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2 □ LIMITS

2.1 Limits of Sequences

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.

(b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.

(c) The terms a_n become large as n becomes large. In fact, we can make a_n as large as we like by taking n sufficiently large.

2. (a) From Definition 1, a convergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ exists. Examples: {1 }, {1 2 }

(b) A divergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ *does not* exist. Examples: { }, {sin }

3. The graph shows a decline in the world record for the men's 100-meter sprint as n increases. It is tempting to say that this sequence will approach zero, however, it is important to remember that the sequence represents data from a physical competition. Thus, the sequence likely has a nonzero limit as $n \to \infty$ since human physiology will ultimately limit how fast a human can sprint 100-meters. This means that there is a certain world record time which athletes can never surpass.

4. (a) If the sequence does not have a limit as $n \to \infty$, then the world record distances for the women's hammer throw may increase indefinitely as $n \to \infty$. That is, the sequence is divergent.

(b) It seems unlikely that the world record hammer throw distance will increase indefinitely. Human physiology will ultimately limit the maximum distance a woman can throw. Therefore, barring evolutionary changes to human physiology, it seems likely that the sequence will converge.

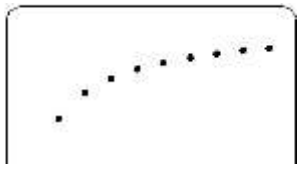
5.

1	0.2000	6	0.3000
2	0.2500	7	0.3043
3	0.2727	8	0.3077
4	0.2857	9	0.3103
5	0.2941	10	0.3125

0.35

0
0.15

11



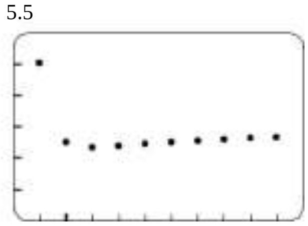
The sequence appears to converge to a number between 0.30 and 0.35. Calculating the limit gives

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2n+3n^2} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{2n}{n^2}+\frac{3n^2}{n^2}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} 3} = \frac{1}{0+3} = \frac{1}{3}$$

This agrees with the value predicted from the data.

6.

n	a_n	n	a_n
1	5.0000	6	3.7500
2	3.7500	7	3.7755
3	3.6667	8	3.7969
4	3.6875	9	3.8148
5	3.7200	10	3.8300



The sequence appears to converge to a number between 3.9 and 4.0. Calculating the limit gives

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(4 - \frac{2}{n} + \frac{3}{n^2}\right) =$$

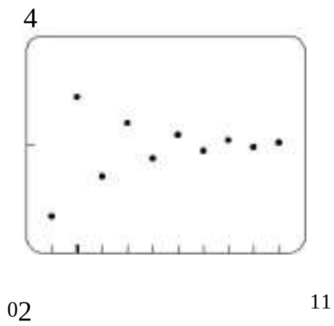
$4 - 0 + 0 = 4$. So we expect the sequence to converge to 4 as we plot more terms.

$0_{2.5}$ 11

71

7.

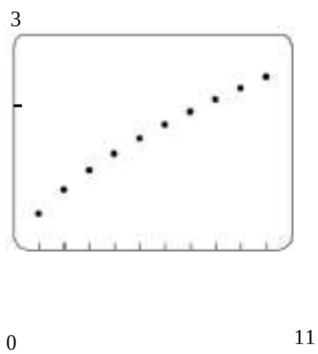
n	a_n
1	2.3333
2	3.4444
3	2.7037
4	3.1975
5	2.8683
6	3.0878
7	2.9415
8	3.0390
9	2.9740
10	3.0173



The sequence appears to converge to approximately 3. Calculating the limit gives $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[3 + \left(-\frac{2}{3}\right)^n\right] = 3 + 0 = 3$. This agrees with the value predicted from the data.

8.

n	a_n
1	0.5000
2	0.8284
3	1.0981
4	1.3333
5	1.5451
6	1.7394
7	1.9200
8	2.0896
9	2.2500
10	2.4025



The sequence does not appear to converge since the values of a_n do not approach a fixed number. We can verify this by trying to calculate the limit:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{\sqrt{n}}{1 + 1/\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}} + \frac{1}{n}}.$$

The denominator approaches 0 while the numerator remains constant so the limit does not exist, as expected.

9. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3n^4} = \frac{1}{3}\lim_{n\to\infty}\frac{1}{n^4} = 0$. Converges

10. $a_n = \frac{5}{3^n}$ is a geometric sequence with $r = \frac{1}{3}$. So $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{5}{3^n} = 5\lim_{n\to\infty}\left(\frac{1}{3}\right)^n = 5\cdot 0 = 0$. Converges

11. $a_n = \frac{2n^2 + n - 1}{n^2} = 2 + \frac{1}{n} - \frac{1}{n^2}$ so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2 + \lim_{n\to\infty}\frac{1}{n} - \lim_{n\to\infty}\frac{1}{n^2} = 2 + 0 - 0 = 2$. Converges

12. $a_n = \frac{n^3 - 1}{n} = n^2 - \frac{1}{n}$ so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 - \lim_{n\to\infty}\frac{1}{n} = \lim_{n\to\infty} n^2$. When n is large, n^2 is large so $\lim_{n\to\infty} n^2 = \infty$ and the sequence diverges.

13. $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3 + 5n^2}{n + n^2} = \lim_{n\to\infty}\frac{(3+5n^2)/n^2}{(2n+7n^2)/n^2} = \lim_{n\to\infty}\frac{\frac{3}{n^2}+5}{\frac{2}{n}+7} = \frac{\lim_{n\to\infty}\frac{3}{n^2} + \lim_{n\to\infty}5}{\lim_{n\to\infty}\frac{2}{n} + \lim_{n\to\infty}7} = \frac{0+5}{0+7} = \frac{5}{7}$. Converges

14. $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^3 - 1}{n^3+1} = \lim_{n\to\infty}\frac{(n^3-1)/n^3}{(n^3+1)/n^3} = \lim_{n\to\infty}\frac{1 - \frac{1}{n^3}}{1 + \frac{1}{n^3}} = \frac{\lim_{n\to\infty}1 - \lim_{n\to\infty}\frac{1}{n^3}}{\lim_{n\to\infty}1 + \lim_{n\to\infty}\frac{1}{n^3}} = \frac{1-0}{1+0} = 1$. Converges

15. = 1 − (0 2) , so lim $_{\to\infty}$ =1−0=1 [by (3) with =02]. Converges

16. =2^{-} +6^{-} = $\frac{1}{2}$ + $\frac{1}{6}$ so lim$_{\to\infty}$ = lim$_{\to\infty}$ $\frac{1}{2}$ + lim$_{\to\infty}$ $\frac{1}{6}$ =0+0=0

by (3) with = $\frac{1}{2}$ and = $\frac{1}{6}$ Converges

17. = $\frac{\sqrt{}^{2}}{\sqrt{3+4}}$ = $\frac{^{2}\sqrt{}}{\sqrt{}\,^{3}\sqrt{}}$ = $\frac{\sqrt{}}{1+4\,^{2}}$, so → ∞ as → ∞ since lim$_{\to\infty}$ $\sqrt{}$ = ∞ and

lim$_{\to\infty}$ $\frac{}{1+4\,^{2}}$ = 1. Diverges

18. = sin(2) ⇒ $_{1}$ = sin(2)=1 $_{2}$ = sin( ) = 0 $_{3}$ = sin(3 2)=−1 $_{4}$ = sin(2) = 0 $_{5}$ = sin(5 2)=1 Observe that cycles between the values 1,0, and -1 as increases. Hence the sequence does not converge.

19. = cos(2) ⇒ $_{1}$ = cos(2)=0 $_{2}$ = cos( )=−1 $_{3}$ = cos(32) = 0 $_{4}$ = cos(2) = 1 $_{5}$ = cos(5 2) = 0. Observe that cycles between the values 1, 0, and −1 as increases. Hence the sequence does not converge.

20. = $\frac{}{33}$ = so lim$_{\to\infty}$ = lim$_{\to\infty}$ $\frac{}{3}$ = ∞ since $\frac{}{3}$ ≈ 105 1 Diverges

21. lim$_{\to\infty}$ $\frac{10}{}$ = lim$_{\to\infty}$ $\frac{10}{1+9}$ = lim$_{\to\infty}$ $\frac{10}{1+9\,10}$ = lim$_{\to\infty}$ $\frac{1}{\frac{1}{10}+\frac{9}{10}}$ = $\frac{\lim_{\to\infty} 1}{\lim \frac{1}{10} + \lim \frac{9}{10}}$ = ∞ because the denominator approaches 0 while the numerator remains constant. Diverges

22. lim$_{\to\infty}$ $\frac{\sqrt[3]{}}{}$ = lim$_{\to\infty}$ $\frac{^{1\,3}}{^{1\,2}+^{1\,4}}$ = lim$_{\to\infty}$ $\frac{^{1\,3}\,^{1\,2}}{^{1\,2}+^{1\,4}}$ = $\frac{\lim_{\to\infty} ^{1\,6}}{\lim_{\to\infty} 1 + \lim_{\to\infty} \frac{1}{^{1\,4}}}$ = $\frac{0}{}$ = 0

Converges

23. = ln(2 ^{2} + 1) − ln(^{2} + 1) = ln $\frac{2\,^{2}+1}{^{2}+1}$ = ln $\frac{2+1\,^{2}}{1+1\,^{2}}$ → ln 2 as → ∞. Converges

24. = $\frac{3^{+2}}{5}$ = $\frac{3^{2}3}{5}$ = 9 $\frac{3}{5}$, so lim$_{\to\infty}$ = 9 lim$_{\to\infty}$ $\frac{3}{5}$ = 9 · 0 = 0 by (3) with = $\frac{3}{5}$. Converges

25. = $\frac{+}{^{2}-1}$ · $\frac{}{}$ = $\frac{1+\ ^{-2}}{-}$ → 0 as → ∞ because 1 + $^{-2}$ → 1 and − → ∞. Converges

26. = ln(+ 1) − ln = ln $\frac{+1}{}$ = ln 1 + $\frac{1}{}$ → ln (1) = 0 as → ∞. Converges

27. The sequence appears to converge to 2 Assume the limit exists so that

lim +1 = lim = then +1 = $\frac{1}{2}$ + 1 ⇒

$\frac{1}{2}$ + 1 ⇒ = $\frac{1}{2}$ + 1 ⇒ = 2

1	1.0000	5	1.9375
2	1.5000	6	1.9688
3	1.7500	7	1.9844

$\lim_{\to\infty}$ +1 = $\lim_{\to\infty}$

4	1.8750	8	1.9922

Therefore, $\lim_{n\to\infty} a_n = 2$.

28.

n	a_n	n	a_n
1	2.0000	5	0.7654
2	0.3333	6	0.7449
3	0.8889	7	0.7517
4	0.7037	8	0.7494

The sequence appears to converge to 0.75. Assume the limit exists so that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$; then $a_{n+1} = 1 - \frac{1}{3}a_n \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(1 - \frac{1}{3}a_n\right) \Rightarrow L = 1 - \frac{1}{3}L \Rightarrow L = \frac{3}{4}$.

Therefore, $\lim_{n\to\infty} a_n = \frac{3}{4}$.

29.

n	a_n	n	a_n
2	3.0000	6	33.0000
3	5.0000	7	65.0000
4	9.0000	8	129.0000

The sequence is divergent.

30.

n	a_n
1	1.0000
2	2.2361
3	3.3437
4	4.0888
5	4.5215
6	4.7547
7	4.8758
8	4.9375

The sequence appears to converge to 5. Assume the limit exists so that

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$; then $a_{n+1} = \sqrt{5a_n} \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\sqrt{5a_n} \Rightarrow$

$L = \sqrt{5L} \Rightarrow L^2 = 5L \Rightarrow L(L-5)=0 \Rightarrow L = 0$ or $L = 5$

Therefore, if the limit exists it will be either 0 or 5. Since the first 8 terms of the sequence appear to approach 5, we surmise that $\lim_{n\to\infty} a_n = 5$.

31.

n	a_n
1	1.0000
2	3.0000
3	1.5000
4	2.4000
5	1.7647
6	2.1702
7	1.8926
8	2.0742

The sequence appears to converge to 2. Assume the limit exists so that

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$; then $a_{n+1} = \frac{6}{1+a_n} \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\frac{6}{1+a_n} \Rightarrow$

$L = \frac{6}{1+L} \Rightarrow L^2 + L - 6 = 0 \Rightarrow (L-2)(L+3)=0 \Rightarrow L = -3$ or $L = 2$

Therefore, if the limit exists it will be either -3 or 2, but since all terms of the sequence are positive, we see that $\lim_{n\to\infty} a_n = 2$.

32.

n	a_n
1	3.0000
2	5.0000
3	3.0000
4	5.0000
5	3.0000
6	5.0000
7	3.0000
8	5.0000

The sequence cycles between 3 and 5; hence it is divergent.

33.

n	a_n
1	1.0000
2	1.7321
3	1.9319
4	1.9829
5	1.9957
6	1.9989
7	1.9997
8	1.9999

The sequence appears to converge to 2. Assume the limit exists so that

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$; then $a_{n+1} = \sqrt{2 + a_n} \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n} \Rightarrow$

$L = \sqrt{2 + L} \Rightarrow L^2 - L - 2 = 0 \Rightarrow (L - 2)(L + 1) = 0 \Rightarrow L = -1$ or $L = 2$

Therefore, if the limit exists it will be either -1 or 2, but since all terms of the sequence are positive, we see that $\lim_{n\to\infty} a_n = 2$.

34.

n	a_n
1	100.0000
2	50.1250
3	25.3119
4	13.1498
5	7.5255
6	5.4238
7	5.0166
8	5.0000

The sequence appears to converge to 5. Assume the limit exists so that

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$; then

$a_{n+1} = \frac{1}{2}\left(a_n + \frac{25}{a_n}\right) \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{25}{a_n}\right) \Rightarrow L =$

$\frac{1}{2}\left(L + \frac{25}{L}\right) \Rightarrow 2L = L + \frac{25}{L} \Rightarrow L^2 = 25 \Rightarrow L = -5$ or $L = 5$

Therefore, if the limit exists it will be either -5 or 5, but since all terms of the sequence are positive, we see that $\lim_{n\to\infty} a_n = 5$.

35. (a) The quantity of the drug in the body after the first tablet is 100 mg. After the second tablet, there is 100 mg plus 20% of the first 100- mg tablet, that is, $[100 + 100(0.20)] = 120$ mg. After the third tablet, the quantity is $[100 + 120(0.20)] = 124$ mg.

(b) After the $(n + 1)$th tablet, there is 100 mg plus 20% of the nth tablet, so that $Q_{n+1} = 100 + (0.20)Q_n$.

(c) From Formula (6), the solution to $Q_{n+1} = 100 + (0.20)Q_n$, $Q_0 = 0$ mg is

$$Q_n = (0.20)^n(0) + 100\,\frac{1 - (0.20)^n}{1 - 0.20} = \frac{100}{0.80}\left(1 - 0.20^n\right) = 125\left(1 - 0.20^n\right)$$

(d) In the long run, we have $\lim_{n\to\infty} Q_n = \lim_{n\to\infty} 125\left(1 - 0.20^n\right) = 125\left(\lim_{n\to\infty} 1 - \lim_{n\to\infty} 0.20^n\right) = 125(1 - 0) = 125$ mg

36. (a) The concentration of the drug in the body after the first injection is 1.5 mg/mL. After the second injection, there is 1.5 mg/mL plus 10% (90% reduction) of the concentration from the first injection, that is, $[1.5 + 1.5(0.10)] = 1.65$ mg/mL. After the third injection, the concentration is $[1.5 + 1.65(0.10)] = 1.665$ mg/mL.

(b) The drug concentration is $0.1C_n$ (90% reduction) just before the $(n + 1)$th injection, after which the concentration increases by 1.5 mg/mL. Hence $C_{n+1} = 0.1C_n + 1.5$.

(c) From Formula (6), the solution to $C_{n+1} = 0.1C_n + 1.5$, $C_0 = 0$ mg/mL is

$$= (01)\ (0) + 15 \quad \frac{1 \quad 0\,1}{1 - 01} \quad = 0\,9 \quad \frac{1\,5}{(1 \qquad 01\ )} = 3\ \frac{5}{(1 - 01\ )}$$

(d) The limiting value of the concentration is

$$\lim_{\square\to\infty} \square_{\square} = \lim_{\square\to\infty} \frac{5}{3}(1-01_{\square}{}^{\square}) = \frac{5}{3}\lim_{\square\to\infty} 1 - \lim_{\square\to\infty} 0\,1_{\square}{}^{\square} = \frac{5}{3}(1-0) = \frac{5}{3} \approx 1\,667\,\text{mg}\ \square\text{mL}.$$

37. (a) The quantity of the drug in the body after the first tablet is 150 mg. After the second tablet, there is 150 mg plus 5% of the first 150- mg tablet, that is, [150 + 150(0 05)] mg. After the third tablet, the quantity is $[150 + 150(0\,05) + 150(0\,05)^2] = 157\,875$ mg. After tablets, the quantity (in mg) is

$150 + 150(0\,05) + \cdots + 150(0\,05)^{-1}$. We can use Formula 5 to write this as $\frac{150(1 - 0\,05)}{1-0\,05} = \frac{3000}{19}(1\ \ 0\,05)$.

(b) The number of milligrams remaining in the body in the long run is $\lim_{\to\infty} \frac{3000}{19}(1-0\,05) = \frac{3000}{19}(1 - 0) \approx 157\,895$, only 0 02 mg more than the amount after 3 tablets.

38. (a) The residual concentration just before the second injection is ; before the third, $+ \ ^{-2}$; before the (+ 1)st, $^{-} + \ ^{-2} + \cdots + \ ^{-}$. This sum is equal to $\frac{1-}{1-}$ [Formula 3].

(b) The limiting pre-injection concentration is $\lim_{\to\infty} \frac{1}{1-} = \frac{(1\ \ 0)}{1-} = \frac{}{-1}$.

(c) $-1 \geq \Rightarrow \geq -1$, so the minimal dosage is $= -1$.

39. (a) Many people would guess that 1, but note that consists of an infinite number of 9s.

(b) $= 0\,99999 = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10,000} + \cdots = \sum_{=1}^{\infty} \frac{9}{10}$, which is a geometric series with $_1 = 0\,9$ and $= 0\,1$. Its sum is $\frac{0\,9}{1-0\,1} = \frac{0\,9}{0\,9} = 1$, that is, $= 1$.

(c) The number 1 has two decimal representations, 1 00000 and 0 99999 .

(d) Except for 0, all rational numbers that have a terminating decimal representation can be written in more than one way. For example, 0 5 can be written as 0 49999 as well as 0 50000 .

40. $=(5-\ )\ _{-1}$, $_1 = 1 \Rightarrow \ _2=(5-2)(1)=3$, $_3=(5-3)(3)=6$, $_4=(5-4)(6)=6$, $_5=(5-5)(6)=0$, $_6 = (5 - 6)(0) = 0$, and so on. Observe that the fifth term and higher will all be zero. So the sum of all the terms in the sequence is found by adding the first four terms: $_1 + \ _2 + \ _3 + \ _4 = 1+3+6+6 = 16$

41. $0\,\overline{8} = \frac{8}{10} + \frac{8}{10^2} + \cdots$ is a geometric series with $= \frac{8}{10}$ and $= \frac{1}{10}$. It converges to $\frac{}{1-} = \frac{8/10}{1-1/10} = \frac{8}{9}$.

42. $0\,\overline{46} = \frac{46}{100} + \frac{46}{100^2} + \cdots$ is a geometric series with $= \frac{46}{100}$ and $= \frac{1}{100}$. It converges to $\frac{}{1-} = \frac{46/100}{1-1/100} = \frac{46}{99}$.

43. $2\,\overline{516} = 2 + \frac{516}{10^3} + \frac{516}{10^6} + \cdots$. Now $\frac{516}{10^3} + \frac{516}{10^6} + \cdots$ is a geometric series with $= \frac{516}{10^3}$ and $= \frac{1}{10^3}$. It converges to $\frac{}{1-} = \frac{516/10^3}{1-1/10^3} = \frac{516/10^3}{999/10^3} = \frac{516}{999}$. Thus, $2\,\overline{516} = 2 + \frac{516}{999} = \frac{2514}{999} = \frac{838}{333}$.

44. $10\,1\overline{35} = 10\,1 + \frac{35}{10^3} + \frac{35}{10^5} + \cdots$. Now $\frac{35}{10^3} + \frac{35}{10^5} + \cdots$ is a geometric series with $= \frac{35}{10^3}$ and $= \frac{1}{10^2}$. It converges to $\dfrac{35/10^3}{1-1/10^2} = \dfrac{35/10^3}{99/10^2} = \dfrac{35}{990}$. Thus, $10\ 135 = 10\ 1 + \dfrac{35}{990} = \dfrac{9999+35}{990} = \dfrac{10{,}034}{990} = \dfrac{5017}{495}$.

45. $15342 = 153 + \dfrac{42}{10^4} + \dfrac{42}{10^6} + \cdots$. Now $\dfrac{42}{10^4} + \dfrac{42}{10^6} + \cdots$ is a geometric series with $a = \dfrac{42}{10^4}$ and $r = \dfrac{1}{10^2}$.

It converges to $\dfrac{\frac{42}{10^4}}{1-\frac{1}{10^2}} = \dfrac{\frac{42}{10^4}}{\frac{99}{10^2}} = \dfrac{42}{9900}$.

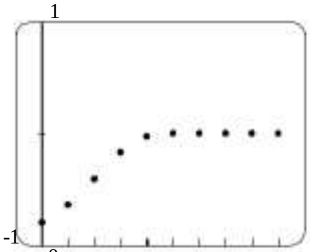
Thus, $1.53\overline{42} = 1.53 + \dfrac{42}{9900} = \dfrac{153}{100} + \dfrac{42}{9900} = \dfrac{15{,}147}{9900} + \dfrac{42}{9900} = \dfrac{15{,}189}{9900}$ or $\dfrac{5063}{3300}$.

46. $7.\overline{12345} = 7 + \frac{12,345}{10^5} + \frac{12,345}{10^{10}} + \cdots$. Now $\frac{12,345}{10^5} + \frac{12,345}{10^{10}} + \cdots$ is a geometric series with $a = \frac{12,345}{10^5}$ and $r = \frac{1}{10^5}$.

It converges to $\frac{a}{1-r} = \frac{12,345/10^5}{1-1/10^5} = \frac{12,345/10^5}{99,999/10^5} = \frac{12,345}{99,999}$.

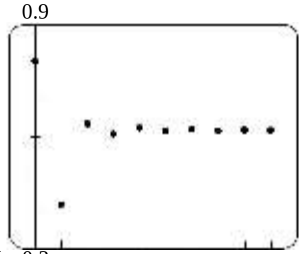
Thus, $7.\overline{12345} = 7 + \frac{12,345}{99,999} = \frac{699,993}{99,999} + \frac{12,345}{99,999} = \frac{712,338}{99,999}$ or $\frac{237,446}{33,333}$.

47.



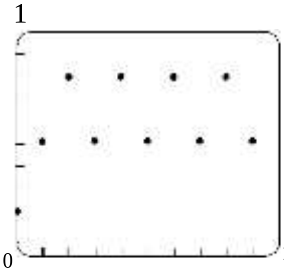
Computer software was used to plot the first 10 points of the recursion equation $a_{n+1} = 2a_n(1-a_n)$, $a_0 = 0.1$. The sequence appears to converge to a value of 0.5. Assume the limit exists so that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$, then $a_{n+1} = 2a_n(1-a_n) \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 2a_n(1-a_n) \Rightarrow L = 2L(1-L) \Rightarrow L(1-2L) = 0 \Rightarrow L = 0$ or $L = 1/2$. Therefore, if the limit exists it will be either 0 or $\frac{1}{2}$. Since the graph of the sequence appears to approach $\frac{1}{2}$, we see that $\lim_{n\to\infty} a_n = \frac{1}{2}$.

48.



Computer software was used to plot the first 10 points of the recursion equation $a_{n+1} = 2.6a_n(1-a_n)$, $a_0 = 0.8$. The sequence appears to converge to a value of 0.6. Assume the limit exists so that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$, then $a_{n+1} = 2.6a_n(1-a_n) \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 2.6a_n(1-a_n) \Rightarrow L = 2.6L(1-L) \Rightarrow L(1.6-2.6L) = 0 \Rightarrow L = 0$ or $L = \frac{8}{13} \approx 0.615$. Therefore, if the limit exists it will be either 0 or $\frac{8}{13}$. Since the graph of the sequence appears to approach $\frac{8}{13}$, we see that $\lim_{n\to\infty} a_n = \frac{8}{13}$.

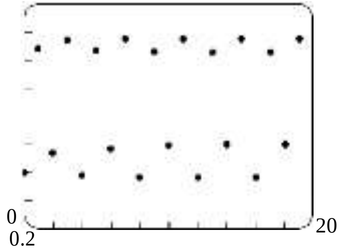
49.

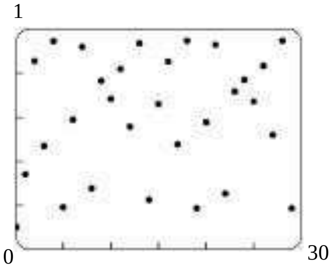


Computer software was used to plot the first 10 points of the recursion equation $a_{n+1} = 3.2a_n(1-a_n)$, $a_0 = 0.2$. The sequence does not appear to converge to a fixed value. Instead, the terms oscillate between values near 0.5 and 0.8.

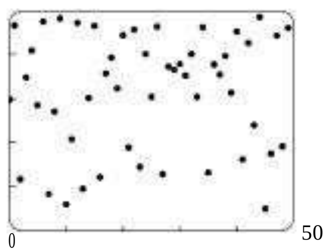
50. 1

Computer software was used to plot the first 20 points of the recursion equation $a_{n+1} = 3.5a_n(1-a_n)$, $a_0 = 0.4$. The sequence does not appear to converge to a fixed value. Instead, the terms oscillate between values near 0.45 and 0.85.



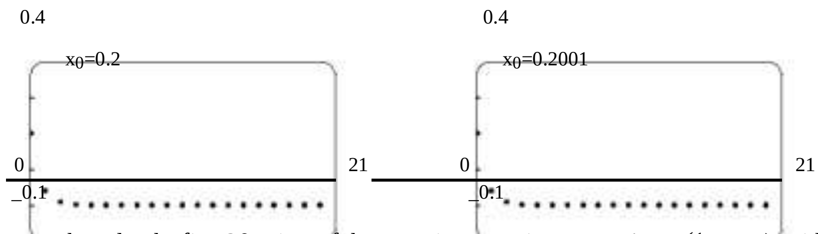
51. 

Computer software was used to plot the first 30 points of the recursion equation $x_{n+1} = 3.8x_n(1 - x_n)$, $x_0 = 0.1$. The sequence does not appear to converge to a fixed value. The terms fluctuate substantially in value exhibiting chaotic behavior.

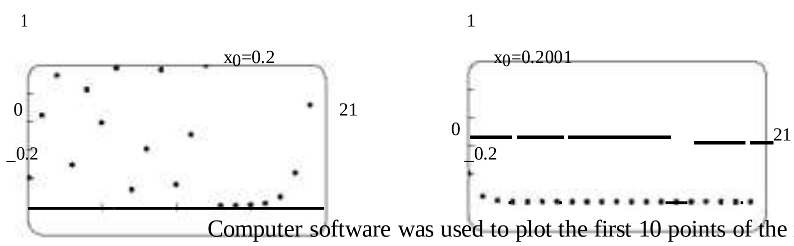
52. 

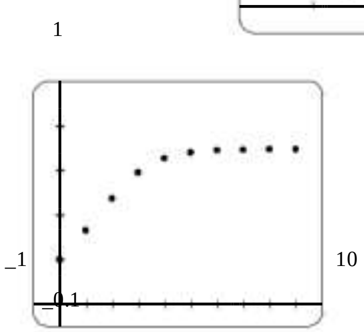
Computer software was used to plot the first 50 points of the recursion equation $x_{n+1} = 3.9x_n(1 - x_n)$, $x_0 = 0.6$. The sequence does not appear to converge to a fixed value. The terms fluctuate substantially in value exhibiting chaotic behavior.

53. Computer software was used to plot the first 20 points of the recursion equation $x_{n+1} = \frac{1}{4}x_n(1 - x_n)$ with $x_0 = 0.2$ and $x_0 = 0.2001$. The plots indicate that the solutions are nearly identical, converging to zero as n increases.



54. Computer software was used to plot the first 20 points of the recursion equation $x_{n+1} = 4x_n(1 - x_n)$ with $x_0 = 0.2$ and $x_0 = 0.2001$. The recursion with $x_0 = 0.2$ behaves chaotically whereas the recursion with $x_0 = 0.2001$ converges to zero. The plots indicate that a small change in initial conditions can significantly impact the behaviour of a recursive sequence.



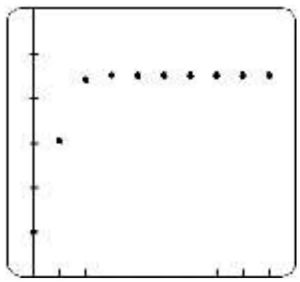
55. 

Computer software was used to plot the first 10 points of the recursion equation $x_{n+1} = 2x_n e^{-x_n}$, $x_0 = 0.2$. The sequence appears to converge to a value near 0.7. Assume the limit exists so that $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = L$, then

$$x_{n+1} = 2x_n e^{-x_n} \Rightarrow \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} 2x_n e^{-x_n} \Rightarrow L = 2Le^{-L} \Rightarrow$$

$L(1 - 2e^{-L}) = 0 \Rightarrow L = 0$ or $L = \ln 2 \approx 0.693$. Therefore, if the limit exists it will be either 0 or ln 2. Since the graph of the sequence appears to approach ln 2, we see that $\lim_{n\to\infty} x_n = \ln 2$.

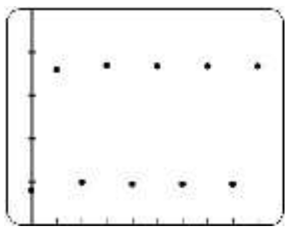
56.



Computer software was used to plot the first 10 points of the recursion equation $a_{n+1} = 3a_n e^{-a_n}$, $a_0 = 0.4$. The sequence appears to converge to a value of 1.1. Assume the limit exists so that $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = L$, then

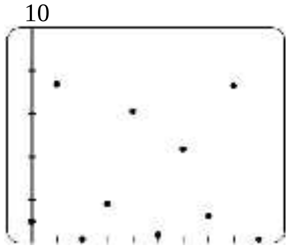
$a_{n+1} = 3a_n e^{-a_n} \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} 3a_n e^{-a_n} \Rightarrow L = 3Le^{-L} \Rightarrow L\left(1-3e^{-L}\right) = 0 \Rightarrow L = 0$ or $L = \ln 3 \approx 1.099$. Therefore, if the limit exists it will be either 0 or ln 3. Since the graph of the sequence appears to approach ln 3, we surmise that $\lim\limits_{n\to\infty} a_n = \ln 3$.

57.



Computer software was used to plot the first 10 points of the recursion equation $a_{n+1} = 10a_n e^{-a_n}$, $a_0 = 0.8$. The sequence does not appear to converge to a fixed value of L. Instead, the terms oscillate between values near 0.9 and 3.7.

58.



Computer software was used to plot the first 10 points of the recursion equation $a_{n+1} = 20a_n e^{-a_n}$, $a_0 = 0.9$. The sequence does not appear to converge to a fixed value of L. The terms fluctuate substantially in value exhibiting chaotic behaviour.

59. Let A_n represent the removed area of the Sierpinski carpet after the nth step of construction. In the first step, one square of area $\frac{1}{9}$ is removed so $A_1 = \frac{1}{9}$. In the second step, 8 squares each of area $\frac{1}{9}\left(\frac{1}{9}\right) = \frac{1}{9^2}$ are removed, so $A_2 = A_1 + \frac{8}{9^2} = \frac{1}{9} + \frac{8}{9^2} = \frac{1}{9}\left(1 + \frac{8}{9}\right)$. In the third step, 8 squares are removed for each of the 8 squares removed in the previous step. So there are a total of $8 \cdot 8 = 8^2$ squares removed each having an area of $\frac{1}{9}\left(\frac{1}{9^2}\right) = \frac{1}{9^3}$. This gives $A_3 = A_2 + \frac{8^2}{9^3} = \frac{1}{9}\left(1 + \frac{8}{9}\right) + \frac{8^2}{9^3} = \frac{1}{9}\left(1 + \frac{8}{9} + \left(\frac{8}{9}\right)^2\right)$. Observing the pattern in the first few terms of the sequence, we deduce the general formula for the nth term to be $A_n = \frac{1}{9}\left(1 + \frac{8}{9} + \left(\frac{8}{9}\right)^2 + \cdots + \left(\frac{8}{9}\right)^{n-1}\right)$. The terms in parentheses represent the sum of a geometric sequence with $a = 1$ and $r = 8/9$. Using Equation (5), we can write $A_n = \frac{1}{9} \cdot \frac{1\left(1-(8/9)^n\right)}{1-8/9} = 1 - \left(\frac{8}{9}\right)^n$. As n increases, $\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty}\left[1 - \left(\frac{8}{9}\right)^n\right] = 1$. Hence the area of the removed squares is 1 implying that the Sierpinski carpet has zero area.

60. $|\ \ | = \ \sin\ ,|\ \ | = |\ \ |\sin\ = \sin^2\ , |\ \ | = |\ \ |\sin\ = \sin^3\ ,$. Therefore,

$|\ \ | + |\ \ | + |\ \ | + |\ \ | + \cdots = \sum_{=1}^{\infty} \sin = \dfrac{\sin}{1 - \sin}$ since this is a geometric series with $=$ sin and $|\sin\ | \ 1$ because $0 \ \ = \frac{}{2}$.

PROJECT Modeling the Dynamics of Viral Infections

1. Viral replication is an example of exponential growth. The exponential growth recursion formula is $(+1) = \ ( )$ where is the growth rate and $( )$ is the number of viral particles at time . In Section 1.6, we saw the general solution of this recursion is $= _0 \cdot$ With $= 3$ and $_0 = 1$, the recursion equation is $_{+1} = 3$ and the general solution is $= 3$

2. Let $_1$ be the amount of time spent in phase 1 of the infection. Solving for $_1$ in the equation $_1 = _0 \cdot {}_1$ using logarithms: $\ln\ _1 = \ln(\ _1\ _0) \Rightarrow\ _1 = \dfrac{\ln(\ _1\ _0)}{\ln(\ )}$. The immune response initiates when $_1 = 2 \cdot 10^6$. Therefore the time it takes for the immune response to kick in is $_1 = \dfrac{\ln(2 \cdot 10^6) - \ln(\ _0)}{\ln(3)} \approx 13.2 - 0.91 \ln(\ _0)$. Hence, the larger the initial viral size the sooner the immune system responds.

3. Let $_2$ be the amount of time since the immune response initiated, $_{\text{immune}}$ be the replication rate during the immune response, and $_{\text{immune}}$ be the number of viruses killed by the immune system at each timestep. The second phase of the infection is

modeled by a two-step recursion. First, the virus replicates producing $^* = \ _{\text{immune}}\ _2$ viruses. Then, the immune system kills viruses leaving $_{2+1} = \ ^* - \ _{\text{immune}}$ leftover. Combining the two steps gives the recursion formula $_{2+1} = \ _{\text{immune}}\ _2 - \ _{\text{immune}}$.

4. The viral population will decrease over time if $\Delta \ \ 0$ at each timestep. Solving this inequality for $_2$:
$_{2+1} - \ _2 \ 0 \Rightarrow (\ _{\text{immune}} - 1)\ _2 - \ _{\text{immune}} \ 0 \Rightarrow \ _2 \ \dfrac{_{\text{immune}}}{(\ _{\text{immune}} - 1)}$ where we assumed $_{\text{immune}} \ 1$

Substituting the constants $_{\text{immune}} = \frac{1}{2} \cdot 3 = 1.5$ and $_{\text{immune}} = 500\,000$ give $_2 \ 1\,000\,000$. Therefore, the immune response will cause the infection to subside over time if the viral count is less than one million. This is not possible since the immune response initiates only once the virus reaches two million copies.

5. The recursion for the third phase can be obtained from the second phase recursion formula by replacing the replication and death rates with the new values. This gives $_{3+1} = \ _{\text{drug}}\ _3 - \ _{\text{drug}}$ where $_3$ is the amount of time since the start of drug treatment.

6. Similar to Problem 4, we solve for $_3$ in the inequality $\Delta \ = \ _{3+1} - \ _3 \ 0$ and find that $_3 \ \dfrac{_{\text{drug}}}{(\ _{\text{drug}} - 1)}$.

Substituting the constants $_{\text{drug}} = 1.25$ and $_{\text{drug}} = 25\,000\,000$ gives $_3 \ 100\,000\,000$. Therefore, the drug and immune system will cause the infection to subside over time if the viral count is less than 100 million. This is possible provided drug treatment begins before the viral count reaches 100 million.

7. From Formula (6), the general solution to the recursion equation $V_{t_2+1} = R_{\text{immune}} V_{t_2} - \alpha_{\text{immune}}$ is given by $V_{t_2} = R_{\text{immune}}^{t_2} V_0 - \alpha_{\text{immune}} \dfrac{1 - R_{\text{immune}}^{t_2}}{1 - R_{\text{immune}}}$. Solving for t_2 in this expression gives

$$V_{t_2} = R_{\text{immune}}^{t_2}\left(V_0 + \frac{\alpha_{\text{immune}}}{1 - R_{\text{immune}}}\right) - \frac{\alpha_{\text{immune}}}{1 - R_{\text{immune}}} \Rightarrow R_{\text{immune}}^{t_2} = \frac{V_{t_2} + \alpha_{\text{immune}}(1 - R_{\text{immune}})^{-1}}{V_0 + \alpha_{\text{immune}}(1 - R_{\text{immune}})^{-1}} \Rightarrow$$

$$t_2 = \ln\left(\frac{V_{t_2} + \alpha_{\text{immune}}(1 - R_{\text{immune}})^{-1}}{V_0 + \alpha_{\text{immune}}(1 - R_{\text{immune}})^{-1}}\right) \Big/ \ln R_{\text{immune}}$$

Note that the number of viral particles at the start of phase two is $V_0 = 2 \cdot 10^6$. Substituting $R_{\text{immune}} = 1.5$, $\alpha_{\text{immune}} = 500\,000$ and the critical viral load $V_{t_2} = 100 \cdot 10^6$ into the equation gives $t_2 = \dfrac{\ln(99)}{\ln(1.5)} \approx 11.33$ h. This is the amount of time spent in phase two after which the infection cannot be controlled.

From Problem 2, phase two begins after $t_1 = \dfrac{\ln(2 \cdot 10^6) - \ln(1)}{\ln(3)} \approx 13.21$ h. Thus, the total time is $t = t_1 + t_2 \approx 24.54$ h.

Hence, drug treatment must be started within approximately one day (24 hours) of the initial infection in order to control the viral count.

8. A general expression for the time it takes to reach the critical viral load is obtained by combining the expressions for t_1 and t_2 from Problems 2 and 7. This gives

$$t = t_1 + t_2 = \frac{\ln(2 \cdot 10^6)}{\ln(R)} - \frac{\ln(V_0)}{\ln(R)} + \frac{\ln\left(\dfrac{V_{t_2} + \alpha_{\text{immune}}(1 - R_{\text{immune}})^{-1}}{2 \cdot 10^6 + \alpha_{\text{immune}}(1 - R_{\text{immune}})^{-1}}\right)}{\ln R_{\text{immune}}}.$$

Substituting $R_{\text{immune}} = 0.5$, $\alpha_{\text{immune}} = 5 \cdot 10^5$, $V_0 = V_0$ and $V_{t_2} = 100 \cdot 10^6$ gives

$$t = \frac{\ln(2 \cdot 10^6)}{\ln(R)} - \frac{\ln(V_0)}{\ln(R)} + \frac{\ln\left(\dfrac{100 \cdot 10^6 + (5 \cdot 10^5)(1 - 0.5)^{-1}}{2 \cdot 10^6 + (5 \cdot 10^5)(1 - 0.5)^{-1}}\right)}{\ln(0.5)}.$$

Note: We have inherently assumed that $V_0 < 2 \cdot 10^6$, so that some time is spent in phase 1.

9. After 24 hours, the infection has been in the immune response phase for $t_2 = 24 - 13.21 = 10.79$ h.

Using the general expression for V_{t_2} from Problem 7 the number of viruses after 24 hours is

$V_{10.79} = (1.5^{10.79})(2 \cdot 10^6) - (5 \cdot 10^5)\dfrac{1 - 1.5^{10.79}}{1 - 1.5} \approx 80\,555\,008$. Since this is less than the critical viral load (100 million), drug intervention will be effective in controlling the virus. Rewriting the equation for t_2 for the drug phase gives

$t_3 = \ln\left(\dfrac{V_{t_3} + \alpha_{\text{drug}}(1 - R_{\text{drug}})^{-1}}{V_0 + \alpha_{\text{drug}}(1 - R_{\text{drug}})^{-1}}\right) \Big/ \ln R_{\text{drug}}$ where t_3 is the amount of time since the drug treatment started. Substituting values $t_3 = 0$, $V_0 = 80{,}555{,}008$, $\eta_{drug} = 1.25$ and $V_{drug} = 25{,}000{,}000$ yields $t_3 = 7.34$ h. Therefore, it takes approximately 7 hours after starting the drug treatment to completely eliminate the virus.

2.2 Limits of Functions at Infinity

1. (a) As x becomes large, the values of $f(x)$ approach 5.

(b) As x becomes large negative, the values of $f(x)$ approach 3.

2. (a) The graph of a function can intersect a horizontal asymptote. It can even intersect its horizontal asymptote an infinite number of times.

(b) The graph of a function can have 0, 1, or 2 horizontal asymptotes. Representative examples are shown.

No horizontal asymptote — One horizontal asymptote — Two horizontal asymptotes

3. If $f(x) = x^2/2^x$, then a calculator gives $f(0) = 0$, $f(1) = 0.5$, $f(2) = 1$, $f(3) = 1.125$, $f(4) = 1$, $f(5) = 0.78125$, $f(6) = 0.5625$, $f(7) = 0.3828125$, $f(8) = 0.25$, $f(9) = 0.158203125$, $f(10) = 0.09765625$, $f(20) \approx 0.00038147$,

$f(50) \approx 2.2204 \times 10^{-12}$, $f(100) \approx 7.8886 \times 10^{-27}$.

It appears that $\lim_{x\to\infty} (x^2/2^x) = 0$.

4. (a) From a graph of $f(x) = (1 - 2/x)^x$ in a window of $[0, 10{,}000]$ by $[0, 0.2]$, we estimate that $\lim_{x\to\infty} f(x) = 0.14$ (to two decimal places.)

(b) From the table, we estimate that $\lim_{x\to\infty} f(x) = 0.1353$ (to four decimal places.)

x	$f(x)$
10,000	0.135308
100,000	0.135333
1,000,000	0.135335

5. $$\lim_{x\to\infty} \frac{1}{2x+3} = \lim_{x\to\infty} \frac{1/x}{(2x+3)/x} = \frac{\lim_{x\to\infty} (1/x)}{\lim_{x\to\infty} (2 + 3/x)} = \frac{\lim_{x\to\infty} (1/x)}{\lim_{x\to\infty} 2 + 3 \lim_{x\to\infty} (1/x)} = \frac{0}{2 + 3(0)} = \frac{0}{2} = 0$$

6. $$\lim_{x\to\infty} \frac{3x+5}{x-4} = \lim_{x\to\infty} \frac{(3x+5)/x}{(x-4)/x} = \lim_{x\to\infty} \frac{3 + 5/x}{1 - 4/x} = \frac{\lim_{x\to\infty} 3 + 5 \lim_{x\to\infty} (1/x)}{\lim_{x\to\infty} 1 - 4 \lim_{x\to\infty} (1/x)} = \frac{3 + 5(0)}{1 - 4(0)} = 3$$

7. $$\lim_{x\to\infty} \frac{3x-2}{2x+1} = \lim_{x\to\infty} \frac{(3x-2)/x}{(2x+1)/x} = \lim_{x\to\infty} \frac{3 - 2/x}{2 + 1/x} = \frac{\lim_{x\to\infty} 3 - 2 \lim_{x\to\infty} (1/x)}{\lim_{x\to\infty} 2 + \lim_{x\to\infty} (1/x)} = \frac{3 - 2(0)}{2 + 0} = \frac{3}{2}$$

→∞ 2 + 1 →∞ + 1) →∞ 2 + 1 lim 2 + lim 1 2 + 0 2

→∞ →∞

2 2 3 3

8. $\lim_{x\to\infty} \frac{1-x}{x^3 - x + 1} = \lim_{x\to\infty} \frac{(1-x)\frac{1}{x^3}}{(x^3 - x + 1)\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{1}{x^3} - \frac{1}{x^2}}{1 - \frac{1}{x^2} + \frac{1}{x^3}}$

$$= \frac{\lim_{x\to\infty} \frac{1}{x^3} - \lim_{x\to\infty} \frac{1}{x^2}}{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{1}{x^2} + \lim_{x\to\infty} \frac{1}{x^3}} = \frac{0-0}{1-0+0} = 0$$

9. lim 1 − □ − □ 2 = lim (1 − □ − □ 2 2) = lim (1 − 2 − 1 − 1)

→−∞ 2 2 − 7 →−∞ (2 2 − 7) 2 lim (2 − 7 2)

→−∞

lim (1 2) − lim (1) − lim 1 0 − 0 − 1 1

→−∞ →−∞ →−∞

= lim 2 − 7 lim (1 2) = 2 − 7(0) = − 2

→−∞ →−∞

10. lim 4 3 + 6 2 − 2 = lim (4 3 + 6 2 − 2) 3 = lim 4 + 6 − 2 4 + 0 − 0 = 2

3 = 4 2 − 0 + 0

→−∞ 2 3 − 4 + 5 →−∞ (2 3 − + 5) 3 →−∞ 2 − 4 2 + 5 3

0

3 5 −

11. lim 6 = lim = lim = ∞ since 5 3 1 and − → ∞ as → −∞

→−∞ →−∞ 5 →−∞ 3

12. lim 5 = 0 since 10 → ∞ as → ∞

→∞ 10

13. lim √ + 2 = lim (√ + 2) 2 = lim 1 32 + 1 = 0 + 1 = −1

→∞ 2 − 2 →∞ (2 − 2) 2 √ →∞ 2 − 1 0 − 1

14. lim − √ = lim − 3 2 = lim 1 1 2 − 1 = 0 − 1 = − 1

→∞ 2 32 + 3 − 5 →∞ (2 3 2 + 3 − 5) 3 2 →∞ 2 + 3 12 − 5 3 2 2 + 0 − 0 2

15. lim (2 2 + 1) 2 = lim (2 2 + 1) 2 4 = lim [(2 2 + 1) 2] 2

2

→∞ (− 1) 2 (2 +) →∞ (− 1) (2 +)] 4 →∞ [(2 − + 1) 2][(2 +) 2]

(

2

+

= lim 1 2) 2 = (2 + 0) 2

→∞ = 4 (1 − 2 + 1 2)(1 + 1) (1 − 0 + 0)(1 + 0)

16. lim √ 2 = lim √ 2 2 = lim 1 [since 2 = √ 4 for 0]

→∞ 4 + 1 →∞ 4 + 1 2 →∞ (4 + 1) 4

= lim 1 = √ 1 = 1

→∞ 1 + 1 4 1 + 0

17. lim √ 9 2 + − 3 = lim √ 9 2 + − 3 √ 9 2 + + 3 = lim √ 9 2 + 2 − (3) 2

→∞ √ 9 2 + + 3 →∞ √ 9 2 + + 3

9

= lim 2 + − 9 2 = lim 1

→∞ √ 9 2 + + 3 →∞ √ 9 2 + + 3 1

= lim = lim 1 = √ 1 = 1 = 1

→∞ 9 2 2 + 2 + 3 →∞ 9 + 1 + 3 9 + 3 3 + 3 6

18. $\lim\limits_{\to\infty}$ √ √ √ $2+$ $2+$ $= \lim\limits_{\to\infty}$ $-\surd$ $2+$ √ √ $2+$ √ $2+$ + √ $2+$

$2+$ + $2+$

$= \lim \dfrac{(^2+\ )-(^2+\ )}{\surd \qquad \surd} = \lim\limits_{\to\infty}$ $-\ )\ ]$ √ √ √ [redacted] [redacted] [redacted]

$2\ ++$ $2+$ +

$= \lim\limits_{\to\infty}$ $-$ $=$ 2 $-$ √ + 2 + $= -$ 2
√

$1+$ $+\ \ 1+$ $1{+}0{+}\ 1{+}0$ 2

19. $\lim_{\to\infty} \frac{6}{3 + {}^{-2}} = \frac{6}{3 + \lim_{\to\infty} {}^{-2}} = \frac{6}{3 + 0} = \frac{6}{3} = 2$

20. For 0, $\sqrt{{}^2 + 1} \quad \sqrt{{}^2} =$. So as $\to \infty$, we have $\sqrt{{}^2 + 1} \to \infty$, that is, $\lim \sqrt{{}^2 + 1} = \infty$.

21. $\lim\limits_{\to\infty} \dfrac{4 - 3\,^2 +}{^3 - \; + 2} = \lim\limits_{\to\infty} \dfrac{(4 - 3\,^2 + \;)/\,^3}{(^3 - \; + 2)/\,^3}$ [divide by the highest power of in the denominator] $= \lim\limits_{\to\infty} \dfrac{3 \;\; + 1 \;\; 2}{1 - 1\,^2 + 2\,^3} = \infty$

since the numerator increases without bound and the denominator approaches 1 as $\to \infty$.

22. $\lim\limits_{\to\infty} (- + 2\cos 3 \;)$ does not exist. $\lim\limits_{\to\infty} - = 0$, but $\lim\limits_{\to\infty} (2\cos 3\;)$ does not exist because the values of 2 cos 3 oscillate between the values of -2 and 2 infinitely often, so the given limit does not exist.

23. $\lim\limits_{\to-\infty} (\,^4 + \,^5) = \lim\limits_{\to-\infty} \,^5(\,^1 + 1)$ [factor out the largest power of] $= -\infty$ because $^5 \to -\infty$ and $1 \; + 1 \to 1$ as $\to -\infty$.

Or: $\lim\limits_{\to-\infty} \,^4 + \,^5 = \lim\limits_{\to-\infty} \,^4(1 + \;) = -\infty$.

24. $\lim\limits_{\to-\infty} \dfrac{1 + \,^6}{^4 + 1} = \lim\limits_{\to-\infty} \dfrac{(1 + \,^6)/\,^4}{(^4 + 1)/\,^4}$ [divide by the highest power of in the denominator] $= \lim\limits_{\to-\infty} \dfrac{1\,^4 + \,^2}{1 + 1\,^4} = \infty$

since the numerator increases without bound and the denominator approaches 1 as $\to -\infty$.

25. As increases, $1\,^2$ approaches zero, so $\lim\limits_{\to\infty} \,^{-1\,^2} = \,^{-(0)} = 1$

26. Divide numerator and denominator by ^{3}: $\lim\limits_{\to\infty} \dfrac{^3 - \,^{-3}}{^3 + \,^{-3}} = \lim\limits_{\to\infty} \dfrac{1 - \,^{-6}}{1 + \,^{-6}} = \dfrac{1 - 0}{1 + 0} = 1$

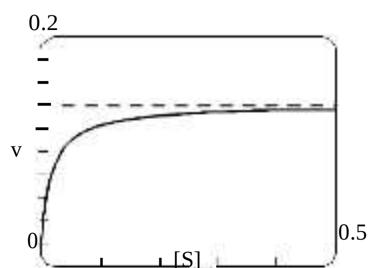
27. $\lim\limits_{\to\infty} \dfrac{1 - \;}{1 + 2\;} = \lim\limits_{\to\infty} \dfrac{(1 - \;)/\;}{(1 + 2\;)/\;} = \lim\limits_{\to\infty} \dfrac{1\; - 1}{1\; + 2} = \dfrac{0 - 1}{0 + 2} = -\dfrac{1}{2}$

28. $\lim\limits_{\to-\infty} [\ln(\,^2) - \ln(\,^2 + 1)] = \lim\limits_{\to-\infty} \ln \dfrac{^2}{^2 + 1} = \lim\limits_{\to-\infty} \ln \dfrac{1}{1 + 1\,^2} = \ln \dfrac{1}{1 + 0} = \ln(1) = 0$

29. $(\;) = (\; + \;) \Rightarrow (\;) = (\; + \;) = \; 2$. Hence, is the nutrient concentration at which the growth rate is half of the maximum possible value. This is often referred to as the half-saturation constant.

30. (a) $\lim\limits_{[S]\to\infty} = \lim\limits_{[S]\to\infty} \dfrac{0.14[S]}{0.015 + [S]} = \lim\limits_{[S]\to\infty} \dfrac{0.14}{0.015\,[S] + 1}$ [divide numerator and denominator by [S]] $= \dfrac{0.14}{0 + 1} = 0.14$. So the line $= 0.14$ is a horizontal asymptote. Therefore, as the concentration increases, the enzymatic reaction rate will approach 0.14. Note, we did not need to consider the limit as $[S] \to -\infty$ because concentrations must be positive in value.

(b)



31. $\lim\limits_{\to\infty} (\;) = \lim\limits_{\to\infty} \dfrac{8\;}{1 + 2\; + \;^2} = \lim\limits_{\to\infty} \dfrac{(8\;)/\;^2}{(1 + 2\; + \;^2)/\;^2}$ [divide by the highest power of in the denominator]

$= \lim\limits_{\to\infty} \dfrac{8\;}{1\;^2 + 2\; + 1} = \dfrac{0}{0 + 0 + 1} = 0$