^{3/2} + 2t^{1/2}\right)\pi$

$V'(t) = \dfrac{1}{2}\left(\dfrac{3}{2}t^{1/2} + t^{-1/2}\right)\pi = \dfrac{3t + 2}{4t^{1/2}}\pi$ in.^{3}/sec

$(x+2)^2 \qquad (x+2)^2$

$2\left(\frac{1}{2} + t^{-1/2}\right)^{\pi} = 4t^{1/2}$

$$g'(x) = \frac{(x+2)5 - (5x+4)1}{(x+2)^2} = \frac{6}{(x+2)^2}$$

$$g(x) = \frac{5x+4}{(x+2)} = \frac{3x}{(x+2)} + \frac{2x+4}{(x+2)} = f(x) + 2$$

f and g differ by a constant.

85. $C = 100\left(\frac{200}{x^2} + \frac{x}{x+30}\right), \; 1 \le x$

(a) When

$$\frac{dC}{dx} = 100\left(-\frac{400}{x^3} + \frac{30}{(x+30)^2}\right)$$

(b) When

$$x = 10: \quad \frac{dC}{dx} = -\$38.13 \text{ thousand/100 components}$$

$$x = 15: \quad \frac{dC}{dx} = -\$10.37 \text{ thousand/100 components}$$

(c) When

$$x = 20: \quad \frac{dC}{dx} = -\$3.80 \text{ thousand/100 components}$$

As the order size increases, the cost per item decreases.

86. $P(t) = 500\left[1 + \frac{4t}{50 + t^2}\right]$

$$P'(t) = 500\left[\frac{(50 + t^2)(4) - (4t)(2t)}{(50 + t^2)^2}\right] = 500\left[\frac{200 - 4t^2}{(50 + t^2)^2}\right] = 2000\left[\frac{50 - t^2}{(50 + t^2)^2}\right]$$

$P'(2) \approx 31.55$ bacteria/h

87. (a) $\sec x = \frac{1}{\cos x}$

$$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos x \cos x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

(b) $\csc x = \frac{1}{\sin x}$

$$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = -\frac{\cos x}{\sin x \sin x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

$\cot x = \frac{\cos x}{\sin x}$

$$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$f(x) = \sec x$

$g(x) = \csc x, \quad [0, 2\pi)$

$f'(x) = g'(x)$

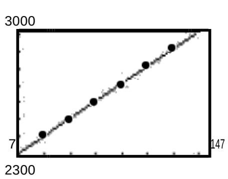
$$\sec x \tan x = -\csc x \cot x \Rightarrow \frac{\sec x \tan x}{\csc x \cot x} = -1 \Rightarrow \frac{\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}}{\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}} = -1 \Rightarrow \frac{\sin^3 x}{\cos^3 x} = -1 \Rightarrow \tan^3 x = -1 \Rightarrow \tan x = -1$$

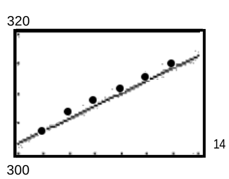
$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

89. (a) $h(t) = 101.7t + 1593$

$p(t) = 2.1t + 287$

(b)





(c) $A = \frac{101.7t + 1593}{2.1t + 287}$

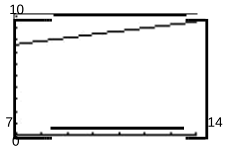
(d) $A'(t) \approx \frac{25{,}842.6}{4.41t^2 + 1205.4t + 82{,}369}$

$A'(t)$ represents the rate of change of the average health care expenditures per person for the given year t.

90. (a) $\sin\theta = \frac{r}{r + h}$

$r + h = r\csc\theta$

$h = r\csc\theta - r = r(\csc\theta - 1)$



A represents the average health care expenditures per person (in thousands of dollars).

(b) $h'(\theta) = r(-\csc\theta \cdot \cot\theta)$

$$h'(30^\circ) = h'\left(\frac{\pi}{6}\right)$$

$$= -4000\left(2 \cdot \sqrt{3}\right) = -8000\ 3 \text{ mi / rad}$$

$f(x) = x^2 + 7x - 4$
$f'(x) = 2x + 7$
$f''(x) = 2$

$f(x) = 4x^5 - 2x^3 + 5x^2$
$f'(x) = 20x^4 - 6x^2 +$
$10x$ $f''(x) = 80x^3 - 12x$
$+ 10$

$f(x) = 4x^{3/2}$ f
$'(x) = 6x^{1/2}$

$f''(x) = 3x^{-1/2} = \frac{3}{\sqrt{x}}$

$f(x) = x^2 + 3x^{-3}$

$f'(x) = 2x - 9x^{-4}$

$f''(x) = 2 + 36x^{-5} = 2 + \frac{36}{x^5}$

95. $f(x) = \frac{x}{x-1}$ $( )$ -1

$f'(x) = \frac{(x-1)1 - x(1)}{(x-1)^2} = \frac{}{(x-1)^2}$

$f''(x) = \frac{2}{(x-1)^3}$

96. $f(x) = \frac{x^2 + 3x}{x - 4}$ $( )$

$f'(x) = \frac{(x-4)(2x+3) - (x^2 + 3x)(1)}{(x-4)^2}$

$= \frac{2x^2 - 5x - 12 - x^2 - 3x}{(x-4)^2} = \frac{x^2 - 8x - 12}{x^2 - 8x + 16}$

$f''(x) = \frac{(x-4)^2(2x-8) - (x^2 - 8x - 12)(2x-8)}{(x-4)^4}$

$= \frac{(x-4)\left[(x-4)(2x-8) - 2(x^2 - 8x - 12)\right]}{(x-4)^4}$

$f(x) = x \sin x$
$f'(x) = x\cos x + \sin x$
$f''(x) = x(-\sin x) + \cos x + \cos x$
$-x\sin x + 2\cos x$

$f(x) = x\cos x$
$f'(x) = \cos x - x\sin x$
$f''(x) = -\sin x - (\sin x + x\cos x) = -x\cos x - 2\sin x$

$f(x) = \csc x$

$f'(x) = -\csc x \cot x$

$f''(x) = -\csc x(-\csc^2 x) - \cot x(-\csc x \cot x)$

$\csc^3 x + \cot^2 x \csc x$

$f(x) = \sec x$

$f'(x) = \sec x \tan x$
$f''(x) = \sec x(\sec^2 x) + \tan x(\sec x \tan x)$

$= \sec x(\sec^2 x + \tan^2 x)$

101. $f'(x) = x^3 - x^{2/5}$

$f''(x) = 3x^2 - \frac{2}{5}x^{-3/5}$
$f'''(x) = 6x + \frac{6}{25}x^{-8/5} = 6x + \frac{6}{25x^{8/5}}$

102. $f^{(3)}(x) = \sqrt[5]{x^4} = x^{4/5}$

$f^{(4)}(x) = \frac{4}{5}x^{-1/5} = \frac{4}{5x^{1/5}}$

$f''(x)$ $f($ $-\sin x$

$3)(x)$ $f($ $-\cos x$

$4)(x)$ $f($ $\sin x$

$5)(x)$ $f($ $\cos x$

$6)(x)$ $f($ $-\sin x$

$)(x)$ f $-\cos x$

$$= \frac{(x-4)(2x-8) - 2(x^2-8x-12)}{(x-4)^3}$$

$$= \frac{2x^2 - 16x + 32 - 2x^2 + 16x + 24}{(x-4)^3}$$

$$= \frac{56}{(x-4)^3}$$

7

$^{(8)}(x)$ sin x

$f^{(4)}(t)$ f $t\cos t$

$\cos t - t\sin t$

$^{(5)}(t)$

$f(x) = 2g(x) + h(x)$

$f'(x) = 2g'(x) + h'(x)$

$f'(2) = 2g'(2) + h'(2)$

$= 2(-2) + 4$

$= 0$

$f(x) = 4 - h(x)$

$f'(x) = -h'(x)$

$f'(2) = -h'(2) = -4$

$f(x) = \dfrac{g(x)}{h(x)}$

$f'(x) = \dfrac{h(x)g'(x) - g(x)h'(x)}{\left[h(x)\right]^2}$

$f'(2) = \dfrac{h(2)g'(2) - g(2)h'(2)}{\left[h(2)\right]^2}$

$= \dfrac{(-1)(-2) - (3)(4)}{(-1)^2}$

$= -10$

$f(x) = g(x)h(x)$

$f'(x) = g(x)h'(x) + h(x)g'(x)$

$f'(2) = g(2)h'(2) + h(2)g'(2)$

$= (3)(4) + (-1)(-2)$

$= 14$

109. Polynomials of degree $n - 1$ (or lower) satisfy $f^{(n)}(x) = 0$. The derivative of a polynomial of the 0th degree (a constant) is 0.

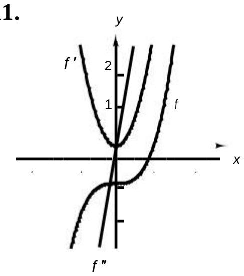
To differentiate a piecewise function, separate the function into its pieces, and differentiate each piece.

If $f(x) = x|x|$, then on $(-\infty, 0)$ you have $f(x) = -x^2$, $f'(x) = -2x$, and $f''(x) = -2$.

On $(0, \infty)$ you have $f(x) = x^2$, $f'(x) = 2x$, and $f''(x) = 2$.

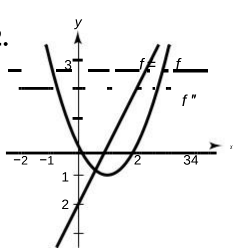
Notice that $f(0) = 0$, $f'(0) = 0$, but $f''(0)$ does not exist (the left-hand limit is -2, whereas the right-hand limit is 2).

111.

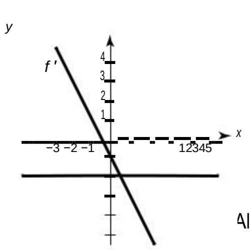


It appears that f is cubic, so f' would be quadratic and f'' would be linear.

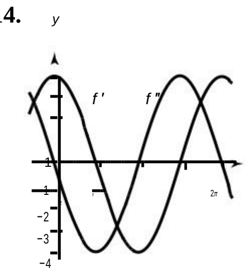
112.



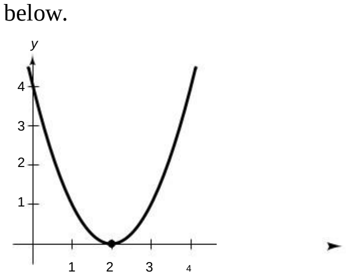
It appears that f is quadratic so f' would be linear and f'' would be constant.



114.



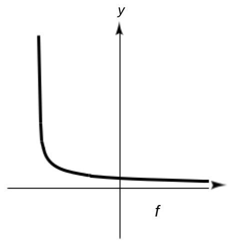
The graph of a differentiable function f such that $f(2) = 0$, $f' < 0$ for $-\infty < x < 2$, and $f' > 0$ for $2 < x < \infty$ would, in general, look like the graph below.



One such function is $f(x) = (x - 2)^2$.

− 3 f''
4
5

The graph of a differentiable function f such that $f > 0$ and $f' < 0$ for all real numbers x would, in general, look like the graph below.



$v(t) = 36 - t^2,\ 0 \le t \le 6$

$a(t) = v'(t) = -2t$

$v(3) = 27$ m/sec

$a(3) = -6$ m/sec^2

The speed of the object is decreasing.

$$v(t) = \frac{100t}{2t + 15}$$

$$a(t) = v'(t) = \frac{(2t + 15)(100) - (100t)(2)}{(2t + 15)^2} = \frac{1500}{(2t + 15)^2}$$

(a) $a(5) = \dfrac{1500}{\left[2(5) + 15\right]^2} = 2.4$ ft/sec^2

(b) $a(10) = \dfrac{1500}{\left[2(10) + 15\right]^2} \approx 1.2$ ft/sec^2

(c) $a(20) = \dfrac{1500}{\left[2(20) + 15\right]^2} \approx 0.5$ ft/sec^2

119. $s(t) = -8.25t^2 + 66t$

$v(t) = s'(t) = -16.50t + 66$

$a(t) = v'(t) = -16.50$

t(sec)	0	1	2	3	4
$s(t)$ (ft)	0	57.75	99	123.75	132
$v(t) = s'(t)$ (ft / sec)	66	49.5	33	16.5	0
$a(t) = v'(t)$ (ft / sec^2)	–16.5	–16.5	–16.5	–16.5	–16.5

Average velocity on:

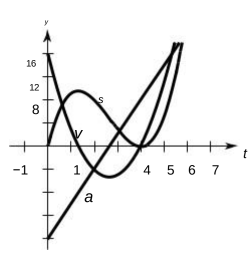
$[0, 1]$ is $\dfrac{57.75 - 0}{1 - 0} = 57.75$

$[1, 2]$ is $\dfrac{99 - 57.75}{2 - 1} = 41.25$

$[2, 3]$ is $\dfrac{123.75 - 99}{3 - 2} = 24.75$

$[3, 4]$ is $\dfrac{132 - 123.75}{4 - 3} = 8.25$

120. (a)

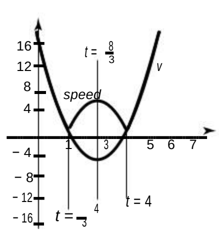


s position function

v velocity function

a acceleration function

The speed of the particle is the absolute value of its velocity. So, the particle's speed is slowing down on the intervals $(0, 4/3)$ and $(8/3, 4)$ and it speeds up on the intervals $(4/3, 8/3)$ and $(4, 6)$.



121. $f(x) = x^n$

$$f^{(n)}(x) = n(n-1)(n-2)\cdots(2)(1) = n!$$

Note: $n! = n(n-1)\cdots 3\cdot 2\cdot 1$ (read "n factorial")

122. $f(x) = \dfrac{1}{x}$

$$f^{(n)}(x) = \frac{(-1)^n n(n-1)(n-2)\cdots(2)(1)}{x^{n+1}} = \frac{(-1)^n n!}{x^{n+1}}$$

123. $f(x) = g(x)h(x)$

$$\begin{aligned}
f'(x) &= g(x)h'(x) + h(x)g'(x)\\
f''(x) &= g(x)h''(x) + g'(x)h'(x) + h(x)g''(x) + h'(x)g'(x)\\
&= g(x)h''(x) + 2g'(x)h'(x) + h(x)g''(x)\\
f'''(x) &= g(x)h'''(x) + g'(x)h''(x) + 2g'(x)h''(x) + 2g''(x)h'(x) + h(x)g'''(x) + h'(x)g''(x)\\
&= g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x)\\
f^{(4)}(x) &= g(x)h^{(4)}(x) + g'(x)h'''(x) + 3g'(x)h'''(x) + 3g''(x)h''(x) + 3g''(x)h''(x) + 3g'''(x)h'(x)\\
&\quad + g'''(x)h'(x) + g^{(4)}(x)h(x)\\
&= g(x)h^{(4)}(x) + 4g'(x)h'''(x) + 6g''(x)h''(x) + 4g'''(x)h'(x) + g^{(4)}(x)h(x)\\
&\ \ \vdots\\
f^{(n)}(x) &= g(x)h^{(n)}(x) + \frac{n(n-1)(n-2)\cdots(2)(1)}{1[(n-1)(n-2)\cdots(2)(1)]}g'(x)h^{(n-1)}(x) + \frac{n(n-1)(n-2)\cdots(2)(1)}{(2)(1)[(n-2)(n-3)\cdots(2)(1)]}g''(x)h^{(n-2)}(x)\\
&\quad + \frac{n(n-1)(n-2)\cdots(2)(1)}{(3)(2)(1)[(n-3)(n-4)\cdots(2)(1)]}g'''(x)h^{(n-3)}(x) + \cdots\\
&\quad + \frac{n(n-1)(n-2)\cdots(2)(1)}{[(n-1)(n-2)\cdots(2)(1)](1)}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)\\
&= g(x)h^{(n)}(x) + \frac{n!}{1!(n-1)!}g'(x)h^{(n-1)}(x) + \frac{n!}{2!(n-2)!}g''(x)h^{(n-2)}(x) + \cdots\\
&\quad + \frac{n!}{(n-1)!1!}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)
\end{aligned}$$

Note: $n! = n(n-1)\cdots 3\cdot 2\cdot 1$ (read "n factorial")

124. $[xf(x)]' = xf'(x) + f(x)$

$$[xf(x)]'' = xf''(x) + f'(x) + f'(x) = xf''(x) + 2f'(x)$$

$$[xf(x)]''' = xf'''(x) + f''(x) + 2f''(x) = xf'''(x) + 3f''(x)$$

In general, $[xf(x)]^{(n)} = xf^{(n)}(x) + nf^{(n-1)}(x)$.

125. $f(x) = x^n \sin x$

$f'(x) = x^n \cos x + nx^{n-1}\sin x$

When $n = 1$: $f'(x) = x\cos x + \sin x$

When $n = 2$: $f'(x) = x^2 \cos x + 2 \sin x$

When $n = 3$: $f'(x) = x^3 \cos x + 3x^2 \sin x$

When $n = 4$: $f'(x) = x^4 \cos x + 4x^3 \sin x$

For general n, $f'(x) = x^n \cos x + nx^{n-1} \sin x$.

126. $f(x) = \dfrac{\cos x}{x^n} = x^{-n}\cos x$

$f'(x) = -x^{-n}\sin x - nx^{-n-1}\cos x$

$= -x^{-n-1}(x\sin x + n\cos x)$

$= -\dfrac{x\sin x + n\cos x}{x^{n+1}}$

When $n = 1$: $f'(x) = -\dfrac{x\sin x + \cos x}{x^2}$

When $n = 2$: $f'(x) = -\dfrac{x\sin x + 2\cos x}{x^3}$

When $n = 3$: $f'(x) = -\dfrac{x\sin x + 3\cos x}{x^4}$

When $n = 4$: $f'(x) = -\dfrac{x\sin x + 4\cos x}{x^5}$

For general n, $f'(x) = -\dfrac{x\sin x + n\cos x}{x^{n+1}}$.

127. $y = \dfrac{1}{x},\ y' = -\dfrac{1}{x^2},\ y'' = \dfrac{2}{x^3}$

$x^3y'' + 2x^2y' = x^3\left[\dfrac{2}{x^3}\right] + 2x^2\left[-\dfrac{1}{x^2}\right] = 2 - 2 = 0$

128. $y = 2x^3 - 6x + 10$

$y' = 6x^2 - 6$

$y'' = 12x$

$y''' = 12$

$-y''' - xy'' - 2y' = -12 - x(12x) - 2(6x^2 - 6) = -24x^2$

129. $y = 2\sin x + 3$

$y' = 2\cos x$

$y'' = -2\sin x$

$y'' + y = -2\sin x + (2\sin x + 3) = 3$

130. $y = 3\cos x + \sin x$

$y' = -3\sin x + \cos x$

$y'' = -3\cos x - \sin x$

$y'' + y = (-3\cos x - \sin x) + (3\cos x + \sin x) = 0$

131. False. If $y = f(x)g(x)$, then

$\dfrac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$.

132. True. y is a fourth-degree polynomial.

$\dfrac{d^ny}{dx^n} = 0$ when $n > 4$.

133. True

$h'(c) = f(c)g'(c) + g(c)f'(c)$

$= f(c)(0) + g(c)(0)$

$= 0$

134. True

135. True

136. True

137. $\dfrac{d}{dx}[f(x)g(x)h(x)] = \dfrac{d}{dx}\big[\big(f(x)g(x)\big)h(x)\big]$

$= \dfrac{d}{dx}[f(x)g(x)]h(x) + f(x)g(x)h'(x)$

$= [f(x)g'(x) + g(x)f'(x)]h(x) + f(x)g(x)h'(x)$

$= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$

138. (a) $(fg' - f'g)' = fg'' + f'g' - f'g' - f''g$

$= fg'' - f''g$ True

(b) $(fg)'' = (fg' + f'g)'$

$= fg'' + f'g' + f'g' + f''g$

$= fg'' + 2f'g' + f''g$

$\neq fg'' + f''g$ False

Section 2.4 The Chain Rule

To find the derivative of the composition of two differentiable functions, take the derivative of the outer function and keep the inner function the same. Then multiply this by the derivative of the inner function.

$$\left[f(g(x))\right]' = f'(g(x))\,g'(x)$$

The (Simple) Power Rule is $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

The General Power Rule uses the Chain Rule:

$$\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}.$$

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
3. $y = (6x - 5)^4$	$u = 6x - 5$	$y = u^4$
4. $y = \sqrt[3]{4x + 3}$	$u = 4x + 3$	$y = u^{1/3}$
5. $y = \dfrac{1}{3x + 5}$	$u = 3x + 5$	$y = u^{-1}$
6. $y = \dfrac{2}{\sqrt{x^2 + 10}}$	$u = x^2 + 10$	$y = \dfrac{2}{\sqrt{u}}$
7. $y = \csc^3 x$	$u = \csc x$	$y = u^3$
8. $y = \sin \dfrac{5x}{2}$	$u = \dfrac{5x}{2}$	$y = \sin u$

9. $y = (2x - 7)^3$

$y' = 3(2x - 7)^2(2)$

$= 6(2x - 7)^2$

$y = 5(2 - x^3)^4$

$y' = 5(4)(2 - x^3)^3(-3x^2) = -60x^2(2 - x^3)^3$

$= 60x^2(x^3 - 2)^3$

$g(x) = 3(4 - 9x)^{5/6}$

$g'(x) = 3\left(\dfrac{5}{6}\right)(4 - 9x)^{-1/6}(-9)$

$= -\dfrac{45}{2}(4 - 9x)^{-1/6}$

$= -\dfrac{45}{2(4 - 9x)^{1/6}}$

$f(t) = (9t + 2)^{2/3}$

$f'(t) = \dfrac{2}{3}(9t + 2)^{-1/3}(9) = \dfrac{6}{\sqrt[3]{9t + 2}}$

13. $h(s) = -2\sqrt{5s^2 + 3} = -2(5s^2 + 3)^{1/2}$

$y = \sqrt[3]{6x^2 + 1} = (6x^2 + 1)^{1/3}$

$y' = \dfrac{1}{3}(6x^2 + 1)^{-2/3}(12x) = \dfrac{4x}{(6x^2 + 1)^{2/3}} = \dfrac{4x}{\sqrt[3]{(6x^2 + 1)^2}}$

16. $y = 2\sqrt[4]{9 - x^2} = 2(9 - x^2)^{1/4}$

$y' = 2\left(\dfrac{1}{4}\right)(9 - x^2)^{-3/4}(-2x)$

$= \dfrac{-x}{(9 - x^2)^{3/4}} = \dfrac{-x}{\sqrt[4]{(9 - x^2)^3}}$

$y = (x - 2)^{-1}$

$y' = -1(x - 2)^{-2}(1) = \dfrac{-1}{(x - 2)^2}$

18. $s(t) = \dfrac{1}{4 - 5t - t^2} = (4 - 5t - t^2)^{-1}$

$s'(t) = -(4 - 5t - t^2)^{-2}(-5 - 2t)$

$= \dfrac{5 + 2t}{(4 - 5t - t^2)^2} = \dfrac{2t + 5}{(t^2 + 5t - 4)^2}$

$\dfrac{6}{\ }$ $\quad ^{-3}$

$$h'(s) = -\frac{1}{2}\left(5s^2+3\right)^{-1/2}(10s)$$

$$= \frac{-10s}{\left(5s^2+3\right)^{1/2}} = -\frac{10s}{\sqrt{5s^2+3}}$$

$$g(x) = \sqrt{4-3x^2} = \left(4-3x^2\right)^{1/2}$$

$$g'(x) = \frac{1}{2}\left(4-3x^2\right)^{-1/2}(-6x) = -\frac{3x}{\sqrt{4-3x^2}}$$

19. $g(s) = \dfrac{6}{\left(s^3-2\right)^3} = 6\left(s^3-2\right)^{-3}$

$$g'(s) = 6(-3)\left(s^3-2\right)^{-4}\left(3s^2\right)$$

$$= -\frac{54s^2}{\left(s^3-2\right)^4}$$

20. $y = -\dfrac{3}{(t-2)^4} = -3(t-2)^{-4}$

$y' = 12(t-2)^{-5} = \dfrac{12}{(t-2)^5}$

21. $y = \dfrac{1}{\sqrt{3x+5}} = (3x+5)^{-1/2}$

$y' = -\dfrac{1}{2}(3x+5)^{-3/2}(3)$

$= \dfrac{-3}{2(3x+5)^{3/2}}$

$= -\dfrac{3}{2\sqrt{(3x+5)^3}}$

22. $g(t) = \dfrac{1}{\sqrt{t^2-2}} = (t^2-2)^{-1/2}$

$g'(t) = -\dfrac{1}{2}(t^2-2)^{-3/2}(2t)$

$= \dfrac{-t}{(t^2-2)^{3/2}}$

$= -\dfrac{t}{\sqrt{(t^2-2)^3}}$

$f(x) = x^2(x-2)^7$

$f'(x) = 2x(x-2)^7 + 7(x-2)^6x^2$

$= x(x-2)^6\left[2(x-2) + 7x\right]$

$= x(x-2)^6(9x-4)$

$f(x) = x(2x-5)^3$

$f'(x) = x(3)(2x-5)^2(2) + (2x-5)^3(1)$

$= (2x-5)^2\left[6x + (2x-5)\right]$

$y = x^2\sqrt{16-x^2} = x^2(16-x^2)^{1/2}$

$y' = 2x(16-x^2)^{1/2} + \dfrac{1}{2}(16-x^2)^{-1/2}(-2x)x^2$

$= \dfrac{x}{(16-x^2)^{1/2}}\left[2(16-x^2) - x^2\right]$

$= \dfrac{x(32-3x^2)}{\sqrt{16-x^2}}$

27. $y = \dfrac{x}{\sqrt{x^2+1}} = \dfrac{x}{(x^2+1)^{1/2}}$

$y' = \dfrac{(x^2+1)^{1/2}(1) - x\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x)}{\left[(x^2+1)^{1/2}\right]^2}$

$= \dfrac{(x^2+1)^{1/2} - x^2(x^2+1)^{-1/2}}{x^2+1}$

$= \dfrac{(x^2+1)^{-1/2}\left[x^2+1-x^2\right]}{x^2+1}$

$= \dfrac{1}{(x^2+1)^{3/2}} = \dfrac{1}{\sqrt{(x^2+1)^3}}$

28. $y = \dfrac{x}{\sqrt{x^4+4}}$

$y' = \dfrac{(x^4+4)^{1/2}(1) - x\left(\frac{1}{2}\right)(x^4+4)^{-1/2}(4x^3)}{x^4+4}$

$= \dfrac{x^4+4-2x^4}{(x^4+4)^{3/2}} = \dfrac{4-x^4}{(x^4+4)^{3/2}} = \dfrac{4-x^4}{\sqrt{(x^4+4)^3}}$

$g(x) = \left(\dfrac{x+5}{x^2+2}\right)^2$

$(2x-5)^2(8x-5)$

$g'(x) = \dfrac{2\left(\frac{x+5}{x^2+2}\right)\left[\frac{(x^2+2)-(x+5)(2x)}{(x^2+2)^2}\right]}{}$

$= \dfrac{2(x+5)(2-10x-x^2)}{(x^2+2)^3}$

$= \dfrac{-2(x+5)(x^2+10x-2)}{(x^2+2)^3}$

25. $y = x\sqrt{1-x^2} = x(1-x^2)^{1/2}$

$y' = x\left[\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right] + (1-x^2)^{1/2}(1)$

$= -x^2(1-x^2)^{-1/2} + (1-x^2)^{1/2}$

$= (1-x^2)^{-1/2}\left[-x^2 + (1-x^2)\right]$

$= \dfrac{1-2x^2}{\sqrt{1-x^2}}$

30. $h(t) = \left(\dfrac{t^2}{t^3+2}\right)^2$

$$h'(t) = 2\left(\frac{t^2}{t^3+2}\right)\left(\frac{(t^3+2)(2t) - t^2(3t^2)}{(t^3+2)^2}\right)$$

$$= \frac{2t^2(4t - t^4)}{(t^3+2)^3} = \frac{2t^3(4 - t^3)}{(t^3+2)^3}$$

31. $s(t) = \left(\dfrac{1+t}{t+3}\right)^4$

$$s'(t) = 4\left(\frac{1+t}{t+3}\right)^3\left[\frac{(t+3)(1) - (1+t)(1)}{(t+3)^2}\right]$$

$$= \frac{4(1+t)^3(2)}{(t+3)^5}$$

$$= \frac{8(t+1)^3}{(t+3)^5}$$

32. $g(x) = \left(\dfrac{3x^2-2}{2x+3}\right)^{-2} = \left(\dfrac{2x+3}{3x^2-2}\right)^2$

$$g'(x) = 2\left(\frac{2x+3}{3x^2-2}\right)\left[\frac{(3x^2-2)(2) - (2x+3)(6x)}{(3x^2-2)^2}\right]$$

$$= \frac{2(2x+3)(-6x^2-18x-4)}{(3x^2-2)^3}$$

$$= \frac{-4(2x+3)(3x^2+9x+2)}{(3x^2-2)^3}$$

$$f(x) = \left((x^2+3)^5 + x\right)^2$$

$$f'(x) = 2\left((x^2+3)^5 + x\right)\left(5(x^2+3)^4(2x)+1\right)$$

$$= 2\left[10x(x^2+3)^9 + (x^2+3)^5 + 10x^2(x^2+3)^4 + x\right] = 20x(x^2+3)^9 + 2(x^2+3)^5 + 20x^2(x^2+3)^4 + 2x$$

$$g(x) = \left(2 + (x^2+1)^4\right)^3$$

$$g'(x) = 3\left(2 + (x^2+1)^4\right)^2\left(4(x^2+1)^3(2x)\right) = 24x(x^2+1)^3\left(2 + (x^2+1)^4\right)^2$$

$y = \cos 4x$

$\dfrac{dy}{dx} = -4\sin 4x$

$y = \csc(1-2x)^2$

$y' = -\csc(1-2x)^2 \cot(1-2x)^2\left[2(1-2x)(-2)\right]$

$y = \sin \pi x$

$\dfrac{dy}{dx} = \pi \cos \pi x$

$y = \sin(\pi x)^2 = \sin(\pi^2 x^2)$

$g(x) = 5\tan 3x$

$g'(x) = 15\sec^2 3x$

$h(x) = \sec 6x$

$h'(x) = \sec 6x \tan 6x\,(6)$

$= 6\sec 6x \tan 6x$

$= 4(1-2x)\csc(1-2x)^2\cot(1-2x)$

$h(x) = \sin 2x \cos 2x$

$h'(x) = \sin 2x(-2\sin 2x) + \cos 2x(2\cos 2x)$

$y' = \cos(\pi^2 x^2)\left[2\pi^2 x\right] = 2\pi^2 x\cos(\pi^2 x^2)$

$= 2\pi^2 x\cos \pi x^2$

$2\cos^2 2x - 2\sin^2 2x$

$2\cos 4x$

Alternate solution: $h(x) = \frac{1}{2}\sin 4x$

$h'(x) = \frac{1}{2}\cos 4x(4) = 2\cos 4x$

$g(\theta) = \sec\frac{1}{2}\theta\tan\frac{1}{2}\theta$

$g'(\theta) = \sec\left(\frac{1}{2}\theta\right)\sec^2\left(\frac{1}{2}\theta\right)\frac{1}{2} + \tan\left(\frac{1}{2}\theta\right)\sec\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{2}\theta\right)\frac{1}{2} = \frac{1}{2}\sec\left(\frac{1}{2}\theta\right)\left[\sec^2\left(\frac{1}{2}\theta\right) + \tan^2\left(\frac{1}{2}\theta\right)\right]$

$$f(x) = \frac{\cot x}{\sin x} = \frac{\cos x}{\sin^2 x}$$

$$f'(x) = \frac{\sin^2 x(-\sin x) - \cos x(2\sin x\cos x)}{\sin^4 x}$$

$$= \frac{-\sin^2 x - 2\cos^2 x}{\sin^3 x} = \frac{-1-\cos^2 x}{\sin^3 x}$$

$$h(t) = 2\cot^2(\pi t + 2)$$

$$h'(t) = 4\cot(\pi t + 2)\left[-\csc^2(\pi t + 2)(\pi)\right]$$

$$-4\pi \cot(\pi t + 2)\csc^2(\pi t + 2)$$

$$g(v) = \frac{\cos v}{\csc v} = \cos v \cdot \sin v$$

$$g'(v) = \cos v(\cos v) + \sin v(-\sin v)$$

$$\cos^2 v - \sin^2 v = \cos 2v$$

$$f(t) = 3\sec(\pi t - 1)^2$$

$$f'(t) = 3\sec(\pi t - 1)^2\tan(\pi t - 1)^2(2)(\pi t - 1)(\pi)$$

$$6\pi(\pi t - 1)\sec(\pi t - 1)^2\tan(\pi t - 1)^2$$

$$y = 4\sec^2 x$$

$$y' = 8\sec x \cdot \sec x\tan x = 8\sec^2 x\tan x$$

$$y = 5\cos(\pi x)^2$$

$$y' = -5\sin(\pi x)^2(2)(\pi x)(\pi)$$

$$-10\pi^2 x\sin(\pi x)^2$$

$$g(t) = 5\cos^2\pi t = 5(\cos\pi t)^2$$

$$g'(t) = 10\cos\pi t(-\sin\pi t)(\pi)$$

$$-10\pi(\sin\pi t)(\cos\pi t)$$

$$-5\pi\sin 2\pi t$$

$$y = \sin(3x^2 + \cos x)$$

$$y' = \cos(3x^2 + \cos x)(6x - \sin x)$$

$$y = \cos(5x + \csc x)$$

$$y' = -\sin(5x + \csc x)(5 - \csc x\cot x)$$

$$f(\theta) = \frac{1}{4}\sin^2 2\theta = \frac{1}{4}(\sin 2\theta)^2$$

$$f'(\theta) = 2\left(\frac{1}{4}\right)(\sin 2\theta)(\cos 2\theta)(2)$$

$$\sin 2\theta\cos 2\theta = \frac{1}{2}\sin 4\theta$$

$$y = \sin\sqrt{\cot 3\pi x} = \sin(\cot 3\pi x)^{1/2}$$

$$y' = \cos(\cot 3\pi x)^{1/2}\left[\frac{1}{2}(\cot 3\pi x)^{-1/2}(-\csc^2 3\pi x)(3\pi)\right]$$

$$-\frac{3\pi\cos\left(\sqrt{\cot 3\pi x}\right)\csc^2(3\pi x)}{2\sqrt{\cot 3\pi x}}$$

54. $y = \cos\sqrt{\sin(\tan\pi x)}$

$$y' = -\sin\sqrt{\sin(\tan\pi x)} \cdot \frac{1}{2}(\sin(\tan\pi x))^{-1/2}\cos(\tan\pi x)\sec^2\pi x(\pi) = \frac{-\pi\sin\sqrt{\sin(\tan\pi x)}\cos(\tan\pi x)\sec^2\pi x}{2\sqrt{\sin(\tan\pi x)}}$$

$$y = \frac{x+1}{\sqrt{x^2+1}}$$

$$\sqrt{x}(x^2+1)^2$$

The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.

2

y

−1 5

2

$$y = \sqrt{\frac{2x}{x+1}}$$

7

y

$$y' = \frac{1}{\sqrt{2x}(x+1)^{3/2}}$$

−6 6

y' has no zeros.

1

$$y = \sqrt{\frac{x+1}{x}}$$

4

$$y' = -\frac{\sqrt{(x+1)/x}}{2x(x+1)}$$

y

−5 4

−2

y' has no zeros.

58. $g(x) = \sqrt{x-1} + \sqrt{x+1}$

$g'(x) = \frac{1}{2\sqrt{x-1}} + \frac{1}{2\sqrt{x+1}}$

g' has no zeros.

6

g

′

−2 10

2

$y = \frac{\cos \pi x + 1}{x}$

$\frac{dy}{dx} = \frac{-\pi x \sin \pi x - \cos \pi x - 1}{x^2}$

$= -\frac{\pi x \sin \pi x + \cos \pi x + 1}{x^2}$

The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.

y

3

−5 5

′

3

$y = x^2 \tan \frac{1}{x}$

$\frac{dy}{dx} = 2x \tan \frac{1}{x} - \sec^2 \frac{1}{x}$

The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.

6

y

−4 5

′

6

$y = \sin 3x$ $y' = 3 \cos 3x$

$y'(0) = 3$

3 cycles in $[0, 2\pi]$

$y = \sin \frac{x}{2}$

$y' = \frac{1}{2} \cos \frac{x}{2}$

$\frac{1}{2}$ cycle in $[0, 2\pi]$

$y = \sqrt{x^2 + 8x} = (x^2 + 8x)^{1/2}$, $(1, 3)$

$y' = \frac{1}{2}(x^2 + 8x)^{-1/2}(2x + 8) = \frac{2(x+4)}{2(x^2+8x)^{1/2}} = \frac{x + 4}{\sqrt{x^2 + 8x}}$

$y'(1) = \frac{1 + 4}{\sqrt{1^2 + 8(1)}} = \frac{5}{\sqrt{9}} = \frac{5}{3}$

64. $y = (3x^3 + 4x)^{1/5}$, $(2, 2)$

$y' = \frac{1}{5}(3x^3 + 4x)^{-4/5}(9x^2 + 4)$

$= \frac{9x^2 + 4}{5(3x^3 + 4x)^{4/5}}$

$y'(2) = \frac{1}{2}$

65. $f(x) = 5(x^3 - 2)^{-1}$, $\left(-2, -\frac{1}{2}\right)$

$f'(x) = -5(x^3 - 2)^{-2}(3x^2) = \frac{-15x^2}{(x^3-2)^2}$

66. $f(x) = \frac{1}{(x^2 - 3x)^2} = (x^2 - 3x)^{-2}$, $\left(4, \frac{1}{16}\right)$

$f'(x) = -2(x^2 - 3x)^{-3}(2x - 3) = \frac{-2(2x - 3)}{(x^2 - 3x)^3}$

$f'(4) = -\frac{5}{32}$

67. $y = \frac{4}{(x + 2)^2} = 4(x + 2)^{-2}$, $(0, 1)$

$y' = -8(x + 2)^{-3} = \frac{-8}{(x + 2)^3}$

$$f'(-2) = -\frac{60}{100} = -\frac{3}{5}$$

$$y'(0) = \frac{-8}{8} = -1$$

$y = \frac{4}{(x^2 - 2x)^3} = 4(x^2 - 2x)^{-3}, \ (1, -4)$

68. $y' = -12(x^2 - 2x)^{-4}(2x - 2)$

$y'(1) = -12(-1)^{-4}(0) = 0$

69. $y = 26 - \sec^3 4x, \quad (0, 25)$

$y' = -3\sec^2 4x \sec 4x \tan 4x(4)$
$\quad = -12\sec^3 4x \tan 4x$

$y'(0) = 0$

70. $y = \frac{1}{x} + \sqrt{\cos x} = x^{-1} + (\cos x)^{1/2}, \ \left(\frac{\pi}{2}, \frac{2}{\pi}\right)$

$y' = -x^{-2} + \frac{1}{2}(\cos x)^{-1/2}(-\sin x) = -\frac{1}{x^2} - \frac{\sin x}{2\sqrt{\cos x}}$

$y'(\pi/2)$ is undefined.

71. (a) $f(x) = (2x^2 - 7)^{1/2}, \quad (4, 5)$

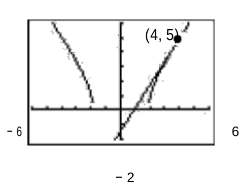
$f'(x) = \frac{1}{2}(2x^2 - 7)^{-1/2}(4x) = \frac{2x}{\sqrt{2x^2 - 7}}$

$f'(4) = \frac{8}{5}$

Tangent line:

$y - 5 = \frac{8}{5}(x - 4) \Rightarrow 8x - 5y - 7 = 0$

(b)



72. (a) $f(x) = \frac{1}{3}x\sqrt{x^2 + 5} = \frac{1}{3}x(x^2 + 5)^{1/2}, \ (2, 2)$

$f'(x) = \frac{1}{3}x\left[\frac{1}{2}(x^2 + 5)^{-1/2}(2x)\right] + \frac{1}{3}(x^2 + 5)^{1/2}$

73. (a) $y = (4x^3 + 3)^2, \quad (-1, 1)$

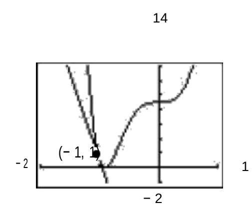
$y' = 2(4x^3 + 3)(12x^2) = 24x^2(4x^3 + 3)$

$y'(-1) = -24$

Tangent line:

$y - 1 = -24(x + 1) \Rightarrow 24x + y + 23 = 0$

(b)



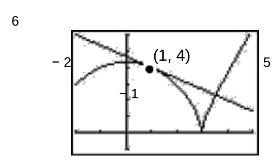
74. (a) $f(x) = (9 - x^2)^{2/3}, \quad (1, 4)$

$f'(x) = \frac{2}{3}(9 - x^2)^{-1/3}(-2x) = \frac{-4x}{3(9 - x^2)^{1/3}}$

$f'(1) = \frac{-4}{3(8)^{1/3}} = -\frac{2}{3}$

Tangent line:

$y - 4 = -\frac{2}{3}(x - 1) \Rightarrow 2x + 3y - 14 = 0$



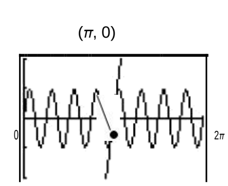
75. (a) $f(x) = \sin 8x, \quad (\pi, 0)$

$f'(x) = 8\cos 8x$

$f'(\pi) = 8$

Tangent line: $y = 8(x - \pi) = 8x - 8\pi$

(b)

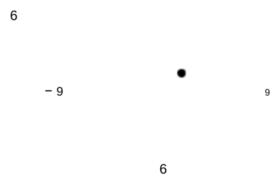


$$= \frac{x^2}{2\sqrt{x^2+5}} + \frac{1}{3}\sqrt{x^2+5}$$

$$f'(2) = \frac{4}{3(3)} + \frac{1}{3}(3) = \frac{13}{9}$$

Tangent line:

$$y - 2 = \frac{13}{9}(x - 2) \Rightarrow 13x - 9y - 8 = 0$$



76. (a) $y = \cos 3x, \left(\frac{\pi}{4}, -\frac{\sqrt{2}}{2}\right)$

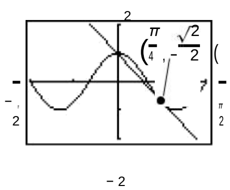
$y' = -3\sin 3x$

$y'\left(\frac{\pi}{4}\right) = -3\sin\left(\frac{3\pi}{4}\right) = \frac{-3\sqrt{2}}{2}$

Tangent line: $y + \frac{\sqrt{2}}{2} = \frac{-3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$

$y = \frac{-3\sqrt{2}}{2}x + \frac{3\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2}$

(b)



77. (a) $f(x) = \tan^2 x, \left(\frac{\pi}{4}, 1\right)$

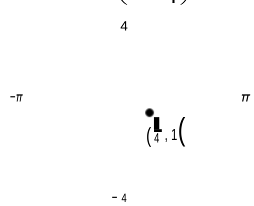
$f'(x) = 2\tan x\sec^2 x$

$f'\left(\frac{\pi}{4}\right) = 2(1)(2) = 4$

Tangent line:

$y - 1 = 4\left(x - \frac{\pi}{4}\right) \Rightarrow 4x - y + (1 - \pi) = 0$

(b)



78. (a) $y = 2\tan^3 x, \left(\frac{\pi}{4}, 2\right)$

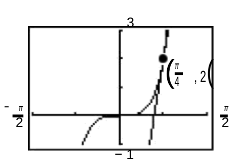
$y' = 6\tan^2 x \cdot \sec^2 x$

$y'\left(\frac{\pi}{4}\right) = 6(1)(2) = 12$

Tangent line:

$y - 2 = 12\left(x - \frac{\pi}{4}\right) \Rightarrow 12x - y + (2 - 3\pi) = 0$

(b)



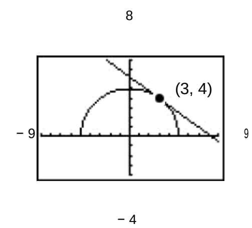
79. $f(x) = \sqrt{25 - x^2} = (25 - x^2)^{1/2}, (3, 4)$

$f'(x) = \frac{1}{2}(25 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{25 - x^2}}$

$f'(3) = -\frac{3}{4}$

Tangent line:

$y - 4 = -\frac{3}{4}(x - 3) \Rightarrow 3x + 4y - 25 = 0$

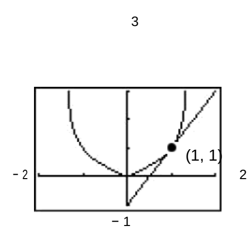


80. $f(x) = \frac{|x|}{\sqrt{2 - x^2}} = |x|(2 - x^2)^{-1/2}, (1, 1)$

$f'(x) = \frac{2}{(2 - x^2)^{3/2}}$ for $x > 0$

$f'(1) = 2$

Tangent line: $y - 1 = 2(x - 1) \Rightarrow 2x - y - 1 = 0$



81. $f(x) = 2\cos x + \sin 2x, \quad 0 < x < 2\pi$

$f'(x) = -2\sin x + 2\cos 2x$

$= -2\sin x + 2 - 4\sin^2 x = 0$

$2\sin^2 x + \sin x - 1 = 0$

$(\sin x + 1)(2\sin x - 1) = 0$

$\sin x = -1 \Rightarrow x = \frac{3\pi}{2}$

$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

Horizontal tangents at $x = \frac{\pi}{6}, \frac{3\pi}{2}, \frac{5\pi}{6}$

Horizontal tangent at the points $\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right), \left(\frac{3\pi}{2}, 0\right)$, and $\left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2}\right)$

82. $f(x) = \frac{-4x}{\sqrt{2x-1}}$

$f'(x) = \frac{(2x-1)^{1/2}(-4) + 4x\left(\frac{1}{2}\right)(2x-1)^{-1/2}(2)}{2x-1}$

$= \frac{(2x-1)(-4) + 4x}{(2x-1)^{3/2}}$

$= \frac{4-4x}{(2x-1)^{3/2}}$

$f'(x) = 0 = \frac{4-4x}{(2x-1)^{3/2}} \Rightarrow x = 1$

Horizontal tangent at $(1, -4)$

$f(x) = 5(2-7x)^4$

$f'(x) = 20(2-7x)^3(-7) = -140(2-7x)^3$

$f''(x) = -420(2-7x)^2(-7) = 2940(2-7x)^2$

$f(x) = 6(x^3+4)^3$

$f'(x) = 18(x^3+4)^2(3x^2) = 54x^2(x^3+4)^2$

$f''(x) = 54x^2(2)(x^3+4)(3x^2) + 108x(x^3+4)^2$

$= 108x(x^3+4)\left[3x^3 + x^3 + 4\right]$

$= 432x(x^3+4)(x^3+1)$

85. $f(x) = \frac{1}{11x-6} = (11x-6)^{-1}$

$f'(x) = -(11x-6)^{-2}(11)$

$f''(x) = -22(11x-6)^{-3}(11)$

$= 242(11x-6)^{-3}$

$= \frac{242}{(11x-6)^3}$

86. $f(x) = \frac{8}{(x-2)^2} = 8(x-2)^{-2}$

$f'(x) = -16(x-2)^{-3}$

$f''(x) = 48(x-2)^{-4} = \frac{48}{(x-2)^4}$

87. $f(x) = \sin x^2$

$f'(x) = 2x\cos x^2$

$f''(x) = 2x\left[2x\left(-\sin x^2\right)\right] + 2\cos x^2$

$= 2\left(\cos x^2 - 2x^2\sin x^2\right)$

$f(x) = \sec^2 \pi x$

$f'(x) = 2\sec \pi x(\pi \sec \pi x \tan \pi x)$

$$2\pi \sec^2 \pi x \tan \pi x$$

$$f''(x) = 2\pi \sec^2 \pi x(\sec^2 \pi x)(\pi) + 2\pi \tan\pi x(2\pi$$

$$2\pi^2 \sec^4 \pi x + 4\pi^2 \sec^2 \pi x \tan^2 \pi x$$

$$2\pi^2 \sec^2 \pi x(\sec^2 \pi x + 2 \tan^2 \pi x) \quad \sec^2 \pi x \tan \pi x)$$

$$2\pi^2 \sec^2 \pi x(3 \sec^2 \pi x - 2)$$

89. $h(x) = \frac{1}{9}(3x+1)^3, \left(1, \frac{64}{9}\right)$

$h'(x) = \frac{1}{9}3(3x+1)^2(3) = (3x+1)^2$

$h''(x) = 2(3x+1)(3) = 18x + 6$

$h''(1) = 24$

90. $f(x) = \frac{1}{\sqrt{x+4}} = (x+4)^{-1/2}, \left(0, \frac{1}{2}\right)$

$f'(x) = -\frac{1}{2}(x+4)^{-3/2}$

$f''(x) = \frac{3}{4}(x+4)^{-5/2} = \frac{3}{4(x+4)^{5/2}}$

$f''(0) = \frac{3}{128}$

91. $f(x) = \cos x^2, (0, 1)$

$f'(x) = -\sin(x^2)(2x) = -2x\sin(x^2)$

$f''(x) = -2x\cos(x^2)(2x) - 2\sin(x^2)$

$= -4x^2\cos(x^2) - 2\sin(x^2)$

$f''(0) = 0$

92. $g(t) = \tan 2t, \left(\frac{\pi}{6}, \sqrt{3}\right)$

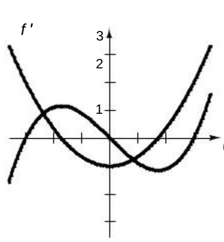
$g'(t) = 2\sec^2(2t)$

$g''(t) = 4\sec(2t)\left(\sec(2t)\tan(2t)\right)(2)$

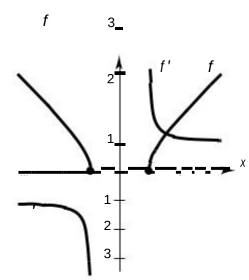
$= 8\sec^2(2t)\tan(2t)$

$g''\left(\frac{\pi}{6}\right) = 32\sqrt{3}$

93.



94.



f is decreasing on $(-\infty, -1)$ so f' must be negative there.

f is increasing on $(1, \infty)$ so f' must be positive there.

95. (a) $g(x) = f(3x)$

$g'(x) = f'(3x)(3) \Rightarrow g'(x) = 3f'(3x)$

The rate of change of g is three times as fast as the rate of change of f.

$g(x) = f(x^2)$

$g'(x) = f'(x^2)(2x) \Rightarrow g'(x) = 2xf'(x^2)$

The rate of change of g is $2x$ times as fast as the rate of change of f.

96. $r(x) = \frac{2x-5}{(3x+1)^2}$

If $h(x) = \frac{f(x)}{g(x)}$, then write $h(x) = f(x)(g(x))^{-1}$ and use the Product Rule.

$r(x) = (2x-5)(3x+1)^{-2}$

$r'(x) = (2x-5)(-2)(3x+1)^{-3}(3) + (3x+1)^{-2}(2)$

$= \frac{-6(2x-5) + 2(3x+1)}{(3x+1)^3}$

$= \frac{-6x+32}{(3x+1)^3}$

f

−2

−3

The zeros of f' correspond to the points where the graph of f has horizontal tangents.

$$r'(x) = \frac{(3x+1)^2(2) - (2x-5)(2)(3x+1)(3)}{(3x+1)^4}$$

$$= \frac{(3x+1)(2) - 6(2x-5)}{(3x+1)^3}$$

$$= \frac{-6x+32}{(3x+1)^3}$$

(d) Answers will vary.

97. (a) $g(x)=f(x)-2\Rightarrow g'(x)=f'(x)$

$h(x)=2f(x)\Rightarrow h'(x)=2f'(x)$

(c) $r(x)=f(-3x)\Rightarrow r'(x)=f'(-3x)(-3)=-3f'(-3x)$

So, you need to know $f'(-3x)$.

$r'(0)=-3f'(0)=(-3)\left(-\frac{1}{3}\right)=1$

$r'(-1)=-3f'(3)=(-3)(-4)=12$

(d) $s(x)=f(x+2)\Rightarrow s'(x)=f'(x+2)$

So, you need to know $f'(x+2)$.

$s'(-2)=f'(0)=-\frac{1}{3}$, etc.

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$h'(x)$		$\frac{4}{3}$	$-\frac{2}{3}$			
$r'(x)$						
$s'(x)$	$-\frac{1}{3}$	-1	-2	-4		

(a) $f(x)=g(x)h(x)$

$f'(x)=g(x)h'(x)+g'(x)h(x)$

$f'(5)=(-3)(-2)+(6)(3)=24$

$f(x)=g(h(x))$

$f'(x)=g'(h(x))h'(x)$

$f'(5)=g'(3)(-2)=-2g'(3)$

Not possible, you need $g'(3)$ to find $f'(5)$.

$f(x)=\frac{g(x)}{h(x)}$

$f'(x)=\frac{h(x)g'(x)-g(x)h'(x)}{[h(x)]^2}$

$f'(5)=\frac{(3)(6)-(-3)(-2)}{(3)^2}=\frac{12}{9}=\frac{4}{3}$

$f(x)=[g(x)]^3$

$f'(x)=3[g(x)]^2g'(x)$

$f'(5)=3(-3)^2(6)=162$

99. (a) $h(x)=f(g(x)),\ g(1)=4,\ g'(1)=-\frac{1}{2},\ f'(4)=-1$

$h'(x)=f'(g(x))g'(x)$

$h'(1)=f'(g(1))g'(1)=f'(4)g'(1)=(-1)\left(-\frac{1}{2}\right)=\frac{1}{2}$

$s(x)=g(f(x)),\ f(5)=6,\ f'(5)=-1,\ g'(6)$ does not exist. $s'(x)=g'(f(x))f'(x)$

$s'(5)=g'(f(5))f'(5)=g'(6)(-1)$

$s'(5)$ does not exist because g is not differentiable at 6.

(a) $h(x) = f(g(x))$

$h'(x) = f'(g(x))g'(x)$

$h'(3) = f'(g(3))g'(3) \quad f'(5)(1) = \frac{1}{2}$

$s(x) = g(f(x))$

$s'(x) = g'(f(x))f'(x)$

$s'(9) = g'(f(9))f'(9) \quad g'(8)(2) = (-1)(2) = -2$

(a) $F = 132{,}400(331 - v)^{-1}$

$F' = (-1)(132{,}400)(331 - v)^{-2}(-1) = \frac{132{,}400}{(331 - v)^2}$

When $v = 30$, $F' \approx 1.461$.

(b) $F = 132{,}400(331 + v)^{-1}$

$F' = (-1)(132{,}400)(331 + v)^{-2}(1) = \frac{-132{,}400}{(331 + v)^2}$

When $v = 30$, $F' \approx -1.016$.

102. $y = \frac{1}{3}\cos 12t - \frac{1}{4}\sin 12t$

$v = y' = \frac{1}{3}[-12\sin 12t] - \frac{1}{4}[12\cos 12t]$

$= -4\sin 12t - 3\cos 12t$

When $t = \pi/8$, $y = 0.25$ ft and $v = 4$ ft/sec.

$\theta = 0.2\cos 8t$

The maximum angular displacement is $\theta = 0.2$ (because $-1 \le \cos 8t \le 1$).

$\frac{d\theta}{dt} = 0.2[-8\sin 8t] = -1.6\sin 8t$

When $t = 3$, $d\theta/dt = -1.6\sin 24 \approx 1.4489$ rad/sec.

$y = A\cos\omega t$

Amplitude: $A = \frac{3.5}{2} = 1.75$

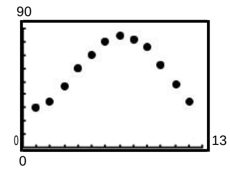
$y = 1.75\cos\omega t$

Period: 10 $\Rightarrow \omega = \frac{2\pi}{10} = \frac{\pi}{5}$

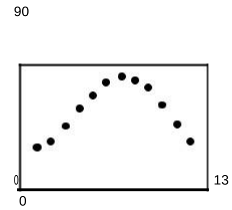
$y = 1.75\cos\frac{\pi t}{5}$

(b) $v = y' = 1.75\left[-\frac{\pi}{5}\sin\frac{\pi t}{5}\right] = -0.35\pi\sin\frac{\pi t}{5}$

105. (a) $T(t) = 27.3\sin(0.49t - 1.90) + 57.1$

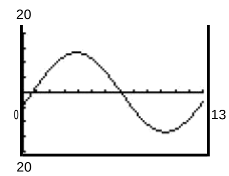


(b)



The model is a good fit.

$T'(t) = 13.377\cos(0.49t - 1.90)$



The temperature changes most rapidly around spring (March–May) and fall (Oct.–Nov.). The temperature changes most slowly around winter (Dec.–Feb.) and summer (Jun.–Aug.). Yes. Explanations will vary.

(a) According to the graph $C'(4) > C'(1)$.

Answers will vary.

107. $N = 400\left[1 - \frac{3}{(t^2 + 2)^2}\right] = 400 - 1200(t^2 + 2)^{-2}$

$N'(t) = 2400(t^2 + 2)^{-3}(2t) = \frac{4800t}{(t^2 + 2)^3}$

(a) $N'(0) = 0$ bacteria/day

(b) $N'(1) = \frac{4800(1)}{(1 + 2)^3} = \frac{4800}{27} \approx 177.8$ bacteria/day

(c) $N'(2) = \frac{4800(2)}{(4 + 2)^3} = \frac{9600}{216} \approx 44.4$ bacteria/day

$\frac{4800(3)}{} \quad \frac{14{,}400}{}$

(d) $N'(3) = (9 + 2)^3$

⌊ 5 5 ⌋ 5

$= 1331$ ≈ 10.8 bacteria/day

(e) $N'(4) = \frac{4800(4)}{(16+2)^3} = \frac{19{,}200}{5832} \approx 3.3$ bacteria/day

(f) The rate of change of the population is decreasing as $t \to \infty$.

108. (a) $V = \dfrac{k}{\sqrt{t+1}}$

$V(0) = 10{,}000 = \dfrac{k}{\sqrt{0+1}} = k$

$V = \dfrac{10{,}000}{\sqrt{t+1}} = 10{,}000(t+1)^{-1/2}$

(b) $\dfrac{dV}{dt} = 10{,}000\left(-\dfrac{1}{2}\right)(t+1)^{-3/2} = \dfrac{-5000}{(t+1)^{3/2}}$

$V'(1) = \dfrac{-5000}{2^{3/2}} \approx -1767.77$ dollars/year

(c) $V'(3) = \dfrac{-5000}{4^{3/2}} = \dfrac{-5000}{8} = -625$ dollars/year

$f(x) = \sin \beta x$

(a) $f'(x) = \beta \cos \beta x$

$f''(x) = -\beta^2 \sin \beta x$

$f'''(x) = -\beta^3 \cos \beta x$

$f^{(4)} = \beta^4 \sin \beta x$

$f''(x) + \beta^2 f(x) = -\beta^2 \sin \beta x + \beta^2(\sin \beta x) = 0$

(c) $f^{(2k)}(x) = (-1)^k \beta^{2k} \sin \beta x$

$^{(2k-1)}(x) = (-1)^{k+1}\beta^{2k-1}\cos \beta x$

110. (a) Yes, if $f(x+p) = f(x)$ for all x, then

$f'(x+p) = f'(x)$, which shows that f is periodic as well.

Yes, if $g(x) = f(2x)$, then $g'(x) = 2f'(2x)$. Because f' is periodic, so is g'.

(a) $r'(x) = f'(g(x))g'(x)$

$r'(1) = f'(g(1))g'(1)$

Note that $g(1) = 4$ and $f'(4) = \dfrac{5-0}{6-2} = \dfrac{5}{4}$.

112. (a) $g(x) = \sin^2 x + \cos^2 x = 1 \Rightarrow g'(x) = 0$

$g'(x) = 2 \sin x \cos x + 2 \cos x(-\sin x) = 0$

$\tan^2 x + 1 = \sec^2 x$

$g(x) + 1 = f(x)$

Taking derivatives of both sides, $g'(x) = f'(x)$. Equivalently,

$f'(x) = 2 \sec x \cdot \sec x \tan x = 2\sec^2 x \tan x$ and

$g'(x) = 2 \tan x \cdot \sec^2 x = 2 \sec^2 x \tan x$, which are the same.

113. (a) If $f(-x) = -f(x)$, then

$\dfrac{d}{dx}\left[f(-x)\right] = \dfrac{d}{dx}\left[-f(x)\right]$

$'(-x)(-1) = -f'(x)$

$f'(-x) = f'(x)$.

So, $f'(x)$ is even.

(b) If $f(-x) = f(x)$, then

$\dfrac{d}{dx}\left[f(-x)\right] = \dfrac{d}{dx}\left[f(x)\right]$

$f'(-x)(-1) = f'(x)$

$f'(-x) = -f'(x)$.

So, f' is odd.

114. $|u| = \sqrt{u^2}$

$\dfrac{d}{dx}\left[|u|\right] = \dfrac{d}{dx}\left[\sqrt{u^2}\right] = \dfrac{1}{2}(u^2)^{-1/2}(2uu')$

$= \dfrac{uu'}{\sqrt{u^2}} = u'\dfrac{u}{|u|}, u \neq 0$

$g'(x) = 3\left(\dfrac{3x-5}{|3x-5|}\right), x \neq \dfrac{5}{3}$

Also, $g'(1) = 0$. So, $r'(1) = 0$.

$s'(x) = g'(f(x))f'(x)$

$s'(4) = g'(f(4))f'(4)$

Note that $f(4) = \frac{5}{2}$, $g'\left(\frac{5}{2}\right) = \frac{6-4}{6-2} = \frac{1}{2}$ and

$f'(4) = \frac{5}{4}$. So, $s'(4) = \frac{1}{2}\left(\frac{5}{4}\right) = \frac{5}{8}$.

$f(x) = |x^2 - 9|$

$$f'(x) = 2x\left(\frac{x^2-9}{|x^2-9|}\right), \quad x \neq \pm 3$$

117. $h(x) = |x| \cos x$

$$h'(x) = -|x| \sin x + \frac{x}{|x|}\cos x, \quad x \neq 0$$

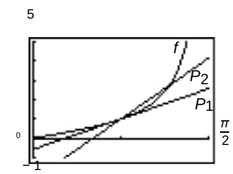
$f(x) = |\sin x|$

$$f'(x) = \cos x\left(\frac{\sin x}{|\sin x|}\right), \quad x \neq k\pi$$

119. (a)

$$f(x) = \tan x \qquad f(\pi/4) = 1$$
$$f'(x) = \sec^2 x \qquad f'(\pi/4) = 2$$
$$f''(x) = 2\sec^2 x \tan x \qquad f''(\pi/4) = 4$$
$$P_1(x) = 2(x - \pi/4) + 1$$
$$P_2(x) = \frac{1}{2}(4)(x - \pi/4)^2 + 2(x - \pi/4) + 1$$
$$= 2(x - \pi/4)^2 + 2(x - \pi/4) + 1$$

(b)



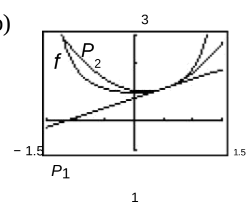
(c) P_2 is a better approximation than P_1.

(d) The accuracy worsens as you move away from $x = \pi/4$.

120. (a)

$$f(x) = \sec x \qquad f(\pi/6) = \frac{2}{\sqrt{3}}$$
$$f'(x) = \sec x \tan x \qquad f'(\pi/6) = \frac{2}{3}$$
$$f''(x) = \sec x(\sec^2 x) + \tan x(\sec x \tan x) \qquad f''(\pi/6) = \frac{10\sqrt{3}}{9}$$
$$= \sec^3 x + \sec x \tan^2 x$$
$$P_1(x) = \frac{2}{3}\left(x - \frac{\pi}{6}\right) + \frac{2}{\sqrt{3}}$$
$$P_2(x) = \frac{1}{2}\cdot\left(\frac{10}{3\sqrt{3}}\right)\left(x - \frac{\pi}{6}\right)^2 + \frac{2}{3}\left(x - \frac{\pi}{6}\right) + \frac{2}{\sqrt{3}}$$
$$= \left(\frac{5}{3\sqrt{3}}\right)\left(x - \frac{\pi}{6}\right)^2 + \frac{2}{3}\left(x - \frac{\pi}{6}\right) + \frac{2}{\sqrt{3}}$$

(b)



P_2 is a better approximation than P_1.

The accuracy worsens as you move away from $x = \pi/6$.

121. True

122. False. $f'(x) = -b\sin x$ and $f'(0) = 0$

123. True

124. True

125.

$$f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$$
$$f'(x) = a_1 \cos x + 2a_2 \cos 2x + \cdots + na_n \cos nx$$
$$f'(0) = a_1 + 2a_2 + \cdots + na_n$$

$$|a_1 + 2a_2 + \cdots + na_n| = |f'(0)| = \lim_{x\to 0}\left|\frac{f(x) - f(0)}{x - 0}\right| = \lim_{x\to 0}\left|\frac{f(x)}{\sin x}\right|\cdot\left|\frac{\sin x}{x}\right| = \lim_{x\to 0}\left|\frac{f(x)}{\sin x}\right| \le 1$$

126. $$\frac{d}{dx}\left[\frac{P_n(x)}{(x^k-1)^{n+1}}\right] = \frac{(x^k-1)^{n+1}P_n'(x) - P_n(x)(n+1)kx^{k-1}(x^k-1)^n}{(x^k-1)^{2n+2}} = \frac{(x^k-1)P_n'(x) - (n+1)kx^{k-1}P_n(x)}{(x^k-1)^{n+2}}$$

$$P_n(x) = (x^k-1)^{n+1}\frac{d^n}{dx^n}\left[\frac{1}{x^k-1}\right] \Rightarrow$$

$$P_{n+1}(x) = (x^k-1)^{n+2}\frac{d}{dx}\left[\frac{d^n}{dx^n}\left[\frac{1}{x^k-1}\right]\right] = (x^k-1)P_n'(x) - (n+1)kx^{k-1}P_n(x)$$

$$P_{n+1}(1) = -(n+1)kP_n(1)$$

For $n = 1$, $\frac{d}{dx}\left[\frac{1}{x^k-1}\right] = \frac{-kx^{k-1}}{(x^k-1)^2} = \frac{P_1(x)}{(x^k-1)^2} \Rightarrow P_1(1) = -k$. Also, $P_0(1) = 1$.

You now use mathematical induction to verify that $P_n(1) = (-k)^n n!$ for $n \geq 0$. Assume true for n. Then

$$P_{n+1}(1) = -(n+1)kP_n(1) = -(n+1)k(-k)^n n! = (-k)^{n+1}(n+1)!.$$

Section 2.5 Implicit Differentiation

Answers will vary. *Sample answer:* In the explicit form of a function, the variable is explicitly written as a function of x. In an implicit equation, the function is only implied by an equation. An example of an implicit function is $x^2 + xy = 5$. In explicit form it would be $y = (5 - x^2)/x$.

Answers will vary. *Sample answer:* Given an implicit equation, first differentiate both sides with respect to x. Collect all terms involving y' on the left, and all other terms to the right. Factor out y' on the left side. Finally, divide both sides by the left-hand factor that does not contain y'.

You use implicit differentiation to find the derivative y' when it is difficult to express y explicitly as a function of x.

If y is an implicit function of x, then to compute y', you differentiate the equation with respect to x. For example, if $xy^2 = 1$, then $y^2 + 2xyy' = 0$. Here, the derivative of y^2 is $2yy'$.

$$x^2 + y^2 = 9$$
$$2x + 2yy' = 0$$
$$y' = -\frac{x}{y}$$

$$x^5 + y^5 = 16$$
$$5x^4 + 5y^4y' = 0$$
$$5y^4y' = -5x^4$$
$$y' = -\frac{x^4}{y^4}$$

$$2x^3 + 3y^3 = 64$$
$$6x^2 + 9y^2y' = 0$$
$$9y^2y' = -6x^2$$
$$y' = \frac{-6x^2}{9y^2} = -\frac{2x^2}{3y^2}$$

$$x^3 - xy + y^2 = 7$$
$$3x^2 - xy' - y + 2yy' = 0$$
$$(2y - x)y' = y - 3x^2$$
$$y' = \frac{y - 3x^2}{2y - x}$$

$$x^2y + y^2x = -2$$
$$x^2y' + 2xy + y^2 + 2yxy' = 0$$

$$x^2 - y^2 = 25$$
$$2x - 2yy' = 0$$
$$y' = \frac{x}{y}$$

$x^2 + 2xy)y' = -(y^2 + 2xy)$

$x(x + 2y)$

$y' = -\frac{y(y+2x)}{}$

$x^3y^3 - y - x = 0$ $3x^3y^2y' + 3x^2y^3 - y' - 1 = 0$

$(3x^3y^2 - 1)y' = 1 - 3x^2y^3$

$$y' = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}$$

12. $\sqrt{xy} = x^2y + 1$

$\frac{1}{2}(xy)^{-1/2}(xy' + y) = 2xy + x^2y'$

$\frac{x}{2\sqrt{xy}}y' + \frac{y}{2\sqrt{xy}} = 2xy + x^2y'$

$\left(\frac{x}{2\sqrt{xy}} - x^2\right)y' = 2xy - \frac{y}{2\sqrt{xy}}$

$y' = \frac{2xy - \frac{y}{2\sqrt{xy}}}{\frac{x}{2\sqrt{xy}} - x^2}$

$y' = \frac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$

14. $x^4y - 8xy + 3xy^2 = 9$

$x^4y' + 4x^3y - 8xy' - 8y + 6xyy' + 3y^2 = 0$

$(x^4 - 8x + 6xy)y' = 8y - 4x^3y - 3y^2$

$y' = \frac{8y - 4x^3y - 3y^2}{x^4 - 8x + 6xy}$

$\sin x + 2\cos 2y = 1$

$\cos x - 4(\sin 2y)y' = 0$

$y' = \frac{\cos x}{4\sin 2y}$

16. $(\sin \pi x + \cos \pi y)^2 = 2$

$2(\sin \pi x + \cos \pi y)[\pi \cos \pi x - \pi (\sin \pi y) y'] = 0$

$\pi \cos \pi x - \pi(\sin \pi y)y' = 0$

$y' = \frac{\cos \pi x}{\sin \pi y}$

$\csc x = x(1 + \tan y)$

$\csc x \cot x = (1 + \tan y) + x(\sec^2 y)y'$

$y' = -\frac{\csc x \cot x \pm 1 \pm \tan y}{x\sec^2 y}$

$\cot y = x - y$

$(-\csc^2 y)y' = 1 - y'$

$y' = \frac{1}{1 - \csc^2 y} = \frac{1}{-\cot^2 y} = -\tan^2 y$

$y = \sin xy$

$x^3 - 3x^2y + 2xy^2 = 12$

$3x^2 - 3x^2y' - 6xy + 4xyy' + 2y^2 = 0$

$(4xy - 3x^2)y' = 6xy - 3x^2 - 2y^2$

$y' = \frac{6xy - 3x^2 - 2y^2}{4xy - 3x^2}$

$x = \sec \frac{1}{y}$

$1 = -\frac{y'}{y^2}\sec\frac{1}{y}\tan\frac{1}{y}$

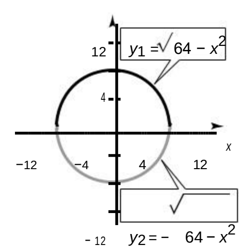
$y' = \frac{-y^2}{\sec(1/y)\tan(1/y)} = -y^2\cos\left(\frac{1}{y}\right)\cot\left(\frac{1}{y}\right)$

(a) $x^2 + y^2 = 64$

$y^2 = 64 - x^2$

$y = \pm\sqrt{64 - x^2}$

(b)



(c) Explicitly:

$\frac{dy}{dx} = \pm\frac{1}{2}(64 - x^2)^{-1/2}(-2x) = \mp\frac{x}{\sqrt{64 - x^2}}$

$= \frac{-x}{\pm\sqrt{64 - x^2}} = -\frac{x}{y}$

$$y' = [xy' + y]\cos(xy)$$

$$y' - x\cos(xy)\,y' = y\cos(xy)$$

$$y = \frac{y\cos(xy)}{1 - x\cos(xy)}$$

Implicitly: $2x + 2yy' = 0$ $y' = -\frac{x}{y}$

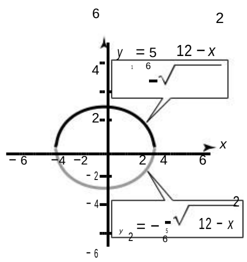
(a) $25x^2 + 36y^2 = 300$

$$36y^2 = 300 - 25x^2 = 25(12 - x^2)$$

$$y^2 = \frac{25}{36}(12 - x^2)$$

$$y = \pm\frac{5}{6}\sqrt{12 - x^2}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{5}{6}\left(\frac{1}{2}\right)(12 - x^2)^{-1/2}(-2x)$$

$$= \frac{-5x}{6\sqrt{12 - x^2}}$$

$$= -\frac{25x}{36y}$$

(d) Implicitly: $50x + 72y \cdot y' = 0$

$$y' = \frac{-50x}{72y} = -\frac{25x}{36y}$$

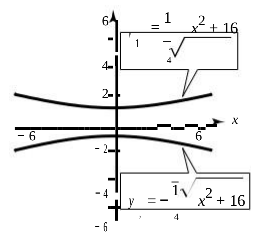
(a) $16y^2 - x^2 = 16$

$$16y^2 = x^2 + 16$$

$$y^2 = \frac{x^2}{16} + 1 = \frac{x^2 + 16}{16}$$

$$y = \frac{\pm\sqrt{x^2 + 16}}{4}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \frac{\pm\frac{1}{2}(x^2 + 16)^{-1/2}(2x)}{4}$$

$$= \frac{\pm x}{4\sqrt{x^2 + 16}} = \frac{\pm x}{4(\pm 4y)} = \frac{x}{16y}$$

Implicitly: $16y^2 - x^2 = 16$
$32yy' - 2x = 0$
$32yy' = 2x$

$$y' = \frac{2x}{32y} = \frac{x}{16y}$$

24. (a) $x^2 + y^2 - 4x + 6y + 9 = 0$

$$(x^2 - 4x + 4) + (y^2 + 6y + 9) = -9 + 4 + 9$$

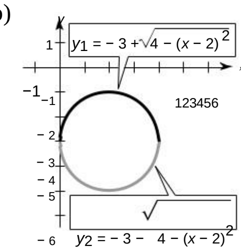
$$(x - 2)^2 + (y + 3)^2 = 4$$

$$(y + 3)^2 = 4 - (x - 2)^2$$

$$y + 3 = \pm\sqrt{4 - (x - 2)^2}$$

$$= -3 \pm \sqrt{4 - (x - 2)^2}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{1}{2}\left[4 - (x - 2)^2\right]^{-1/2}\left[-2(x - 2)\right]$$

$$= \mp\frac{x - 2}{\sqrt{4 - (x - 2)^2}}$$

$$= -\frac{x - 2}{y + 3}$$

(d) Implicitly:

$$2x + 2yy' - 4 + 6y' = 0$$

$$2yy' + 6y' = -2x + 4$$

$$y'(2y + 6) = -2(x - 2)$$

$$y' = \frac{-2(x - 2)}{2(y + 3)} = -\frac{x - 2}{y + 3}$$

$xy = 6$

$xy' + y(1) = 0$

$xy' = -y$

$y' = -\frac{y}{x}$

At $(-6, -1)$: $y' = -\frac{1}{6}$

$3x^3 y = 6 \quad x^3 y = 2$

$3x^2 y + x^3 y' = 0$

$x^3 y' = -3x^2 y$

$y' = \frac{-3x^2 y}{x^3} = -\frac{3y}{x}$

At $(1, 2)$: $y' = \frac{-3(2)}{1} = -6$

$$y^2 = \frac{x^2 - 49}{x^2 + 49}$$

$$2yy' = \frac{(x^2 + 49)(2x) - (x^2 - 49)(2x)}{(x^2 + 49)^2}$$

$$2yy' = \frac{196x}{(x^2 + 49)^2}$$

$$y' = \frac{98x}{y(x^2 + 49)^2}$$

At $(7, 0)$: y' is undefined.

$$4y^3 = \frac{x^2 - 36}{x^3 + 36}$$

$$\frac{(x^3 + 36)(2x) - (x^2 - 36)3x^2}{(x^3 + 36)^2}$$

$$y' = \frac{72x + 108x^2 - x^4}{12y^2(x^3 + 36)^2}$$

At $(6, 0)$: y' is undefined (division by 0).

29. $(x + y)^3 = x^3 + y^3$

$x^3 + 3x^2 y + 3xy^2 + y^3 = x^3 + y^3$

$3x^2 y + 3xy^2 = 0$

$x^2 y + xy^2 = 0$

$x^2 y' + 2xy + 2xyy' + y^2 = 0$

$(x^2 + 2xy)y' = -(y^2 + 2xy)$

$$y' = -\frac{y(y + 2x)}{x(x + 2y)}$$

$x^3 + y^3 = 6xy - 1$

$3x^2 + 3y^2 y' = 6xy' + 6y$

$(3y^2 - 6x)y' = 6y - 3x^2$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x}$$

At $(2, 3)$: $y' = \frac{18 - 12}{27 - 12} = \frac{6}{15} = \frac{2}{5}$

$\tan(x + y) = x$

$(1 + y')\sec^2(x + y) = 1$

$$y' = \frac{1 - \sec^2(x + y)}{\sec^2(x + y)}$$

$$= \frac{-\tan^2(x + y)}{\tan^2(x + y) + 1}$$

$$= -\sin^2(x + y)$$

$$= -\frac{x^2}{x^2 + 1}$$

At $(0, 0)$: $y' = 0$

$x \cos y = 1$

$x[-y' \sin y] + \cos y = 0$

$$y' = \frac{\cos y}{x \sin y}$$

$$= \frac{1}{x}\cot y$$

$$= \frac{\cot y}{x}$$

At $\left(2, \frac{\pi}{3}\right)$: $y' = \frac{1}{2\sqrt{3}}$

33. $(x^2 + 4)y = 8$

$(x^2 + 4)y' + y(2x) = 0$

$$y' = \frac{-2xy}{x^2 + 4}$$

$$= \frac{-2x[8/(x^2 + 4)]}{x^2 + 4}$$

$$= \frac{-16x}{(x^2 + 4)^2}$$

At $(2, 1)$: $y' = \frac{-32}{64} = -\frac{1}{2}$

$\left(\quad \frac{8}{\quad} \right)$

At $(-1, 1): y' = -1$

$\left(\text{Or, you could just solve for } y: y = \qquad x \quad + 4 \right)$

34. $(4 - x)y^2 = x^3$

$(4 - x)(2yy') + y^2(-1) = 3x^2$

$y' = \dfrac{3x^2 + y^2}{2y(4 - x)}$

At $(2, 2)$: $y' = 2$

35. $(x^2 + y^2)^2 = 4x^2y$

$2(x^2 + y^2)(2x + 2yy') = 4x^2y' + y(8x)$

$4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 4x^2y' + 8xy$

$4x^2yy' + 4y^3y' - 4x^2y' = 8xy - 4x^3 - 4xy^2$

$4y'(x^2y + y^3 - x^2) = 4(2xy - x^3 - xy^2)$

$y' = \dfrac{2xy - x^3 - xy^2}{x^2y + y^3 - x^2}$

At $(1, 1)$: $y' = 0$

$x^3 + y^3 - 6xy = 0$
$3x^2 + 3y^2y' - 6xy' - 6y = 0$

$y'(3y^2 - 6x) = 6y - 3x^2$

$y' = \dfrac{6y - 3x^2}{3y^2 - 6x} = \dfrac{2y - x^2}{y^2 - 2x}$

At $\left(\dfrac{4}{3}, \dfrac{8}{3}\right)$: $y' = \dfrac{(16/3) - (16/9)}{(64/9) - (8/3)} = \dfrac{32}{40} = \dfrac{4}{5}$

37. $(y - 3)^2 = 4(x - 5)$, $(6, 1)$

$2(y - 3)y' = 4$

$y' = \dfrac{2}{y - 3}$

At $(6, 1)$: $y' = \dfrac{2}{1 - 3} = -1$

Tangent line: $y - 1 = -1(x - 6)$

$y = -x + 7$

38. $(x + 2)^2 + (y - 3)^2 = 37$, $(4, 4)$

39. $x^2y^2 - 9x^2 - 4y^2 = 0$, $(-4, 2\sqrt{3})$

$x^2 2yy' + 2xy^2 - 18x - 8yy' = 0$

$y' = \dfrac{18x - 2xy^2}{2x^2y - 8y}$

At $(-4, 2\sqrt{3})$: $y' = \dfrac{18(-4) - 2(-4)(12)}{2(16)(2\sqrt{3}) - 16\sqrt{3}}$

$= \dfrac{24}{48\sqrt{3}} = \dfrac{1}{2\sqrt{3}} = \dfrac{\sqrt{3}}{6}$

Tangent line: $y - 2\sqrt{3} = \dfrac{\sqrt{3}}{6}(x + 4)$

$y = \dfrac{\sqrt{3}}{6}x + \dfrac{8}{3}\sqrt{3}$

40. $x^{2/3} + y^{2/3} = 5$, $(8, 1)$

$\dfrac{2}{3}x^{-1/3} + \dfrac{2}{3}y^{-1/3}y' = 0$

$y' = \dfrac{-x^{-1/3}}{y^{-1/3}} = -\left(\dfrac{y}{x}\right)^{1/3}$

At $(8, 1)$: $y' = -\dfrac{1}{2}$

Tangent line: $y - 1 = -\dfrac{1}{2}(x - 8)$

$y = -\dfrac{1}{2}x + 5$

41. $3(x^2 + y^2)^2 = 100(x^2 - y^2)$, $(4, 2)$

$6(x^2 + y^2)(2x + 2yy') = 100(2x - 2yy')$

At $(4, 2)$:

$6(16 + 4)(8 + 4y') = 100(8 - 4y')$

$960 + 480y' = 800 - 400y'$

$880y' = -160$

$y' = -\dfrac{2}{11}$

$2(x + 2) + 2(y - 3)y' = 0$

$y - 3)y' = -(x + 2)$

$$y' = -\frac{(x + 2)}{y - 3}$$

At $(4, 4)$: $y' = -\frac{6}{1} = -6$

Tangent line: $y - 4 = -6(x - 4)$

$$y = -6x + 28$$

Tangent line: $y - 2 = -\frac{2}{11}(x - 4)$

$$11y + 2x - 30 = 0$$

$$y = -\frac{2}{11}x + \frac{30}{11}$$

42. $y^2(x^2 + y^2) = 2x^2$, $(1, 1)$

$$y^2x^2 + y^4 = 2x^2$$
$$2yy'x^2 + 2xy^2 + 4y^3y' = 4x$$

At $(1, 1)$:

$$2y' + 2 + 4y' = 4$$
$$6y' = 2$$
$$y' = \frac{1}{3}$$

Tangent line: $y - 1 = \frac{1}{3}(x - 1)$

$$= \frac{1}{3}x + \frac{2}{3}$$

Answers will vary. *Sample answers*:

$$xy = 2 \Rightarrow y = \frac{2}{x}$$
$$yx^2 + x = 2 \Rightarrow y = \frac{2 - x}{x^2}$$
$$x^2 + y^2 = 4$$
$$xy + y^2 = 2$$

The equation $x^2 + y^2 + 2 = 1$ implies $x^2 + y^2 = -1$, which has no real solutions.

45. (a) $\frac{x^2}{2} + \frac{y^2}{8} = 1$, $(1, 2)$

$$x + \frac{yy'}{4} = 0$$
$$y' = -\frac{4x}{y}$$

At $(1, 2)$: $y' = -2$

Tangent line: $y - 2 = -2(x - 1)$

$$= -2x + 4$$

(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \Rightarrow y' = \frac{-b^2x}{a^2y}$

$y - y_0 = \frac{-b^2x_0}{a^2y_0}(x - x_0)$, Tangent line at (x_0, y_0)

$$\frac{y_0y}{b^2} - \frac{y_0^2}{b^2} = -\frac{x_0x}{a^2} + \frac{x_0^2}{a^2}$$

Because $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, you have $\frac{y_0y}{b^2} + \frac{x_0x}{a^2} = 1$.

46. (a) $\frac{x^2}{6} - \frac{y^2}{8} = 1$, $(3, -2)$

$$\frac{x}{3} - \frac{y}{4}y' = 0$$
$$\frac{y}{4}y' = \frac{x}{3}$$
$$y' = \frac{4x}{3y}$$

At $(3, -2)$: $y' = \frac{4(3)}{3(-2)} = -2$

Tangent line: $y + 2 = -2(x - 3)$

$$= -2x + 4$$

(b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \Rightarrow y' = \frac{xb^2}{ya^2}$

$y - y_0 = \frac{x_0b^2}{y_0a^2}(x - x_0)$, Tangent line at (x_0, y_0)

$$\frac{yy_0}{b^2} - \frac{y_0^2}{b^2} = \frac{x_0x}{a^2} - \frac{x_0^2}{a^2}$$

Because $\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$, you have $\frac{x_0x}{a^2} - \frac{yy_0}{b^2} = 1$.

Note: From part (a),

$\frac{3x}{6} - \frac{(-2)y}{8} = 1 \Rightarrow \frac{1}{2}x + \frac{y}{4} = 1 \Rightarrow y = -2x + 4$, Tangent line.

$$\tan y = x$$
$$y' \sec^2 y = 1$$
$$y' = \frac{1}{\sec^2 y} = \cos^2 y, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$
$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$
$$y' = \frac{1}{1 + x^2}$$

$$\cos y = x$$
$$-\sin y \cdot y' = 1$$
$$y' = \frac{-1}{\sin y}, \quad 0 < y < \pi$$
$$\sin^2 y + \cos^2 y = 1$$
$$\sin^2 y = 1 - \cos^2 y$$

a^2 b^2 b^2 a^2

Note: From part (a),

$$\frac{1x}{2} + \frac{2y}{8} = 1 \Rightarrow \frac{1}{4}y = -\frac{1}{2}x + 1 \Rightarrow y = -2x + 4,$$

Tangent line.

$$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$$

$$y' = \frac{-1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$$

49. $x^2 + y^2 = 4$

$2x + 2yy' = 0$

$y' = \dfrac{-x}{y}$

$y'' = \dfrac{y(-1) + xy'}{y^2}$

$= \dfrac{-y + x(-x/y)}{y^2}$

$= \dfrac{-y^2 - x^2}{y^3}$

$= -\dfrac{4}{y^3}$

50. $x^2y - 4x = 5$

$x^2y' + 2xy - 4 = 0$

$y' = \dfrac{4 - 2xy}{x^2}$

$x^2y'' + 2xy' + 2xy' + 2y = 0$

$x^2y'' + 4x\left[\dfrac{4 - 2xy}{x^2}\right] + 2y = 0$

$x^4y'' + 4x(4 - 2xy) + 2x^2y = 0$

$x^4y'' + 16x - 8x^2y + 2x^2y = 0$

$x^4y'' = 6x^2y - 16x$

$y'' = \dfrac{6xy - 16}{x^3}$

51. $x^2y - 2 = 5x + y$

$2xy + x^2y' = 5 + y'$

$(x^2 - 1)y' = 5 - 2xy$

$y' = \dfrac{5 - 2xy}{x^2 - 1}$

$2xy' + (x^2 - 1)y'' = -2y - 2xy'$

$(x^2 - 1)y'' = -2y - 4xy' = -2y - 4x\left(\dfrac{5 - 2xy}{x^2 - 1}\right)$

$y'' = \dfrac{-2y}{x^2 - 1} - \dfrac{4x(5 - 2xy)}{(x^2 - 1)^2}$

$= \dfrac{-2y(x^2 - 1) - 20x + 8x^2y}{(x^2 - 1)^2} = \dfrac{6x^2y - 20x + 2y}{(x^2 - 1)^2}$

$xy - 1 = 2x + y^2$

$xy' + y = 2 + 2yy'$

$xy' - 2yy' = 2 - y$

$(x - 2y)y' = 2 - y$

$y' = \dfrac{2 - y}{x - 2y}$

$xy'' + y' + y' = 2yy'' + 2(y')^2$

$xy'' - 2yy'' = 2(y')^2 - 2y'$

$(x - 2y)y'' = 2(y')^2 - 2y' = 2\left(\dfrac{2 - y}{x - 2y}\right)^2 - 2\left(\dfrac{2 - y}{x - 2y}\right)$

$y'' = \dfrac{2(2 - y)\left[2 - y - (x - 2y)\right]}{(x - 2y)^3} \quad \dfrac{2 \cdots 2y}{(x - 2y)^3}$

$= \dfrac{2(4 - 2x + 2y - 2y + xy - y^2)}{} = \dfrac{2(y^2 - xy + 2x - 4)}{}$

$(\quad) \quad (x-2y)^3 \qquad\qquad (2y-x)^3$

$2 \quad -5$

$= (2y-x)^3 \quad = (x-2y)^3$

$7xy + \sin x = 2$

$7xy' + 7y + \cos x = 0$

$y' = \dfrac{-7y - \cos x}{7x}$

$7xy'' + 7y' + 7y' - \sin x = 0$

$7xy'' = \sin x - 14y' = \sin x - 14\left(\dfrac{-7y - \cos x}{7x}\right)$

$7xy'' = \sin x + \dfrac{14y + 2\cos x}{x}$

$y'' = \dfrac{\sin x}{7x} + \dfrac{14y + 2\cos x}{7x^2}$

$y'' = \dfrac{x\sin x + 14y + 2\cos x}{7x^2}$

$3xy - 4\cos x = -6 \quad 3xy' + 3y + 4\sin x = 0$

$y' = \dfrac{-4\sin x - 3y}{3x}$

$3xy'' + 3y' + 3y' + 4\cos x = 0$

$3xy'' = -6y' - 4\cos x = -6\left(\dfrac{-4\sin x - 3y}{3x}\right) - 4\cos x$

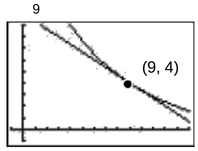
$= \dfrac{8\sin x + 6y - 4x\cos x}{x}$

$y'' = \dfrac{8\sin x + 6y - 4x\cos x}{3x^2}$

55. $\sqrt{x} + \sqrt{y} = 5$

$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$

$y' = \dfrac{-\sqrt{y}}{\sqrt{x}}$



At $(9, 4)$: $y' = -\dfrac{2}{3}$

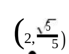
Tangent line: $y - 4 = -\dfrac{2}{3}(x - 9)$

$2x + 3y - 30 = 0$

56. $y^2 = \dfrac{x-1}{x^2+1}$

$2yy' = \dfrac{(x^2+1)(1) - (x-1)(2x)}{(x^2+1)^2} = \dfrac{x^2 + 1 - 2x^2 + 2x}{(x^2+1)^2}$

$y' = \dfrac{1 + 2x - x^2}{2y(x^2+1)^2}$



At $\left(2, \dfrac{\sqrt{5}}{5}\right)$: $y' = \dfrac{1 + 4 - 4}{2\left(\dfrac{\sqrt{5}}{5}\right)(4+1)^2} = \dfrac{1}{10\sqrt{5}}$

Tangent line: $y - \dfrac{\sqrt{5}}{5} = \dfrac{1}{10\sqrt{5}}(x - 2)$

$$10\sqrt{5}y - 10 = x - 2$$
$$x - 10\sqrt{5}y + 8 = 0$$

$x^2 + y^2 = 25$ $2x$

$+ 2yy' = 0$

$$y' = -\frac{x}{y}$$

At $(4, 3)$:

Tangent line:

$y - 3 = -\frac{4}{3}(x - 4) \Rightarrow 4x + 3y - 25 = 0$ Normal

line: $y - 3 = \frac{3}{4}(x - 4) \Rightarrow 3x - 4y = 0$

6

(4, 3)

−9 9

6

At $(-3, 4)$:

Tangent line:

$y - 4 = \frac{3}{4}(x + 3) \Rightarrow 3x - 4y + 25 = 0$ Normal

line: $y - 4 = -\frac{4}{3}(x + 3) \Rightarrow 4x + 3y = 0$

6

(−3, 4)

−9 9

6

$x^2 + y^2 = 36$ $2x$

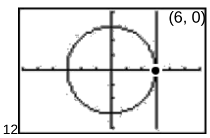
$+ 2yy' = 0$

$$y' = -\frac{x}{y}$$

At $(6, 0)$; slope is undefined.

Tangent line: $x = 6$

Normal line: $y = 0$



At $(5, \sqrt{11})$, slope is $\frac{-5}{\sqrt{11}}$.

Tangent line: $y - \sqrt{11} = \frac{-5}{\sqrt{11}}(x - 5)$

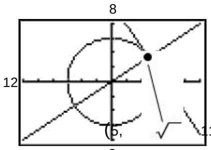
$$\sqrt{11}y - 11 = -5x + 25$$

$$5x + \sqrt{11}y - 36 = 0$$

Normal line: $y - \sqrt{11} = \frac{\sqrt{11}}{5}(x - 5)$

$$5y - 5\sqrt{11} = \sqrt{11}x - 5\sqrt{11}$$

$$5y - \sqrt{11}x = 0$$



$x^2 + y^2 = r^2$ $2x$

$+ 2yy' = 0$

$y' = -\frac{x}{y} =$ slope of tangent line

$\frac{y}{x} =$ slope of normal line

Let (x_0, y_0) be a point on the circle. If $x_0 = 0$, then the tangent line is horizontal, the normal line is vertical and, hence, passes through the origin. If $x_0 \neq 0$, then the equation of the normal line is

$$y - y_0 = \frac{y_0}{x_0}(x - x_0)$$

$$y = \frac{y_0}{x_0}x$$

which passes through the origin.

$$y^2 = 4x$$
$$2yy' = 4$$
$$y' = \frac{2}{y} = 1 \text{ at } (1, 2)$$

Equation of normal line at $(1, 2)$ is $y - 2 = -1(x - 1)$, $y = 3 - x$. The centers of the circles must be on the normal line and at a distance of 4 units from $(1, 2)$. Therefore,

$$(x - 1)^2 + [(3 - x) - 2]^2 = 16$$
$$2(x - 1)^2 = 16$$
$$x = 1 \pm 2\sqrt{2}.$$

Centers of the circles: $(1 + 2\sqrt{2}, 2 - 2\sqrt{2})$ and $(1 - 2\sqrt{2}, 2 + 2\sqrt{2})$

Equations: $(x - 1 - 2\sqrt{2})^2 + (y - 2 + 2\sqrt{2})^2 = 16$

$(x - 1 + 2\sqrt{2})^2 + (y - 2 - 2\sqrt{2})^2 = 16$

$$4x^2 + y^2 - 8x + 4y + 4 = 0$$
$$8x + 2yy' - 8 + 4y' = 0$$
$$y' = \frac{8 - 8x}{2y + 4} = \frac{4 - 4x}{y + 2}$$

Horizontal tangents occur when $x = 1$:

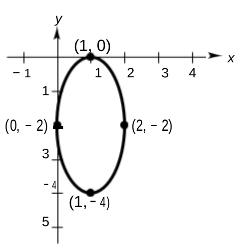
$$4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$$
$$y^2 + 4y = y(y + 4) = 0 \Rightarrow y = 0, -4$$

Horizontal tangents: $(1, 0)$, $(1, -4)$

Vertical tangents occur when $y = -2$:

$$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$$
$$4x^2 - 8x = 4x(x - 2) = 0 \Rightarrow x = 0, 2$$

Vertical tangents: $(0, -2)$, $(2, -2)$



$$25x^2 + 16y^2 + 200x - 160y + 400 = 0$$
$$50x + 32yy' + 200 - 160y' = 0$$
$$y' = \frac{200 + 50x}{160 - 32y}$$

Horizontal tangents occur when $x = -4$:

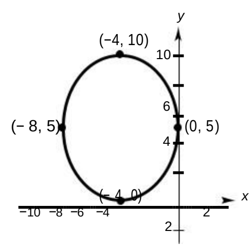
$$25(16) + 16y^2 + 200(-4) - 160y + 400 = 0$$
$$y(y - 10) = 0 \Rightarrow y = 0, 10$$

Horizontal tangents: $(-4, 0)$, $(-4, 10)$

Vertical tangents occur when $y = 5$:

$$25x^2 + 400 + 200x - 800 + 400 = 0$$
$$25x(x + 8) = 0 \Rightarrow x = 0, -8$$

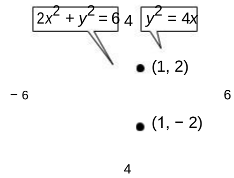
Vertical tangents: $(0, 5)$, $(-8, 5)$



63. Find the points of intersection by letting $y^2 = 4x$ in the equation $2x^2 + y^2 = 6$.

$2x^2 + 4x = 6$ and $(x + 3)(x - 1) = 0$

The curves intersect at $(1, \pm 2)$.



Ellipse:	*Parabola*:
$4x + 2yy' = 0$	$2yy' = 4$
$y' = -\frac{2x}{y}$	$y' = \frac{2}{y}$

At $(1, 2)$, the slopes are:

$y' = -1$	$y' = 1$

At $(1, -2)$, the slopes are:

$y' = 1$	$y' = -1$

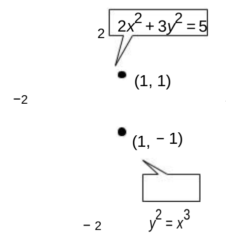
Tangents are perpendicular.

Find the points of intersection by letting $y^2 = x^3$ in the equation $2x^2 + 3y^2 = 5$.

$2x^2 + 3x^3 = 5$ and $3x^3 + 2x^2 - 5 = 0$

Intersect when $x = 1$.

Points of intersection: $(1, \pm 1)$



$y^2 = x^3$:	$2x^2 + 3y^2 = 5$:
$2yy' = 3x^2$	$4x + 6yy' = 0$
$y' = \frac{3x^2}{2y}$	$y' = -\frac{2x}{3y}$

At $(1, 1)$, the slopes are:

$y' = \frac{3}{2}$	$y' = -\frac{2}{3}$

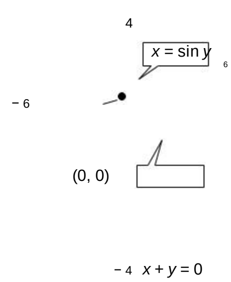
At $(1, -1)$, the slopes are:

$y' = -\frac{3}{2}$	$y' = \frac{2}{3}$

Tangents are perpendicular.

$y = -x$ and $x = \sin y$

Point of intersection: $(0, 0)$



$y = -x$:	$x = \sin y$:
$y' = -1$	$1 = y' \cos y$
	$y' = \sec y$

At $(0, 0)$, the slopes are:

$y' = -1$	$y' = 1$

Tangents are perpendicular.

Rewriting each equation and differentiating:

$$x^3 = 3(y - 1) \qquad x(3y - 29) = 3$$

$$y = \frac{x^3}{3} + 1 \qquad y = \frac{1}{3}\left(\frac{3}{x} + 29\right)$$

$$y' = x^2 \qquad y' = -\frac{1}{x^2}.$$

$x(3y - 29) = 3$ 15 $x^3 = 3y - 3$

−15 12

3

For each value of x, the derivatives are negative reciprocals of each other.
So, the tangent lines are orthogonal at both points of intersection.

67. $xy = C \qquad x^2 - y^2 = K$

$$xy' + y = 0 \qquad 2x - 2yy' = 0$$

$$y' = -\frac{y}{x} \qquad y' = \frac{x}{y}$$

At any point of intersection (x, y) the product of the slopes is $(-y/x)(x/y) = -1$. The curves are orthogonal.

2

−3 3

$C = 1$

$= -1$

2

2

$C = 4$

−3 3

$K = 2$

2

(a) The slope is greater at $x = -3$.

The graph has vertical tangent lines at about $(-2, 3)$ and $(2, 3)$.

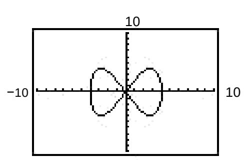
The graph has a horizontal tangent line at about $(0, 6)$.

(a) $x^4 = 4(4x^2 - y^2)$

$$4y^2 = 16x^2 - x^4$$

$$y^2 = 4x^2 - \frac{1}{4}x^4$$

$$y = \pm\sqrt{4x^2 - \frac{1}{4}x^4}$$



68. $x^2 + y^2 = C^2 \qquad y = Kx$

$$2x + 2yy' = 0 \qquad y' = K$$

$$y' = -\frac{x}{y}$$

At the point of intersection (x, y), the product of the slopes is $(-x/y)(K) = (-x/Kx)(K) = -1$. The curves *are orthogonal.*

2

$K = 1$

−3 3

$C = 2$

2

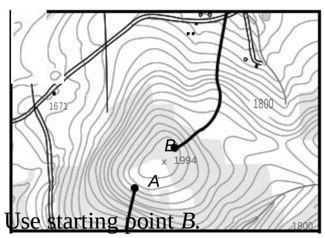
2

$K = -1$

$C = 1$

−3 3

2



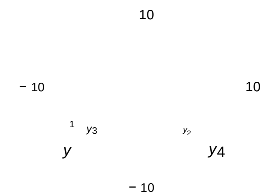
Use starting point B.

$y = 3 \Rightarrow 9 = 4x^2 - \frac{1}{4}x^4$

$36 = 16x^2 - x^4$

$x^4 - 16x^2 + 36 = 0$

$x^2 = \frac{16 \pm \sqrt{256 - 144}}{2}$



Note that $x^2 = 8 \pm \sqrt{28} = 8 \pm 2\sqrt{7} = \left(1 \pm \sqrt{7}\right)^2$. So, there are four values of x:

$-1 - \sqrt{7},\ 1 - \sqrt{7},\ -1 + \sqrt{7},\ 1 + \sqrt{7}$

To find the slope, $2yy' = 8x - x^3 \Rightarrow y' = \frac{x\left(8 - x^2\right)}{(\ )}$.

For $x = -1 - \sqrt{7}$, $y' = \frac{1}{3}\left(\sqrt{7} + 7\right)$, and the line is

$y_1 = \frac{1}{3}\left(\sqrt{7} + 7\right)\left(x + 1 + \sqrt{7}\right) + 3 = \frac{1}{3}\left[\left(\sqrt{7} + 7\right)x + 8\sqrt{7} + 23\right]$.

For $x = 1 - \sqrt{7}$, $y' = \frac{1}{3}\left(\sqrt{7} - 7\right)$, and the line is

$y_2 = \frac{1}{3}\left(\sqrt{7} - 7\right)\left(x - 1 + \sqrt{7}\right) + 3 = \frac{1}{3}\left[\left(\sqrt{7} - 7\right)x + 23 - 8\sqrt{7}\right]$.

For $x = -1 + \sqrt{7}$, $y' = -\frac{1}{3}\left(\sqrt{7} - 7\right)$, and the line is

$y_3 = -\frac{1}{3}\left(\sqrt{7} - 7\right)\left(x + 1 - \sqrt{7}\right) + 3 = -\frac{1}{3}\left[\left(\sqrt{7} - 7\right)x - \left(23 - 8\sqrt{7}\right)\right]$.

For $x = 1 + \sqrt{7}$, $y' = -\frac{1}{3}\left(\sqrt{7} + 7\right)$, and the line is

$y_4 = -\frac{1}{3}\left(\sqrt{7} + 7\right)\left(x - 1 - \sqrt{7}\right) + 3 = -\frac{1}{3}\left[\left(\sqrt{7} + 7\right)x - \left(8\sqrt{7} + 23\right)\right]$.

(c) Equating y_3 and y_4:

$$-\frac{1}{3}\left(\sqrt{7} - 7\right)\left(x + 1 - \sqrt{7}\right) + 3 = -\frac{1}{3}\left(\sqrt{7} + 7\right)\left(x - 1 - \sqrt{7}\right) + 3$$

$$\left(\sqrt{7} - 7\right)\left(x + 1 - \sqrt{7}\right) = \left(\sqrt{7} + 7\right)\left(x - 1 - \sqrt{7}\right)$$

$$\sqrt{7}x + \sqrt{7} - 7 - 7x - 7 + 7\sqrt{7} = \sqrt{7}x - \sqrt{7} - 7 + 7x - 7 - 7\sqrt{7}$$

$$\sqrt{7} = 14x$$

$$= \frac{8\sqrt{7}}{7}$$

If $x = \frac{8\sqrt{7}}{7}$, then $y =$ 5 and the lines intersect at $\left(\frac{8\sqrt{7}}{7}, 5\right)$.

72. $\sqrt{x} + \sqrt{y} = \sqrt{c}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Tangent line at (x_0, y_0): $y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$

x-intercept: $\left(x_0 + \sqrt{x_0}\sqrt{y_0}, 0\right)$

y-intercept: $\left(0, y_0 + \sqrt{x_0}\sqrt{y_0}\right)$

Sum of intercepts:

$$\left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = \left(\sqrt{c}\right)^2 = c$$

73. $y = x^{p/q}$; p, q integers and $q > 0$

$$y^q = x^p$$

$$qy^{q-1}y' = px^{p-1}$$

$$y' = \frac{px^{p-1}}{q \cdot y^{q-1}} = \frac{px^{p-1}}{q} \cdot \frac{y}{y^q}$$

$$= \frac{p}{q} \cdot \frac{x^{p-1}}{x^p}x^{p/q} = \frac{p}{q}x^{p/q-1}$$

So, if $y = x^n$, $n = p/q$, then $y' = nx^{n-1}$.

74. $x^2 + y^2 = 100$, slope $= \frac{3}{4}$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y} = \frac{3}{4} \Rightarrow y = -\frac{4}{3}x$$

$$x^2 + \left(\frac{16}{9}x^2\right) = 100$$

$$\frac{25}{9}x^2 = 100$$

$$x = \pm 6$$

Points: $(6, -8)$ and $(-6, 8)$

75. $\frac{x^2}{4} + \frac{y^2}{9} = 1$, $(4, 0)$

$$\frac{2x}{4} + \frac{2yy'}{9} = 0$$

$$y' = \frac{-9x}{4y}$$

$$\frac{-9x}{4y} = \frac{y - 0}{x - 4}$$

$$-9x(x - 4) = 4y^2$$

But, $9x^2 + 4y^2 = 36 \Rightarrow 4y^2 = 36 - 9x^2$. So, $-9x^2 + 36x = 4y^2 = 36 - 9x^2 \Rightarrow x = 1$.

Points on ellipse: $\left(1, \pm\frac{3}{2}\sqrt{3}\right)$

At $\left(1, \frac{3}{2}\sqrt{3}\right)$: $y' = \frac{-9x}{4y} = \frac{-9}{4\left[(3/2)\sqrt{3}\right]} = -\frac{\sqrt{3}}{2}$

At $\left(1, -\frac{3}{2}\sqrt{3}\right)$: $y' = \frac{\sqrt{3}}{2}$

Tangent lines: $y = -2(x-4) = -2x + 2\ 3$

$$y = \frac{\sqrt{3}}{2}(x-4) = \frac{\sqrt{3}}{2}x - 2\sqrt{3}$$

$x = y^2$

$1 = 2yy'$

$y' = \frac{1}{2y}$, slope of tangent line

Consider the slope of the normal line joining $(x_0, 0)$ and $(x, y) = (y^2, y)$ on the parabola.

$$-2y = \frac{y - 0}{y^2 - x_0}$$

$$y^2 - x_0 = -\frac{1}{2}$$

$$y^2 = x_0 - \frac{1}{2}$$

(a) If $x_0 = \frac{1}{4}$, then $y^2 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$, which is impossible. So, the only normal line is the x-axis $(y = 0)$.

(b) If $x_0 = \frac{1}{2}$, then $y^2 = 0 \Rightarrow y = 0$. Same as part (a).

If $x_0 = 1$, then $y^2 = \frac{1}{2} = x$ and there are three normal lines.

The x-axis, the line joining $(x_0, 0)$ and $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ and the line joining $(x_0, 0)$ and $\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}\right)$

If two normals are perpendicular, then their slopes are –1 and 1. So,

$$-2y = -1 = \frac{y - 0}{y^2 - x_0} \Rightarrow y = \frac{1}{2}$$

and

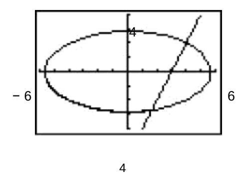
$$\frac{1/2}{(1/4) - x_0} = -1 \Rightarrow \frac{1}{4} - x_0 = -\frac{1}{2} \Rightarrow x_0 = \frac{3}{4}.$$

The perpendicular normal lines are $y = -x + \frac{3}{4}$ and $y = x - \frac{3}{4}$.

77. (a) $\frac{x^2}{32} + \frac{y^2}{8} = 1$

$$\frac{2x}{32} + \frac{2yy'}{8} = 0 \Rightarrow y' = \frac{-x}{4y}$$

(b)



At $(4, 2)$: $y' = \frac{-4}{4(2)} = -\frac{1}{2}$

Slope of normal line is 2.

$$y - 2 = 2(x - 4)$$
$$= 2x - 6$$

$$\frac{x^2}{32} + \frac{(2x - 6)^2}{8} = 1$$

$$x^2 + 4(4x^2 - 24x + 36) = 32$$

$$17x^2 - 96x + 112 = 0$$

$$(17x - 28)(x - 4) = 0 \Rightarrow x = 4, \frac{28}{17}$$

Second point: $\left(\frac{28}{17}, -\frac{46}{17}\right)$

Section 2.6 Related Rates

A related-rate equation is an equation that relates the rates of change of various quantities.

Answers will vary. See page 153.

$y = \sqrt{x}$

$$\frac{dy}{dt} = \left(\frac{1}{2\sqrt{x}}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = 2\sqrt{x}\,\frac{dy}{dt}$$

When $x = 4$ and $dx/dt = 3$: $\frac{dy}{dt}$

$$= 2\,\frac{1}{4}\left(\sqrt{3}\right) = \frac{3}{4}$$

When $x = 25$ and $dy/dt = 2$: $\frac{dx}{dt}$

$$= 2\sqrt{25}(2) = 20$$

$y = 3x^2 - 5x$

$$\frac{dy}{dt} = (6x - 5)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{1}{6x - 5}\frac{dy}{dt}$$

(a) When $x = 3$ and $\frac{dx}{dt} = 2$:

$$\frac{dy}{dt} = [6(3) - 5]2 = 26$$

(b) When $x = 2$ and $\frac{dy}{dt} = 4$:

$$\frac{dx}{dt} = \frac{1}{6(2) - 5}(4) = \frac{4}{7}$$

$xy = 4$

$$x\frac{dy}{dt} + y\frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{y}{x}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{x}{y}\right)\frac{dy}{dt}$$

(a) When $x = 8$, $y = 1/2$, and $dx/dt = 10$:

$$\frac{dy}{dt} = -\frac{1/2}{8}(10) = -\frac{5}{8}$$

When $x = 1$, $y = 4$, and $dy/dt = -6$:

$$\frac{dx}{dt} = -\frac{1}{4}(-6) = \frac{3}{2}$$

$x^2 + y^2 = 25$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{x}{y}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{y}{x}\right)\frac{dy}{dt}$$

(a) When $x = 3$, $y = 4$, and $dx/dt = 8$:

$$\frac{dy}{dt} = -\frac{3}{4}(8) = -6$$

$y = 2x^2 + 1$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = 4x\frac{dx}{dt}$$

(a) When $x = -1$:

$$\frac{dy}{dt} = 4(-1)(2) = -8 \text{ cm/sec}$$

(b) When $x = 0$:

$$\frac{dy}{dt} = 4(0)(2) = 0 \text{ cm/sec}$$

(c) When $x = 1$:

$$\frac{dy}{dt} = 4(1)(2) = 8 \text{ cm/sec}$$

8. $y = \frac{1}{1 + x^2}$, $\frac{dx}{dt} = 6$

$$\frac{dy}{dt} = \left(\frac{-2x}{(1 + x^2)^2}\right)\frac{dx}{dt}$$

$$= \frac{-2x}{(1 + x^2)^2}(6) = \frac{-12x}{(1 + x^2)^2}$$

(a) When $x = -2$:

$$\frac{dy}{dt} = \frac{-12(-2)}{\left[1 + (-2)^2\right]^2} = \frac{24}{25} \text{ in./sec}$$

(b) When $x = 0$:

$$\frac{dy}{dt} = \frac{-12(0)}{(1 + 0^2)^2} = 0 \text{ in./sec}$$

(c) When $x = 2$:

$$\frac{dy}{dt} = \frac{-12(2)}{(1 + 2^2)^2} = -\frac{24}{25} \text{ in./sec}$$

(b) When $x = 4$, $y = 3$, and $dy/dt = -2$:

$$\frac{dx}{dt} = -\frac{3}{4}(-2) = \frac{3}{2}$$

$y = \tan x, \frac{dx}{dt} = 3$

$\frac{dy}{dt} = \sec^2 x \cdot \frac{dx}{dt} = \sec^2 x\,(3) = 3\sec^2 x$

(a) When $x = -\frac{\pi}{3}$:

$\frac{dy}{dt} = 3\sec^2\left(-\frac{\pi}{3}\right) = 3(2)^2 = 12$ ft/sec

(b) When $x = -\frac{\pi}{4}$:

$\frac{dy}{dt} = 3\sec^2\left(-\frac{\pi}{4}\right) = 3\left(\sqrt{2}\right)^2 = 6$ ft/sec

(c) When $x = 0$:

$\frac{dy}{dt} = 3\sec^2(0) = 3$ ft/sec

$y = \cos x, \frac{dx}{dt} = 4$

$\frac{dy}{dt} = -\sin x \cdot \frac{dx}{dt} = -\sin x\,(4) = -4\sin x$

(a) When $x = \frac{\pi}{6}$:

$\frac{dy}{dt} = -4\sin\left(\frac{\pi}{6}\right) = -4\left(\frac{1}{2}\right) = -2$ cm/sec

(b) When $x = \frac{\pi}{4}$:

$\frac{dy}{dt} = -4\sin\left(\frac{\pi}{4}\right) = -4\left(\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$ cm/sec

(c) When $x = \frac{\pi}{3}$:

$\frac{dy}{dt} = -4\sin\left(\frac{\pi}{3}\right) = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$ cm/sec

$A = \pi r^2$

$\frac{dr}{dt} = 4$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

When $r = 37$, $\frac{dA}{dt} = 2\pi(37)(4) = 296\pi\ \text{cm}^2/\text{min}.$

$V = \frac{4}{3}\pi r^3$

$\frac{dr}{dt} = 3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

(a) When $r = 9$,

$\frac{dV}{dt} = 4\pi(9)^2(3) = 972\pi\ \text{in.}^3/\text{min.}$

When $r = 36$,

$\frac{dV}{dt} = 4\pi(36)^2(3) = 15{,}552\pi\ \text{in.}^3/\text{min.}$

If dr/dt is constant, dV/dt is proportional to r^2.

$V = \frac{4}{3}\pi r^3, \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\frac{dV}{dt} = 800$

$\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}$

At $r = 30$, $\frac{dr}{dt} = \frac{800}{4\pi(30)^2} = \frac{2}{9\pi}$ cm/min.

At $r = 85$, $\frac{dr}{dt} = \frac{800}{4\pi(85)^2} = \frac{8}{289\pi}$ cm/min.

$\frac{dr}{dt}$ depends on r^2, not r.

$V = x^3$

$\frac{dx}{dt} = 6$

$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

(a) When $x = 2$,

$\frac{dV}{dt} = 3(2)^2(6) = 72\ \text{cm}^3/\text{sec}.$

(b) When $x = 10$,

$\frac{dV}{dt} = 3(10)^2(6) = 1800\ \text{cm}^3/\text{sec}.$

$s = 6x^2$

$\frac{dx}{dt} = 6$

$\sqrt{\ }$ $s_2 3$

$A = \frac{\sqrt{3}}{4} s^2$

$\frac{ds}{dt} = 13$

$\frac{dA}{dt} = \frac{\sqrt{3}}{4}(2s)\frac{ds}{dt} = \frac{\sqrt{3}s}{2}\frac{ds}{dt}$

When $s = 41$, $\frac{dA}{dt} = \frac{\sqrt{3}}{2}(41)(13) = \frac{533\sqrt{3}}{2}$ ft^2/hr.

$\frac{ds}{dt} = 12x\frac{dx}{dt}$

(a) When $x = 2$,

$\frac{ds}{dt} = 12(2)(6) = 144 \text{ cm}^2/\text{sec}$.

(b) When $x = 10$,

$\frac{ds}{dt} = 12(10)(6) = 720 \text{ cm}^2/\text{sec}$.

17. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{9}{4}h^2\right)h$ $\quad [\text{because } 2r = 3h]$

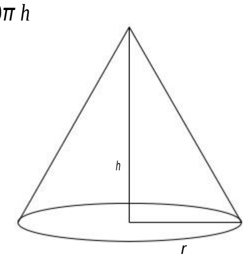
$= \frac{3}{4}\pi h^3$

$\frac{dV}{dt} = 10$

$\frac{dV}{dt} = \frac{9\pi}{4}h^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4(dV/dt)}{9\pi h^2}$

When $h = 15$,

$\frac{dh}{dt} = \frac{4(10)}{9\pi(15)^2} = \frac{8}{405\pi}$ ft/min.



$V = \pi r^2 h$

$\frac{dV}{dt} = 150$

$h = 10r \Rightarrow r = \frac{h}{10}$

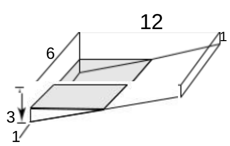
$V = \pi\left(\frac{h}{10}\right)^2 h = \frac{\pi}{100}h^3$

$\frac{dV}{dt} = \frac{3\pi}{100}h^2\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{100}{3\pi h^2}\frac{dV}{dt}$

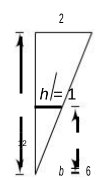
When $h = 35$, $\frac{dh}{dt} = \frac{100}{3\pi(35)^2}(150) = \frac{200}{49\pi}$ in./sec.

19.



(a) Total volume of pool $= \frac{1}{2}(2)(12)(6) + (1)(6)(12) = 144 \text{ m}^3$

Volume of 1 m of water $= \frac{1}{2}(1)(6)(6) = 18 \text{ m}^3$ (see similar triangle diagram)



% pool filled $= \frac{18}{144}(100\%) = 12.5\%$

(b) Because for $0 \le h \le 2$, $b = 6h$, you have

$V = \frac{1}{2}bh(6) = 3bh = 3(6h)h = 18h^2$

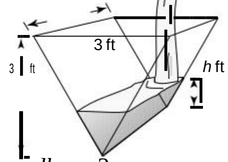
$\frac{dV}{dt} = 36h\frac{dh}{dt} = \frac{1}{4} \Rightarrow \frac{dh}{dt} = \frac{1}{144h} = \frac{1}{144(1)} = \frac{1}{144}$ m/min.

20. $V = \frac{1}{2}bh(12) = 6bh = 6h^2$ (since $b = h$)

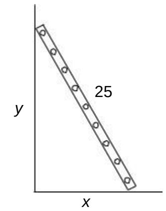
(a) $\frac{dV}{dt} = 12h\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{12h}\frac{dV}{dt}$

When $h = 1$ and $\frac{dV}{dt} = 2$, $\frac{dh}{dt} = \frac{1}{12(1)}(2) = \frac{1}{6}$ ft/min.

12 ft



(b) If $\frac{dh}{dt} = \frac{3}{8}$ in./min $= \frac{1}{32}$ ft/min and $h = 2$ ft, then $\frac{dV}{dt} = 12(2)\left(\frac{1}{32}\right) = \frac{3}{4}$ ft^3/min.



$$x^2 + y^2 = 25^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x}{y}\cdot\frac{dx}{dt} = \frac{-2x}{y} \quad \text{because } \frac{dx}{dt} = 2.$$

(a) When $x = 7$, $y = \sqrt{576} = 24$, $\frac{dy}{dt} = \frac{-2(7)}{24} = -\frac{7}{12}$ ft/sec.

When $x = 15$, $y = \sqrt{400} = 20$, $\frac{dy}{dt} = \frac{-2(15)}{20} = -\frac{3}{2}$ ft/sec.

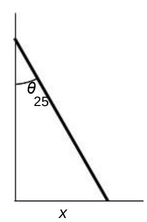
When $x = 24$, $y = 7$, $\frac{dy}{dt} = \frac{-2(24)}{7} = -\frac{48}{7}$ ft/sec.

$$A = \frac{1}{2}xy$$

$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$$

From part (a) you have $x = 7$, $y = 24$, $\frac{dx}{dt} = 2$, and $\frac{dy}{dt} = -\frac{7}{12}$. So,

$$\frac{dA}{dt} = \frac{1}{2}\left[7\left(-\frac{7}{12}\right) + 24(2)\right] = \frac{527}{24}\ \text{ft}^2/\text{sec}.$$



$$\tan\theta = \frac{x}{y}$$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{y}\frac{dx}{dt} - \frac{x}{y^2}\cdot\frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \cos^2\theta\left[\frac{1}{y}\cdot\frac{dx}{dt} - \frac{x}{y^2}\cdot\frac{dy}{dt}\right]$$

Using $x = 7$, $y = 24$, $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = -\frac{7}{12}$ and $\cos\theta = \frac{24}{25}$, you have

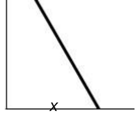
$$\frac{d\theta}{dt} = \left(\frac{24}{25}\right)^2\left[\frac{1}{24}(2) - \frac{7}{(24)^2}\left(-\frac{7}{12}\right)\right] = \frac{1}{12}\ \text{rad/sec}.$$

$$x^2 + y^2 = 25 \qquad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

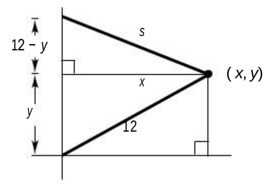
$$\frac{dx}{dt} = -\frac{y}{x}\cdot\frac{dy}{dt} = -\frac{0.15y}{x}\quad\left(\text{because } \frac{dy}{dt} = 0.15\right)$$

When $x = 2.5$, $y = \sqrt{18.75}$, $\frac{dx}{dt} = -\frac{\sqrt{18.75}}{2.5}(0.15) \approx -0.26$ m/sec.

5


x

23. When $y = 6$, $x = \sqrt{12^2 - 6^2} = 6\sqrt{3}$, and $s = \sqrt{x^2 + (12 - y)^2} = \sqrt{108 + 36} = 12$.



$$x^2 + (12 - y)^2 = s^2$$

$$2x\frac{dx}{dt} + 2(12 - y)(-1)\frac{dy}{dt} = 2s\frac{ds}{dt}$$

$$x\frac{dx}{dt} + (y - 12)\frac{dy}{dt} = s\frac{ds}{dt}$$

Also, $x^2 + y^2 = 12^2$.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$$

So, $x\frac{dx}{dt} + (y - 12)\left(\frac{-x}{y}\frac{dx}{dt}\right) = s\frac{ds}{dt}$.

$$\frac{dx}{dt}\left[x - x + \frac{12x}{y}\right] = s\frac{ds}{dt} \Rightarrow \frac{dx}{dt} = \frac{sy}{12x} \cdot \frac{ds}{dt} = \frac{(12)(6)}{(12)(6\sqrt{3})}(-0.2) = \frac{-1}{5\sqrt{3}} = \frac{-\sqrt{3}}{15} \text{ m/sec (horizontal)}$$

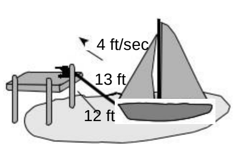
$$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt} = \frac{-6\sqrt{3}}{6} \cdot \left(\frac{-\sqrt{3}}{15}\right) = \frac{1}{5} \text{ m/sec (vertical)}$$

Let L be the length of the rope.

(a) $L^2 = 144 + x^2$

$$2L\frac{dL}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{L}{x} \cdot \frac{dL}{dt} = -\frac{4L}{x} \quad \left(\text{since } \frac{dL}{dt} = -4 \text{ ft/sec}\right)$$



When $L = 13$:

$$x = \sqrt{L^2 - 144} = \sqrt{169 - 144} = 5$$

$$\frac{dx}{dt} = -\frac{4(13)}{5} = -\frac{52}{5} = -10.4 \text{ ft/sec}$$

Speed of the boat increases as it approaches the dock.

(b) If $\frac{dx}{dt} = -4$, and $L = 13$:

$$\frac{dL}{dt} = \frac{x}{L}\frac{dx}{dt} = \frac{5}{13}(-4) = \frac{-20}{13} \text{ ft/sec}$$

$$\frac{dL}{dt} = \frac{x}{L}\frac{dx}{dt} = \frac{\sqrt{L^2 - 144}}{L}(-4)$$

$$\lim_{L \to 12^+} \frac{dL}{dt} = \lim_{L \to 12^+} \frac{-4}{L}\sqrt{L^2 - 144} = 0$$

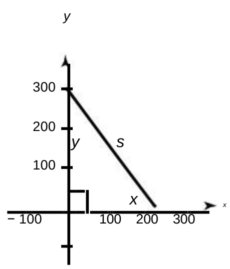
(a) $s^2 = x^2 + y^2$

$\frac{dx}{dt} = -450$

$\frac{dy}{dt} = -600$

$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$\frac{ds}{dt} = \frac{x(dx/dt) + y(dy/dt)}{s}$



When $x = 225$ and $y = 300$, $s = 375$ and

$\frac{ds}{dt} = \frac{225(-450) + 300(-600)}{375} = -750$ mi/h.

(b) $t = \frac{375}{750} = \frac{1}{2}$ h $= 30$ min

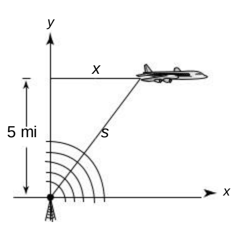
26. $x^2 + y^2 = s^2$

$2x\frac{dx}{dt} + 0 = 2s\frac{ds}{dt} \quad \left(\text{because } \frac{dy}{dt} = 0\right)$

$\frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}$

When $s = 10$, $x = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}$,

$\frac{dx}{dt} = \frac{10}{5\sqrt{3}}(240) = \frac{480}{\sqrt{3}} = 160\sqrt{3} \approx 277.13$ mi/h.



27. $s^2 = 90^2 + x^2$

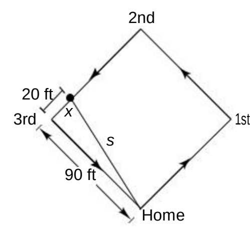
$x = 20$

$\frac{dx}{dt} = -25$

$2s\frac{ds}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{ds}{dt} = \frac{x}{s}\cdot\frac{dx}{dt}$

When $x = 20$, $s = \sqrt{90^2 + 20^2} = 10\sqrt{85}$,

$\frac{ds}{dt} = \frac{20}{10\sqrt{85}}(-25) = \frac{-50}{\sqrt{85}} \approx -5.42$ ft/sec.



28. $s^2 = 90^2 + x^2$

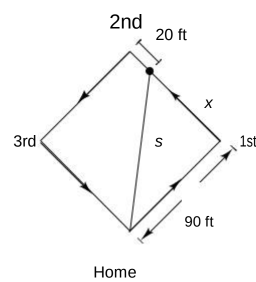
$x = 90 - 20 = 70$

$\frac{dx}{dt} = 25$

$\frac{ds}{dt} = \frac{x}{s}\cdot\frac{dx}{dt}$

When $x = 70$, $s = \sqrt{90^2 + 70^2} = 10\sqrt{130}$,

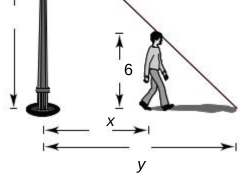
$\frac{ds}{dt} = \frac{70}{10\sqrt{130}}(25) = \frac{175}{\sqrt{130}} \approx 15.35$ ft/sec.



29. (a) $\frac{15}{6} = \frac{y}{y - x} \Rightarrow 15y - 15x = 6y$

$y = \frac{5}{3}x$

$\frac{dx}{dt} = 5$



$\frac{dy}{dt} = \frac{5}{3} \cdot \frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3}$ ft/sec

(b) $\frac{d(y-x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = \frac{25}{3} - 5 = \frac{10}{3}$ ft/sec

30. (a) $\dfrac{20}{6} = \dfrac{y}{y - x}$

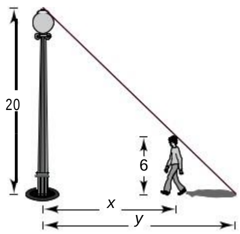
$20y - 20x = 6y$

$14y = 20x$

$y = \dfrac{10}{7}x$

$\dfrac{dx}{dt} = -5$

$\dfrac{dy}{dt} = \dfrac{10}{7}\dfrac{dx}{dt} = \dfrac{10}{7}(-5) = \dfrac{-50}{7}$ ft/sec



(b) $\dfrac{d(y - x)}{dt} = \dfrac{dy}{dt} - \dfrac{dx}{dt}$

$= \dfrac{-50}{7} - (-5)$

$= \dfrac{-50}{7} + \dfrac{35}{7} = \dfrac{-15}{7}$ ft/sec

$x(t) = \frac{1}{2}\sin\frac{\pi t}{6}$, $x^2 + y^2 = 1$

Period: $\dfrac{2\pi}{\pi/6} = 12$ seconds

(b) When $x = \frac{1}{2}$, $y = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$ m.

Lowest point: $\left(0, \frac{\sqrt{3}}{2}\right)$

(c) When $x = \frac{1}{4}$,

$y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and $t = 1$:

$\dfrac{dx}{dt} = \dfrac{1}{2}\left(\dfrac{\pi}{6}\right)\cos\dfrac{\pi t}{6} = \dfrac{\pi}{12}\cos\dfrac{\pi t}{6}$

$x^2 + y^2 = 1$

$2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0 \Rightarrow \dfrac{dy}{dt} = \dfrac{-x}{y}\dfrac{dx}{dt}$

So, $\dfrac{dy}{dt} = -\dfrac{1/4}{\sqrt{15}/4}\cdot\dfrac{\pi}{12}\cos\left(\dfrac{\pi}{6}\right)$

$= \dfrac{-\pi}{\sqrt{15}}\left(\dfrac{1}{12}\right)\dfrac{\sqrt{3}}{2} = \dfrac{-\pi}{24}\dfrac{1}{\sqrt{5}} = \dfrac{-\sqrt{5}\pi}{120}.$

Speed $= \left|\dfrac{-\sqrt{5}\pi}{120}\right| = \dfrac{\sqrt{5}\pi}{120}$ m/sec

$x(t) = \frac{3}{5}\sin\pi t$, $x^2 + y^2 = 1$

Period: $\dfrac{2\pi}{\pi} = 2$ seconds

(c) When $x = \frac{3}{10}$, $y = \sqrt{1 - \left(\frac{3}{10}\right)^2} = \frac{\sqrt{91}}{10}$ and

$\frac{3}{10} = \frac{3}{5}\sin\pi t \Rightarrow \sin\pi t = \frac{1}{2} \Rightarrow t = \frac{1}{6}$

$\dfrac{dx}{dt} = \dfrac{3}{5}\pi\cos\pi t$

$x^2 + y^2 = 1$

$2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0 \Rightarrow \dfrac{dy}{dt} = \dfrac{-x}{y}\dfrac{dx}{dt}$

So, $\dfrac{dy}{dt} = \dfrac{-3/10}{\sqrt{91}/10}\cdot\dfrac{3}{5}\pi\cos\left(\dfrac{\pi}{6}\right)$

$= \dfrac{-9\pi}{25\sqrt{5}} = \dfrac{-9\sqrt{5}\pi}{125}.$

Speed $= \left|\dfrac{-9\sqrt{5}\pi}{125}\right| \approx 0.5058$ m/sec

33. Because the evaporation rate is proportional to the surface area, $\dfrac{dV}{dt} = k(4\pi r^2)$. However, because $V = (4/3)\pi r^3$, you have $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. Therefore, $k(4\pi r^2) = 4\pi r^2\dfrac{dr}{dt} \Rightarrow k = \dfrac{dr}{dt}$.

34. (i) (a) $\frac{dx}{dt}$ negative $\Rightarrow$ $\frac{dy}{dt}$ positive

(b) $\frac{dy}{dt}$ positive $\Rightarrow$ $\frac{dx}{dt}$ negative

(ii) (a) $\frac{dx}{dt}$ negative $\Rightarrow$ $\frac{dy}{dt}$ negative

(b) $\frac{dy}{dt}$ positive $\Rightarrow$ $\frac{dx}{dt}$ positive

35. (a) $dy/dt = 3(dx/dt)$ means that y changes three times as fast as x changes.

y changes slowly when $x \approx 0$ or $x \approx L$. y changes more rapidly when x is near the middle of the interval.

No. $V = s^3$, $\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$

If $\frac{ds}{dt}$ is constant, then $\frac{dV}{dt}$ is $3s^2$ times that constant.

37. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

$\frac{dR_1}{dt} = 1$

$\frac{dR_2}{dt} = 1.5$

$$\frac{1}{R^2} \cdot \frac{dR}{dt} = \frac{1}{R_1^2} \cdot \frac{dR_1}{dt} + \frac{1}{R_2^2} \cdot \frac{dR_2}{dt}$$

When $R_1 = 50$ and $R_2 = 75$:

$R = 30$

$$\frac{dR}{dt} = (30)^2\left[\frac{1}{(50)^2}(1) + \frac{1}{(75)^2}(1.5)\right] = 0.6 \text{ ohm/sec}$$

38. $V = IR$

$$\frac{dV}{dt} = I\frac{dR}{dt} + R\frac{dI}{dt}$$

$$\frac{dI}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{I}{R}\frac{dR}{dt}$$

When $V = 12, R = 4, \frac{dV}{dt} = 3$, and $\frac{dR}{dt} = 2$, $I = \frac{V}{R} = \frac{12}{4} = 3$ and

$$\frac{dI}{dt} = \frac{1}{4}(3) - \frac{3}{4}(2) = -\frac{3}{4} \text{ amperes/sec.}$$

$\sin 18° = \frac{x}{y}$

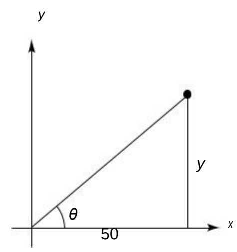
$0 = -\frac{x}{y^2} \cdot \frac{dy}{dt} + \frac{1}{y} \cdot \frac{dx}{dt}$

$\tan\theta = \frac{y}{50}$

$\frac{dy}{dt} = 4$ m/sec

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{50}\frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{50}\cos^2\theta \cdot \frac{dy}{dt}$$



When $y = 50$, $\theta = \frac{\pi}{4}$, and $\cos\theta = \frac{\sqrt{2}}{2}$. So,

$$\frac{d\theta}{dt} = \frac{1}{50}\left(\frac{\sqrt{2}}{2}\right)^2(4) = \frac{1}{25} \text{ rad/sec.}$$

41. $\sin\theta = \frac{10}{x}$

$\frac{dx}{dt} = -1$ ft/sec

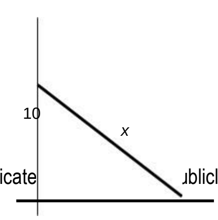
$$\cos\theta\left(\frac{d\theta}{dt}\right) = \frac{(-10)}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-10}{x^2}\frac{dx}{dt}(\sec\theta)$$

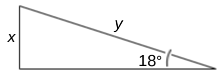
$$= \frac{-10}{25^2}(-1)\frac{25}{\sqrt{25^2 - 10^2}}$$

$$= \frac{10}{25}\frac{1}{5\sqrt{21}} = \frac{2}{25\sqrt{21}}$$

$$= \frac{2\sqrt{21}}{525} \approx 0.017 \text{ rad/sec}$$



$$\frac{dx}{dt} = \frac{x}{y} \cdot \frac{dy}{dt} = (\sin 18°)(275) \approx 84.9797 \text{ mi/hr}$$



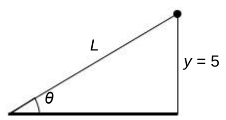
$\tan\theta = \dfrac{y}{x},\ y = 5$

$\dfrac{dx}{dt} = -600$ mi/h

$\left(\sec^2\theta\right)\dfrac{d\theta}{dt} = -\dfrac{5}{x^2}\cdot\dfrac{dx}{dt}$

$\dfrac{d\theta}{dt} = \cos^2\theta\left(-\dfrac{5}{x^2}\right)\dfrac{dx}{dt} = \dfrac{x^2}{L^2}\left(-\dfrac{5}{x^2}\right)\dfrac{dx}{dt}$

$= \left(-\dfrac{5^2}{L^2}\right)\left(\dfrac{1}{5}\right)\dfrac{dx}{dt} = \left(-\sin^2\theta\right)\left(\dfrac{1}{5}\right)(-600) = 120\sin^2\theta$



(a) When $\theta = 30°$,

$\dfrac{d\theta}{dt} = \dfrac{120}{4} = 30$ rad/h $= \dfrac{1}{2}$ rad/min.

(b) When $\theta = 60°$,

$\dfrac{d\theta}{dt} = 120\left(\dfrac{3}{4}\right) = 90$ rad/h $= \dfrac{3}{2}$ rad/min.

(c) When $\theta = 75°$,

$\dfrac{d\theta}{dt} = 120\sin^2 75° \approx 111.96$ rad/h ≈ 1.87 rad/min.

$\tan\theta = \dfrac{x}{50}$

$\dfrac{d\theta}{dt} = 30(2\pi) = 60\pi$ rad/min $= \pi$ rad/sec

$\sec^2\theta\left(\dfrac{d\theta}{dt}\right) = \dfrac{1}{50}\left(\dfrac{dx}{dt}\right)$

$\dfrac{dx}{dt} = 50\sec^2\theta\left(\dfrac{d\theta}{dt}\right)$

$\dfrac{d\theta}{dt} = (10 \text{ rev/sec})(2\pi \text{ rad/rev}) = 20\pi$ rad/sec

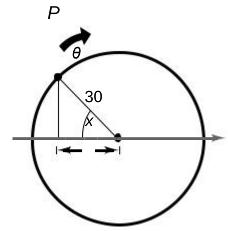
(a) $\cos\theta = \dfrac{x}{30}$

$-\sin\theta\dfrac{d\theta}{dt} = \dfrac{1}{30}\dfrac{dx}{dt}$

$\dfrac{dx}{dt} = -30\sin\theta\dfrac{d\theta}{dt}$

$= -30\sin\theta(20\pi)$

$= -600\pi\sin\theta$



(b) 2000

0 4π

−2000

$|\sin\theta| = 1 \Rightarrow \theta = \dfrac{\pi}{2} + n\pi$ (or $90° + n\cdot 180°$).

$\left|dx/dt\right|$ is least when $\theta = n\pi$ (or $n\cdot 180°$).

(d) For $\theta = 30°$,

$\dfrac{dx}{dt} = -600\pi\sin(30°) = -600\pi\dfrac{1}{2} = -300\pi$ cm/sec.

For $\theta = 60°$,

$\dfrac{dx}{dt} = -600\pi\sin(60°)$

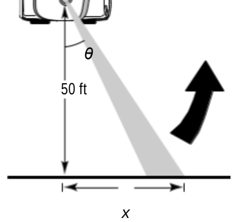
$= -600\pi\dfrac{\sqrt{3}}{2} = -300\sqrt{3}\pi$ cm/sec.

45. (a) $\sin\dfrac{\theta}{2} = \dfrac{(1/2)b}{s} \Rightarrow b = 2s\sin\dfrac{\theta}{2}$.

$\cos\dfrac{\theta}{2} = \dfrac{h}{s} \Rightarrow h = s\cos\dfrac{\theta}{2}$

$A = \dfrac{1}{2}bh = \dfrac{1}{2}\left(2s\sin\dfrac{\theta}{2}\right)\left(s\cos\dfrac{\theta}{2}\right)$

$= \dfrac{s^2}{2}\left(2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\right) = \dfrac{s^2}{2}\sin\theta$.



When $\theta = 30°$, $\frac{dx}{dt} = \frac{200\pi}{3}$ ft/sec.

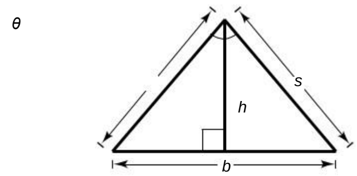
When $\theta = 60°$, $\frac{dx}{dt} = 200\pi$ ft/sec.

When $\theta = 70°$, $\frac{dx}{dt} \approx 427.43\pi$ ft/sec.

(b) $\frac{dA}{dt} = \frac{s}{2} \cos \theta \frac{d\theta}{dt}$ where $\frac{d\theta}{dt} = \frac{1}{2}$ rad/min.

When $\theta = \frac{\pi}{6}$, $\frac{dA}{dt} = \frac{s^2}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}s^2}{8}$.

When $\theta = \frac{\pi}{3}$, $\frac{dA}{dt} = \frac{s^2}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{s^2}{8}$.



(c) If s and $\frac{d\theta}{dt}$ is constant, $\frac{dA}{dt}$ is proportional to $\cos \theta$.

46. $\tan\theta = \frac{x}{50} \Rightarrow x = 50\tan\theta$

$$\frac{dx}{dt} = 50\sec^2\theta\,\frac{d\theta}{dt}$$

$$2 = 50\sec^2\theta\,\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{25}\cos^2\theta, \quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

47. (a) Using a graphing utility,

$$r(f) = 0.0096f^3 - 0.559f^2 + 10.54f - 61.5.$$

(b) $\frac{dr}{dt} = \frac{dr}{df}\frac{df}{dt} = \left(0.0288f^2 - 1.118f + 10.54\right)\frac{df}{dt}$

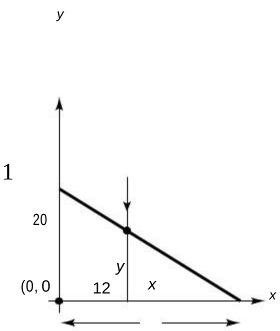
For $t = 9$, $f = 16.3$ from the table under the year 2009.

Also, $\frac{df}{dt} = 1.25$, so you have

$$\frac{dr}{dt} = \left(0.0288(16.3)^2 - 1.118(16.3) + 10.54\right)(1.25)$$

$$= -0.03941 \text{ million participants per year.}$$

48. $y(t) = -4.9t^2 + 20$



$$\frac{dy}{dt} = -9.8t$$

$$y(1) = -4.9 + 20 = 15.1$$

$$y'(1) = -9.8$$

By similar triangles: $\frac{20}{x} = \frac{y}{x - 12}$

$$20x - 240 = xy$$

When $y = 15.1$: $20x - 240 = x(15.1)$

$$(20 - 15.1)x = 240$$

$$x = \frac{240}{4.9}$$

$$20x - 240 = xy$$

$$20\frac{dx}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{20 - y}\frac{dy}{dt}$$

At $t = 1$, $\frac{dx}{dt} = \frac{240/4.9}{20 - 15.1}(-9.8) \approx -97.96$ m/sec.

49. $x^2 + y^2 = 25$; acceleration of the top of the ladder $= \frac{d^2y}{dt^2}$

First derivative: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Second derivative: $x\frac{d^2x}{dt^2} + \frac{dx}{dt}\cdot\frac{dx}{dt} + y\frac{d^2y}{dt^2} + \frac{dy}{dt}\cdot\frac{dy}{dt} = 0$

$$\frac{d^2y}{dt^2} = \left(\frac{1}{y}\right)\left[-x\frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2\right]$$

When $x = 7$, $y = 24$, $\frac{dy}{dt} = -\frac{7}{12}$, and $\frac{dx}{dt} = 2$ (see Exercise 25). Because $\frac{dx}{dt}$ is constant, $\frac{d^2x}{dt^2} = 0$.

$$\frac{d^2y}{dt^2} = \frac{1}{24}\left[-7(0) - (2)^2 - \left(-\frac{7}{12}\right)^2\right] = \frac{1}{24}\left[-4 - \frac{49}{144}\right] = \frac{1}{24}\left[-\frac{625}{144}\right] \approx -0.1808 \text{ ft/sec}^2$$

50. $L^2 = 144 + x^2$; acceleration of the boat $= \dfrac{d^2x}{dt^2}$

First derivative: $2L\dfrac{dL}{dt} = 2x\dfrac{dx}{dt}$

$$L\frac{dL}{dt} = x\frac{dx}{dt}$$

Second derivative: $L\dfrac{d^2L}{dt^2} + \dfrac{dL}{dt}\cdot\dfrac{dL}{dt} = x\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt}\cdot\dfrac{dx}{dt}$

$$\frac{d^2x}{dt^2} = \left(\frac{1}{x}\right)\left[L\frac{d^2L}{dt^2} + \left(\frac{dL}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2\right]$$

When $L = 13$, $x = 5$, $\dfrac{dx}{dt} = -10.4$, and $\dfrac{dL}{dt} = -4$ (see Exercise 28). Because $\dfrac{dL}{dt}$ is constant, $\dfrac{d^2L}{dt^2} = 0$.

$$\frac{d^2x}{dt^2} = \frac{1}{5}\left[13(0) + (-4)^2 - (-10.4)^2\right]$$

$$= \frac{1}{5}[16 - 108.16] = \frac{1}{5}[-92.16] = -18.432 \text{ ft/sec}^2$$

Review Exercises for Chapter 2

1. $f(x) = 12$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{12 - 12}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

2. $f(x) = 5x - 4$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[5(x + \Delta x) - 4] - (5x - 4)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{5x + 5\Delta x - 4 - 5x + 4}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{5\Delta x}{\Delta x} = 5$$

3. $f(x) = x^3 - 2x + 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^3 - 2(x + \Delta x) + 1\right] - \left[x^3 - 2x + 1\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - 2x - 2(\Delta x) + 1 - x^3 + 2x - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - 2(\Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{3x^2 + 3x(\Delta x) + (\Delta x)^2 - 2}{1}\right]$$

$$= 3x^2 - 2$$

$$f(x) = \frac{6}{x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{6}{x+\Delta x} - \frac{6}{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{6x - (6x + 6\Delta x)}{x(x+\Delta x)\Delta x} = \lim_{\Delta x \to 0} \frac{-6\Delta x}{x(x+\Delta x)\Delta x} = \lim_{\Delta x \to 0} \frac{-6}{(x+\Delta x)x} = \frac{-6}{x^2}$$

$$g(x) = 2x^2 - 3x, \; c = 2$$

$$g'(2) = \lim_{x \to 2} \frac{g(x) - g(2)}{x - 2}$$

$$= \lim_{x \to 2} \frac{(2x^2 - 3x) - 2}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x - 2)(2x + 1)}{x - 2}$$

$$= \lim_{x \to 2} (2x + 1) = 2(2) + 1 = 5$$

$$f(x) = \frac{1}{x+4}, \; c = 3$$

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$$

$$= \lim_{x \to 3} \frac{\frac{1}{x+4} - \frac{1}{7}}{x - 3}$$

$$= \lim_{x \to 3} \frac{7 - x - 4}{(x - 3)(x + 4)7}$$

$$= \lim_{x \to 3} \frac{-1}{(x + 4)7} = -\frac{1}{49}$$

f is differentiable for all $x \neq 3$.

f is differentiable for all $x \neq -1$.

$y = 25$

$y' = 0$

$f(t) = \frac{\pi}{6}$

$f'(t) = 0$

$f(x) = x^3 - 11x^2$ f

14. $f(x) = x^{1/2} - x^{-5/6}$

$$f'(x) = \frac{1}{2}x^{-1/2} + \frac{5}{6}x^{-11/6}$$

$$g(t) = \frac{2}{3}t^{-2}$$

$$g'(t) = \frac{-4}{3}t^{-3} = -\frac{4}{3t^3}$$

16. $h(x) = \dfrac{8}{5x^4} = \dfrac{8}{5}x^{-4}$

$$h'(x) = -\frac{32}{5}x^{-5} = -\frac{32}{5x^5}$$

17. $f(\theta) = 4\theta - 5\sin\theta$

$$f'(\theta) = 4 - 5\cos\theta$$

18. $g(\alpha) = 4\cos\alpha + 6$

$$g'(\alpha) = -4\sin\alpha$$

19. $f(\theta) = 3\cos\theta - \dfrac{\sin\theta}{4}$

$$f'(\theta) = -3\sin\theta - \frac{\cos\theta}{4}$$

20. $g(\alpha) = \dfrac{5\sin\alpha}{3} - 2\alpha$

$$g'(\alpha) = \frac{5\cos\alpha}{3} - 2$$

$f(x) = \dfrac{27}{x^3} = 27x^{-3}, \; (3, 1)$

$$f'(x) = 27(-3)x^{-4} = -\frac{81}{x^4}$$

$$f'(3) = -\frac{81}{81} = -1$$

$'(x) = 3x^2 - 22x$

$g(s) = 3s^5 - 2s^4$

$g'(s) = 15s^4 - 8s^3$

13. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x} = 6x^{1/2} + 3x^{1/3}$

$h'(x) = 3x^{-1/2} + x^{-2/3} = \frac{3}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^2}}$

22. $f(x) = 3x^2 - 4x, (1, -1)$

$f'(x) = 6x - 4$

$f'(1) = 6 - 4 = 2$

23. $f(x) = 4x^5 + 3x - \sin x, (0, 0)$

$f'(x) = 20x^4 + 3 - \cos x$

$f'(0) = 3 - 1 = 2$

24. $f(x) = 5\cos x - 9x, \quad (0, 5)$

$f'(x) = -5\sin x - 9$

$f'(0) = -5\sin 0 - 9 = -9$

25. $F = 200\sqrt{T}$

$F'(t) = \frac{100}{\sqrt{T}}$

When $T = 4$, $F'(4) = 50$ vibrations/sec/lb.

When $T = 9$, $F'(9) = 33\frac{1}{3}$ vibrations/sec/lb.

$S = 6x^2$

$\frac{dS}{dx} = 12x$

When $x = 4$, $\frac{dS}{dx} = 12(4) = 48$ in.^{2}/ in.

$s(t) = -16t^2 + v_0 t + s_0$; $s_0 = 600$, $v_0 = -30$

$s(t) = -16t^2 - 30t + 600$

$s'(t) = v(t) = -32t - 30$

Average velocity $= \frac{s(3) - s(1)}{3 - 1}$

$\frac{366 - 554}{2}$

-94 ft/sec

$v(1) = -32(1) - 30 = -62$ ft/sec

$v(3) = -32(3) - 30 = -126$ ft/sec

$s(t) = 0 = -16t^2 - 30t + 600$

Using a graphing utility or the Quadratic Formula,

≈ 5.258 seconds.

When

≈ 5.258, $v(t) \approx -32(5.258) - 30 \approx -198.3$ ft/sec.

28. $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$

$f(x) = (5x^2 + 8)(x^2 - 4x - 6)$

$f'(x) = (5x^2 + 8)(2x - 4) + (x^2 - 4x - 6)(10x)$

$10x^3 + 16x - 20x^2 - 32 + 10x^3 - 40x^2 - 60x$

$20x^3 - 60x^2 - 44x - 32$

$4(5x^3 - 15x^2 - 11x - 8)$

$g(x) = (2x^3 + 5x)(3x - 4)$

$g'(x) = (2x^3 + 5x)(3) + (3x - 4)(6x^2 + 5)$

$6x^3 + 15x + 18x^3 - 24x^2 + 15x - 20$

$24x^3 - 24x^2 + 30x - 20$

$f(x) = (9x - 1)\sin x$

$f'(x) = (9x - 1)\cos x + 9\sin x$

$9x\cos x - \cos x + 9\sin x$

$f(t) = 2t^5 \cos t$

$f'(t) = 2t^5(-\sin t) + \cos t(10t^4)$

$-2t^5 \sin t + 10t^4 \cos t$

$f(x) = \frac{x^2 + x - 1}{x^2 - 1}$

$f'(x) = \frac{(x^2 - 1)(2x + 1) - (x^2 + x - 1)(2x)}{(x^2 - 1)^2}$

$= \frac{-x^2 + 1}{(x^2 - 1)^2}$

34. $f(x) = \frac{2x + 7}{x^2 + 4}$

$f'(x) = \frac{(x^2 + 4)(2) - (2x + 7)(2x)}{(x^2 + 4)^2}$

$= \frac{2x^2 + 8 - 4x^2 - 14x}{(x^2 + 4)^2}$

$-2x^2 + 7x - 4$

$= -2x^2 - 14x + 8$

6 t^2 + 450

−1

$v(t) = s'(t) = -32t$

$v(2) = -32(2) = -64$ ft/sec

$v(5) = -32(5) = -160$ ft/sec

$(x^2 + 4)^2$ $(x^2 + 4)^2$

$y = \frac{x^4}{\cos x}$

$y' = \frac{(\cos x)4x^3 - x^4(-\sin x)}{\cos^2 x} = \frac{4x^3 \cos x + x^4 \sin x}{\cos^2 x}$

$y = \frac{\sin x}{x^4}$

$y' = \frac{(x^4)\cos x - (\sin x)(4x^3)}{(x^4)^2} = \frac{x\cos x - 4\sin x}{x^5}$

$y = 3x^2 \sec x$

$y' = 3x^2 \sec x \tan x + 6x \sec x$

$y = -x^2 \tan x$

$y' = -x^2 \sec^2 x - 2x \tan x$

$y = x\cos x - \sin x$

$y' = -x\sin x + \cos x - \cos x = -x\sin x$

$g(x) = x^4 \cot x + 3x\cos x$

$g'(x) = 4x^3 \cot x + x^4(-\csc^2 x) + 3\cos x - 3x\sin x$

$4x^3 \cot x - x^4 \csc^2 x + 3\cos x - 3x\sin x$

41. $f(x) = (x+2)(x^2+5),\ (-1, 6)$

$f'(x) = (x+2)(2x) + (x^2+5)(1)$

$= 2x^2 + 4x + x^2 + 5 = 3x^2 + 4x + 5$

$f'(-1) = 3 - 4 + 5 = 4$

Tangent line: $y - 6 = 4(x+1)$

$= 4x + 10$

42. $f(x) = (x-4)(x^2 + 6x - 1),\ (0, 4)$

$f'(x) = (x-4)(2x+6) + (x^2 + 6x - 1)(1)$

$= 2x^2 - 2x - 24 + x^2 + 6x - 1$

$= 3x^2 + 4x - 25$

$f'(0) = 0 + 0 - 25 = -25$

Tangent line: $y - 4 = -25(x - 0)$

$= -25x + 4$

43. $f(x) = \frac{x+1}{x-1},\ \left(\frac{1}{2}, -3\right)$

-2

$f(x) = \frac{1+\cos x}{1-\cos x},\ \left(\frac{\pi}{2}, 1\right)$

$f'(x) = \frac{(1-\cos x)(-\sin x) - (1+\cos x)(\sin x)}{(1-\cos x)^2}$

$= \frac{-2\sin x}{(1-\cos x)^2}$

$f'\left(\frac{\pi}{2}\right) = \frac{-2}{1} = -2$

Tangent line: $y - 1 = -2\left(x - \frac{\pi}{2}\right)$

$= -2x + 1 + \pi$

$g(t) = -8t^3 - 5t + 12$

$g'(t) = -24t^2 - 5$

$g''(t) = -48t$

$h(x) = 6x^{-2} + 7x^2$

$h'(x) = -12x^{-3} + 14x$

$h''(x) = 36x^{-4} + 14 = \frac{36}{x^4} + 14$

$f(x) = 15x^{5/2}$

$f'(x) = \frac{75}{2}x^{3/2}$

$f''(x) = \frac{225}{4}x^{1/2} = \frac{225}{4}\sqrt{x}$

48. $f(x) = 20\sqrt[5]{x} = 20x^{1/5}$

$f'(x) = 4x^{-4/5}$

$f''(x) = \frac{-16}{5}x^{-9/5} = -\frac{16}{5x^{9/5}}$

49. $f(\theta) = 3\tan\theta$

$f'(\theta) = 3\sec^2\theta$

$f''(\theta) = 6\sec\theta \sec\theta \tan\theta$

$6\sec^2\theta \tan\theta$

$h(t) = 10\cos t - 15\sin t$

$$f'(x) = \frac{(x-1) - (x+1)}{(x-1)^2} = \frac{\quad}{(x-1)^2}$$

$$f'\left(\frac{1}{2}\right) = \frac{-2}{(1/4)} = -8$$

Tangent line: $y + 3 = -8\left(x - \frac{1}{2}\right)$

$$y = -8x + 1$$

$$h'(t) = -10 \sin t - 15 \cos t$$

$$h''(t) = -10 \cos t + 15 \sin t$$

$$g(x) = 4 \cot x$$

$$g'(x) = -4 \csc^2 x$$

$$g''(x) = -8 \csc x(-\csc x \cot x)$$

$$= 8 \csc^2 x \cot x$$

$h(t) = -12\csc t$

$h'(t) = -12(-\csc t\cot t) = 12\csc t\cot t$

$h''(t) = 12\csc t(-\csc^2 t) + 12\cot t(-\csc t\cot t)$

$= -12(\csc^3 t + \csc t\cot^2 t)$

53. $v(t) = 20 - t^2,\ 0 \le t \le 6$

$a(t) = v'(t) = -2t$

$v(3) = 20 - 3^2 = 11$ m/sec

$a(3) = -2(3) = -6$ m/sec^2

54. $v(t) = \dfrac{90t}{4t + 10}$

$a(t) = \dfrac{(4t + 10)90 - 90t(4)}{(4t + 10)^2}$

$= \dfrac{900}{(4t + 10)^2} = \dfrac{225}{(2t + 5)^2}$

(a) $v(1) = \dfrac{90}{14} \approx 6.43$ ft/sec

$a(1) = \dfrac{225}{49} \approx 4.59$ ft/sec^2

(b) $v(5) = \dfrac{90(5)}{30} = 15$ ft/sec

$a(5) = \dfrac{225}{15^2} = 1$ ft/sec^2

$v(10) = \dfrac{90(10)}{50} = 18$ ft/sec

$a(10) = \dfrac{225}{25^2} = 0.36$ ft/sec^2

$y = (7x + 3)^4$

$y' = 4(7x + 3)^3(7) = 28(7x + 3)^3$

$y = (x^2 - 6)^3$

$y' = 3(x^2 - 6)^2(2x) = 6x(x^2 - 6)^2$

57. $y = \dfrac{1}{(x^2 + 5)^3} = (x^2 + 5)^{-3}$

58. $f(x) = \dfrac{1}{(5x + 1)^2} = (5x + 1)^{-2}$

$f'(x) = -2(5x + 1)^{-3}(5) = -\dfrac{10}{(5x + 1)^3}$

$y = 5\cos(9x + 1)$

$y' = -5\sin(9x + 1)(9) = -45\sin(9x + 1)$

$y = -6\sin 3x^4$

$y' = -6\cos(3x^4)(12x^3) = -72x^3\cos 3x^4$

$y = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$

$y' = \dfrac{1}{2} - \dfrac{1}{4}(\cos 2x)(2) = \dfrac{1}{2}(1 - \cos 2x) = \sin^2 x$

62. $y = \dfrac{\sec^7 x}{7} - \dfrac{\sec^5 x}{5}$

$y' = \sec^6 x(\sec x\tan x) - \sec^4 x(\sec x\tan x)$

$\sec^5 x\tan x(\sec^2 x - 1)$

$\sec^5 x\tan^3 x$

$y = x(6x + 1)^5$

$y' = x\,5(6x + 1)^4(6) + (6x + 1)^5(1)$

$30x(6x + 1)^4 + (6x + 1)^5$

$(6x + 1)^4(30x + 6x + 1)$

$(6x + 1)^4(36x + 1)$

64. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

$f'(s) = (s^2 - 1)^{5/2}(3s^2) + (s^3 + 5)\left(\frac{5}{2}\right)(s^2 - 1)^{3/2}(2s)$

$= s(s^2 - 1)^{3/2}\left[3s(s^2 - 1) + 5(s^3 + 5)\right]$

$s(s^2 - 1)^{3/2}(8s^3 - 3s + 25)$

65. $f(x) = \left(\dfrac{x}{\sqrt{x + 5}}\right)^3$

$$y' = -3(x^2+5)^{-4}(2x)$$

$$= -\frac{6x}{(x^2+5)^4}$$

$$f'(x) = 3\left(\frac{x}{\sqrt{x+5}}\right)^2\left[\frac{(x+5)^{1/2}(1) - x\left(\frac{1}{2}\right)(x+5)^{-1/2}}{x+5}\right]$$

$$= \frac{3x^2}{x+5}\left[\frac{2(x+5)-x}{2(x+5)^{3/2}}\right]$$

$$= \frac{3x^2(x+10)}{2(x+5)^{5/2}}$$

$h(x) = \left(\frac{x+5}{x^2+3}\right)^2$

$h'(x) = 2\left(\frac{x+5}{x^2+3}\right)\left(\frac{(x^2+3)(1)-(x+5)(2x)}{(x^2+3)^2}\right)$

$= \frac{2(x+5)(-x^2-10x+3)}{(x^2+3)^3}$

67. $f(x) = \sqrt{1-x^3}, \quad (-2, 3)$

$f'(x) = \frac{1}{2}(1-x^3)^{-1/2}(-3x^2) = \frac{-3x^2}{2\sqrt{1-x^3}}$

$f'(-2) = \frac{-12}{2(3)} = -2$

68. $f(x) = \sqrt[3]{x^2-1}, \quad (3, 2)$

$f'(x) = \frac{1}{3}(x^2-1)^{-2/3}(2x) = \frac{2x}{3(x^2-1)^{2/3}}$

$f'(3) = \frac{2(3)}{3(4)} = \frac{1}{2}$

69. $f(x) = \frac{x+8}{(3x+1)^{1/2}}, \quad (0, 8)$

$f'(x) = \frac{(3x+1)^{1/2}(1) - (x+8)\left(\frac{1}{2}\right)(3x+1)^{-1/2}(3)}{3x+1}$

$f'(0) = \frac{1-4(3)}{1} = -11$

70. $f(x) = \frac{3x+1}{(4x-3)^3}, \quad (1, 4)$

$f'(x) = \frac{(4x-3)^3(3) - (3x+1)3(4x-3)^2(4)}{(4x-3)^6}$

$f'(1) = \frac{3-4(3)(4)}{1} = -45$

$y = x \sin^2 x$

$y = \frac{1}{2}\csc 2x, \quad \left(\frac{\pi}{4}, \frac{1}{2}\right)$

$y' = -\csc 2x \cot 2x$

$y'\left(\frac{\pi}{4}\right) = 0$

72. $y = \csc 3x + \cot 3x, \quad \left(\frac{\pi}{6}, 1\right)$

$y' = -3\csc 3x \cot 3x - 3\csc^2 3x$

$y'\left(\frac{\pi}{6}\right) = 0 - 3 = -3$

$y = (8x+5)^3$

$y' = 3(8x+5)^2(8) = 24(8x+5)^2$

$y'' = 24(2)(8x+5)(8) = 384(8x+5)$

74. $y = \frac{1}{5x+1} = (5x+1)^{-1}$

$y' = -1(5x+1)^{-2}(5) = -5(5x+1)^{-2}$

$y'' = -5(-2)(5x+1)^{-3}(5) = \frac{50}{(5x+1)^3}$

$f(x) = \cot x$

$f'(x) = -\csc^2 x$

$f''(x) = -2\csc x(-\csc x \cdot \cot x)$

$= 2\csc^2 x \cot x$

$$y' = \sin^2 x + 2x \sin x \cos x$$

$$y'' = 2 \sin x \cos x + 2 \sin x \cos x + 2x \cos^2 x - 2x \sin^2 x$$

$$= 4 \sin x \cos x + 2x \left(\cos^2 x - \sin^2 x \right)$$

77. $T = \frac{700}{t^2 + 4t + 10}$

$= 700(t^2 + 4t + 10)^{-1}$

$T' = \frac{-1400(t+2)}{(t^2 + 4t + 10)^2}$

When $t = 1$,

$T' = \frac{-1400(1+2)}{(1+4+10)^2} \approx -18.667$ deg/h.

When $t = 3$,

$T' = \frac{-1400(3+2)}{(9+12+10)^2} \approx -7.284$ deg/h.

(c) When $t = 5$,

$T' = \frac{-1400(5+2)}{(25+20+10)^2} \approx -3.240$ deg/h.

(d) When $t = 10$,

$T' = \frac{-1400(10+2)}{(100+40+10)^2} \approx -0.747$ deg/h.

78. $y = \frac{1}{4}\cos 8t - \frac{1}{4}\sin 8t$

$y' = \frac{1}{4}(-\sin 8t)8 - \frac{1}{4}(\cos 8t)8$

$= -2\sin 8t - 2\cos 8t$

At time $t = \frac{\pi}{4}$,

$y\left(\frac{\pi}{4}\right) = \frac{1}{4}\cos\left[8\left(\frac{\pi}{4}\right)\right] - \frac{1}{4}\sin\left[8\left(\frac{\pi}{4}\right)\right]$

$= \frac{1}{4}(1) = \frac{1}{4}$ ft.

$v(t) = y'\left(\frac{\pi}{4}\right) = -2\sin\left[8\left(\frac{\pi}{4}\right)\right] - 2\cos\left[8\left(\frac{\pi}{4}\right)\right]$

$-2(0) - 2(1) = -2$ ft/sec

$x^2 + y^2 = 64$

$2x + 2yy' = 0$

$2yy' = -2x$

x

81. $x^3y - xy^3 = 4$

$x^3y' + 3x^2y - x3y^2y' - y^3 = 0$

$x^3y' - 3xy^2y' = y^3 - 3x^2y$

$y'(x^3 - 3xy^2) = y^3 - 3x^2y$

$y' = \frac{y^3 - 3x^2y}{x^3 - 3xy^2}$

$y' = \frac{y(y^2 - 3x^2)}{x(x^2 - 3y^2)}$

$xy = x - 4\sqrt{y}$

$\frac{\sqrt{x}}{2\sqrt{y}}y' + \frac{\sqrt{y}}{2\sqrt{x}} = 1 - 4y'$

$xy' + y = 2\sqrt{xy} - 8\sqrt{xy}\,y'$

$x + 8\sqrt{xy}\,y' = 2\sqrt{xy} - y$

$y' = \frac{2\sqrt{xy} - y}{x + 8\sqrt{xy}}$

$= \frac{2(x - 4y) - y}{x + 8(x - 4y)}$

$= \frac{2x - 9y}{9x - 32y}$

$x\sin y = y\cos x$

$(x\cos y)y' + \sin y = -y\sin x + y'\cos x$

$y'(x\cos y - \cos x) = -y\sin x - \sin y$

$y' = \frac{y\sin x + \sin y}{\cos x - x\cos y}$

$\cos(x + y) = x$

$-(1 + y')\sin(x + y) = 1$

$y'\sin(x + y) = 1 + \sin(x + y)$

$y' = -\frac{1 + \sin(x + y)}{\sin(x + y)} = -\csc(x + 1) - 1$

$$y' = -\frac{}{}y$$

$$x^2 + 4xy - y^3 = 6$$

$$2x + 4xy' + 4y - 3y^2 y' = 0$$

$$\left(4x - 3y^2\right)y' = -2x - 4y$$

$$y' = \frac{2x + 4y}{3y^2 - 4x}$$

85. $x^2 + y^2 = 10$

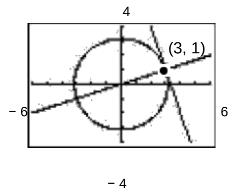
$2x + 2yy' = 0$

$y' = \dfrac{-x}{y}$

At $(3, 1)$, $y' = -3$

Tangent line: $y - 1 = -3(x - 3) \Rightarrow 3x + y - 10 = 0$

Normal line: $y - 1 = \frac{1}{3}(x - 3) \Rightarrow x - 3y = 0$



$x^2 - y^2 = 20$

$2x - 2yy' = 0$

$y' = \dfrac{x}{y}$

At $(6, 4)$, $y' = \frac{3}{2}$

Tangent line: $y - 4 = \frac{3}{2}(x - 6)$

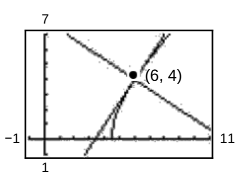
$y = \frac{3}{2}x - 5$

$2y - 3x + 10 = 0$

Normal line: $y - 4 = -\frac{2}{3}(x - 6)$

$= -\frac{2}{3}x + 8$

$3y + 2x - 24 = 0$



$y = \sqrt{x}$

$\dfrac{dy}{dt} = 2$ units/sec

$\dfrac{dy}{dt} = \dfrac{1}{2\sqrt{x}}\dfrac{dx}{dt} \Rightarrow \dfrac{dx}{dt} = 2\sqrt{x}\dfrac{dy}{dt} = 4\sqrt{x}$

(a) When $x = \frac{1}{2}$, $\dfrac{dx}{dt} = 2\sqrt{2}$ units/sec.

When $x = 1$, $\dfrac{dx}{dt} = 4$ units/sec.

When $x = 4$, $\dfrac{dx}{dt} = 8$ units/sec.

Surface area $= A = 6x^2$, $x =$ length of edge

$\dfrac{dx}{dt} = 8$

$\dfrac{dA}{dt} = 12x\dfrac{dx}{dt} = 12(6.5)(8) = 624\ \text{cm}^2/\text{sec}$

$\tan\theta = x$

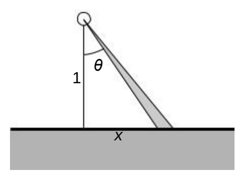
$\dfrac{d\theta}{dt} = 3(2\pi)$ rad/min

$\sec^2\theta\left(\dfrac{d\theta}{dt}\right) = \dfrac{dx}{dt}$

$\dfrac{dx}{dt} = (\tan^2\theta + 1)(6\pi) = 6\pi(x^2 + 1)$

When $x = \frac{1}{2}$,

$\dfrac{dx}{dt} = 6\pi\left(\dfrac{1}{4} + 1\right) = \dfrac{15\pi}{2}$ km/min $= 450\pi$ km/h.



$s(t) = 60 - 4.9t^2$

$s'(t) = -9.8t$

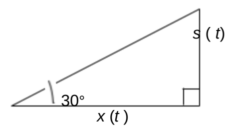
$s = 35 = 60 - 4.9t^2$

$4.9t^2 = 25$

$t = \frac{5}{\sqrt{4.9}}$

$\tan 30 = \frac{1}{\sqrt{3}} = \frac{s(t)}{x(t)}$

$x(t) = \sqrt{3}s(t)$



$\frac{dx}{dt} = \sqrt{3}\frac{ds}{dt} = \sqrt{3}\left(-9.8\frac{5}{\sqrt{4.9}}\right) \approx -38.34$ m/sec

Problem Solving for Chapter 2

(a) $x^2 + (y - r)^2 = r^2$, Circle

$x^2 = y$, Parabola

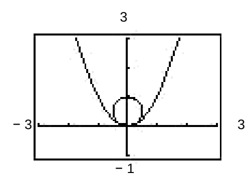
Substituting:

$y - r)^2 = r^2 - y$

$y^2 - 2ry + r^2 = r^2 - y$

$y^2 - 2ry + y = 0$

$y(y - 2r + 1) = 0$



Because you want only one solution, let $1 - 2r = 0 \Rightarrow r = \frac{1}{2}$. Graph $y = x^2$ and $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$.

(b) Let (x, y) be a point of tangency:

$x^2 + (y - b)^2 = 1 \Rightarrow 2x + 2(y - b)y' = 0 \Rightarrow y' = \frac{x}{b - y}$, Circle

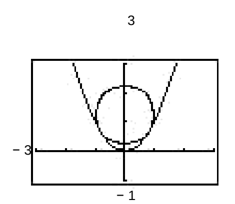
$= x^2 \Rightarrow y' = 2x$, Parabola

Equating:

$2x = \frac{x}{b - y}$

$2(b - y) = 1$

$b - y = \frac{1}{2} \Rightarrow b = y + \frac{1}{2}$



Also, $x^2 + (y - b)^2 = 1$ and $y = x^2$ imply:

$y + (y - b)^2 = 1 \Rightarrow y + \left[y - \left(y + \frac{1}{2}\right)\right]^2 = 1 \Rightarrow y + \frac{1}{4} = 1 \Rightarrow y = \frac{3}{4}$ and $b = \frac{5}{4}$

Center: $\left(0, \frac{5}{4}\right)$

Graph $y = x^2$ and $x^2 + \left(y - \frac{5}{4}\right)^2 = 1$.

( ) ( )

2. Let (a, a^2) and $(b, -b^2 + 2b - 5)$ be the points of tangency. For $y = x^2$, $y' = 2x$ and for $y = -x^2 + 2x - 5$,

$y' = -2x + 2$. So, $2a = -2b + 2 \Rightarrow a + b = 1$, or $a = 1 - b$. Furthermore, the slope of the common tangent line is

$$\frac{a^2 - (-b^2 + 2b - 5)}{a - b} = \frac{(1-b)^2 + b^2 - 2b + 5}{(1 - b) - b} = -2b + 2$$

$$\frac{1 - 2b + b^2 + b^2 - 2b + 5}{1 - 2b} = -2b + 2$$

$$2b^2 - 4b + 6 = 4b^2 - 6b + 2$$

$$2b^2 - 2b - 4 = 0$$

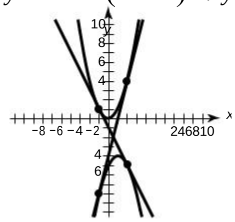
$$b^2 - b - 2 = 0$$

$$(b - 2)(b + 1) = 0$$

$$= 2, -1$$

For $b = 2$, $a = 1 - b = -1$ and the points of tangency are $(-1, 1)$ and $(2, -5)$. The tangent line has slope -2: $y - 1 = -2(x + 1) \Rightarrow y = -2x - 1$

For $b = -1$, $a = 1 - b = 2$ and the points of tangency are $(2, 4)$ and $(-1, -8)$. The tangent line has slope $y - 4 = 4(x - 2) \Rightarrow y = 4x - 4$



Let $p(x) = Ax^3 + Bx^2 + Cx + D$, $p'(x) = 3Ax^2 + 2Bx + C$.

At $(1, 1)$:

$A + B + C + D = 1$ Equation 1

$3A + 2B + C = 14$ Equation 2

At $(-1, -3)$:

$-A + B - C + D = -3$ Equation 3

$3A - 2B + C = -2$ Equation 4

Adding Equations 1 and 3: $2B + 2D = -2$

Subtracting Equations 1 and 3: $2A + 2C = 4$ $D = \frac{1}{2}(-2 - 2B) = -5$.

Adding Equations 2 and 4: $6A + 2C = 12$

Subtracting Equations 2 and 4: $4B = 16$

So, $B = 4$ and $D = \frac{1}{2}(-2 - 2B) = -5$. Subtracting $2A + 2C = 4$ and $6A + 2C = 12$,

you obtain $4A = 8 \Rightarrow A = 2$. Finally, $C = \frac{1}{2}(4 - 2A) = 0$. So, $p(x) = 2x^3 + 4x^2 - 5$.

$f(x) = a + b\cos cx$

$f'(x) = -bc\sin cx$

At $(0, 1)$: $a + b = 1$ Equation 1

At $\left(\frac{\pi}{4}, \frac{3}{2}\right)$: $a + b\cos\left(\frac{c\pi}{4}\right) = \frac{3}{2}$ Equation 2

$-bc\sin\left(\frac{c\pi}{4}\right) = 1$ Equation 3

From Equation 1, $a = 1 - b$. Equation 2 becomes

$$(1 - b) + b\cos\left(\frac{c\pi}{4}\right) = \frac{3}{2} \Rightarrow -b + b\cos\frac{c\pi}{4} = \frac{1}{2}.$$

From Equation 3, $b = \dfrac{-1}{c\sin(c\pi/4)}$. So:

$$\frac{1}{c\sin(c\pi/4)} + \frac{-1}{c\sin(c\pi/4)}\cos\left(\frac{c\pi}{4}\right) = \frac{1}{2}$$

$$1 - \cos\left(\frac{c\pi}{4}\right) = \frac{1}{2}c\sin\left(\frac{c\pi}{4}\right)$$

Graphing the equation

$$g(c) = \frac{1}{2}c\sin\left(\frac{c\pi}{4}\right) + \cos\left(\frac{c\pi}{4}\right) - 1,$$

you see that many values of c will work. One answer:

$c = 2, b = -\frac{1}{2}, a = \frac{3}{2} \Rightarrow f(x) = \frac{3}{2} - \frac{1}{2}\cos 2x$

(a) $y = x^2, y' = 2x$, Slope $= 4$ at $(2, 4)$

Tangent line: $y - 4 = 4(x - 2)$ $y = 4x - 4$

Slope of normal line: $-\frac{1}{4}$

Normal line: $y - 4 = -\frac{1}{4}(x - 2)$

$$y = -\frac{1}{4}x + \frac{9}{2}$$

$$y = -\frac{1}{4}x + \frac{9}{2} = x^2$$

$$\Rightarrow 4x^2 + x - 18 = 0$$

$$\Rightarrow (4x + 9)(x - 2) = 0$$

$x = 2, -\frac{9}{4}$

Second intersection point: $\left(-\frac{9}{4}, \frac{81}{16}\right)$

Tangent line: $y = 0$

Normal line: $x = 0$

(d) Let be a point on the parabola

Tangent line at is

Normal line at is

To find points of intersection, solve:

The normal line intersects a second time at

6. (a) $f(x) = \cos x$ $P_1(x) = a_0 + a_1 x$

$f(0) = 1$ $P_1(0) = a_0 \Rightarrow a_0 = 1$

$f'(0) = 0$ $P_1'(0) = a_1 \Rightarrow a_1 = 0$

$P_1(x) = 1$

(b) $f(x) = \cos x$ $P_2(x) = a_0 + a_1x + a_2x^2$

$f(0) = 1$ $P_2(0) = a_0 \Rightarrow a_0 = 1$

$f'(0) = 0$ $P_2'(0) = a_1 \Rightarrow a_1 = 0$

$f''(0) = -1$ $P_2''(0) = 2a_2 \Rightarrow a_2 = -\frac{1}{2}$

$P_2(x) = 1 - \frac{1}{2}x^2$

(c)

x	−1.0	−0.1	−0.001	0	0.001	0.1	1.0
$\cos x$	0.5403	0.9950	≈1	1	≈1	0.9950	0.5403
$P_2(x)$	0.5	0.9950	≈1	1	≈1	0.9950	0.5

$P_2(x)$ is a good approximation of $f(x) = \cos x$ when x is near 0.

$(a, a^2), a \neq 0,$ $y = x^2.$

(a, a^2) $y = 2a(x - a) + a^2.$

(a, a^2) $y = -(1/2a)(x - a) + a^2.$

$$x^2 = -\frac{1}{2a}(x - a) + a^2$$

$$x^2 + \frac{1}{2a}x = a^2 + \frac{1}{2}$$

$$x^2 + \frac{1}{2a}x + \frac{1}{16a^2} = a^2 + \frac{1}{2} + \frac{1}{16a^2}$$

$$\left(x + \frac{1}{4a}\right)^2 = \left(a + \frac{1}{4a}\right)^2$$

$$x + \frac{1}{4a} = \pm\left(a + \frac{1}{4a}\right)$$

$$x + \frac{1}{4a} = a + \frac{1}{4a} \Rightarrow x = a \quad \text{(Point of tangency)}$$

$$x + \frac{1}{4a} = -\left(a + \frac{1}{4a}\right) \Rightarrow x = -a - \frac{1}{2a} = -\frac{2a^2 + 1}{2a}$$

$$x = -\frac{2a^2 + 1}{2a}.$$

$f(x) = \sin x$ $\qquad P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$f(0) = 0 \qquad P_3(0) = a_0 \Rightarrow a_0 = 0$

$f'(0) = 1 \qquad P_3'(0) = a_1 \Rightarrow a_1 = 1$

$f''(0) = 0 \qquad P_3''(0) = 2a_2 \Rightarrow a_2 = 0$

$f'''(0) = -1 \qquad P_3'''(0) = 6a_3 \Rightarrow a_3 = -\frac{1}{6}$

$P_3(x) = x - \frac{1}{6}x^3$

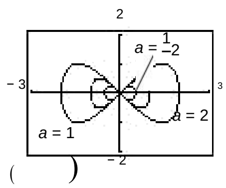
7. (a) $x^4 = a^2x^2 - a^2y^2$

$a^2y^2 = a^2x^2 - x^4$

$y = \dfrac{\pm\sqrt{a^2x^2 - x^4}}{a}$

Graph: $y_1 = \dfrac{\sqrt{a^2x^2 - x^4}}{a}$ and $y_2 = -\dfrac{\sqrt{a^2x^2 - x^4}}{a}$.

(b)



$(\pm a, 0)$ are the x-intercepts, along with $(0, 0)$.

(c) Differentiating implicitly:

$4x^3 = 2a^2x - 2a^2yy'$

$y' = \dfrac{2a^2x - 4x^3}{2a^2y} = \dfrac{x(a^2 - 2x^2)}{a^2y} = 0 \Rightarrow 2x^2 = a^2 \Rightarrow x = \dfrac{\pm a}{\sqrt{2}}$

$\left(\dfrac{a^2}{2}\right)^2 = a^2\left(\dfrac{a^2}{2}\right) - a^2y^2$

$\dfrac{a^4}{4} = \dfrac{a^4}{2} - a^2y^2$

$a^2y^2 = \dfrac{a^4}{4}$

$y^2 = \dfrac{a^2}{4}$

$y = \pm\dfrac{a}{2}$

Four points: $\left(\dfrac{a}{\sqrt{2}}, \dfrac{a}{2}\right), \left(\dfrac{a}{\sqrt{2}}, -\dfrac{a}{2}\right), \left(-\dfrac{a}{\sqrt{2}}, \dfrac{a}{2}\right), \left(-\dfrac{a}{\sqrt{2}}, -\dfrac{a}{2}\right)$

8. (a) $b^2y^2 = x^3(a - x);\ a, b > 0$

$y^2 = \dfrac{x^3(a - x)}{b^2}$

Graph $y_1 = \dfrac{\sqrt{x^3(a - x)}}{b}$ and $y_2 = -\dfrac{\sqrt{x^3(a - x)}}{b}$.

a determines the x-intercept on the right: $(a, 0)$. b affects the height.

Differentiating implicitly:

$2b^2yy' = 3x^2(a - x) - x^3 = 3ax^2 - 4x^3$

$y' = \dfrac{3ax^2 - 4x^3}{2b^2y} = 0$

$3ax^2 = 4x^3$

$3a = 4x$

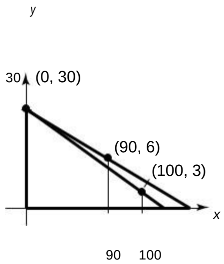
$\dfrac{3a}{4} = x$

$b^2y^2 = \left(\dfrac{3a}{4}\right)^3\left(a - \dfrac{3a}{4}\right) = \dfrac{27a^3}{64}\left(\dfrac{1}{4}a\right)$

$y^2 = \dfrac{27a^4}{256b^2} \Rightarrow y = \pm\dfrac{3\sqrt{3}a^2}{16b}$

Two points: $\left(\frac{3a}{4}, \frac{3\sqrt{3}a^2}{16b}\right), \left(\frac{3a}{4}, -\frac{3\sqrt{3}a^2}{16b}\right)$

9. (a)



Not drawn to scale

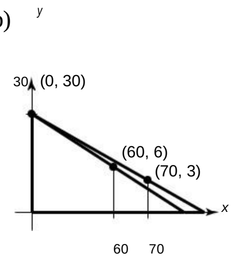
Line determined by $(0, 30)$ and $(90, 6)$:

$$y - 30 = \frac{30-6}{0-90}(x - 0) = -\frac{24}{90}x = -\frac{4}{15}x \Rightarrow y = -\frac{4}{15}x + 30$$

When $x = 100$: $y = -\frac{4}{15}(100) + 30 = \frac{10}{3} > 3$

As you can see from the figure, the shadow determined by the man extends beyond the shadow determined by the child.

(b)



Not drawn to scale

Line determined by $(0, 30)$ and $(60, 6)$:

$$y - 30 = \frac{30-6}{0-60}(x - 0) = -\frac{2}{5}x \Rightarrow y = -\frac{2}{5}x + 30$$

When $x = 70$: $y = -\frac{2}{5}(70) + 30 = 2 < 3$

As you can see from the figure, the shadow determined by the child extends beyond the shadow determined by the man.

(c) Need $(0, 30)$, $(d, 6)$, $(d + 10, 3)$ collinear.

$$\frac{30-6}{0-d} = \frac{6-3}{d-(d+10)} \Rightarrow \frac{24}{d} = \frac{3}{10} \Rightarrow d = 80 \text{ feet}$$

Let y be the distance from the base of the street light to the tip of the shadow. You know that $dx/dt = -5$.

For $x > 80$, the shadow is determined by the man.

$$\frac{y}{30} = \frac{y-x}{6} \Rightarrow y = \frac{5}{4}x \text{ and } \frac{dy}{dt} = \frac{5}{4}\frac{dx}{dt} = \frac{-25}{4}$$

For $x < 80$, the shadow is determined by the child.

$$\frac{y}{30} = \frac{y-x-10}{3} \Rightarrow y = \frac{10}{9}x + \frac{100}{9} \text{ and } \frac{dy}{dt} = \frac{10}{9}\frac{dx}{dt} = -\frac{50}{9}$$

Therefore:

$$\frac{dy}{dt} = \begin{cases} -\frac{25}{4}, & x > 80 \\ -\frac{50}{9}, & 0 < x < 80 \end{cases}$$

dy/dt is not continuous at $x = 80$.

ALTERNATE SOLUTION for parts (a) and (b):

(a) As before, the line determined by the man's shadow is

$$y_m = -\frac{4}{15}x + 30$$

The line determined by the child's shadow is obtained by finding the line through $(0, 30)$ and $(100, 3)$:

$$y - 30 = \frac{30-3}{0-100}(x - 0) \Rightarrow y_c = -\frac{27}{100}x + 30$$

By setting $y_m = y_c = 0$, you can determine how far the shadows extend:

Man: $y_m = 0 \Rightarrow \frac{4}{15}x = 30 \Rightarrow x = 112.5 = 112\frac{1}{2}$

Child: $y_c = 0 \Rightarrow \frac{27}{100}x = 30 \Rightarrow x = 111.\overline{11} = 111\frac{1}{9}$

The man's shadow is $112\frac{1}{2} - 111\frac{1}{9} = 1\frac{7}{18}$ ft beyond the child's shadow.

(b) As before, the line determined by the man's shadow is

$$y_m = -\frac{2}{5}x + 30$$

For the child's shadow,

$$y - 30 = \frac{30-3}{0-70}(x-0) \Rightarrow y_c = -\frac{27}{70}x + 30$$

Man: $y_m = 0 \Rightarrow \frac{2}{5}x = 30 \Rightarrow x = 75$

Child: $y_c = 0 \Rightarrow \frac{27}{70}x = 30 \Rightarrow x = \frac{700}{9} = 77\frac{7}{9}$

So the child's shadow is $77\frac{7}{9} - 75 = 2\frac{7}{9}$ ft beyond the man's shadow.

10. (a) $y = x^{1/3} \Rightarrow \frac{dy}{dt} = \frac{1}{3}x^{-2/3}\frac{dx}{dt}$

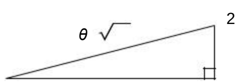
$$1 = \frac{1}{3}(8)^{-2/3}\frac{dx}{dt}$$

$$\frac{dx}{dt} = 12 \text{ cm/sec}$$

(b) $D = \sqrt{x^2 + y^2} \Rightarrow \frac{dD}{dt} = \frac{1}{2}(x^2 + y^2)^{-1/2}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right) = \frac{x(dx/dt) + y(dy/dt)}{\sqrt{x^2 + y^2}}$

$$= \frac{8(12) + 2(1)}{\sqrt{64+4}} = \frac{98}{\sqrt{68}} = \frac{49}{\sqrt{17}} \text{ cm/sec}$$

(c) $\tan\theta = \frac{y}{x} \Rightarrow \sec^2\theta \cdot \frac{d\theta}{dt} = \frac{x(dy/dt) - y(dx/dt)}{x^2}$



From the triangle, $\sec\theta = \sqrt{68}/8$. So $\frac{d\theta}{dt} = \frac{64}{68}\left(\frac{8(1) - 2(12)}{64}\right) = \frac{-16}{68} = -\frac{4}{17}$ rad/sec.

(a) $v(t) = -\frac{27}{5}t + 27$ ft/sec

$a(t) = -\frac{27}{5}$ ft/sec^2

$v(t) = -\frac{27}{5}t + 27 = 0 \Rightarrow \frac{27}{5}t = 27 \Rightarrow t = 5$ seconds

$s(5) = -\frac{27}{10}(5)^2 + 27(5) + 6 = 73.5$ feet

The acceleration due to gravity on Earth is greater in magnitude than that on the moon.

12. $E'(x) = \lim\limits_{\Delta x \to 0} \frac{E(x + \Delta x) - E(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{E(x)E(\Delta x) - E(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} E(x)\frac{(E(\Delta x) - 1)}{\Delta x} = E(x)\lim\limits_{\Delta x \to 0}\frac{E(\Delta x) - 1}{\Delta x}$

$x \to 0$ x $x \to 0$ x $x \to 0$ x $x \to 0$ x

But, $E'(0) = \lim_{x \to 0} \frac{E(x) - E(0)}{x} = \lim_{x \to 0} \frac{E(x) - 1}{x} = 1$. So, $E'(x) = E(x)E'(0) = E(x)$ exists for all x.

For example: $E(x) = e^x$.

13. $L'(x) = \lim_{\Delta x \to 0} \frac{L(x + \Delta x) - L(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{L(x) + L(\Delta x) - L(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{L(\Delta x)}{\Delta x}$

Also, $L'(0) = \lim_{\Delta x \to 0} \frac{L(\Delta x) - L(0)}{\Delta x}$. But, $L(0) = 0$ because $L(0) = L(0 + 0) = L(0) + L(0) \Rightarrow L(0) = 0.$

So, $L'(x) = L'(0)$ for all x. The graph of L is a line through the origin of slope $L'(0)$.

14. (a)

z (degrees)	0.1	0.01	0.0001
$\frac{\sin z}{z}$	0.0174524	0.0174533	0.0174533

(b) $\lim_{z \to 0} \frac{\sin z}{z} \approx 0.0174533$

In fact, $\lim_{z \to 0} \frac{\sin z}{z} = \frac{\pi}{180}$.

(c) $\frac{d}{dz}(\sin z) = \lim_{\Delta z \to 0} \frac{\sin(z + \Delta z) - \sin z}{\Delta z}$

$= \lim_{\Delta z \to 0} \frac{\sin z \cdot \cos \Delta z + \sin \Delta z \cdot \cos z - \sin z}{\Delta z}$

$= \lim_{\Delta z \to 0} \left[\sin z \left(\frac{\cos \Delta z - 1}{\Delta z} \right) \right] + \lim_{\Delta z \to 0} \left[\cos z \left(\frac{\sin \Delta z}{\Delta z} \right) \right]$

$= (\sin z)(0) + (\cos z)\left(\frac{\pi}{180} \right) = \frac{\pi}{180} \cos z$

$S(90) = \sin\left(\frac{\pi}{180} 90 \right) = \sin \frac{\pi}{2} = 1$

$C(180) = \cos\left(\frac{\pi}{180} 180 \right) = -1$

$\frac{d}{dz} S(z) = \frac{d}{dz} \sin(cz) = c \cdot \cos(cz) = \frac{\pi}{180} C(z)$

The formulas for the derivatives are more complicated in degrees.

$j(t) = a'(t)$

$j(t)$ is the rate of change of acceleration.

$s(t) = -8.25t^2 + 66t$
$v(t) = -16.5t + 66$
$a(t) = -16.5$
$a'(t) = j(t) = 0$

The acceleration is constant, so $j(t) = 0$.

a is position.
b is acceleration.
c is jerk.
d is velocity.

Chapter 2 Differentiation

Chapter Comments

The material presented in Chapter 2 forms the basis for the remainder of calculus. Much of it needs to be memorized, beginning with the definition of a derivative of a function found on page 103. Students need to have a thorough understanding of the tangent line problem and they need to be able to find an equation of a tangent line. Frequently, students will use the function $f'(x)$ as the slope of the tangent line. They need to understand that $f'(x)$ is the formula for the slope and the actual value of the slope can be found by substituting into $f'(x)$ the appropriate value for x On pages 105–106 of Section 2.1, you will find a discussion of situations where the derivative fails to exist. These examples (or similar ones) should be discussed in class.

As you teach this chapter, vary your notations for the derivative. One time write y'; another time write $dy\ dx$ or $f'(x)$. Terminology is also important. Instead of saying "find the derivative," sometimes say, "differentiate." This would be an appropriate time, also, to talk a little about Leibnitz and Newton and the discovery of calculus.

Sections 2.2, 2.3, and 2.4 present a number of rules for differentiation. Have your students memorize the Product Rule and the Quotient Rule (Theorems 2.7 and 2.8) in words rather than symbols. Students tend to be lazy when it comes to trigonometry and therefore, you need to impress upon them that the formulas for the derivatives of the six trigonometric functions need to be memorized also. You will probably not have enough time in class to prove every one of these differentiation rules, so choose several to do in class and perhaps assign a few of the other proofs as homework.

The Chain Rule, in Section 2.4, will require two days of your class time. Students need a lot of practice with this and the algebra involved in these problems. Many students can find the derivative of $f(x) = x^2\sqrt{1 - x^2}$ without much trouble, but simplifying the answer is often difficult for them. Insist that they learn to factor and write the answer without negative exponents. Strive to get the answer in the form given in the back of the book. This will help them later on when the derivative is set equal to zero.

Implicit differentiation is often difficult for students. Have students think of as a function of x and therefore y^3 is $[f(x)]^3$. This way they can relate implicit differentiation to the Chain Rule studied in the previous section.

Try to get your students to see that related rates, discussed in Section 2.6, are another use of the Chain Rule.

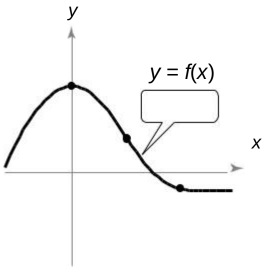
Section 2.1 The Derivative and the Tangent Line Problem

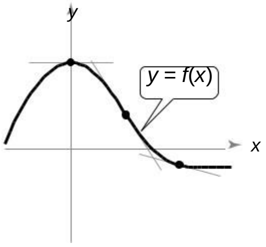
Section Comments

2.1 **The Derivative and the Tangent Line Problem**—Find the slope of the tangent line to a curve at a point. Use the limit definition to find the derivative of a function. Understand the relationship between differentiability and continuity.

Teaching Tips

Ask students what they think "the line tangent to a curve" means. Draw a curve with tangent lines to show a visual picture of tangent lines. For example:





When talking about the tangent line problem, use the suggested example of finding the equation of the tangent line to the parabola $y = x^2$ at the point $(1, 1)$.

Compute an approximation of the slope m by choosing a nearby point $Q(x, x^2)$ on the parabola and computing the slope m_{PQ} of the secant line PQ.

After going over Examples 1–3, return to Example 2 where $f(x) = x^2 + 1$ and note that $f'(x) = 2x$. How can we find the equation of the line tangent to f and parallel to $4x - y = 0$? Because the slope of the line is 4,

$2x = 4$

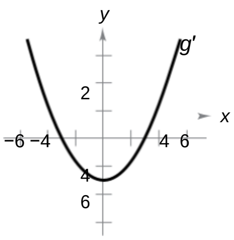
$x = 2.$

So, at the point $(2, 5)$, the tangent line is parallel to $4x - y = 0$. The equation of the tangent line is $y - 5 = 4(x - 2)$ or $y = 4x - 3$.

Be sure to find the derivatives of various types of functions to show students the different types of techniques for finding derivatives. Some suggested problems are $f(x) = 4x^3 - 3x^2$, $g(x) = 2 (x - 1)$, and $h(x) = 2x + 5$.

How Do You See It? Exercise

Page 108, Exercise 64 The figure shows the graph of g'.



(a) $g'(0) =$

(b) $g'(3) =$

(c) What can you conclude about the graph of g knowing that $g'(1) = -\frac{8}{3}$?

(d) What can you conclude about the graph of g knowing that $g'(-4) = \frac{7}{3}$?

(e) Is $g(6) - g(4)$ positive or negative? Explain.

(f) Is it possible to find $g(2)$ from the graph? Explain.

Solution

(a) $g'(0) = -3$

(b) $g'(3) = 0$

(c) Because $g'(1) = -\frac{8}{3}$, g is decreasing (falling) at $x = 1$.

(d) Because $g'(-4) = \frac{7}{3}$, g is increasing (rising) at $x = -4$.

(e) Because $g'(4)$ and $g'(6)$ are both positive, $g(6)$ is greater than $g(4)$ and $g(6) - g(4) > 0$.

(f) No, it is not possible. All you can say is that g is decreasing (falling) at $x = 2$.

Suggested Homework Assignment

Pages 107–109: 1, 3, 7, 11, 21–27 odd, 37, 43–47 odd, 53, 57, 61, 77, 87, 93, and 95.

Section 2.2 Basic Differentiation Rules and Rates of Change

Section Comments

2.2 **Basic Differentiation Rules and Rates of Change**—Find the derivative of a function using the Constant Rule. Find the derivative of a function using the Power Rule. Find the derivative of a function using the Constant Multiple Rule. Find the derivative of a function using the Sum and Difference Rules. Find the derivatives of the sine function and of the cosine function. Use derivatives to find rates of change.

Teaching Tips

Start by showing proofs of the Constant Rule and the Power Rule. Students who are mathematics majors need to start seeing proofs early on in their college careers as they will be taking Functions of a Real Variable at some point.

Go over an example in class like $f(x) = \frac{5x^2 + x}{x}$. Show students that before differentiating they can rewrite the function as $f(x) = 5x + 1$. Then they can differentiate to obtain $f'(x) = 5$. Use this example to emphasize the prudence of examining the function first before differentiating. Rewriting the function in a simpler, equivalent form can expedite the differentiating process.

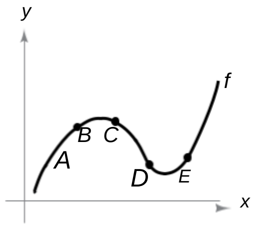
Give mixed examples of finding derivatives. Some suggested examples are:

$f(x) = 3x^6 - x^{2/3} + 3\sin x$ and $g(x) = \frac{4}{\sqrt[3]{x}} + \frac{2}{(3x)^2} - 3\cos x + 7x + \pi^3$.

This will test students' understanding of the various differentiation rules of this section.

How Do You See It? Exercise

Page 119, Exercise 76 Use the graph of f to answer each question. To print an enlarged copy of the graph, go to *MathGraphs.com.*



(a) Between which two consecutive points is the average rate of change of the function greatest?

(b) Is the average rate of change of the function between *A* and *B* greater than or less than the instantaneous rate of change at *B*?

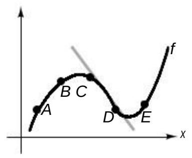
(c) Sketch a tangent line to the graph between *C* and *D* such that the slope of the tangent line is the same as the average rate of change of the function between *C* and *D*.

Solution

(a) The slope appears to be steepest between *A* and *B*.

(b) The average rate of change between *A* and *B* is **greater** than the instantaneous rate of change at *B*.

(c)



Suggested Homework Assignment

Pages 118–120: 1, 3, 5, 7–29 odd, 35, 39–53 odd, 55, 59, 65, 75, 85–89 odd, 91, 95, and 97.

Section 2.3 Product and Quotient Rules and Higher-Order Derivatives

Section Comments

2.3 **Product and Quotient Rules and Higher-Order Derivatives**—Find the derivative of a function using the Product Rule. Find the derivative of a function using the Quotient Rule. Find the derivative of a trigonometric function. Find a higher-order derivative of a function.

Teaching Tips

Some students have difficulty simplifying polynomial and rational expressions. Students should review these concepts by studying Appendices A.2–A.4 and A.7 in *Precalculus*, 10th edition, by Larson.

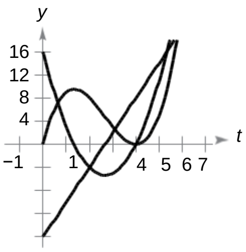
When teaching the Product and Quotient Rules, give proofs of each rule so that students can see where the rules come from. This will provide mathematics majors a tool for writing proofs, as each proof requires subtracting and adding the same quantity to achieve the desired results. For the Project Rule, emphasize that there are many ways to write the solution. Remind students that there must be one derivative in each term of the solution. Also, the Product Rule can be extended to more that just the product of two functions. Simplification is up to the discretion of the instructor. Examples such as $f(x) = (2x^2 - 3x)(5x^3 + 6)$ can be done with or without the Product Rule. Show the class both ways.

After the Quotient Rule has been proved to the class, give students the memorization tool of LO d HI – HI d LO. This will give students a way to memorize what goes in the numerator of the Quotient Rule.

Some examples to use are $f(x) = \dfrac{2x - 1}{x^2 + 7x}$ and $g(x) = \dfrac{4 - (1\ x)}{3 - x^2}$. Save $f(x)$ for the next section as this will be a good example for the Chain Rule. $g(x)$ is a good example for first finding the least common denominator.

How Do You See It? Exercise

Page 132, Exercise 120 The figure shows the graphs of the position, velocity, and acceleration functions of a particle.

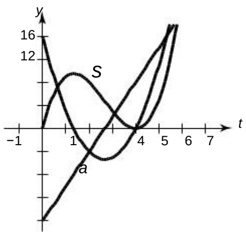


(a) Copy the graphs of the functions shown. Identify each graph. Explain your reasoning. To print an enlarged copy of the graph, go to *MathGraphs.com.*

(b) On your sketch, identify when the particle speeds up and when it slows down. Explain your reasoning.

Solution

(a)

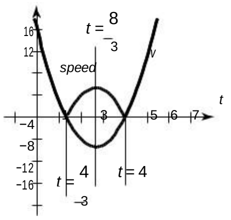


s position function

v velocity function

a acceleration function

(b) The speed of the particle is the absolute value of its velocity. So, the particle's speed is slowing down on the intervals (0, 4 3), and (8 3, 4) and it speeds up on the intervals (4 3, 8 3) and (4, 6).



Suggested Homework Assignment

Pages 129–132: 1, 3, 9, 13, 19, 23, 29–55 odd, 59, 61, 63, 75, 77, 91–107 odd, 111, 113, 117, and 131–135 odd.

Section 2.4 The Chain Rule

Section Comments

2.4 **The Chain Rule**—Find the derivative of a composite function using the Chain Rule. Find the derivative of a function using the General Power Rule. Simplify the derivative of a function using algebra. Find the derivative of a trigonometric function using the Chain Rule.

Teaching Tips

Begin this section by asking students to consider finding the derivative of $F(x) = \sqrt{x^2 + 1}$. F is a composite function. Letting $y = f(u) = u$ and $u = g(x) = x^2 + 1$, then $y = F(x) = f(g(x))$ or $F = f \circ g$. When stating the Chain Rule, be sure to state it using function notation and using Leibniz notation as students will see both forms when studying other courses with other texts. Following the definition, be sure to prove the Chain Rule as done on page 134.

Be sure to give examples that involve all rules discussed so far. Some examples include: $f(x) = (\sin(6x))^4$, $g(x) = \left(\frac{3 + \sin(2x)}{\sqrt[3]{x + 3}}\right)^2$, and $h(x) = \sqrt{x - \frac{2}{x}} \cdot [8x + \cos(x^2 + 1)]^3$.

You can use Exercise 98 on page 141 to review the following concepts:

- Product Rule
- Chain Rule
- Quotient Rule
- General Power Rule

Students need to understand these rules because they are the foundation of the study of differentiation.

Use the solution to show students how to solve each problem. As you apply each rule, give the definition of the rule verbally. Note that part (b) is not possible because we are not given $g'(3)$.

Solution

(a) $f(x) = g(x)h(x)$

$f'(x) = g(x)h'(x) + g'(x)h(x)$

$f'(5) = (-3)(-2) + (6)(3) = 24$

(b) $f(x) = g(h(x))$

$f'(x) = g'(h(x))h'(x)$

$f'(5) = g'(3)(-2) = -2g'(3)$

Not possible. You need $g'(3)$ to find $f'(5)$.

(c) $f(x) = \frac{g(x)}{h(x)}$

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(x) = \frac{(3)(6) - (-3)(-2)}{(3)^2} = \frac{12}{9} = \frac{4}{3}$$

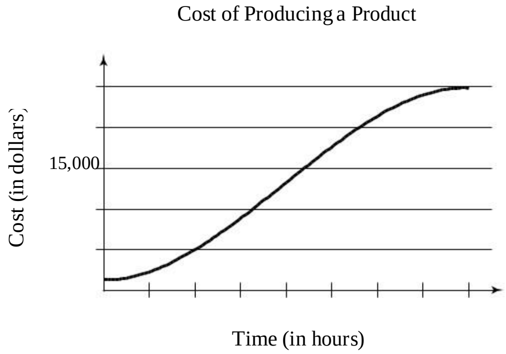
(d) $f(x) = [g(x)]^3$

$f'(x) = 3[g(x)]^2 g'(x)$

$f'(5) = 3(-3)^2(6) = 162$

How Do You See It? Exercise

Page 142, Exercise 106 The cost C (in dollars) of producing x units of a product is $C = 60x + 1350$. For one week, management determined that the number of units produced x at the end of t hours can be modeled by $x = -1.6t^3 + 19t^2 - 0.5t - 1$. The graph shows the cost C in terms of the time t.



(a) Using the graph, which is greater, the rate of change of the cost after 1 hour or the rate of change of the cost after 4 hours?

(b) Explain why the cost function is not increasing at a constant rate during the eight-hour shift.

Solution

(a) According to the graph, $C'(4) > C'(1)$.

(b) Answers will vary.

Suggested Homework Assignment

Pages 140–143: 1–53 odd, 63, 67, 75, 81, 83, 91, 97, 121, and 123.

Section 2.5 Implicit Differentiation

Section Comments

2.5 **Implicit Differentiation**—Distinguish between functions written in implicit form and explicit form. Use implicit differentiation to find the derivative of a function.

Teaching Tips

Material learned in this section will be vital for students to have for related rates. Be sure to ask students to find $\frac{dy}{dx}$ when $x = c$.

You can use the exercise below to review the following concepts:

- Finding derivatives when the variables agree and when they disagree
- Using implicit differentiation to find the derivative of a function

Determine if the statement is true. If it is false, explain why and correct it. For each statement, assume is a function of x

(a) $\frac{d}{dx}\cos(x^2) = -2x\sin(x^2)$

(b) $\frac{d}{dy}\cos(y^2) = 2y\sin(y^2)$

(c) $\frac{d}{dx}\cos(y^2) = -2y\sin(y^2)$

Implicit differentiation is often difficult for students, so as you review this concept remind students to think of y as a function of x. Part (a) is true, and part (b) can be corrected as shown below. Part (c) requires implicit differentiation. Note that the result can also be written as $-2y\sin(y^2)\frac{dy}{dx}$.

Solution

(a) True

(b) False. $\frac{d}{dy}\cos(y^2) = -2y\sin(y^2)$.

(c) False. $\frac{d}{dx}\cos(y^2) = -2yy'\sin(y^2)$.

A good way to teach students how to understand the differentiation of a mix of variables in part (c) is to let g y. Then $g' = y'$. So,

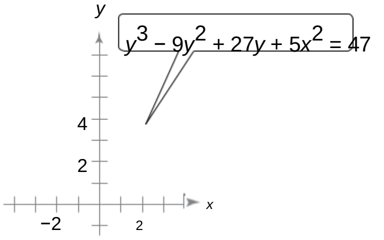
$$\frac{d}{dx}\cos(y^2) = \frac{d}{dx}\cos(g^2)$$

$$-\sin(g^2) \cdot 2gg'$$

$$-\sin(y^2) \cdot 2yy'$$

How Do You See It? Exercise

Page 151, Exercise 70 Use the graph to answer the questions.



(a) Which is greater, the slope of the tangent line at $x = -3$ or the slope of the tangent line at $= -1$?

(b) Estimate the point(s) where the graph has a vertical tangent line.

(c) Estimate the point(s) where the graph has a horizontal tangent line.

Solution

(a) The slope is greater at $x = -3$.

(b) The graph has vertical tangent lines at about $(-2, 3)$ and $(2, 3)$.

(c) The graph has a horizontal tangent line at about $(0, 6)$.

Suggested Homework Assignment

Pages 149–150: 1–17 odd, 25–35 odd, 53, and 61.

Section 2.6 Related Rates

Section Comments

2.6 **Related Rates**—Find a related rate. Use related rates to solve real-life problems.

Teaching Tips

Begin this lesson with a quick review of implicit differentiation with an implicit function in terms of x and differentiated with respect to time. Follow this with an example similar to Example 1 on page 152, outlining the step-by-step procedure at the top of page 153 along with the guidelines at the bottom of page 153. Be sure to tell students, that for every related rate problem, to write down the given information, the equation needed, and the unknown quantity. A suggested problem to work out with the students is as follows:

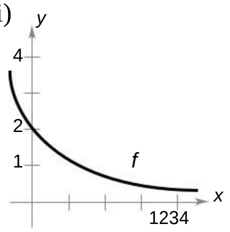
A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 foot per second, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?

Be sure to go over a related rate problem similar to Example 5 on page 155 so that students are exposed to working with related rate problems involving trigonometric functions.

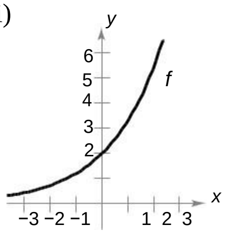
How Do You See It? Exercise

Page 159, Exercise 34 Using the graph of f, (a) determine whether *dy dt* is positive or negative given that *dx dt* is negative, and (b) determine whether *dx dt* is positive or negative given that *dy dt* is positive. Explain.

(i)



(ii)



Solution

(a) $\frac{dx}{dt}$ negative ⇒ $\frac{dy}{dt}$ positive

$\frac{dy}{dt}$ positive ⇒ $\frac{dx}{dt}$ negative (ii)

(a) $\frac{dx}{dt}$ negative ⇒ $\frac{dy}{dt}$ negative

$\frac{dy}{dt}$ positive ⇒ $\frac{dx}{dt}$ positive

Suggested Homework Assignment

Pages 157–160: 1, 7, 11, 13, 15, 17, 21, 25, 29, and 41.

Chapter 2 Project

Timing a Handoff

You are a competitive bicyclist. During a race, you bike at a constant velocity of *k* meters per second. A chase car waits for you at the ten-mile mark of a course. When you cross the ten-mile mark, the car immediately accelerates to catch you. The position function of the chase car is given by the equation $s(t) = \frac{15}{4}t^2 - \frac{5}{12}t^3$, for $0 \le t \le 6$, where *t* is the time in seconds and *s* is the distance traveled in meters. When the car catches you, you and the car are traveling at the same velocity, and the driver hands you a cup of water while you continue to bike at *k* meters per second.

Exercises

1. Write an equation that represents your position *s* (in meters) at time *t* (in seconds).

2. Use your answer to Exercise 1 and the given information to write an equation that represents the velocity *k* at which the chase car catches you in terms of *t*.

3. Find the velocity function of the car.

4. Use your answers to Exercises 2 and 3 to find how many seconds it takes the chase car to catch you.

5. What is your velocity when the car catches you?

6. Use a graphing utility to graph the chase car's position function and your position function in the same viewing window.

7. Find the point of intersection of the two graphs in Exercise 6. What does this point represent in the context of the problem?

8. Describe the graphs in Exercise 6 at the point of intersection. Why is this important for a successful handoff?

9. Suppose you bike at a constant velocity of 9 meters per second and the chase car's position function is unchanged.

(a) Use a graphing utility to graph the chase car's position function and your position function in the same viewing window.

(b) In this scenario, how many times will the chase car be in the same position as you after the 10-mile mark?

(c) In this scenario, would the driver of the car be able to successfully handoff a cup of water to you? Explain.

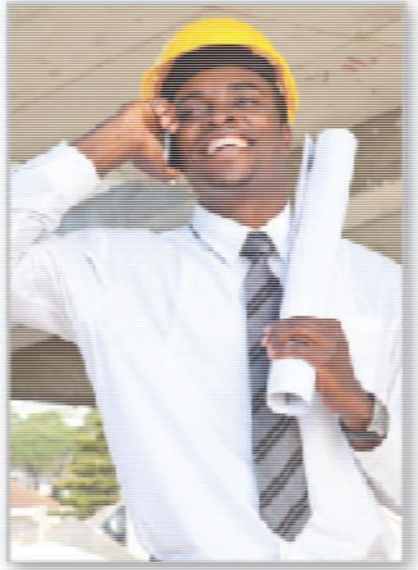
10. Suppose you bike at a constant velocity of 8 meters per second and the chase car's position function is unchanged.

(a) Use a graphing utility to graph the chase car's position function and your position function in the same viewing window.

(b) In this scenario, how many times will the chase car be in the same position as you after the ten-mile mark?

(c) In this scenario, why might it be difficult for the driver of the chase car to successfully handoff a cup of water to you? Explain.

P Preparation for Calculus



P.2

Objectives

Find the slope of a line passing through two points.

Write the equation of a line with a given point and slope.

Interpret slope as a ratio or as a rate in a real-life application.

Sketch the graph of a linear equation in slope-intercept form.

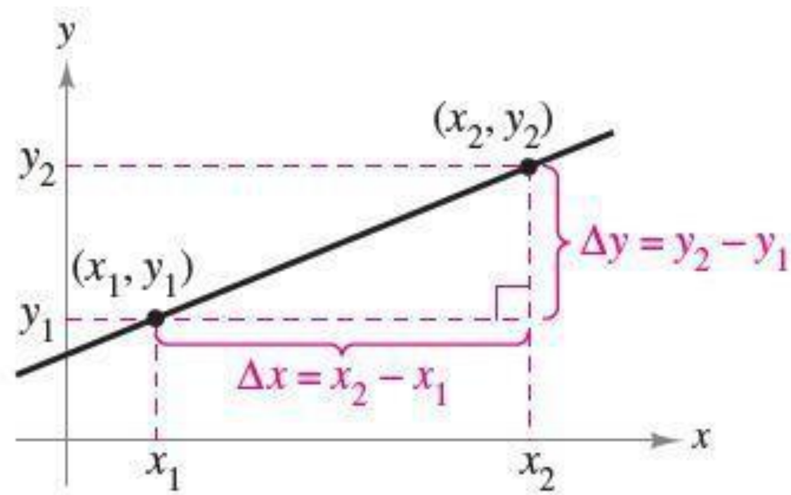
Write equations of lines that are parallel or perpendicular to a given line.

The Slope of a Line

The Slope of a Line

The **slope** of a nonvertical line is a measure of the number of units the line rises (or falls) vertically for each unit of horizontal change from left to right.

Consider the two points (x_1, y_1) and (x_2, y_2) on the line in Figure P.12.



$\Delta y = y_2 - y_1$ = change in y

$\Delta x = x_2 - x_1$ = change in x

Figure P.12

The Slope of a Line

As you move from left to right along this line, a vertical change of

$$\Delta y = y_2 - y_1 \qquad \text{Change in } y$$

units corresponds to a horizontal change of

$$\Delta x = x_2 - x_1 \qquad \text{Change in } x$$

units. (The symbol Δ is the uppercase Greek letter delta, and the symbols Δy and Δx are read “delta *y*” and “delta *x*.”)

The Slope of a Line

Definition of the Slope of a Line

The **slope** m of the nonvertical line passing through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2.$$

Slope is not defined for vertical lines.

The Slope of a Line

When using the formula for slope, note that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{-(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2}.$$

So, it does not matter in which order you subtract *as long as* you are consistent and both “subtracted coordinates” come from the same point.

The Slope of a Line

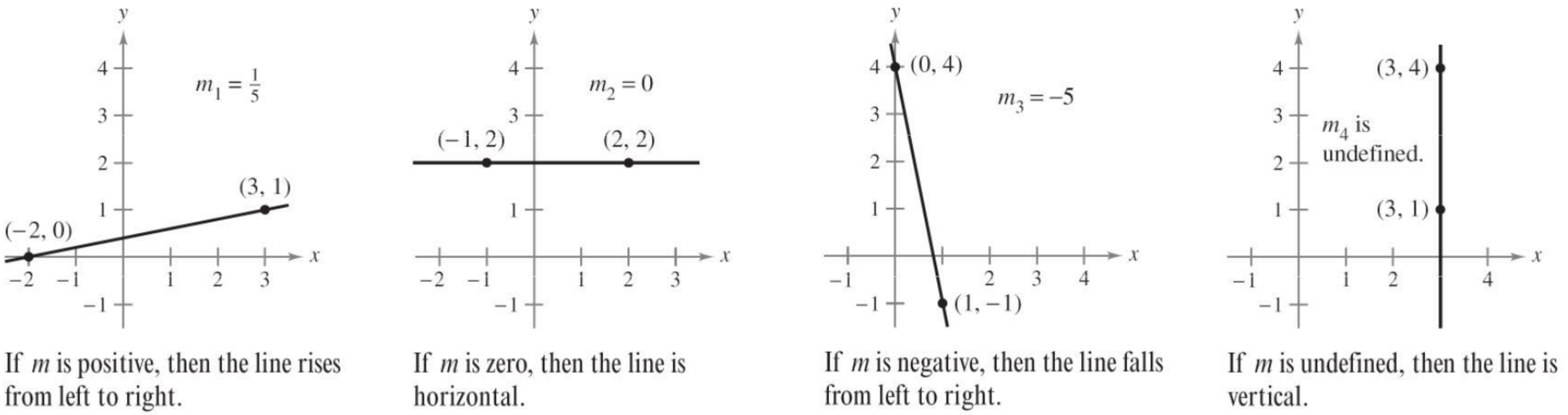


Figure P.13

Figure P.13 shows four lines: one has a positive slope, one has a slope of zero, one has a negative slope, and one has an “undefined” slope.

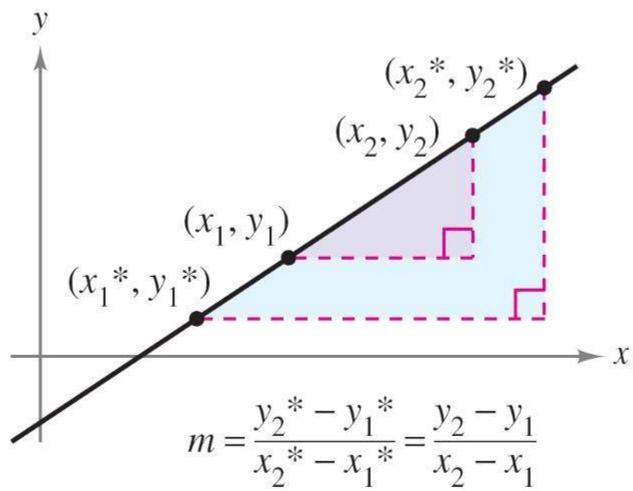
In general, the greater the absolute value of the slope of a line, the steeper the line.

Equations of Lines

Equations of Lines

Any two points on a nonvertical line can be used to calculate its slope.

This can be verified from the similar triangles shown in Figure P.14.



Any two points on a nonvertical line can be used to determine its slope.

Figure P.14

Equations of Lines

If (x_1, y_1) is a point on a nonvertical line that has a slope of m and (x, y) is *any other* point on the line, then

$$\frac{y - y_1}{x - x_1} = m.$$

This equation in the variables x and y can be rewritten in the form

$$- y_1 = m(x - x_1)$$

which is called the **point-slope form** of the equation of a line.

Equations of Lines

Point-Slope Form of the Equation of a Line

The **point-slope form** of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is

$$y - y_1 = m(x - x_1).$$

Example 1 – *Finding an Equation of a Line*

Find an equation of the line that has a slope of 3 and passes through the point (1, –2). Then sketch the line.

Solution:

$$y - y_1 = m(x - x_1)$$ Point-slope form

$$y - (-2) = 3(x - 1)$$ Substitute -2 for y_1, 1 for x_1, and 3 for m.

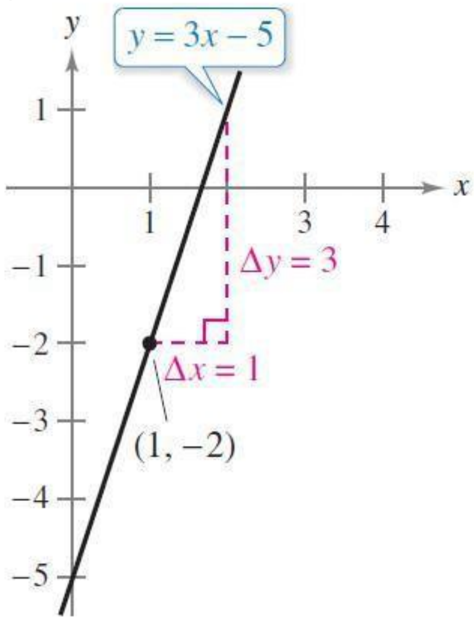
$$y + 2 = 3x - 3$$ Simplify.

$$y = 3x - 5$$ Solve for y.

Example 1 – *Solution*

cont'd

To sketch the line, first plot the point (1, –2). Then, because the slope is $m = 3$, you can locate a second point on the line by moving one unit to the right and three units upward, as shown in Figure P.15.



The line with a slope of 3 passing through the point (1, –2)

Figure P.15

Ratios and Rates of Change

Ratios and Rates of Change

The slope of a line can be interpreted as either a *ratio* or a *rate*.

If the *x*- and *y*-axes have the same unit of measure, then the slope has no units and is a **ratio.**

If the *x*- and *y*-axes have different units of measure, then the slope is a rate or **rate of change.**

Example 2 – *Using Slope as a Ratio*

The maximum recommended slope of a wheelchair ramp is 1/12. A business installs a wheelchair ramp that rises to a height of 22 inches over a length of 24 feet, as shown in Figure P.16. Is the ramp steeper than recommended?

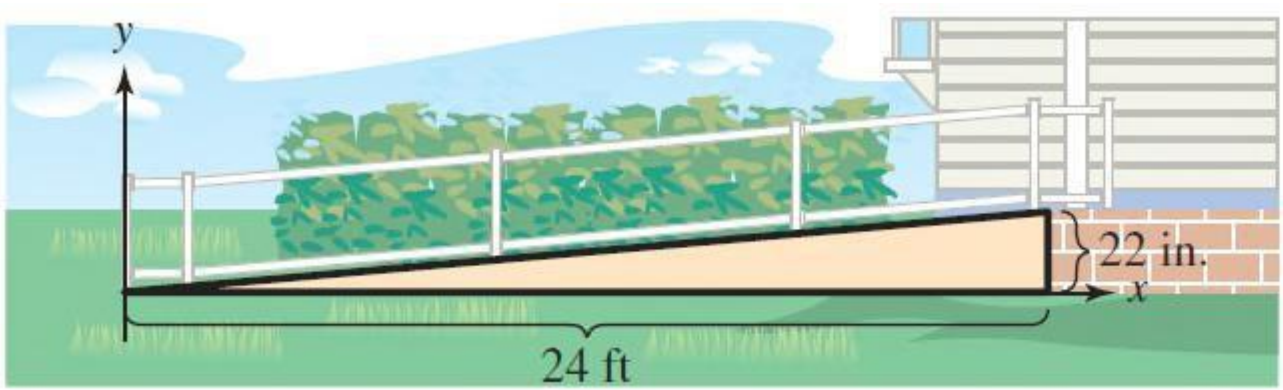


Figure P.16

Example 2 – *Solution*

The length of the ramp is 24 feet or 12 (24) = 288 inches. The slope of the ramp is the ratio of its height (the rise) to its length (the run).

$$\text{Slope of ramp} = \frac{\text{rise}}{\text{run}}$$

$$= \frac{22 \text{ in.}}{288 \text{ in.}}$$

$$\approx 0.076$$

Because the slope of the ramp is less than ½ ≈ 0.083, the ramp is not steeper than recommended. Note that the slope is a ratio and has no units.

Example 3 – *Using Slope as a Rate of Change*

The population of Oregon was about 3,831,000 in 2010 and about 3,970,000 in 2014. Find the average rate of change of the population over this four-year period. What will the population of Oregon be in 2024?

Solution:

Over this four-year period, the average rate of change of the population of Oregon was

$$\text{Rate of change} = \frac{\text{change in population}}{\text{change in years}}$$

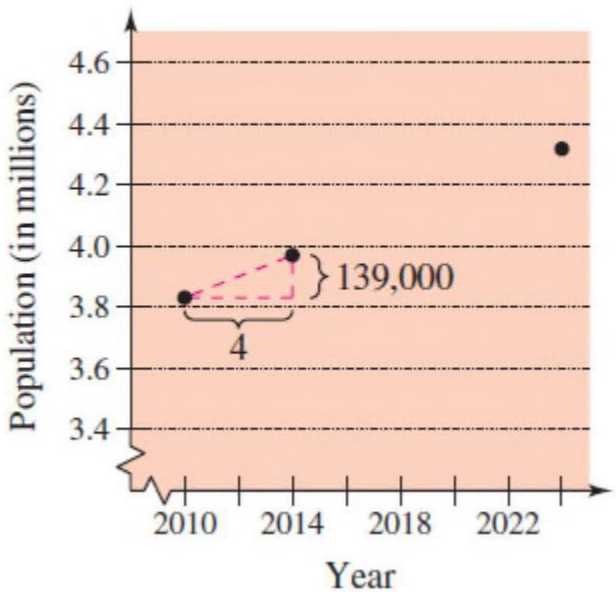
Example 3 – *Solution*

cont'd

$$= \frac{3{,}970{,}000 - 3{,}831{,}000}{2014 - 2010}$$

$$= 34{,}750 \text{ people per year.}$$

Assuming that Oregon's population continues to increase at this same rate for the next 10 years, it will have a 2024 population of about 4,318,000. (See Figure P.17.)



Population of Oregon

Figure P.17

Ratios and Rates of Change

The rate of change found in Example 3 is an **average rate of change**. An average rate of change is always calculated over an interval.

Graphing Linear Models

Graphing Linear Models

Many problems in analytic geometry can be classified in two basic categories:

Given a graph (or parts of it), find its equation.

Given an equation, sketch its graph.

For lines, problems in the first category can be solved by using the point-slope form. The point-slope form, however, is not especially useful for solving problems in the second category.

Graphing Linear Models

The form that is better suited to sketching the graph of a line is the **slope-intercept** form of the equation of a line.

The Slope-Intercept Form of the Equation of a Line

The graph of the linear equation

$$y = mx + b \qquad \text{Slope-intercept form}$$

is a line whose slope is m and whose y-intercept is $(0, b)$.

Example 4 – *Sketching Lines in the Plane*

Sketch the graph of each equation.

a. $y = 2x + 1$

b. $y = 2$

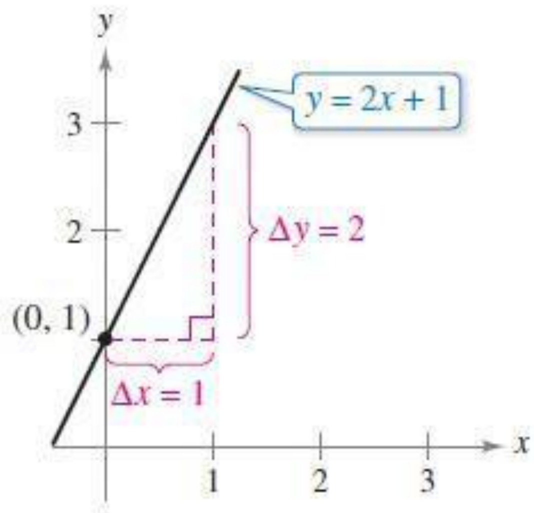
c. $3y + x - 6 = 0$

Example 4(a) – *Solution*

cont'd

Because $b = 1$, the y-intercept is (0, 1).

Because the slope is $m = 2$, you know that the line rises two units for each unit it moves to the right, as shown in Figure P.18(a).



(a) $m = 2$; line rises

Figure P.18(a)

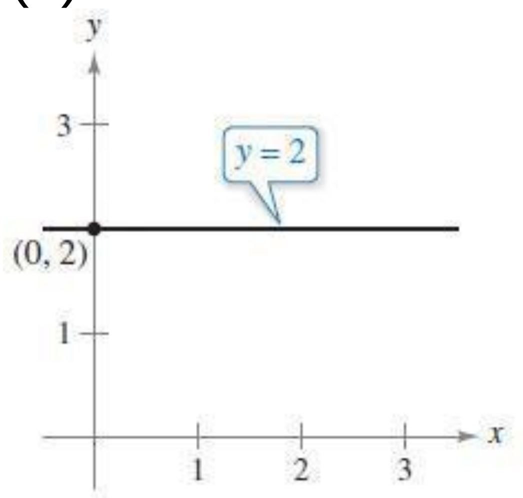
Example 4(b) – *Solution*

cont'd

By writing the equation $y = 2$ in slope-intercept form

$$y = (0)x + 2$$

you can see that the slope is $m = 0$ and the y-intercept is (0,2). Because the slope is zero, you know that the line is horizontal, as shown in Figure P.18(b).



(b) $m = 0$; line is horizontal

Figure P.18(b)

Example 4(c) – *Solution*

cont'd

Begin by writing the equation in slope-intercept form.

$$3y + x - 6 = 0 \qquad \text{Write original equation.}$$

$$3y = -x + 6 \qquad \text{Isolate } y\text{-term on the left.}$$

$$y = -\frac{1}{3}x + 2 \qquad \text{Slope-intercept form}$$

In this form, you can see that the *y*-intercept is (0, 2) and the slope is $m = -\frac{1}{3}$. This This means that the line one falls unit for every three units it moves to the right.

Example 4(c) – *Solution*

cont'd

This is shown in Figure P.18(c).

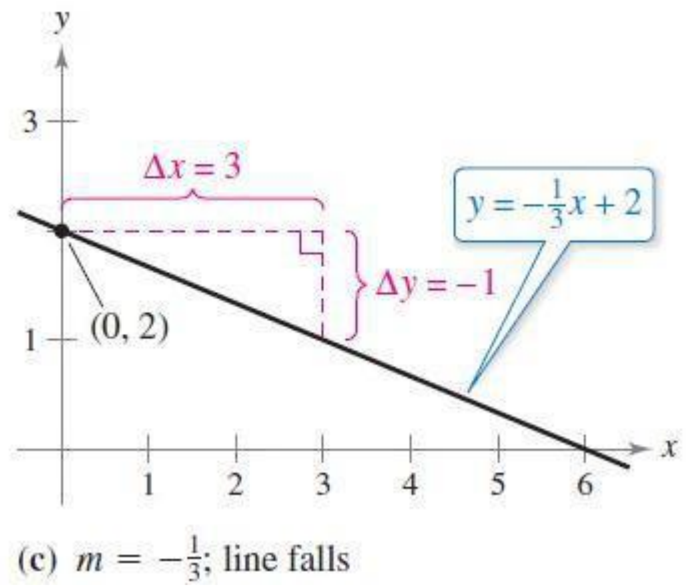


Figure P.18(c)

Graphing Linear Models

Because the slope of a vertical line is not defined, its equation cannot be written in the slope-intercept form. However, the equation of any line can be written in the **general form**

$$Ax + By + C = 0 \qquad \text{General form of the equation of a line}$$

where *A* and *B* are not *both* zero. For instance, the vertical line

$$x = a \qquad \text{Vertical line}$$

can be represented by the general form

$$x - a = 0. \qquad \text{General form}$$

Graphing Linear Models

SUMMARY OF EQUATIONS OF LINES

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Slope-intercept form: $y = mx + b$
5. Point-slope form: $y - y_1 = m(x - x_1)$

Parallel and Perpendicular Lines

Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular, as shown in Figure P.19.

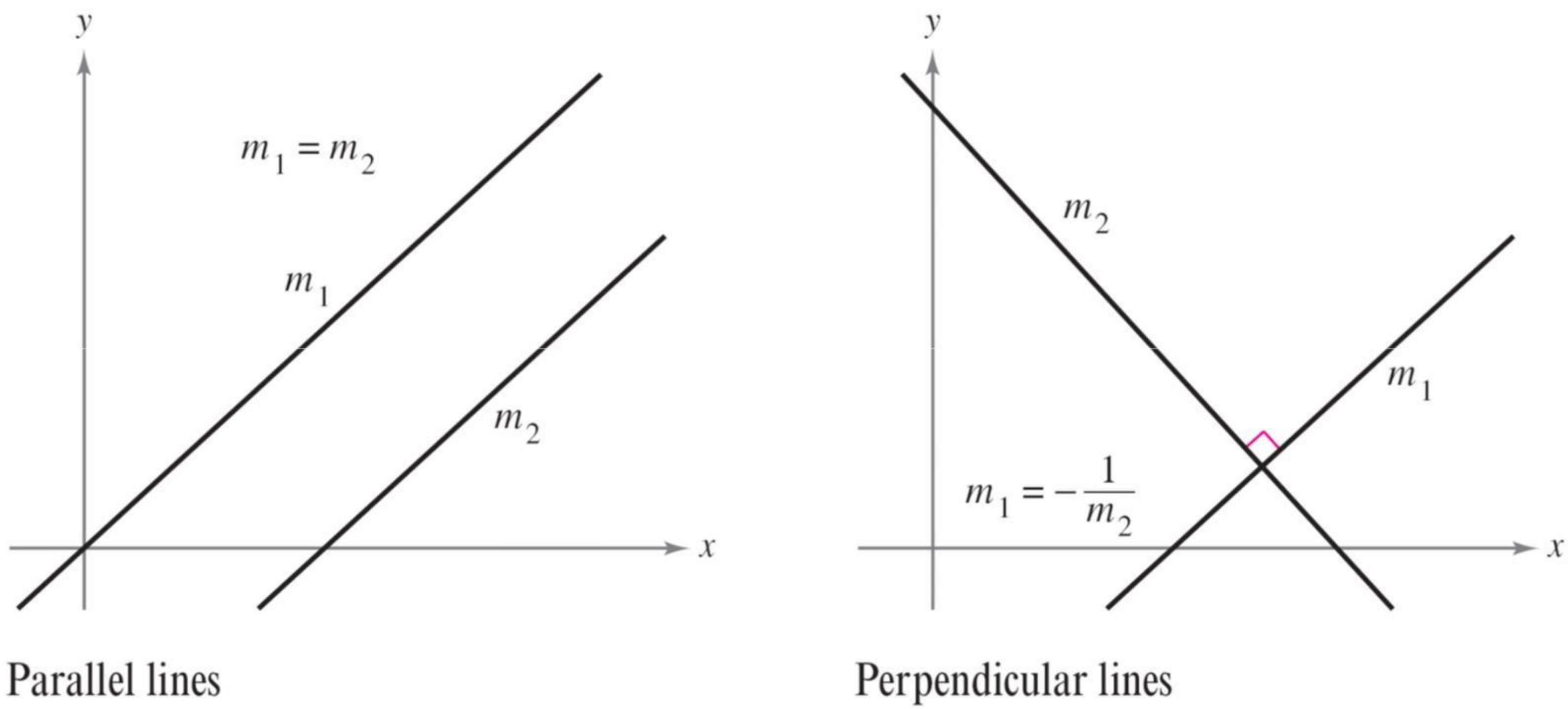


Figure P.19

Parallel and Perpendicular Lines

Parallel and Perpendicular Lines

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal—that is, if and only if

$$m_1 = m_2.$$ Parallel ⟺ Slopes are equal.

2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other—that is, if and only if

$$m_1 = -\frac{1}{m_2}.$$ Perpendicular ⟺ Slopes are negative reciprocals.

Example 5 – *Finding Parallel and Perpendicular Lines*

Find the general forms of the equations of the lines that pass through the point (2, –1) and are

parallel to the line

$2x - 3y = 5$

perpendicular to the line

$2x - 3y = 5.$

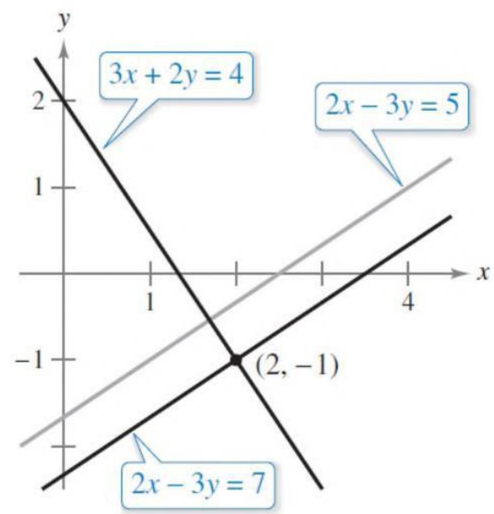
Example 5 – *Solution*

Begin by writing the linear equation 2*x* – 3*y* = 5 in slope-intercept form.

$$2x - 3y = 5 \qquad \text{Write original equation.}$$

$$y = \frac{2}{3}x - \frac{5}{3} \qquad \text{Slope-intercept form}$$

So, the given line has a slope of $m = \frac{2}{3}$. (See Figure P.20.)



Lines parallel and perpendicular to $2x - 3y = 5$

Figure P.20

Example 5 – *Solution*

cont'd

a. The line through (2, –1) that is parallel to the given line also has a slope of 2/3.

$$y - y_1 = m(x - x_1)$$ Point-slope form

$$y - (-1) = \tfrac{2}{3}(x - 2)$$ Substitute.

$$3(y + 1) = 2(x - 2)$$ Simplify.

$$3y + 3 = 2x - 4$$ Distributive Property

$$2x - 3y - 7 = 0$$ General form

Note the similarity to the equation of the given line, $2x - 3y = 5$.

b. Using the negative reciprocal of the slope of **the** given line, you can determine that the slope of a line perpendicular to the given line is –3/2.

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-1) = -\frac{3}{2}(x - 2)$	Substitute.
$2(y + 1) = -3(x - 2)$	Simplify.
$2y + 2 = -3x + 6$	Distributive Property
$3x + 2y - 4 = 0$	General form