Solutions Manual for Calculus 10th Edition by Anton Bivens Davis ISBN 0470647701 9780470647707

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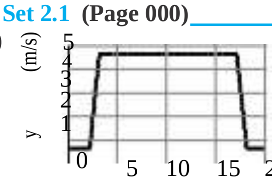
Exercise Set 2.1

1. (a) $m_{tan} = (50 - 10)/(15 - 5) = 40/10 = 4$ m/s.

Exercise Set 2.1 (Page 000)

1. **(a)** 4 m/s **(b)**



(b)

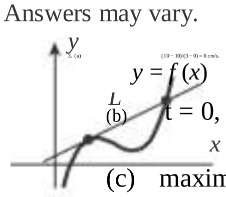
3. **(a)** 0 cm/s **(b)** $t = 0$, $t = 2$, and $t = 4.2$ **(c)** maximum: $t = 1$; minimum: $t = 3$ **(d)** -7.5 cm/s

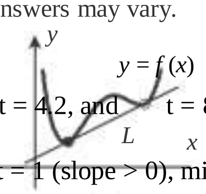
2. At $t = 4$ s, $m_{tan} \approx (90 - 0)/(10 - 2) = 90/8 = 11.25$ m/s.

At $t = 8$ s, $m_{tan} \approx (140 - 0)/(10 - 4) = 140/6 \approx 23.33$ m/s.

5. straight line with slope equal to the velocity

7. Answers may vary. **9.** Answers may vary.





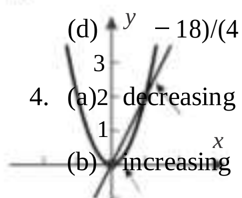
3. (a) $(10 - 10)/(3 - 0) = 0$ cm/s.

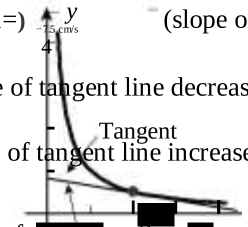
(b) $t = 0$, $t = 2$, $t = 4.2$, and $t = 8$ (horizontal tangent line).

(c) maximum: $t = 1$ (slope > 0), minimum: $t = 3$ (slope < 0).

(d) $(3 - 18)/(4 - 2) = -7.5$ cm/s (slope of estimated tangent line to curve at $t = 3$).

(d)





4. (a) decreasing (slope of tangent line decreases with increasing time)

(b) increasing (slope of tangent line increases with increasing time)

(c) increasing (slope of t[redacted] line [redacted] reases with increasing time)

15. **(a)** $2x_0$ (d) decreasing $\frac{1}{2}$ x_0 (slope of tangent line decreases with increasing time)

Responses to True–False questions may be abridged to save space.

5. It is a straight line with slope equal to the velocity.

19. True; set $h = x - 1$, so $x = 1 + h$ and $h \to 0$ is equivalent to $x \to 1$.

21. False; velocity is a ratio of change in position to change in time.

6. The velocity increases from time 0 to time t_0, so the slope of the curve increases during that time. From time t_0 to time t_1, the velocity, and hence the slope, decrease. At time t_1, the velocity, and hence the slope, instantaneously drop to zero, so there is a sharp bend in the curve at that point.

23. **(a)** 72° F at about 4:30 p.m. **(b)** 4 $^\circ$F/h **(c)** -7° F/h at about 9 p.m.

25. **(a)** first year **(b)** 6 cm/year **(c)** 10 cm/year at about age 14 **(d)** Growth rate (cm/year)

40

30

20

10

t (yr)

t

27. **(a)** 19,200 ft **(b)** 480 ft/s **(c)** 66.94 ft/s **(d)** 1440 ft/s

/ t t /

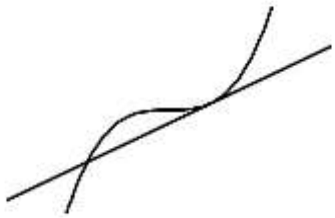
29. **(a)** 720 ft min **(b)** 192$_0$ ft$_1$ min

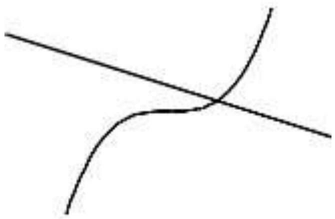
! **Exercise Set 2.2 (Page 000)**

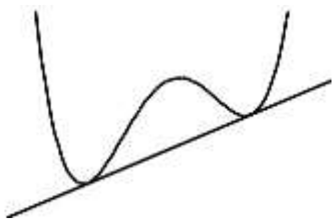
1. 2, 0, −2, −1 **5.** y **7.** y = 5x − 16

1

3. **(b)** 3 **(c)** 3 **9.** 4x, y = 4x − 2

7. 

8. 

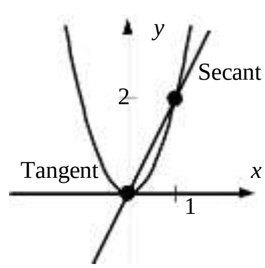
9. 

10. 

11. (a) $m_{sec} = \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{2}{1} = 2$

(b) $m_{tan} = \lim\limits_{x_1 \to 0} \dfrac{f(x_1) - f(0)}{x_1 - 0} = \lim\limits_{x_1 \to 0} \dfrac{2x_1^2 - 0}{x_1 - 0} = \lim\limits_{x_1 \to 0} 2x_1 = 0$

(c) $m_{tan} = \lim\limits_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim\limits_{x_1 \to x_0} \dfrac{2x_1^2 - 2x_0^2}{x_1 - x_0} = \lim\limits_{x_1 \to x_0} (2x_1 + 2x_0) = 4x_0$

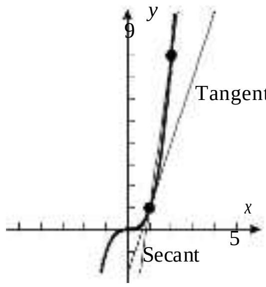


(d) The tangent line is the x-axis.

12. (a) $m_{sec} = \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{2^3 - 1^3}{1} = 7$

(b) $m_{\tan} = \lim_{x_1 \to 1} \dfrac{f(x_1) - f(1)}{x_1 - 1} = \lim_{x_1 \to 1} \dfrac{x_1^3 - 1}{x_1 - 1} = \lim_{x_1 \to 1} \dfrac{(x_1 - 1)(x_1^2 + x_1 + 1)}{x_1 - 1} = \lim_{x_1 \to 1} (x_1^2 + x_1 + 1) = 3$

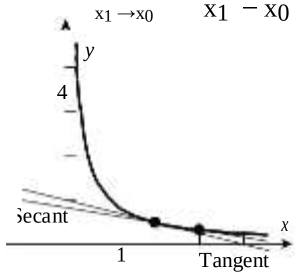
(c) $m_{tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{x_1^3 - x_0^3}{x_1 - x_0} = \lim_{x_1 \to x_0} (x_1^2 + x_1 x_0 + x_0^2) = 3x_0^2$



13. (a) $m_{sec} = \frac{f(3) - f(2)}{3 - 2} = \frac{1/3 - 1/2}{1} = \frac{1}{6}$

(b) $m_{tan} = \lim_{x_1 \to 2} \frac{f(x_1) - f(2)}{x_1 - 2} = \lim_{x_1 \to 2} \frac{1/x_1 - 1/2}{x_1 - 2} = \lim_{x_1 \to 2} \frac{2 - x_1}{2x_1(x_1 - 2)} = \lim_{x_1 \to 2} \frac{-1}{2x_1} = \frac{1}{4}$

(c) $m_{tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{1/x_1 - 1/x_0}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{x_0 - x_1}{x_0 x_1 (x_1 - x_0)} = \lim_{x_1 \to x_0} \frac{-1}{x_0 x_1} = -\frac{1}{x_0^2}$

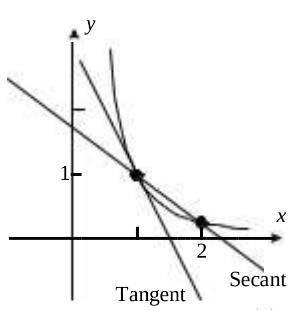


(d)

14. (a) $m_{sec} = \frac{f(2) - f(1)}{2 - 1} = \frac{1/4 - 1}{1} = \frac{3}{4}$

(b) $m_{tan} = \lim_{x_1 \to 1} \frac{f(x_1) - f(1)}{x_1 - 1} = \lim_{x_1 \to 1} \frac{1/x_1^2 - 1}{x_1 - 1} = \lim_{x_1 \to 1} \frac{1 - x_1^2}{x_1^2(x_1 - 1)} = \lim_{x_1 \to 1} \frac{-(x_1 + 1)}{x_1^2} = -2$

(c) $m_{tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{1/x_1^2 - 1/x_0^2}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{x_0^2 - x_1^2}{x_0^2 x_1^2 (x_1 - x_0)} = \lim_{x_1 \to x_0} \frac{-(x_1 + x_0)}{x_0^2 x_1^2} = -\frac{2}{x_0^3}$



(d)

15. (a) $m_{tan} = \lim \frac{f(x_1) - f(x_0)}{(x}$

$$m_{\tan} = \lim_{x_1 \to x_0} \frac{(x_1^2 - 1) - (x_0^2 - 1)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{x_1^2 - x_0^2}{x_1 - x_0} = \lim_{x_1 \to x_0} (x_1 + x_0) = 2x_0$$

(b) $m_{\tan} = 2(-1) = -2$

16. (a) $m_{\tan} = \lim\limits_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim\limits_{x_1 \to x_0} \dfrac{(x_1^2 + 3x_1 + 2) - (x_0^2 + 3x_0 + 2)}{x_1 - x_0} = \lim\limits_{x_1 \to x_0} \dfrac{(x_1^2 - x_0^2) + 3(x_1 - x_0)}{x_1 - x_0} =$
$= \lim\limits_{x_1 \to x_0} (x_1 + x_0 + 3) = 2x_0 + 3$

(b) $m_{\tan} = 2(2) + 3 = 7$

17. (a) $m_{\tan} = \lim\limits_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim\limits_{x_1 \to x_0} \dfrac{(x_1 + \sqrt{x_1}) - (x_0 + \sqrt{x_0})}{x_1 - x_0} = \lim\limits_{x_1 \to x_0} 1 + \dfrac{1}{\sqrt{x_1} + \sqrt{x_0}} = 1 + \dfrac{1}{2\sqrt{x_0}}$

(b) $m_{\tan} = 1 + \dfrac{1}{2\sqrt{1}} = \dfrac{3}{2}$

18. (a) $m_{\tan} = \lim\limits_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim\limits_{x_1 \to x_0} \dfrac{1/\sqrt{x_1} - 1/\sqrt{x_0}}{x_1 - x_0} = \lim\limits_{x_1 \to x_0} \dfrac{\sqrt{x_0} - \sqrt{x_1}}{\sqrt{x_0}\sqrt{x_1}\,(x_1 - x_0)} =$
$= \lim\limits_{x_1 \to x_0} \dfrac{-1}{\sqrt{x_0}\sqrt{x_1}\,(\sqrt{x_1} + \sqrt{x_0})} = -\dfrac{1}{2x_0^{3/2}}$

(b) $m_{\tan} = -\dfrac{1}{2(4)^{3/2}} = -\dfrac{1}{16}$

19. True. Let $x = 1 + h$.

20. False. A secant line meets the curve in at least two places, but a tangent line might meet it only once.

21. False. Velocity represents the rate at which position changes.

22. True. The units of the rate of change are obtained by dividing the units of $f(x)$ (inches) by the units of x (tons).

23. (a) 72° F at about 4:30 P.M. (b) About $(67 - 43)/6 = 4^\circ$ F/h.

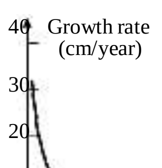
(c) Decreasing most rapidly at about 9 P.M.; rate of change of temperature is about -7° F/h (slope of estimated tangent line to curve at 9 P.M.).

24. For $V = 10$ the slope of the tangent line is about $(0 - 5)/(20 - 0) = -0.25$ atm/L, for $V = 25$ the slope is about $(1 - 2)/(25 - 0) = -0.04$ atm/L.

25. (a) During the first year after birth.

(b) About 6 cm/year (slope of estimated tangent line at age 5).

(c) The growth rate is greatest at about age 14; about 10 cm/year.



10

t (yrs)

(d) 5 10 15 20

26. (a) The object falls until s = 0. This happens when $1250 - 16t^2 = 0$, so $t = \sqrt{1250/16} = \sqrt{78.125} > \sqrt{25} = 5$; hence the object is still falling at t = 5 sec.

(b) $\frac{f(6) - f(5)}{6 - 5} = \frac{674 - 850}{}$

1

$= -176$. The average velocity is -176 ft/s.

(c) $v_{inst} = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h\to 0} \frac{[1250 - 16(5+h)^2] - 850}{h} = \lim_{h\to 0} \frac{-160h - 16h^2}{h} = \lim_{h\to 0} (-160 - 16h) =$ -160 ft/s.

27. (a) $0.3 \cdot 40^3 = 19{,}200$ ft (b) $v_{ave} = 19{,}200/40 = 480$ ft/s

(c) Solve $s = 0.3t^3 = 1000$; $t \approx 14.938$ so $v_{ave} \approx 1000/14.938 \approx 66.943$ ft/s.

(d) $v_{inst} = \lim_{h\to 0} \frac{0.3(40+h)^3 - 0.3 \cdot 40^3}{h} = \lim_{h\to 0} \frac{0.3(4800h + 120h^2 + h^3)}{h} = \lim_{h\to 0} 0.3(4800 + 120h + h^2) = 1440$ ft/s

28. (a) $v_{ave} = \frac{4.5(12)^2 - 4.5(0)^2}{12 - 0} = 54$ ft/s

(b) $v_{inst} = \lim_{t_1\to 6} \frac{4.5t_1^2 - 4.5(6)^2}{t_1 - 6} = \lim_{t_1\to 6} \frac{4.5(t_1^2 - 36)}{t_1 - 6} = \lim_{t_1\to 6} \frac{4.5(t_1 + 6)(t_1 - 6)}{t_1 - 6} = \lim_{t_1\to 6} 4.5(t_1 + 6) = 54$ ft/s

29. (a) $v_{ave} = \frac{6(4)^4 - 6(2)^4}{4 - 2} = 720$ ft/min

(b) $v_{inst} = \lim_{t_1\to 2} \frac{6t_1^4 - 6(2)^4}{t_1 - 2} = \lim_{t_1\to 2} \frac{6(t_1^4 - 16)}{t_1 - 2} = \lim_{t_1\to 2} \frac{6(t_1^2 + 4)(t_1^2 - 4)}{t_1 - 2} = \lim_{t_1\to 2} 6(t_1^2 + 4)(t_1 + 2) = 192$ ft/min

30. See the discussion before Definition 2.1.1.

31. The instantaneous velocity at $t = 1$ equals the limit as $h \to 0$ of the average velocity during the interval between $t = 1$ and $t = 1 + h$.

Exercise Set 2.2

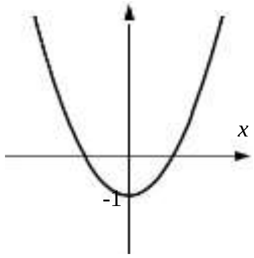
1. $f'(1) = 2.5$, $f'(3) = 0$, $f'(5) = -2.5$, $f'(6) = -1$.

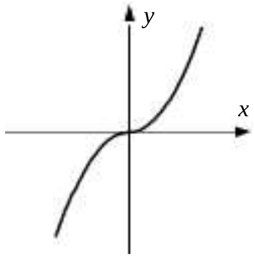
2. $f'(4) < f'(0) < f'(2) < 0 < f'(-3)$.

3. (a) $f'(a)$ is the slope of the tangent line. (b) $f'(2) = m = 3$ (c) The same, $f'(2) = 3$.

4. $f'(1) = \frac{2 - (-1)}{1 - (-1)} = \frac{3}{2}$

y

5. 



6.

7. $y - (-1) = 5(x - 3)$, $y = 5x - 16$

8. $y - 3 = -4(x + 2)$, $y = -4x - 5$

9. $f'(x) = \lim_{h\to0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to0} \frac{2(x+h)^2 - 2x^2}{h} = \lim_{h\to0} \frac{4xh + 2h^2}{h} = 4x$; $f'(1) = 4$ so the tangent line is given by $y - 2 = 4(x - 1)$, $y = 4x - 2$.

10. $f'(x) = \lim_{h\to0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to0} \frac{1/(x+h)^2 - 1/x^2}{h} = \lim_{h\to0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2} = \lim_{h\to0} \frac{-2xh - h^2}{hx^2(x+h)^2} = \lim_{h\to0} \frac{-2x - h}{x^2(x+h)^2} =$ 2

11. $f'(x) = \lim_{h\to0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to0} (3x^2 + 3xh + h^2) = 3x^2$; $f'(0) = 0$ so the tangent line is given by $y - 0 = 0(x - 0)$, $y = 0$.

12. $f'(x) = \lim_{h\to0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to0} \frac{[2(x+h)^3 + 1] - [2x^3 + 1]}{h} = \lim_{h\to0} (6x^2 + 6xh + 2h^2) = 6x^2$; $f(-1) = 2(-1)^3 + 1 = -1$ and $f'(-1) = 6$ so the tangent line is given by $y + 1 = 6(x + 1)$, $y = 6x + 5$.

13. $f'(x) = \lim_{h\to0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to0}$ [redacted] $= \lim_{h\to0}$ [redacted] $=$ $\lim_{h\to0} \frac{h}{h(\sqrt{x+1+h} + \sqrt{x+1})} = \frac{1}{2\sqrt{x+1}}$; $f(8) = \sqrt{8+1} = 3$ and $f'(8) = \frac{1}{6}$ so the tangent line is given by $y - 3 = \frac{1}{6}(x - 8)$, $y = \frac{1}{6}x + \frac{5}{3}$.

14. $f'(x) = \lim_{h\to0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to0} \frac{\sqrt{2x+2h+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2x+2h+1} + \sqrt{2x+1}}{\sqrt{2x+2h+1} + \sqrt{2x+1}} = \lim_{h\to0} \frac{2h}{h(\sqrt{2x+2h+1} + \sqrt{2x+1})} = \lim_{h\to0} \frac{2}{\sqrt{2x+2h+1} + \sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$; $f(4) = \sqrt{2 \cdot 4 + 1} = \sqrt{9} = 3$ and $f'(4) = 1/3$ so the tangent line is given by $y - 3 = \frac{1}{3}(x - 4)$, $y = \frac{1}{3}x + \frac{5}{3}$.

15. $f'(x) = \lim_{\Delta x\to0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} = \lim_{\Delta x\to0} \frac{\frac{x - (x+\Delta x)}{x(x+\Delta x)}}{\Delta x} = \lim_{\Delta x\to0} \frac{-\Delta x}{\cdots} = \lim_{\Delta x\to0} -\frac{1}{\cdots} = -\frac{1}{\cdots}$

$\lim_{\Delta x\to 0}$ ... Δx ... $\lim_{\Delta x\to 0}$ ... Δx ... $\lim_{\Delta x\to 0} x\Delta x(x+\Delta x)$ ... $\lim_{\Delta x\to 0}$ ... $x(x+\Delta x)$ ... x^2

16. $$f'(x) = \lim_{\Delta x\to 0}\frac{\frac{1}{(x+\Delta x)+1}-\frac{1}{x+1}}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{(x+1)-(x+\Delta x+1)}{(x+1)(x+\Delta x+1)}}{\Delta x} = \lim_{\Delta x\to 0}\frac{x+1-x-\Delta x-1}{\Delta x(x+1)(x+\Delta x+1)} =$$

$$= \lim_{\Delta x\to 0}\frac{-\Delta x}{\Delta x(x+1)(x+\Delta x+1)} = \lim_{\Delta x\to 0}\frac{-1}{(x+1)(x+\Delta x+1)} = -\frac{1}{(x+1)^2}.$$

17. $$f'(x) = \lim_{\Delta x \to 0} \frac{\left[(x+\Delta x)^2 - (x+\Delta x)\right] - (x^2 - x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 - \Delta x}{\Delta x} = \lim_{\Delta x \to 0} (2x - 1 + \Delta x) = 2x - 1.$$

18. $f'(x) = \lim_{\Delta x\to 0} \frac{(x+\Delta x)^4 - x^4}{\Delta x} = \lim_{\Delta x\to 0} \frac{4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4}{\Delta x}$

$= \lim_{\Delta x\to 0} (4x^3 + 6x^2\Delta x + 4x(\Delta x)^2 + (\Delta x)^3) = 4x^3.$

19. $f'(x) = \lim_{\Delta x\to 0} \frac{\frac{1}{\sqrt{x+\Delta x}} - \frac{1}{\sqrt{x}}}{\Delta x} = \lim_{\Delta x\to 0} \frac{\sqrt{x} - \sqrt{x+\Delta x}}{\Delta x\sqrt{x}\sqrt{x+\Delta x}} = \lim_{\Delta x\to 0} \frac{x-(x+\Delta x)}{\Delta x\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x}+\sqrt{x+\Delta x})} =$

$= \lim_{\Delta x\to 0} \frac{-1}{\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x}+\sqrt{x+\Delta x})} = -\frac{1}{2x^{3/2}}.$

20. $f'(x) = \lim_{\Delta x\to 0} \frac{\frac{1}{\sqrt{x+\Delta x-1}} - \frac{1}{\sqrt{x-1}}}{\Delta x} = \lim_{\Delta x\to 0} \frac{\sqrt{x-1} - \sqrt{x+\Delta x-1}}{\Delta x\sqrt{x-1}\sqrt{x+\Delta x-1}} =$

$= \lim_{\Delta x\to 0} \frac{-\Delta x}{\Delta x\sqrt{x-1}\sqrt{x+\Delta x-1}(\sqrt{x-1}+\sqrt{x+\Delta x-1})} = \lim_{\Delta x\to 0} \frac{-1}{\sqrt{x-1}\sqrt{x+\Delta x-1}(\sqrt{x-1}+\sqrt{x+\Delta x-1})} =$

$-\frac{1}{2(x-1)^{3/2}}.$

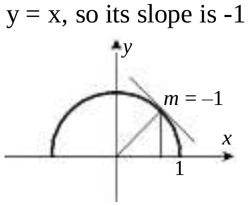
21. $f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h\to 0} \frac{[4(t+h)^2 + (t+h)] - [4t^2 + t]}{h} = \lim_{h\to 0} \frac{4t^2 + 8th + 4h^2 + t + h - 4t^2 - t}{h} =$

$\lim_{h\to 0} \frac{8th + 4h^2 + h}{h} = \lim_{h\to 0} (8t + 4h + 1) = 8t + 1.$

22. $\frac{dV}{dr} = \lim_{h\to 0} \frac{\frac{4}{3}\pi(r+h)^3 - \frac{4}{3}\pi r^3}{h} = \lim_{h\to 0} \frac{\frac{4}{3}\pi(r^3 + 3r^2h + 3rh^2 + h^3 - r^3)}{h} = \lim_{h\to 0} \frac{4}{3}\pi(3r^2 + 3rh + h^2) = 4\pi r^2.$

23. (a) D (b) F (c) B (d) C (e) A (f) E

24. $f'(\sqrt{2}/2)$ is the slope of the tangent line to the unit circle at $(\sqrt{2}/2, \sqrt{2}/2)$. This line is perpendicular to the line $y = x$, so its slope is -1.

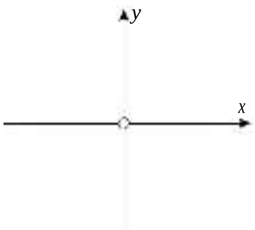


x x 1 2

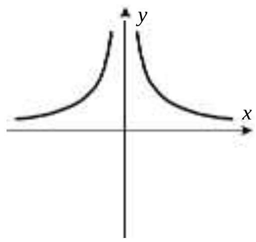
–1

25. (a) (b) (c)

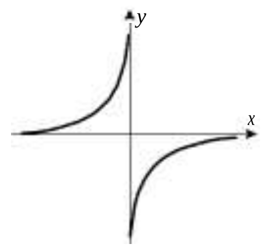
26. (a)



(b)



(c)



27. False. If the tangent line is horizontal then $f'(a) = 0$.

28. True. $f'(-2)$ equals the slope of the tangent line.

29. False. E.g. $|x|$ is continuous but not differentiable at $x = 0$.

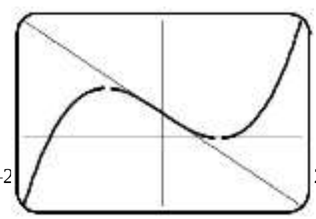
30. True. See Theorem 2.2.3.

31. (a) $f(x) = \sqrt{x}$ and $a = 1$ (b) $f(x) = x^2$ and $a = 3$

32. (a) $f(x) = \cos x$ and $a = \pi$ (b) $f(x) = x^7$ and $a = 1$

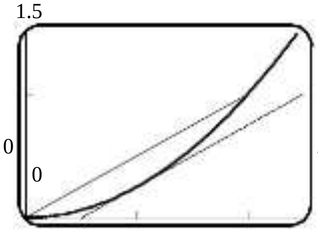
33. $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{(1-(x+h)^2) - (1-x^2)}{h} = \lim\limits_{h\to 0} \dfrac{-2xh - h^2}{h} = \lim\limits_{h\to 0} (-2x - h) = -2x$, and $\left.\dfrac{dy}{dx}\right|_{x=1} = -2$.

34. $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{\dfrac{x+2+h}{x+h} - \dfrac{x+2}{x}}{h} = \lim\limits_{h\to 0} \dfrac{x(x+2+h) - (x+2)(x+h)}{hx(x+h)} = \lim\limits_{h\to 0} \dfrac{-2}{x(x+h)} = \dfrac{-2}{x^2}$, and $\left.\dfrac{dy}{dx}\right|_{x=-2} = -\dfrac{1}{2}$.



35. $y = -2x + 1$

36.



37. (b)

w	1.5	1.1	1.01	1.001	1.0001	1.00001
$\dfrac{f(w) - f(1)}{w - 1}$	1.6569	1.4355	1.3911	1.3868	1.3863	1.3863

w	0.5	0.9	0.99	0.999	0.9999	0.99999
$\dfrac{f(w) - f(1)}{w - 1}$	1.1716	1.3393	1.3815	1.3858	1.3863	1.3863

38. (b)

w	$\frac{\pi}{4} + 0.5$	$\frac{\pi}{4} + 0.1$	$\frac{\pi}{4} + 0.01$	$\frac{\pi}{4} + 0.001$	$\frac{\pi}{4} + 0.0001$	$\frac{\pi}{4} + 0.00001$
$\dfrac{f(w) - f(\pi/4)}{w - \pi/4}$	0.50489	0.67060	0.70356	0.70675	0.70707	0.70710

w	$\frac{\pi}{4} - 0.5$	$\frac{\pi}{4} - 0.1$	$\frac{\pi}{4} - 0.01$	$\frac{\pi}{4} - 0.001$	$\frac{\pi}{4} - 0.0001$	$\frac{\pi}{4} - 0.00001$
$\frac{f(w) - f(\pi/4)}{w - \pi/4}$	0.85114	0.74126	0.71063	0.70746	0.70714	0.70711

39. (a) $\frac{f(3)-f(1)}{3-1} = \frac{2.2-2.12}{2} = 0.04$; $\frac{f(2)-f(1)}{2-1} = \frac{2.34-2.12}{1} = 0.22$; $\frac{f(2)-f(0)}{2-0} = \frac{2.34-0.58}{2} = 0.88$.

(b) The tangent line at $x = 1$ appears to have slope about 0.8, so $\frac{f(2)-f(0)}{2-0}$ gives the best approximation and $\frac{f(3)-f(1)}{3-1}$ gives the worst.

40. (a) $f'(0.5) \approx \frac{f(1)-f(0)}{1-0} = \frac{2.12-0.58}{1} = 1.54$.

(b) $f'(2.5) \approx \frac{f(3)-f(2)}{3-2} = \frac{2.2-2.34}{1} = -0.14$.

41. (a) dollars/ft

(b) $f'(x)$ is roughly the price per additional foot.

(c) If each additional foot costs extra money (this is to be expected) then $f'(x)$ remains positive.

(d) From the approximation $1000 = f'(300) \approx \frac{f(301)-f(300)}{301-300}$ we see that f (301)f (300) + 1000, so the extra foot will cost around \$1000.

42. (a) $\frac{\text{gallons}}{\text{dollars/gallon}} = \text{gallons}^2/\text{dollar}$

(b) The increase in the amount of paint that would be sold for one extra dollar per gallon.

(c) It should be negative since an increase in the price of paint would decrease the amount of paint sold.

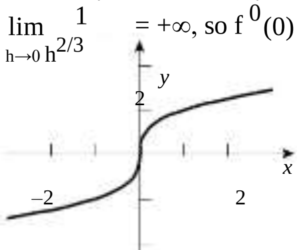
(d) From $-100 = f'(10) \approx \frac{f(11)-f(10)}{11-10}$ we see that $f(11) \approx f(10) - 100$, so an increase of one dollar per gallon would decrease the amount of paint sold by around 100 gallons.

43. (a) $F \approx 200$ lb, $dF/d\theta \approx 50$ (b) $\mu = (dF/d\theta)/F \approx 50/200 = 0.25$

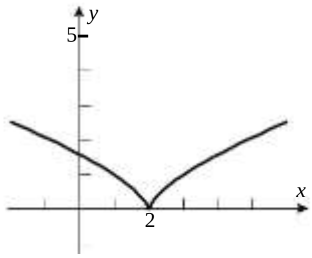
44. The derivative at time $t = 100$ of the velocity with respect to time is equal to the slope of the tangent line, which is approximately $m \approx \frac{12500-0}{140-40} = 125$ ft/s^2. Thus the mass is approximately $M(100) \approx \frac{T}{dv/dt} = \frac{7680982 \text{ lb}}{125 \text{ ft/s}^2} \approx$ 61000 slugs.

45. (a) $T \approx 115^\circ$ F, $dT/dt \approx -3.35^\circ$ F/min (b) $k = (dT/dt)/(T - T_0) \approx (-3.35)/(115 - 75) = -0.084$

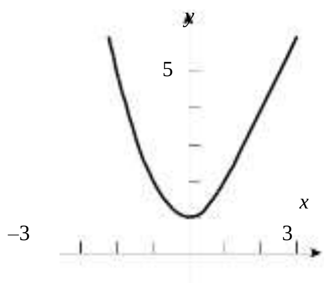
46. (a) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \sqrt[3]{x} = 0 = f(0)$, so f is continuous at $x = 0$. $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h}-0}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = +\infty$, so $f'(0)$ does not exist.



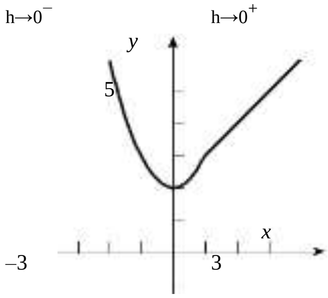
(b) $\lim_{x\to 2} f(x) = \lim_{x\to 2}(x-2)^{2/3} = 0 = f(2)$ so f is continuous at $x = 2$. $\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{h^{2/3}}{h} = \lim_{h\to 0}\frac{1}{h^{1/3}}$ which does not exist so $f'(2)$ does not exist.



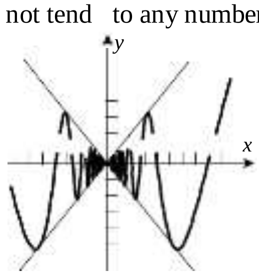
47. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$, so f is continuous at $x = 1$. $\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-}\frac{[(1+h)^2+1]-2}{h} = \lim_{h\to 0^-}(2+h) = 2$; $\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+}\frac{2(1+h)-2}{h} = \lim_{h\to 0^+} 2 = 2$, so $f'(1) = 2$.



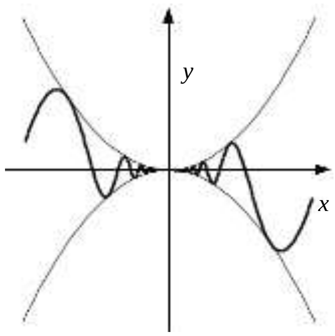
48. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$ so f is continuous at $x = 1$. $\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-}\frac{[(1+h)^2+2]-3}{h} = \lim_{h\to 0^-}(2+h) = 2$; $\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+}\frac{[(1+h)+2]-3}{h} = \lim_{h\to 0^+} 1 = 1$, so $f'(1)$ does not exist.



49. Since $-|x| \le x\sin(1/x) \le |x|$ it follows by the Squeezing Theorem (Theorem 1.6.4) that $\lim_{x\to 0} x\sin(1/x) = 0$. The derivative cannot exist: consider $\frac{f(x)-f(0)}{x} = \sin(1/x)$. This function oscillates between -1 and $+1$ and does not tend to any number as x tends to zero.



50. For continuity, compare with $\pm x^2$ to establish that the limit is zero. The difference quotient is $x \sin(1/x)$ and (see Exercise 49) this has a limit of zero at the origin.



51. Let $\; = |f'(x_0)/2|$. Then there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then $\left|\dfrac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < \;$. Since $f'(x_0) > 0$ and $\; = f'(x_0)/2$ it follows that $\dfrac{f(x) - f(x_0)}{x - x_0} > \; > 0$. If $x = x_1 < x_0$ then $f(x_1) < f(x_0)$ and if $x = x_2 > x_0$ then $f(x_2) > f(x_0)$.

52. $g'(x_1) = \lim\limits_{h\to 0} \dfrac{g(x_1 + h) - g(x_1)}{h} = \lim\limits_{h\to 0} \dfrac{f(m(x_1 + h) + b) - f(mx_1 + b)}{h} = m \lim\limits_{h\to 0} \dfrac{f(x_0 + mh) - f(x_0)}{mh} = mf'(x_0)$.

53. (a) Let $\; = |m|/2$. Since $m = 0$, $\; > 0$. Since $f(0) = f'(0) = 0$ we know there exists $\delta > 0$ such that $\left|\dfrac{f(0 + h) - f(0)}{h}\right| < \;$ whenever $0 < |h| < \delta$. It follows that $|f(h)| < \frac{1}{2}|hm|$ for $0 < |h| < \delta$. Replace h with x to get the result.

(b) For $0 < |x| < \delta$, $|f(x)| < \frac{1}{2}|mx|$. Moreover $|mx| = |mx - f(x) + f(x)| \le |f(x) - mx| + |f(x)|$, which yields $|f(x) - mx| \ge |mx| - |f(x)| > \frac{1}{2}|mx| > |f(x)|$, i.e. $|f(x) - mx| > |f(x)|$.

(c) If any straight line $y = mx + b$ is to approximate the curve $y = f(x)$ for small values of x, then $b = 0$ since $f(0) = 0$. The inequality $|f(x) - mx| > |f(x)|$ can also be interpreted as $|f(x) - mx| > |f(x) - 0|$, i.e. the line $y = 0$ is a better approximation than is $y = mx$.

54. Let $g(x) = f(x) - [f(x_0) + f'(x_0)(x - x_0)]$ and $h(x) = f(x) - [f(x_0) + m(x - x_0)]$; note that $h(x) - g(x) = (f'(x_0) - m)(x - x_0)$. If $m = f'(x_0)$ then there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $\left|\dfrac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < \frac{1}{2}|f'(x_0) - m|$. Multiplying by $|x - x_0|$ gives $|g(x)| < \frac{1}{2}|h(x) - g(x)|$. Hence $2|g(x)| < |h(x) + (-g(x))| \le |h(x)| + |g(x)|$, so $|g(x)| < |h(x)|$. In words, $f(x)$ is closer to $f(x_0) + f'(x_0)(x - x_0)$ than it is to $f(x_0) + m(x - x_0)$. So the tangent line gives a better approximation to $f(x)$ than any other line through $(x_0, f(x_0))$. Clearly any line not passing through that point gives an even worse approximation for x near x_0, so the tangent line gives the best linear approximation.

55. See discussion around Definition 2.2.2.

56. See Theorem 2.2.3.

Exercise Set 2.3

1. $28x^6$, by Theorems 2.3.2 and 2.3.4.

2. $-36x^{11}$, by Theorems 2.3.2 and 2.3.4.

3. $24x^{7} + 2$, by Theorems 2.3.1, 2.3.2, 2.3.4, and 2.3.5.

4. $2x^{3}$, by Theorems 2.3.1, 2.3.2, 2.3.4, and 2.3.5.

5. 0, by Theorem 2.3.1.

6. $\sqrt{2}$, by Theorems 2.3.1, 2.3.2, 2.3.4, and 2.3.5.

7. $-\frac{1}{3}(7x^6 + 2)$, by Theorems 2.3.1, 2.3.2, 2.3.4, and 2.3.5.

8. $\frac{2}{5}x$, by Theorems 2.3.1, 2.3.2, 2.3.4, and 2.3.5.

9. $-3x^{-4} - 7x^{-8}$, by Theorems 2.3.3 and 2.3.5.

10. $\frac{1}{2\sqrt{x}} - \frac{1}{x^2}$, by Theorems 2.3.3 and 2.3.5.

11. $24x^{-9} + 1/\sqrt{x}$, by Theorems 2.3.3, 2.3.4, and 2.3.5.

12. $-42x^{-7} - \frac{5}{2\sqrt{x}}$, by Theorems 2.3.3, 2.3.4, and 2.3.5.

13. $f'(x) = \pi x^{\pi-1} - \sqrt{10}\,x^{-1-\sqrt{10}}$, by Theorems 2.3.3 and 2.3.5.

14. $f'(x) = -\frac{2}{3}x^{-4/3}$, by Theorems 2.3.3 and 2.3.4. $2x$

15. $(3x^2 + 1)^2 = 9x^4 + 6x^2 + 1$, so $f'(x) = 36x^3 + 12x$, by Theorems 2.3.1, 2.3.2, 2.3.4, and 2.3.5.

16. $3ax^2 + 2bx + c$, by Theorems 2.3.1, 2.3.2, 2.3.4, and 2.3.5.

17. $y' = 10x - 3$, $y'(1) = 7$.

18. $y' = \frac{1}{2\sqrt{x}} - \frac{2}{x^2}$, $y'(1) = -3/2$.

19. $2t - 1$, by Theorems 2.3.2 and 2.3.5.

20. $\frac{1}{3} - \frac{1}{3t^2}$, by Theorems 2.3.3, 2.3.4, and 2.3.5.

21. $dy/dx = 1 + 2x + 3x^2 + 4x^3 + 5x^4$, $dy/dx|_{x=1} = 15$.

22. $\frac{dy}{dx} = \frac{-3}{x^4} - \frac{2}{x^3} - \frac{1}{x^2} + 1 + 2x + 3x^2$, $\left.\frac{dy}{dx}\right|_{x=1} = 0$.

23. $y = (1 - x^2)(1 + x^2)(1 + x^4) = (1 - x^4)(1 + x^4) = 1 - x^8$, $\frac{dy}{dx} = 8x^7$, $dy/dx\,|_{x=1} = -8$.

24. $dy/dx = 24x^{23} + 24x^{11} + 24x^7 + 24x^5$, $dy/dx|_{x=1} = 96$.

25. $f'(1) \approx \frac{f(1.01) - f(1)}{0.01} = \frac{-0.999699 - (-1)}{0.01} = 0.0301$, and by differentiation, $f'(1) = 3(1)^2 - 3 = 0$.

26. $f'(1) \approx \frac{f(1.01) - f(1)}{0.01} \approx \frac{0.980296 - 1}{0.01} \approx -1.9704$, and by differentiation, $f'(1) = -2/1^3 = -2$.

27. The estimate will depend on your graphing utility and on how far you zoom in. Since $f'(x) = 1 - \frac{1}{x}$, the exact value is $f'(1) = 0$.

28. The estimate will depend on your graphing utility and on how far you zoom in. Since $f^0(x) = \frac{1}{2\sqrt{x}} + 2$, the exact value is $f^0(1) = 5/2$.

29. 32t, by Theorems 2.3.2 and 2.3.4.

30. 2π, by Theorems 2.3.2 and 2.3.4.

31. $3\pi r^2$, by Theorems 2.3.2 and 2.3.4.

32. $-2\alpha^{-2} + 1$, by Theorems 2.3.2, 2.3.4, and 2.3.5.

33. True. By Theorems 2.3.4 and 2.3.5, $\frac{d}{dx}[f(x) - 8g(x)] = f^0(x) - 8g^0(x)$; substitute x = 2 to get the result.

34. True. $\frac{d}{dx}[ax^3 + bx^2 + cx + d] = 3ax^2 + 2bx + c$.

35. False. $\frac{d}{dx}[4f(x) + x^3]\Big|_{x=2} = (4f^0(x) + 3x^2)\Big|_{x=2} = 4f^0(2) + 3 \cdot 2^2 = 32$

36. False. $f(x) = x^6 - x^3$ so $f^0(x) = 6x^5 - 3x^2$ and $f^{00}(x) = 30x^4 - 6x$, which is not equal to $2x(4x^3 - 1) = 8x^4 - 2x$.

37. (a) $\frac{dV}{dr} = 4\pi r^2$ (b) $\frac{dV}{dr}\Big|_{r=5} = 4\pi(5)^2 = 100\pi$

38. $\frac{d}{d\lambda}\left[\frac{\lambda\lambda_0 + \lambda^6}{2 - \lambda_0}\right] = \frac{1}{2 - \lambda_0}\frac{d}{d\lambda}(\lambda\lambda_0 + \lambda^6) = \frac{1}{2-\lambda_0}(\lambda_0 + 6\lambda^5) = \frac{\lambda_0 + 6\lambda^5}{2-\lambda_0}$.

39. $y - 2 = 5(x + 3)$, $y = 5x + 17$.

40. $y + 2 = -(x - 2)$, $y = -x$.

41. (a) $dy/dx = 21x^2 - 10x + 1$, $d^2y/dx^2 = 42x - 10$ (b) $dy/dx = 24x - 2$, $d^2y/dx^2 = 24$

(c) $dy/dx = -1/x^2$, $d^2y/dx^2 = 2/x^3$ (d) $dy/dx = 175x^4 - 48x^2 - 3$, $d^2y/dx^2 = 700x^3 - 96x$

42. (a) $y^0 = 28x^6 - 15x^2 + 2$, $y^{00} = 168x^5 - 30x$ (b) $y^0 = 3$, $y^{00} = 0$

(c) $y^0 = \frac{2}{5x^2}$, $y^{00} = -\frac{4}{5x^3}$ (d) $y^0 = 8x^3 + 9x^2 - 10$, $y^{00} = 24x^2 + 18x$

43. (a) $y^0 = -5x^{-6} + 5x^4$, $y^{00} = 30x^{-7} + 20x^3$, $y^{000} = -210x^{-8} + 60x^2$

(b) $y = x^{-1}$, $y^0 = -x^{-2}$, $y^{00} = 2x^{-3}$, $y^{000} = -6x^{-4}$

(c) $y^0 = 3ax^2 + b$, $y^{00} = 6ax$, $y^{000} = 6a$

44. (a) $dy/dx = 10x - 4$, $d^2y/dx^2 = 10$, $d^3y/dx^3 = 0$

(b) $dy/dx = -6x^{-3} - 4x^{-2} + 1$, $d^2y/dx^2 = 18x^{-4} + 8x^{-3}$, $d^3y/dx^3 = -72x^{-5} - 24x^{-4}$

(c) $dy/dx = 4ax^3 + 2bx$, $d^2y/dx^2 = 12ax^2 + 2b$, $d^3y/dx^3 = 24ax$

45. (a) $f^0(x) = 6x$, $f^{00}(x) = 6$, $f^{000}(x) = 0$, $f^{000}(2) = 0$

(b) $\frac{dy}{dx} = 30x^4 \quad 8x, \; \frac{d^2y}{dx^2} = 120x^3 - 8, \; \left.\frac{d^2y}{dx^2}\right|_{x=1} = 112$

(c) $\frac{d}{dx}x^{-3} = 3x^{-4}, \; \frac{d^2}{dx^2}x^{-3} = 12x^{-5}, \; \frac{d^3}{dx^3}x^{-3} = -60x^{-6}, \; \frac{d^4}{dx^4}x^{-3} = 360x^{-7}, \; \left.\frac{d^4}{dx^4}x^{-3}\right|_{x=1} = 360$

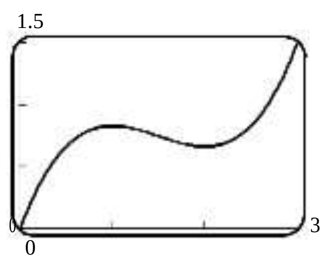
46. (a) $y^{0} = 16x^3 + 6x^2, \; y^{00} = 48x^2 + 12x, \; y^{000} = 96x + 12, \; y^{000}(0) = 12$

(b) $y = 6x^{-4}, \; \frac{dy}{dx} = -24x^{-5}, \; \frac{d^2y}{dx^2} = 120x^{-6}, \; \frac{d^3y}{dx^3} = -720x^{-7}, \; \frac{d^4y}{dx^4} = 5040x^{-8}, \; \left.\frac{d^4y}{dx^4}\right|_{x=1} = 5040$

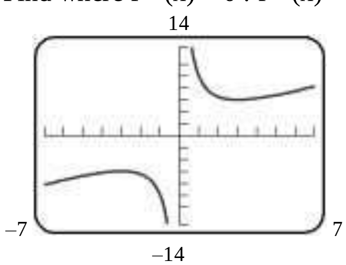
47. $y^{0} = 3x^2 + 3, \; y^{00} = 6x$, and $y^{000} = 6$ so $y^{000} + xy^{00} - 2y^{0} = 6 + x(6x) - 2(3x^2 + 3) = 6 + 6x^2 - 6x^2 - 6 = 0$.

48. $y = x^{-1}, \; y^{0} = -x^{-2}, \; y^{00} = 2x^{-3}$ so $x^3y^{00} + x^2y^{0} - xy = x^3(2x^{-3}) + x^2(-x^{-2}) - x(x^{-1}) = 2 - 1 - 1 = 0$.

49. The graph has a horizontal tangent at points where $\frac{dy}{dx} = 0$, but $\frac{dy}{dx} = x^2 - 3x + 2 = (x-1)(x-2) = 0$ if $x = 1, 2$.



50. Find where $f^{0}(x) = 0$: $f^{0}(x) = 1 - 9/x^2 = 0, \; x^2 = 9, \; x = \pm 3$. The tangent line is horizontal at $(3, 6)$ and $(-3, -6)$.



51. The y-intercept is -2 so the point $(0, -2)$ is on the graph; $-2 = a(0)^2 + b(0) + c$, $c = -2$. The x-intercept is 1 so the point $(1,0)$ is on the graph; $0 = a + b - 2$. The slope is $dy/dx = 2ax + b$; at $x = 0$ the slope is b so $b = -1$, thus $a = 3$. The function is $y = 3x^2 - x - 2$.

52. Let $P(x_0, y_0)$ be the point where $y = x^2 + k$ is tangent to $y = 2x$. The slope of the curve is $\frac{dy}{dx} = 2x$ and the slope of the line is 2 thus at P, $2x_0 = 2$ so $x_0 = 1$. But P is on the line, so $y_0 = 2x_0 = 2$. Because P is also on the curve we get $y_0 = x_0^2 + k$ so $k = y_0 - x_0^2 = 2 - (1)^2 = 1$.

53. The points $(-1, 1)$ and $(2, 4)$ are on the secant line so its slope is $(4-1)/(2+1) = 1$. The slope of the tangent line to $y = x^2$ is $y^{0} = 2x$ so $2x = 1$, $x = 1/2$.

54. The points $(1, 1)$ and $(4, 2)$ are on the secant line so its slope is $1/3$. The slope of the tangent line to $y = \sqrt{x}$ is $y^{0} = 1/(2\sqrt{x})$ so $1/(2\sqrt{x}) = 1/3$, $2\sqrt{x} = 3$, $x = 9/4$.

55. $y^{0} = -2x$, so at any point (x_0, y_0) on $y = 1 - x^2$ the tangent line is $y - y_0 = -2x_0(x - x_0)$, or $y = -2x_0x + x_0^2 + 1$. The point $(2,0)$ is to be on the line, so $0 = -4x_0 + x_0^2 + 1$, $x_0^2 - 4x_0 + 1 = 0$. Use the quadratic formula to get $x_0 = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$. The points are $(2 + \sqrt{3}, -6 - 4\sqrt{3})$ and $(2 - \sqrt{3}, -6 + 4\sqrt{3})$.

56. Let $P_1(x_1, ax_1^2)$ and $P_2(x_2, ax_2^2)$ be the points of tangency. $y' = 2ax$ so the tangent lines at P_1 and P_2 are $y - ax_1^2 = 2ax_1(x - x_1)$ and $y - ax_2^2 = 2ax_2(x - x_2)$. Solve for x to get $x = \frac{1}{2}(x_1 + x_2)$ which is the x-coordinate of a point on the vertical line halfway between P_1 and P_2.

57. $y' = 3ax^2 + b$; the tangent line at $x = x_0$ is $y - y_0 = (3ax_0^2 + b)(x - x_0)$ where $y_0 = ax_0^3 + bx_0$. Solve with $y = ax^3 + bx$ to get

$$ax^3 + bx - ax_0^3 - bx_0 = 3ax_0^2 x - 3ax_0^3 + bx - bx_0$$
$$x^3 - 3x_0^2 x + 2x_0^3 = 0$$
$$(x - x_0)(x^2 + xx_0 - 2x_0^2) = 0$$
$$(x - x_0)^2 (x + 2x_0) = 0, \text{ so } x = -2x_0.$$

58. Let (x_0, y_0) be the point of tangency. Note that $y_0 = 1/x_0$. Since $y' = -1/x^2$, the tangent line has the equation $y - y_0 = (-1/x_0^2)(x - x_0)$, or $y - \frac{1}{x_0} = -\frac{1}{x_0^2}x + \frac{1}{x_0}$ or $y = -\frac{1}{x_0^2}x + \frac{2}{x_0}$, with intercepts at $\left(0, \frac{2}{x_0}\right) = (0, 2y_0)$ and $(2x_0, 0)$. The distance from the y-intercept to the point of tangency is $\sqrt{(x_0 - 0)^2 + (y_0 - 2y_0)^2}$, and the distance from the x-intercept to the point of tangency is $\sqrt{(x_0 - 2x_0)^2 + (y_0 - 0)^2}$ so that they are equal (and equal the distance $\sqrt{x_0^2 + y_0^2}$ from the point of tangency to the origin).

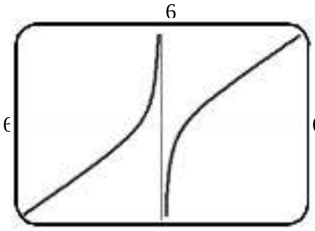
59. $y' = -\frac{1}{x^2}$; the tangent line at $x = x_0$ is $y - y_0 = -\frac{1}{x_0^2}(x - x_0)$, or $y = -\frac{x}{x_0^2} + \frac{2}{x_0}$. The tangent line crosses the x-axis at $2x_0$, the y-axis at $2/x_0$, so that the area of the triangle is $\frac{1}{2}(2/x_0)(2x_0) = 2$.

60. $f'(x) = 3ax^2 + 2bx + c$; there is a horizontal tangent where $f'(x) = 0$. Use the quadratic formula on $3ax^2 + 2bx + c = 0$ to get $x = (-b \pm \sqrt{b^2 - 3ac})/(3a)$ which gives two real solutions, one real solution, or none if

(a) $b^2 - 3ac > 0$ (b) $b^2 - 3ac = 0$ (c) $b^2 - 3ac < 0$

61. $F = GmMr^{-2}$, $\frac{dF}{dr} = -2GmMr^{-3} = -\frac{2GmM}{r^3}$

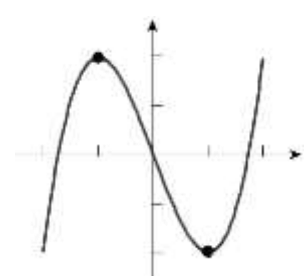
62. $dR/dT = 0.04124 - 3.558 \times 10^{-5}\,T$ which decreases as T increases from 0 to 700. When T = 0, dR/dT = 0.04124 $\Omega/^\circ$ C; when T = 700, dR/dT = 0.01633 $\Omega/^\circ$ C. The resistance is most sensitive to temperature changes at T = 0° C, least sensitive at T = 700° C.



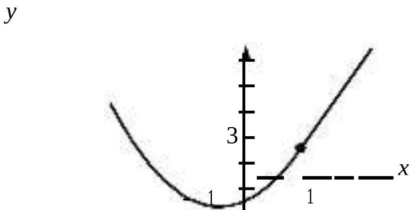
63. $f'(x) = 1 + 1/x^2 > 0$ for all $x \neq 0$

y
(–1, 2) 2

64. $f^{0}(x) = 3x^2 - 3 = 0$ when $x = \pm 1$; $f^{0}(x) > 0$ for $-\infty < x < -1$ and $1 < x < +\infty$



65. f is continuous at 1 because $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1)$; also $\lim_{x\to1^-} f'(x) = \lim_{x\to1^-}(2x+1) = 3$ and $\lim_{x\to1^+} f'(x) = \lim_{x\to1^+} 3 = 3$ so f is differentiable at 1, and the derivative equals 3.



66. f is not continuous at x = 9 because $\lim_{x\to9^-} f(x) = -63$ and $\lim_{x\to9^+} f(x) = 3$. f cannot be differentiable at x = 9, for if it were, then f would also be continuous, which it is not.

67. f is continuous at 1 because $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1)$. Also, $\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}$ equals the derivative of x^2 at x = 1, namely $2x\big|_{x=1} = 2$, while $\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}$ equals the derivative of $\sqrt{x}$ at x = 1, namely $\frac{1}{2\sqrt{x}}\Big|_{x=1} = \frac{1}{2}$. Since these are not equal, f is not differentiable at x = 1.

68. f is continuous at 1/2 because $\lim_{x\to1/2^-} f(x) = \lim_{x\to1/2^+} f(x) = f(1/2)$; also $\lim_{x\to1/2^-} f'(x) = \lim_{x\to1/2^-} 3x^2 = 3/4$ and $\lim_{x\to1/2^+} f'(x) = \lim_{x\to1/2^+} 3x/2 = 3/4$ so $f'(1/2) = 3/4$, and f is differentiable at x = 1/2.

69. (a) $f(x) = 3x-2$ if $x \geq 2/3$, $f(x) = -3x+2$ if $x < 2/3$ so f is differentiable everywhere except perhaps at 2/3. f is continuous at 2/3, also $\lim_{x\to2/3^-} f'(x) = \lim_{x\to2/3^-}(-3) = -3$ and $\lim_{x\to2/3^+} f'(x) = \lim_{x\to2/3^+}(3) = 3$ so f is not differentiable at x = 2/3.

(b) $f(x) = x^2-4$ if $|x| \geq 2$, $f(x) = -x^2+4$ if $|x| < 2$ so f is differentiable everywhere except perhaps at ±2. f is continuous at −2 and 2, also $\lim_{x\to2^-} f'(x) = \lim_{x\to2^-}(-2x) = -4$ and $\lim_{x\to2^+} f'(x) = \lim_{x\to2^+}(2x) = 4$ so f is not differentiable at x = 2. Similarly, f is not differentiable at x = −2.

70. (a) $f'(x) = -(1)x^{-2}$, $f''(x) = (2\cdot1)x^{-3}$, $f'''(x) = -(3\cdot2\cdot1)x^{-4}$; $f^{(n)}(x) = (-1)^n\frac{n(n-1)(n-2)\cdots1}{x^{n+1}}$

(b) $f'(x) = -2x^{-3}$, $f''(x) = (3\cdot2)x^{-4}$, $f'''(x) = -(4\cdot3\cdot2)x^{-5}$; $f^{(n)}(x) = (-1)^n\frac{(n+1)(n)(n-1)\cdots2}{x^{n+2}}$

71. (a)

$$\frac{d^2}{dx^2}[cf(x)] = \frac{d}{dx}\frac{d}{dx}[cf(x)] = \frac{d}{dx}c\frac{d}{dx}[f(x)] = c\frac{d}{dx}\frac{d}{dx}[f(x)] = c\frac{d^2}{dx^2}[f(x)]$$

$$\frac{d^2}{dx}■[f(x)+g(x)] = \frac{d}{dx}\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}\frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] = \frac{d^2}{dx}[f(x)] + \frac{d^2}{dx}[g(x)]$$

(b) Yes, by repeated application of the procedure illustrated in part (a).

72. $\lim_{w\to2}\frac{f'(w)-f'(2)}{w-2} = f''(2)$; $f'(x) = 8x^7-2$, $f''(x) = 56x^6$, so $f''(2) = 56(2^6) = 3584$.

73. (a) $f'(x) = nx^{n-1}$, $f''(x) = n(n-1)x^{n-2}$, $f'''(x) = n(n-1)(n-2)x^{n-3}$, . . ., $f^{(n)}(x) = n(n-1)(n-2)\cdots 1$

(b) From part (a), $f^{(k)}(x) = k(k-1)(k-2)\cdots 1$ so $f^{(k+1)}(x) = 0$ thus $f^{(n)}(x) = 0$ if $n > k$.

(c) From parts (a) and (b), $f^{(n)}(x) = a_n n(n-1)(n-2)\cdots 1$.

74. (a) If a function is differentiable at a point then it is continuous at that point, thus f' is continuous on (a, b) and consequently so is f .

(b) f and all its derivatives up to $f^{(n-1)}(x)$ are continuous on (a, b).

75. Let $g(x) = x^n$, $f(x) = (mx + b)^n$. Use Exercise 52 in Section 2.2, but with f and g permuted. If $x_0 = mx_1 + b$ then Exercise 52 says that f is differentiable at x_1 and $f'(x_1) = mg'(x_0)$. Since $g'(x_0) = nx_0^{n-1}$, the result follows.

76. $f(x) = 4x^2 + 12x + 9$ so $f'(x) = 8x + 12 = 2 \cdot 2(2x + 3)$, as predicted by Exercise 75.

77. $f(x) = 27x^3 - 27x^2 + 9x - 1$ so $f'(x) = 81x^2 - 54x + 9 = 3 \cdot 3(3x - 1)^2$, as predicted by Exercise 75.

78. $f(x) = (x - 1)^{-1}$ so $f'(x) = (-1) \cdot 1(x - 1)^{-2} = -1/(x - 1)^2$.

79. $f(x) = 3(2x + 1)^{-2}$ so $f'(x) = 3(-2)2(2x + 1)^{-3} = -12/(2x + 1)^3$.

80. $f(x) = \dfrac{x + 1 - 1}{x + 1} = 1 - (x + 1)^{-1}$, and $f'(x) = -(-1)(x + 1)^{-2} = 1/(x + 1)^2$.

81. $f(x) = \dfrac{2x^2 + 4x + 2 + 1}{(x + 1)^2} = 2 + (x + 1)^{-2}$, so $f'(x) = -2(x + 1)^{-3} = -2/(x + 1)^3$.

82. (a) If n = 0 then $f(x) = x^0 = 1$ so $f'(x) = 0$ by Theorem 2.3.1. This equals $0x^{0-1}$, so the Extended Power Rule holds in this case.

(b) $f'(x) = \lim_{h\to 0} \dfrac{f(x + h) - f(x)}{h} = \lim_{h\to 0} \dfrac{1/(x + h)^m - 1/x^m}{h} = \lim_{h\to 0} \dfrac{x^m - (x + h)^m}{hx^m(x + h)^m} =$

$= \lim_{h\to 0} \dfrac{(x + h)^m - x^m}{h} \cdot \lim_{h\to 0} -\dfrac{1}{x^m(x + h)^m} = \dfrac{d}{dx}(x^m)\left(-\dfrac{1}{x^{2m}}\right) = mx^{m-1} \cdot -\dfrac{1}{x^{2m}} = -mx^{-m-1} = nx^{n-1}$.

Exercise Set 2.4

1. (a) $f(x) = 2x^2 + x - 1$, $f'(x) = 4x + 1$ (b) $f'(x) = (x + 1) \cdot (2) + (2x - 1) \cdot (1) = 4x + 1$

2. (a) $f(x) = 3x^4 + 5x^2 - 2$, $f'(x) = 12x^3 + 10x$ (b) $f'(x) = (3x^2 - 1) \cdot (2x) + (x^2 + 2) \cdot (6x) = 12x^3 + 10x$

3. (a) $f(x) = x^4 - 1$, $f'(x) = 4x^3$ (b) $f'(x) = (x^2 + 1) \cdot (2x) + (x^2 - 1) \cdot (2x) = 4x^3$

4. (a) $f(x) = x^3 + 1$, $f'(x) = 3x^2$ (b) $f'(x) = (x + 1)(2x - 1) + (x^2 - x + 1) \cdot (1) = 3x^2$

5. $f'(x) = (3x^2 + 6)\dfrac{d}{dx}\left(2x - \dfrac{1}{4}\right) + \left(2x - \dfrac{1}{4}\right)\dfrac{d}{dx}(3x^2 + 6) = (3x^2 + 6)(2) + \left(2x - \dfrac{1}{4}\right)(6x) = 18x^2 - \dfrac{3}{2}x + 12$

6. $f'(x) = (2 - x - 3x^3)\dfrac{d}{dx}(7 + x^5) + (7 + x^5)\dfrac{d}{dx}(2 - x - 3x^3) = (2 - x - 3x^3)(5x^4) + (7 + x^5)(-1 - 9x^2) = -24x^7 - 6x^5 + 10x^4 - 63x^2 - 7$

7. $f'(x) = (x^3 + 7x^2 - 8)\dfrac{d}{dx}(2x^{-3} + x^{-4}) + (2x^{-3} + x^{-4})\dfrac{d}{dx}(x^3 + 7x^2 - 8) = (x^3 + 7x^2 - 8)(-6x^{-4} - 4x^{-5}) +$

$+(2x^{-3} + x^{-4})(3x^2 + 14x) = -15x^{-2} - 14x^{-3} + 48x^{-4} + 32x^{-5}$

8. $f'(x) = (x^{-1} + x^{-2})\frac{d}{dx}(3x^3 + 27) + (3x^3 + 27)\frac{d}{dx}(x^{-1} + x^{-2}) = (x^{-1} + x^{-2})(9x^2) + (3x^3 + 27)(-x^{-2} - 2x^{-3}) =$
$3 + 6x - 27x^{-2} - 54x^{-3}$

9. $f'(x) = 1 \cdot (x^2 + 2x + 4) + (x - 2) \cdot (2x + 2) = 3x^2$

10. $f^0(x) = (2x+1)(x^2-x) + (x^2+x)(2x-1) = 4x^3 - 2x$

11. $f^0(x) = \dfrac{(x^2+1)\frac{d}{dx}(3x+4) - (3x+4)\frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \dfrac{(x^2+1)\cdot 3 - (3x+4)\cdot 2x}{(x^2+1)^2} = \dfrac{-3x^2 - 8x + 3}{(x^2+1)^2}$

[redacted]

12. $f^0(x) = \dfrac{(x^4+x+1)\frac{d}{dx}(x-2) - (x-2)\frac{d}{dx}(x^4+x+1)}{(x^4+x+1)^2} = \dfrac{(x^4+x+1)\cdot 1 - (x-2)\cdot(4x^3+1)}{(x^4+x+1)^2} = \dfrac{-3x^4+8x^3+3}{(x^4+x+1)^2}$

[redacted]

13. $f^0(x) = \dfrac{(3x-4)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(3x-4)}{(3x-4)^2} = \dfrac{(3x-4)\cdot 2x - x^2\cdot 3}{(3x-4)^2} = \dfrac{3x^2-8x}{(3x-4)^2}$.

14. $f^0(x) = \dfrac{(3x-4)\frac{d}{dx}(2x^2+5) - (2x^2+5)\frac{d}{dx}(3x-4)}{(3x-4)^2} = \dfrac{(3x-4)\cdot 4x - (2x^2+5)\cdot 3}{(3x-4)^2} = \dfrac{6x^2-16x-15}{(3x-4)^2}$

15. $f(x) = \dfrac{2x^{3/2} + x - 2x^{1/2} - 1}{x+3}$, so

$$f^0(x) = \frac{(x+3)\frac{d}{dx}(2x^{3/2}+x-2x^{1/2}-1) - (2x^{3/2}+x-2x^{1/2}-1)\frac{d}{dx}(x+3)}{(x+3)^2} =$$

$$= \frac{(x+3)\cdot(3x^{1/2}+1-x^{-1/2}) - (2x^{3/2}+x-2x^{1/2}-1)\cdot 1}{(x+3)^2} = \frac{x^{3/2}+10x^{1/2}+4-3x^{-1/2}}{(x+3)^2}$$

16. $f(x) = \dfrac{-2x^{3/2} - x + 4x^{1/2} + 2}{x^2+3x}$, so

$$f^0(x) = \frac{(x^2+3x)\frac{d}{dx}(-2x^{3/2}-x+4x^{1/2}+2) - (-2x^{3/2}-x+4x^{1/2}+2)\frac{d}{dx}(x^2+3x)}{(x^2+3x)^2} =$$

$$= \frac{(x^2+3x)\cdot(-3x^{1/2}-1+2x^{-1/2}) - (-2x^{3/2}-x+4x^{1/2}+2)\cdot(2x+3)}{(x^2+3x)^2} =$$

$$= \frac{x^{5/2}+x^2-9x^{3/2}-4x-6x^{1/2}-6}{(x^2+3x)^2}$$

17. This could be computed by two applications of the product rule, but it's simpler to expand $f(x)$: $f(x) = 14x + 21 + 7x^{-1} + 2x^{-2} + 3x^{-3} + x^{-4}$, so $f^0(x) = 14 - 7x^{-2} - 4x^{-3} - 9x^{-4} - 4x^{-5}$.

18. This could be computed by two applications of the product rule, but it's simpler to expand $f(x)$: $f(x) = -6x^7 - 4x^6 + 16x^5 - 3x^{-2} - 2x^{-3} + 8x^{-4}$, so $f^0(x) = -42x^6 - 24x^5 + 80x^4 + 6x^{-3} + 6x^{-4} - 32x^{-5}$.

19. In general, $\frac{d}{dx} g(x)^2 = 2g(x)g^0(x)$ and $\frac{d}{dx} g(x)^3 = \frac{d}{dx} g(x)^2 g(x) = g(x)^2 g^0(x) + g(x)\frac{d}{dx} g(x)^2 = g(x)^2 g^0(x) + g(x)\cdot 2g(x)g^0(x) = 3g(x)^2 g^0(x)$.
Letting $g(x) = x^7 + 2x - 3$, we have $f^0(x) = 3(x^7 + 2x - 3)^2(7x^6 + 2)$.

20. In general, $\frac{d}{dx} g(x)^2 = 2g(x)g^0(x)$, so $\frac{d}{dx} g(x)^4 = \frac{d}{dx}\left[g(x)^2\right]^2 = 2g(x)^2\cdot\frac{d}{dx} g(x)^2 = 2g(x)^2\cdot 2g(x)g^0(x) = 4g(x)^3 g^0(x)$

Letting $g(x) = x^2 + 1$, we have $f'(x) = 4(x^2 + 1)^3 \cdot 2x = 8x(x^2 + 1)^3$.

21. $\dfrac{dy}{dx} = \dfrac{(x+3)\cdot 2 - (2x-1)\cdot 1}{(x+3)^2} = \dfrac{7}{(x+3)^2}$, so $\left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{7}{16}$.

22. $\dfrac{dy}{dx} = (x^2 - 5)\cdot 4 - (4x+1)\cdot(2x) = -4x^2 - 2x - 20$, so $dy = 26 = \dfrac{1}{3}$.

$dx \quad (x^2 - 5)^2 \quad (x^2 - 5)^2 \quad dx_{x=1} \quad 16 \quad 8$

23. $\dfrac{dy}{dx} = \dfrac{3x+2}{x}\dfrac{d}{dx}\left(x^{-5}+1\right) + \left(x^{-5}+1\right)\dfrac{d}{dx}\left(\dfrac{3x+2}{x}\right) = \dfrac{3x+2}{x}\left(-5x^{-6}\right) + \left(x^{-5}+1\right)\dfrac{x(3)-(3x+2)(1)}{x^2} = \dfrac{3x+2}{x}\left(-5x^{-6}\right) + \left(x^{-5}+1\right)\left(-\dfrac{2}{x^2}\right)$; so $\left.\dfrac{dy}{dx}\right|_{x=1} = 5(-5)+2(-2) = -29$.

24. $\dfrac{dy}{dx} = (2x^7-x^2)\dfrac{d}{dx}\left(\dfrac{x-1}{x+1}\right) + \dfrac{x-1}{x+1}\dfrac{d}{dx}(2x^7-x^2) = (2x^7-x^2)\dfrac{(x+1)(1)-(x-1)(1)}{(x+1)^2} + \dfrac{x-1}{x+1}(14x^6-2x) = (2x^7-x^2)\cdot\dfrac{2}{(x+1)^2} + \dfrac{x-1}{x+1}(14x^6-2x)$; so $\left.\dfrac{dy}{dx}\right|_{x=1} = (2-1)\dfrac{2}{4} + 0(14-2) = \dfrac{1}{2}$.

25. $f'(x) = \dfrac{(x^2+1)\cdot 1 - x\cdot 2x}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$, so $f'(1) = 0$.

26. $f'(x) = \dfrac{(x^2+1)\cdot 2x - (x^2-1)\cdot 2x}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$, so $f'(1) = 1$.

27. (a) $g'(x) = \sqrt{x}\,f'(x) + \dfrac{1}{2\sqrt{x}}f(x)$, $g'(4) = (2)(-5) + \dfrac{1}{4}(3) = -37/4$.

(b) $g'(x) = \dfrac{xf'(x) - f(x)}{x^2}$, $g'(4) = \dfrac{(4)(-5)-3}{16} = -23/16$.

28. (a) $g'(x) = 6x - 5f'(x)$, $g'(3) = 6(3) - 5(4) = -2$.

(b) $g'(x) = \dfrac{2f(x) - (2x+1)f'(x)}{f^2(x)}$, $g'(3) = \dfrac{2(-2)-7(4)}{(-2)^2} = -8$.

29. (a) $F'(x) = 5f'(x) + 2g'(x)$, $F'(2) = 5(4) + 2(-5) = 10$.

(b) $F'(x) = f'(x) - 3g'(x)$, $F'(2) = 4 - 3(-5) = 19$.

(c) $F'(x) = f(x)g'(x) + g(x)f'(x)$, $F'(2) = (-1)(-5) + (1)(4) = 9$.

(d) $F'(x) = [g(x)f'(x) - f(x)g'(x)]/g^2(x)$, $F'(2) = [(1)(4) - (-1)(-5)]/(1)^2 = -1$.

30. (a) $F'(x) = 6f'(x) - 5g'(x)$, $F'(\pi) = 6(-1) - 5(2) = -16$.

(b) $F'(x) = f(x) + g(x) + x(f'(x) + g'(x))$, $F'(\pi) = 10 - 3 + \pi(-1+2) = 7 + \pi$.

(c) $F'(x) = 2f(x)g'(x) + 2f'(x)g(x) = 2(20) + 2(3) = 46$.

(d) $F'(x) = \dfrac{(4+g(x))f'(x) - f(x)g'(x)}{(4+g(x))^2} = \dfrac{(4-3)(-1) - 10(2)}{(4-3)^2} = -21$.

31. $\dfrac{dy}{dx} = \dfrac{2x(x+2) - (x^2-1)}{(x+2)^2}$, $\dfrac{dy}{dx} = 0$ if $x^2 + 4x + 1 = 0$. By the quadratic formula, $x = \dfrac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$.

The tangent line is horizontal at $x = -2 \pm \sqrt{3}$.

32. $\frac{dy}{dx} = \frac{2x(x-1) - (x^2+1)}{(x-1)^2} = \frac{x^2-2x-1}{(x-1)^2}$. The tangent line is horizontal when it has slope 0, i.e. $x^2 - 2x - 1 = 0$ which, by the quadratic formula, has solutions $x = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$, the tangent line is horizontal when $x = 1 \pm \sqrt{2}$.

33. The tangent line is parallel to the line $y = x$ when it has slope 1. $\frac{dy}{dx} = \frac{2x(x+1) - (x^2+1)}{(x+1)^2} = \frac{x^2+2x-1}{(x+1)^2} = 1$ if $x^2 + 2x - 1 = (x+1)^2$, which reduces to $-1 = +1$, impossible. Thus the tangent line is never parallel to the line $y = x$.

34. The tangent line is perpendicular to the line $y = x$ when the tangent line has slope -1. $y = \frac{x+2+1}{x+2} = 1 + \frac{1}{x+2}$, hence $\frac{dy}{dx} = -\frac{1}{(x+2)^2} = -1$ when $(x+2)^2 = 1$, $x^2 + 4x + 3 = 0$, $(x+1)(x+3) = 0$, $x = -1, -3$. Thus the tangent line is perpendicular to the line $y = x$ at the points $(-1, 2)$, $(-3, 0)$.

35. Fix x_0. The slope of the tangent line to the curve $y = \frac{1}{x+4}$ at the point $(x_0, 1/(x_0+4))$ is given by $\frac{dy}{dx}\Big|_{x=x_0} = \frac{-1}{(x+4)^2}\Big|_{x=x_0} = \frac{-1}{(x_0+4)^2}$. The tangent line to the curve at (x_0, y_0) thus has the equation $y - y_0 = \frac{-(x - x_0)}{(x_0+4)^2}$, and this line passes through the origin if its constant term $y_0 - x_0\frac{-1}{(x_0+4)^2}$ is zero. Then $\frac{1}{x_0+4} = \frac{-x_0}{(x_0+4)^2}$, so $x_0 + 4 = -x_0$, $x_0 = -2$.

36. $y = \frac{2x+5}{x+2} = \frac{2x+4+1}{x+2} = 2 + \frac{1}{x+2}$, and hence $\frac{dy}{dx} = \frac{-1}{(x+2)^2}$, thus the tangent line at the point (x_0, y_0) is given by $y - y_0 = \frac{-1}{(x_0+2)^2}(x - x_0)$, where $y_0 = 2 + \frac{1}{x_0+2}$. If this line is to pass through $(0, 2)$, $2 - y_0 = \frac{-1}{(x_0+2)^2}(-x_0)$, $\frac{-1}{x_0+2} = \frac{x_0}{(x_0+2)^2}$, $-x_0 - 2 = x_0$, so $x_0 = -1$.

37. (a) Their tangent lines at the intersection point must be perpendicular.

(b) They intersect when $\frac{1}{x} = \frac{1}{2-x}$, $x = 2 - x$, $x = 1$, $y = 1$. The first curve has derivative $y = -\frac{1}{x^2}$, so the slope when $x = 1$ is -1. Second curve has derivative $y = \frac{1}{(2-x)^2}$ so the slope when $x = 1$ is 1. Since the two slopes are negative reciprocals of each other, the tangent lines are perpendicular at the point $(1, 1)$.

38. The curves intersect when $a/(x-1) = x^2 - 2x + 1$, or $(x-1)^3 = a$, $x = 1 + a^{1/3}$. They are perpendicular when their slopes are negative reciprocals of each other, i.e. $\frac{-a}{(x-1)^2}(2x-2) = -1$, which has the solution $x = 2a + 1$. Solve $x = 1 + a^{1/3} = 2a + 1$, $2a^{2/3} = 1$, $a = 2^{-3/2}$. Thus the curves intersect and are perpendicular at the point $(2a+1, 1/2)$ provided $a = 2^{-3/2}$.

39. $F'(x) = xf'(x) + f(x)$, $F''(x) = xf''(x) + f'(x) + f'(x) = xf''(x) + 2f'(x)$.

40. (a) $F'''(x) = xf'''(x) + 3f''(x)$.

(b) Assume that $F^{(n)}(x) = xf^{(n)}(x) + nf^{(n-1)}(x)$ for some n (for instance $n = 3$, as in part (a)). Then $F^{(n+1)}(x) = xf^{(n+1)}(x) + (1+n)f^{(n)}(x) = xf^{(n+1)}(x) + (n+1)f^{(n)}(x)$, which is an inductive proof.

41. $R'(p) = p \cdot f'(p) + f(p) \cdot 1 = f(p) + pf'(p)$, so $R'(120) = 9000 + 120 \cdot (-60) = 1800$. Increasing the price by a small amount Δp dollars would increase the revenue by about $1800\Delta p$ dollars.

42. $R'(p) = p \cdot f'(p) + f(p) \cdot 1 = f(p) + pf'(p)$, so $R'(120) = 9000 + 120 \cdot (-80) = -600$. Increasing the price by a small amount Δp dollars would decrease the revenue by about $600\Delta p$ dollars.

43. $f(x) = \frac{1}{x^n}$ so $f'(x) = \frac{x^n \cdot (0) - 1 \cdot (nx^{n-1})}{\quad} = -\frac{n}{\quad} = -nx^{-n-1}$.

Exercise Set 2.5

1. $f'(x) = -4\sin x + 2\cos x$
2. $f'(x) = \dfrac{-10}{x^3} + \cos x$
3. $f'(x) = 4x^2\sin x - 8x\cos x$
4. $f'(x) = 4\sin x\cos x$
5. $f'(x) = \dfrac{\sin x(5+\sin x) - \cos x(5-\cos x)}{(5+\sin x)^2} = \dfrac{1+5(\sin x - \cos x)}{(5+\sin x)^2}$
6. $f'(x) = \dfrac{(x^2+\sin x)\cos x - \sin x(2x+\cos x)}{(x^2+\sin x)^2} = \dfrac{x^2\cos x - 2x\sin x}{(x^2+\sin x)^2}$
7. $f'(x) = \sec x\tan x - \sqrt{2}\sec^2 x$
8. $f'(x) = (x^2+1)\sec x\tan x + (\sec x)(2x) = (x^2+1)\sec x\tan x + 2x\sec x$
9. $f'(x) = -4\csc x\cot x + \csc^2 x$
10. $f'(x) = -\sin x - \csc x + x\csc x\cot x$
11. $f'(x) = \sec x(\sec^2 x) + (\tan x)(\sec x\tan x) = \sec^3 x + \sec x\tan^2 x$
12. $f'(x) = (\csc x)(-\csc^2 x) + (\cot x)(-\csc x\cot x) = -\csc^3 x - \csc x\cot^2 x$
13. $f'(x) = \dfrac{(1+\csc x)(-\csc^2 x) - \cot x(0-\csc x\cot x)}{(1+\csc x)^2} = \dfrac{\csc x(-\csc x - \csc^2 x + \cot^2 x)}{(1+\csc x)^2}$, but $1+\cot^2 x = \csc^2 x$ (identity), thus $\cot^2 x - \csc^2 x = -1$, so $f'(x) = \dfrac{\csc x(-\csc x - 1)}{(1+\csc x)^2} = -\dfrac{\csc x}{1+\csc x}$.
14. $f'(x) = \dfrac{(1+\tan x)(\sec x\tan x) - (\sec x)(\sec^2 x)}{(1+\tan x)^2} = \dfrac{\sec x\tan x + \sec x\tan^2 x - \sec^3 x}{(1+\tan x)^2} =$ $= \dfrac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2} = \dfrac{\sec x(\tan x - 1)}{(1+\tan x)^2}$
15. $f(x) = \sin^2 x + \cos^2 x = 1$ (identity), so $f'(x) = 0$.
16. $f'(x) = 2\sec x\tan x\sec x - 2\tan x\sec^2 x = \dfrac{2\sin x}{\cos^3 x} - 2\dfrac{\sin x}{\cos^3 x} = 0$; also, $f(x) = \sec^2 x - \tan^2 x = 1$ (identity), so $f'(x) = 0$.
17. $f(x) = \dfrac{\tan x}{1+x\tan x}$ (because $\sin x\sec x = (\sin x)(1/\cos x) = \tan x$), so $f'(x) = \dfrac{(1+x\tan x)(\sec^2 x) - \tan x[x(\sec^2 x) + (\tan x)(1)]}{(1+x\tan x)^2} = \dfrac{\sec^2 x - \tan^2 x}{(1+x\tan x)^2} = \dfrac{1}{(1+x\tan x)^2}$ (because $\sec^2 x - \tan^2 x = 1$).

18. $f(x) = \dfrac{(x^2+1)\cot x}{3-\cot x}$ (because $\cos x \csc x = (\cos x)(1/\sin x) = \cot x$), so

$$f'(x) = \frac{(3-\cot x)[2x\cot x - (x^2+1)\csc^2 x] - (x^2+1)\cot x\csc^2 x}{(3-\cot x)^2} = \frac{6x\cot x - 2x\cot^2 x - 3(x^2+1)\csc^2 x}{(3-\cot x)^2}.$$

19. $dy/dx = -x \sin x + \cos x$, $d^2y/dx^2 = -x \cos x - \sin x - \sin x = -x \cos x - 2 \sin x$

20. $dy/dx = -\csc x \cot x$, $d^2y/dx^2 = -[(\csc x)(-\csc^2 x) + (\cot x)(-\csc x \cot x)] = \csc^3 x + \csc x \cot^2 x$

21. $dy/dx = x(\cos x) + (\sin x)(1) - 3(-\sin x) = x \cos x + 4 \sin x$,
$d^2y/dx^2 = x(-\sin x) + (\cos x)(1) + 4 \cos x = -x \sin x + 5 \cos x$

22. $dy/dx = x^2(-\sin x) + (\cos x)(2x) + 4 \cos x = -x^2 \sin x + 2x \cos x + 4 \cos x$,
$d^2y/dx^2 = -[x^2(\cos x) + (\sin x)(2x)] + 2[x(-\sin x) + \cos x] - 4 \sin x = (2 - x^2) \cos x - 4(x + 1) \sin x$

23. $dy/dx = (\sin x)(-\sin x) + (\cos x)(\cos x) = \cos^2 x - \sin^2 x$,
$d^2y/dx^2 = (\cos x)(-\sin x) + (\cos x)(-\sin x) - [(\sin x)(\cos x) + (\sin x)(\cos x)] = -4 \sin x \cos x$

24. $dy/dx = \sec^2 x$, $d^2y/dx^2 = 2 \sec^2 x \tan x$

25. Let $f(x) = \tan x$, then $f^0(x) = \sec^2 x$.

(a) $f(0) = 0$ and $f^0(0) = 1$, so $y - 0 = (1)(x - 0)$, $y = x$.

(b) $f\left(\frac{\pi}{4}\right) = 1$ and $f^0\left(\frac{\pi}{4}\right) = 2$, so $y - 1 = 2\left(x - \frac{\pi}{4}\right)$, $y = 2x - \frac{\pi}{2} + 1$.

(c) $f\left(-\frac{\pi}{4}\right) = -1$ and $f^0\left(-\frac{\pi}{4}\right) = 2$, so $y + 1 = 2\left(x + \frac{\pi}{4}\right)$, $y = 2x + \frac{\pi}{2} - 1$.

26. Let $f(x) = \sin x$, then $f^0(x) = \cos x$.

(a) $f(0) = 0$ and $f^0(0) = 1$, so $y - 0 = (1)(x - 0)$, $y = x$.

(b) $f(\pi) = 0$ and $f^0(\pi) = -1$, so $y - 0 = (-1)(x - \pi)$, $y = -x + \pi$.

(c) $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $f^0\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, so $y - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$, $y = \frac{1}{\sqrt{2}}x - \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}$.

27. (a) If $y = x \sin x$ then $y^0 = \sin x + x \cos x$ and $y^{00} = 2 \cos x - x \sin x$ so $y^{00} + y = 2 \cos x$.

(b) Differentiate the result of part (a) twice more to get $y^{(4)} + y^{00} = -2 \cos x$.

28. (a) If $y = \cos x$ then $y^0 = -\sin x$ and $y^{00} = -\cos x$, so $y^{00} + y = (-\cos x) + (\cos x) = 0$; if $y = \sin x$ then $y^0 = \cos x$ and $y^{00} = -\sin x$ so $y^{00} + y = (-\sin x) + (\sin x) = 0$.

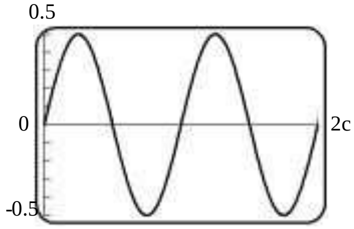
(b) $y^0 = A \cos x - B \sin x$, $y^{00} = -A \sin x - B \cos x$, so $y^{00} + y = (-A \sin x - B \cos x) + (A \sin x + B \cos x) = 0$.

29. (a) $f^0(x) = \cos x = 0$ at $x = \pm\pi/2, \pm 3\pi/2$.

(b) $f^0(x) = 1 - \sin x = 0$ at $x = -3\pi/2, \pi/2$.

(c) $f^{0}(x) = \sec^2 x \geq 1$ always, so no horizontal tangent line.

(d) $f^{0}(x) = \sec x \tan x = 0$ when $\sin x = 0$, $x = \pm 2\pi, \pm\pi, 0$.

30. (a) 

(b) $y = \sin x \cos x = (1/2)\sin 2x$ and $y' = \cos 2x$. So $y' = 0$ when $2x = (2n+1)\pi/2$ for $n = 0, 1, 2, 3$ or $x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

31. $x = 10\sin\theta$, $dx/d\theta = 10\cos\theta$; if $\theta = 60^\circ$, then $dx/d\theta = 10(1/2) = 5$ ft/rad $= \pi/36$ ft/deg ≈ 0.087 ft/deg.

32. $s = 3800\csc\theta$, $ds/d\theta = -3800\csc\theta\cot\theta$; if $\theta = 30^\circ$, then $ds/d\theta = -3800(2)(\sqrt{3}) = -7600\sqrt{3}$ ft/rad $= -380\sqrt{3}\pi/9$ ft/deg ≈ -230 ft/deg.

33. $D = 50\tan\theta$, $dD/d\theta = 50\sec^2\theta$; if $\theta = 45^\circ$, then $dD/d\theta = 50(\sqrt{2})^2 = 100$ m/rad $= 5\pi/9$ m/deg ≈ 1.75 m/deg.

34. (a) From the right triangle shown, $\sin\theta = r/(r+h)$ so $r + h = r\csc\theta$, $h = r(\csc\theta - 1)$.

(b) $dh/d\theta = -r\csc\theta\cot\theta$; if $\theta = 30^\circ$, then $dh/d\theta = -6378(2)(\sqrt{3}) \approx -22{,}094$ km/rad ≈ -386 km/deg.

35. False. $g'(x) = f(x)\cos x + f'(x)\sin x$

36. True, if $f(x)$ is continuous at $x = 0$, then $g'(0) = \lim_{h\to0}\dfrac{g(h)-g(0)}{h} = \lim_{h\to0}\dfrac{f(h)\sin h}{h} = \lim_{h\to0} f(h)\cdot\lim_{h\to0}\dfrac{\sin h}{h} = f(0)\cdot 1 = f(0)$.

37. True. $f(x) = \dfrac{\sin x}{\cos x} = \tan x$, so $f'(x) = \sec^2 x$.

38. False. $g'(x) = f(x)\cdot\dfrac{d}{dx}(\sec x) + f'(x)\sec x = f(x)\sec x\tan x + f'(x)\sec x$, so $g'(0) = f(0)\sec 0\tan 0 + f'(0)\sec 0 = 8\cdot1\cdot0 + (-2)\cdot1 = -2$. The second equality given in the problem is wrong: $\lim_{h\to0}\dfrac{f(h)\sec h - f(0)}{h} = -2$ but $\lim_{h\to0}\dfrac{8(\sec h - 1)}{h} = 0$.

39. $\dfrac{d^4}{dx^4}\sin x = \sin x$, so $\dfrac{d^{4k}}{dx^{4k}}\sin x = \sin x$; $\dfrac{d^{87}}{dx^{87}}\sin x = \dfrac{d^3}{dx^3}\dfrac{d^{4\cdot21}}{dx^{4\cdot21}}\sin x = \dfrac{d^3}{dx^3}\sin x = -\cos x$.

40. $\dfrac{d^{100}}{dx^{100}}\cos x = \dfrac{d^{4k}}{dx^{4k}}\cos x = \cos x$.

41. $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, and $f^{(4)}(x) = \cos x$ with higher order derivatives repeating this pattern, so $f^{(n)}(x) = \sin x$ for $n = 3, 7, 11, \ldots$

42. $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$, and the right-hand sides continue with a period of 4, so that $f^{(n)}(x) = \sin x$ when $n = 4k$ for some k.

43. (a) all x (b) all x (c) $x = \pi/2 + n\pi$, $n = 0, \pm1, \pm2, \ldots$

(d) $x = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$ (e) $x = \pi/2 + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$ (f) $x = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

(g) $x = (2n + 1)\pi, n = 0, \pm 1, \pm 2, \ldots$ (h) $x = n\pi/2, n = 0, \pm 1, \pm 2, \ldots$ (i) all x

44. (a) $\frac{d}{dx}[\cos x] = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} =$

$= \lim_{h\to 0} \cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} = (\cos x)(0) - (\sin x)(1) = -\sin x.$

(b) $\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x.$

(c) $\frac{d}{dx}[\sec x] = \frac{d}{dx}\frac{1}{\cos x} = \frac{0 \cdot \cos x - (1)(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x.$

(d) $\frac{d}{dx}[\csc x] = \frac{d}{dx}\frac{1}{\sin x} = \frac{(\sin x)(0) - (1)(\cos x)}{\sin^2 x} = -\frac{\cos x}{\sin^2 x} = -\csc x \cot x.$

45. $\frac{d}{dx}\sin x = \lim_{w\to x}\frac{\sin w - \sin x}{w - x} = \lim_{w\to x}\frac{2\sin\frac{w-x}{2}\cos\frac{w+x}{2}}{w-x} = \lim_{w\to x}\frac{\sin\frac{w-x}{2}}{\frac{w-x}{2}}\cos\frac{w+x}{2} = 1 \cdot \cos x = \cos x.$

46. $\frac{d}{dx}[\cos x] = \lim_{w\to x}\frac{\cos w - \cos x}{w - x} = \lim_{w\to x}\frac{-2\sin(\frac{w-x}{2})\sin(\frac{w+x}{2})}{w-x} = -\lim_{w\to x}\sin\frac{w+x}{2}\lim_{w\to x}\frac{\sin(\frac{w-x}{2})}{\frac{w-x}{2}} = -\sin x.$

47. (a) $\lim_{h\to 0}\frac{\tan h}{h} = \lim_{h\to 0}\frac{\frac{\sin h}{\cos h}}{h} = \lim_{h\to 0}\frac{\frac{\sin h}{h}}{\cos h} = \frac{1}{1} = 1.$

(b) $\frac{d}{dx}[\tan x] = \lim_{h\to 0}\frac{\tan(x+h) - \tan x}{h} = \lim_{h\to 0}\frac{\frac{\tan x + \tan h}{1 - \tan x \tan h} - \tan x}{h} = \lim_{h\to 0}\frac{\tan x + \tan h - \tan x + \tan^2 x \tan h}{h(1 - \tan x \tan h)} =$

$\lim_{h\to 0}\frac{\tan h(1 + \tan^2 x)}{h(1 - \tan x \tan h)} = \lim_{h\to 0}\frac{\tan h \sec^2 x}{h(1 - \tan x \tan h)} = \sec^2 x \lim_{h\to 0}\frac{\frac{\tan h}{h}}{1 - \tan x \tan h} = \sec^2 x \frac{\lim_{h\to 0}\frac{\tan h}{h}}{\lim_{h\to 0}(1 - \tan x \tan h)} = \sec^2 x.$

48. $\lim_{x\to 0}\frac{\tan(x+y) - \tan y}{x} = \lim_{h\to 0}\frac{\tan(y+h) - \tan y}{h} = \frac{d}{dy}(\tan y) = \sec^2 y.$

49. By Exercises 49 and 50 of Section 1.6, we have $\lim_{h\to 0}\frac{\sin h}{h} = \frac{\pi}{180}$ and $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$. Therefore:

(a) $\frac{d}{dx}[\sin x] = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} = (\sin x)(0) + (\cos x)(\pi/180) = \frac{\pi}{180}\cos x.$

(b) $\frac{d}{dx}[\cos x] = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} = \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} = 0 \cdot \cos x - \frac{\pi}{180} \sin x = -\frac{\pi}{180} \sin x.$

50. If f is periodic, then so is f'. Proof: Suppose $f(x+p) = f(x)$ for all x. Then $f'(x+p) = \lim_{h\to 0} \frac{f(x+p+h) - f(x+p)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$. However, f' may be periodic even if f is not. For example, $f(x) = x + \sin x$ is not periodic, but $f'(x) = 1 + \cos x$ has period 2π.

Exercise Set 2.6

1. $(f \circ g)^0 (x) = f^0 (g(x))g^0 (x)$, so $(f \circ g)^0 (0) = f^0 (g(0))g^0 (0) = f^0 (0)(3) = (2)(3) = 6$.

2. $(f\circ g)^0(2) = f^0(g(2))g^0(2) = 5(-3) = -15$.

3. (a) $(f\circ g)(x) = f(g(x)) = (2x-3)^5$ and $(f\circ g)^0(x) = f^0(g(x))g^0(x) = 5(2x-3)^4(2) = 10(2x-3)^4$.

   (b) $(g\circ f)(x) = g(f(x)) = 2x^5-3$ and $(g\circ f)^0(x) = g^0(f(x))f^0(x) = 2(5x^4) = 10x^4$.

4. (a) $(f\circ g)(x) = 5\sqrt{4+\cos x}$ and $(f\circ g)^0(x) = f^0(g(x))g^0(x) = \frac{5}{2\sqrt{4+\cos x}}(-\sin x)$.

   (b) $(g\circ f)(x) = 4+\cos(5\sqrt{x})$ and $(g\circ f)^0(x) = g^0(f(x))f^0(x) = -\sin(5\sqrt{x})\frac{5}{2\sqrt{x}}$.

5. (a) $F^0(x) = f^0(g(x))g^0(x)$, $F^0(3) = f^0(g(3))g^0(3) = -1(7) = -7$.

   (b) $G^0(x) = g^0(f(x))f^0(x)$, $G^0(3) = g^0(f(3))f^0(3) = 4(-2) = -8$.

6. (a) $F^0(x) = f^0(g(x))g^0(x)$, $F^0(-1) = f^0(g(-1))g^0(-1) = f^0(2)(-3) = (4)(-3) = -12$.

   (b) $G^0(x) = g^0(f(x))f^0(x)$, $G^0(-1) = g^0(f(-1))f^0(-1) = -5(3) = -15$.

7. $f'(x) = 37(x^3+2x)^{36}\frac{d}{dx}(x^3+2x) = 37(x^3+2x)^{36}(3x^2+2)$.

8. $f'(x) = 6(3x^2+2x-1)^5\frac{d}{dx}(3x^2+2x-1) = 6(3x^2+2x-1)^5(6x+2) = 12(3x^2+2x-1)^5(3x+1)$.

9. $f^0(x) = -2\left(x^3-\frac{7}{x}\right)^{-3}\frac{d}{dx}\left(x^3-\frac{7}{x}\right) = -2\left(x^3-\frac{7}{x}\right)^{-3}\left(3x^2+\frac{7}{x^2}\right)$.

10. $f(x) = (x^5-x+1)^{-9}$, $f'(x) = -9(x^5-x+1)^{-10}\frac{d}{dx}(x^5-x+1) = -9(x^5-x+1)^{-10}(5x^4-1) = \frac{-9(5x^4-1)}{(x^5-x+1)^{10}}$.

11. $f(x) = 4(3x^2-2x+1)^{-3}$, $f'(x) = -12(3x^2-2x+1)^{-4}\frac{d}{dx}(3x^2-2x+1) = -12(3x^2-2x+1)^{-4}(6x-2) = \frac{24(1-3x)}{(3x^2-2x+1)^4}$.

12. $f^0(x) = \frac{1}{2\sqrt{x^3-2x+5}}\frac{d}{dx}(x^3-2x+5) = \frac{3x^2-2}{2\sqrt{x^3-2x+5}}$.

13. $f^0(x) = \frac{1}{2\sqrt{4+\sqrt{3x}}}\frac{d}{dx}(4+\sqrt{3x}) = \frac{\sqrt{3}}{4\sqrt{x}\sqrt{4+\sqrt{3x}}}$.

14. $f^0(x) = \frac{1}{3}(12+\sqrt{x})^{-2/3}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{6(12+\sqrt{x})^{2/3}\sqrt{x}}$.

15. $f^0(x) = \cos(1/x^2)\frac{d}{dx}(1/x^2) = -\frac{2}{x^3}\cos(1/x^2)$.

16. $f^0(x) = \sec^2\sqrt{x}\,\frac{d}{dx}\sqrt{x} = \sec^2\sqrt{x}\,\frac{1}{2\sqrt{x}}$.

17. $f'(x) = 20\cos^4 x \frac{d}{dx}(\cos x) = 20\cos^4 x(-\sin x) = -20\cos^4 x \sin x$.

18. $f'(x) = 4 + 20(\sin^3 x)\frac{d}{dx}(\sin x) = 4 + 20\sin^3 x \cos x$.

19. $f'(x) = 2\cos(3\sqrt{x})\frac{d}{dx}[\cos(3\sqrt{x})] = -2\cos(3\sqrt{x})\sin(3\sqrt{x})\frac{d}{dx}(3\sqrt{x}) = -\frac{3\cos(3\sqrt{x})\sin(3\sqrt{x})}{\sqrt{x}}$.

20. $f'(x) = 4\tan^3(x^3)\frac{d}{dx}[\tan(x^3)] = 4\tan^3(x^3)\sec^2(x^3)\frac{d}{dx}(x^3) = 12x^2\tan^3(x^3)\sec^2(x^3)$.

21. $f'(x) = 4\sec(x^7)\frac{d}{dx}[\sec(x^7)] = 4\sec(x^7)\sec(x^7)\tan(x^7)\frac{d}{dx}(x^7) = 28x^6\sec^2(x^7)\tan(x^7)$.

22. $f'(x) = 3\cos^2\left(\frac{x}{x+1}\right)\frac{d}{dx}\cos\left(\frac{x}{x+1}\right) = 3\cos^2\left(\frac{x}{x+1}\right)\left[-\sin\left(\frac{x}{x+1}\right)\right]\frac{(x+1)(1)-x(1)}{(x+1)^2} =$

$= -\frac{3}{(x+1)^2}\cos^2\left(\frac{x}{x+1}\right)\sin\left(\frac{x}{x+1}\right)$.

23. $f'(x) = \frac{1}{2\sqrt{\cos(5x)}}\frac{d}{dx}[\cos(5x)] = -\frac{5\sin(5x)}{2\sqrt{\cos(5x)}}$.

24. $f'(x) = \frac{1}{2\sqrt{3x-\sin^2(4x)}}\frac{d}{dx}[3x-\sin^2(4x)] = \frac{3-8\sin(4x)\cos(4x)}{2\sqrt{3x-\sin^2(4x)}}$.

25. $f'(x) = 3\left[x+\csc(x^3+3)\right]^{-4}\frac{d}{dx}\left[x+\csc(x^3+3)\right] =$

$= -3\left[x+\csc(x^3+3)\right]^{-4}\left[1-\csc(x^3+3)\cot(x^3+3)\frac{d}{dx}(x^3+3)\right] =$

$= -3\left[x+\csc(x^3+3)\right]^{-4}\left[1-3x^2\csc(x^3+3)\cot(x^3+3)\right]$.

26. $f'(x) = -4\left[x^4-\sec(4x^2-2)\right]^{-5}\frac{d}{dx}\left[x^4-\sec(4x^2-2)\right] =$

$= -4\left[x^4-\sec(4x^2-2)\right]^{-5}\left[4x^3-\sec(4x^2-2)\tan(4x^2-2)\frac{d}{dx}(4x^2-2)\right] =$

$= -16x\left[x^4-\sec(4x^2-2)\right]^{-5}\left[x^2-2\sec(4x^2-2)\tan(4x^2-2)\right]$.

27. $\frac{dy}{dx} = x^3(2\sin 5x)\frac{d}{dx}(\sin 5x) + 3x^2\sin^2 5x = 10x^3\sin 5x\cos 5x + 3x^2\sin^2 5x$.

28. $\frac{dy}{dx} = \sqrt{x}\left(3\tan^2(\sqrt{x})\sec^2(\sqrt{x})\frac{1}{2\sqrt{x}}\right) + \frac{1}{2\sqrt{x}}\tan^3(\sqrt{x}) = \frac{3}{2}\tan^2(\sqrt{x})\sec^2(\sqrt{x}) + \frac{1}{2\sqrt{x}}\tan^3(\sqrt{x})$.

29. $\frac{dy}{dx} = x^5\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)\frac{d}{dx}\left(\frac{1}{x}\right) + \sec\left(\frac{1}{x}\right)(5x^4) = x^5\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + 5x^4\sec\left(\frac{1}{x}\right) =$

$= -x^3\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) + 5x^4\sec\left(\frac{1}{x}\right)$.

30. $\frac{dy}{dx} = \frac{\sec(3x+1)\cos x - 3\sin x\sec(3x+1)\tan(3x+1)}{\sec^2(3x+1)} = \cos x\cos(3x+1) - 3\sin x\sin(3x+1)$.

31. $\frac{dy}{dx}$ $\frac{d}{dx}(\cos x) = -\sin(\cos x)(-\sin x) = \sin(\cos x)\sin x$.

$\text{dx} = -\sin(\cos x)\ \text{dx}$

32. $\frac{dy}{dx} = \cos(\tan 3x)\ \frac{d}{dx}(\tan 3x) = 3 \sec^2 3x \cos(\tan 3x)$.

33. $\frac{dy}{dx} = 3 \cos^2 (\sin 2x)\ \frac{d}{dx}[\cos(\sin 2x)] = 3 \cos^2 (\sin 2x)[-\sin(\sin 2x)]\ \frac{d}{dx}(\sin 2x) = -6 \cos^2 (\sin 2x) \sin(\sin 2x) \cos 2x$.

34. $\dfrac{dy}{dx} = \dfrac{(1-\cot x^2)(-2x\csc x^2 \cot x^2) - (1+\csc x^2)(2x\csc^2 x^2)}{(1-\cot x^2)^2} = -2x\csc x^2 \dfrac{1+\cot x^2+\csc x^2}{(1-\cot x^2)^2}$, since $\csc^2 x^2 = 1+\cot^2 x^2$.

35. $\dfrac{dy}{dx} = (5x+8)^7 \dfrac{d}{dx}(1-\sqrt{x})^6 + (1-\sqrt{x})^6 \dfrac{d}{dx}(5x+8)^7 = 6(5x+8)^7(1-\sqrt{x})^5 \dfrac{-1}{2\sqrt{x}} + 7\cdot 5(1-\sqrt{x})^6(5x+8)^6 = \dfrac{-3}{\sqrt{x}}(5x+8)^7(1-\sqrt{x})^5 + 35(1-\sqrt{x})^6(5x+8)^6$.

36. $\dfrac{dy}{dx} = (x^2+x)^5 \dfrac{d}{dx}\sin^8 x + (\sin^8 x)\dfrac{d}{dx}(x^2+x)^5 = 8(x^2+x)^5 \sin^7 x \cos x + 5(\sin^8 x)(x^2+x)^4(2x+1)$.

37. $\dfrac{dy}{dx} = 3\left(\dfrac{x-5}{2x+1}\right)^2 \dfrac{d}{dx}\dfrac{x-5}{2x+1} = 3\left(\dfrac{x-5}{2x+1}\right)^2 \cdot \dfrac{11}{(2x+1)^2} = \dfrac{33(x-5)^2}{(2x+1)^4}$.

38. $\dfrac{dy}{dx} = 17\left(\dfrac{1+x^2}{1-x^2}\right)^{16} \dfrac{d}{dx}\dfrac{1+x^2}{1-x^2} = 17\left(\dfrac{1+x^2}{1-x^2}\right)^{16} \dfrac{(1-x^2)(2x)-(1+x^2)(-2x)}{(1-x^2)^2} = 17\left(\dfrac{1+x^2}{1-x^2}\right)^{16} \dfrac{4x}{(1-x^2)^2} = \dfrac{68x(1+x^2)^{16}}{(1-x^2)^{18}}$.

39. $\dfrac{dy}{dx} = \dfrac{(4x^2-1)^8(3)(2x+3)^2(2) - (2x+3)^3(8)(4x^2-1)^7(8x)}{(4x^2-1)^{16}} = \dfrac{2(2x+3)^2(4x^2-1)^7[3(4x^2-1)-32x(2x+3)]}{(4x^2-1)^{16}} = -\dfrac{2(2x+3)^2(52x^2+96x+3)}{(4x^2-1)^9}$.

40. $\dfrac{dy}{dx} = 12[1+\sin^3(x^5)]^{11} \dfrac{d}{dx}[1+\sin^3(x^5)] = 12[1+\sin^3(x^5)]^{11}\, 3\sin^2(x^5)\dfrac{d}{dx}\sin(x^5) =$
$= 180x^4[1+\sin^3(x^5)]^{11}\sin^2(x^5)\cos(x^5)$.

41. $\dfrac{dy}{dx} = 5\left[x\sin 2x + \tan^4(x^7)\right]^4 \dfrac{d}{dx}\left[x\sin 2x + \tan^4(x^7)\right] =$
$= 5\left[x\sin 2x + \tan^4(x^7)\right]^4 \left[x\cos 2x \dfrac{d}{dx}(2x) + \sin 2x + 4\tan^3(x^7)\dfrac{d}{dx}\tan(x^7)\right] =$
$= 5\left[x\sin 2x + \tan^4(x^7)\right]^4 \left[2x\cos 2x + \sin 2x + 28x^6\tan^3(x^7)\sec^2(x^7)\right]$.

42. $\dfrac{dy}{dx} = 4\tan^3\left(2 + \dfrac{(7-x)\sqrt{3x^2+5}}{x^3+\sin x}\right)\sec^2\left(2 + \dfrac{(7-x)\sqrt{3x^2+5}}{x^3+\sin x}\right) \times \left(\dfrac{-\sqrt{3x^2+5} + 3\dfrac{(7-x)x}{\sqrt{3x^2+5}}}{x^3+\sin x} - \dfrac{(7-x)\sqrt{3x^2+5}\,(3x^2+\cos x)}{(x^3+\sin x)^2}\right)$

43. $\dfrac{dy}{dx} = \cos 3x - 3x\sin 3x$; if $x=\pi$ then $\dfrac{dy}{dx} = -1$ and $y=-\pi$, so the equation of the tangent line is $y+\pi = -(x-\pi)$, or $y=-x$.

44. $\dfrac{dy}{dx} = 3x^2\cos(1+x^3)$; if $x=-3$ then $y=-\sin 26$, $\dfrac{dy}{dx} = 27\cos 26$, so the equation of the tangent line is $y+\sin 26 = 27(\cos 26)(x+3)$, or $y = 27(\cos 26)x + 81\cos 26 - \sin 26$.

45. $\frac{dy}{dx} = -3\sec^3(\pi/2 - x)\tan(\pi/2 - x)$; if $x = -\pi/2$ then $\frac{dy}{dx} = 0$, $y = -1$, so the equation of the tangent line is

$y + 1 = 0$, or $y = -1$

46. $\frac{dy}{dx} = 3\left(x - \frac{1}{x}\right)^2\left(1 + \frac{1}{x^2}\right)$; if $x = 2$ then $y = \frac{27}{8}$, $\frac{dy}{dx} = 3 \cdot \frac{9}{4} \cdot \frac{5}{4} = \frac{135}{16}$, so the equation of the tangent line is $y - 27/8 = (135/16)(x - 2)$, or $y = \frac{135}{16}x - \frac{27}{2}$.

47. $\frac{dy}{dx} = \sec^2(4x^2)\frac{d}{dx}(4x^2) = 8x\sec^2(4x^2)$, $\left.\frac{dy}{dx}\right|_{x=\sqrt{\pi}} = 8\sqrt{\pi}\sec^2(4\pi) = 8\sqrt{\pi}$. When $x = \sqrt{\pi}$, $y = \tan(4\pi) = 0$, so the equation of the tangent line is $y = 8\sqrt{\pi}(x - \sqrt{\pi}) = 8\sqrt{\pi}x - 8\pi$.

48. $\frac{dy}{dx} = 12\cot^3 x\frac{d}{dx}\cot x = -12\cot^3 x\csc^2 x$, $\left.\frac{dy}{dx}\right|_{x=\pi/4} = -24$. When $x = \pi/4$, $y = 3$, so the equation of the tangent line is $y - 3 = -24(x - \pi/4)$, or $y = -24x + 3 + 6\pi$.

49. $\frac{dy}{dx} = 2x\sqrt{5 - x^2} + \frac{x^2}{2\sqrt{5 - x^2}}(-2x)$, $\left.\frac{dy}{dx}\right|_{x=1} = 4 - 1/2 = 7/2$. When $x = 1$, $y = 2$, so the equation of the tangent line is $y - 2 = (7/2)(x - 1)$, or $y = \frac{7}{2}x - \frac{3}{2}$.

50. $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} - \frac{x}{2}(1 - x^2)^{3/2}(-2x)$, $\left.\frac{dy}{dx}\right|_{x=0} = 1$. When $x = 0$, $y = 0$, so the equation of the tangent line is $y = x$.

51. $\frac{dy}{dx} = x(-\sin(5x))\frac{d}{dx}(5x) + \cos(5x) - 2\sin x\frac{d}{dx}(\sin x) = -5x\sin(5x) + \cos(5x) - 2\sin x\cos x =$
$= -5x\sin(5x) + \cos(5x) - \sin(2x)$,
$\frac{d^2y}{dx^2} = -5x\cos(5x)\frac{d}{dx}(5x) - 5\sin(5x) - \sin(5x)\frac{d}{dx}(5x) - \cos(2x)\frac{d}{dx}(2x) = -25x\cos(5x) - 10\sin(5x) - 2\cos(2x)$.

52. $\frac{dy}{dx} = \cos(3x^2)\frac{d}{dx}(3x^2) = 6x\cos(3x^2)$, $\frac{d^2y}{dx^2} = 6x(-\sin(3x^2))\frac{d}{dx}(3x^2) + 6\cos(3x^2) = -36x^2\sin(3x^2) + 6\cos(3x^2)$.

53. $\frac{dy}{dx} = \frac{(1 - x) + (1 + x)}{(1 - x)^2} = \frac{2}{(1 - x)^2} = 2(1 - x)^{-2}$ and $\frac{d^2y}{dx^2} = -2(2)(-1)(1 - x)^{-3} = 4(1 - x)^{-3}$.

54. $\frac{dy}{dx} = x\sec^2\frac{1}{x}\frac{d}{dx}\frac{1}{x} + \tan\frac{1}{x} = -\frac{1}{x}\sec^2\frac{1}{x} + \tan\frac{1}{x}$,
$\frac{d^2y}{dx^2} = -\frac{2}{x}\sec\frac{1}{x}\frac{d}{dx}\sec\frac{1}{x} + \frac{1}{x^2}\sec^2\frac{1}{x} + \sec^2\frac{1}{x}\frac{d}{dx}\frac{1}{x} = \frac{2}{x^3}\sec^2\frac{1}{x}\tan\frac{1}{x}$.

55. $y = \cot^3(\pi - \theta) = -\cot^3\theta$ so $dy/dx = 3\cot^2\theta\csc^2\theta$.

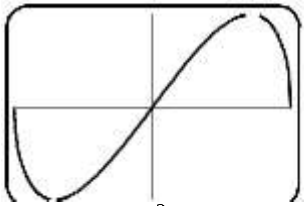
56. $6\left(\frac{au + b}{cu + d}\right)^5\frac{ad - bc}{(cu + d)^2}$.

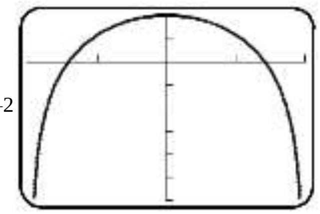
57. $\frac{d}{d\omega}[a\cos^2\pi\omega + b\sin^2\pi\omega] = -2\pi a\cos\pi\omega\sin\pi\omega + 2\pi b\sin\pi\omega\cos\pi\omega = \pi(b - a)(2\sin\pi\omega\cos\pi\omega) = \pi(b - a)\sin 2\pi\omega$.

58. $2\csc^2(\pi/3 - y)\cot(\pi/3 - y)$.

2

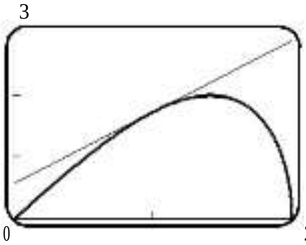
–2 2

59. (a) 

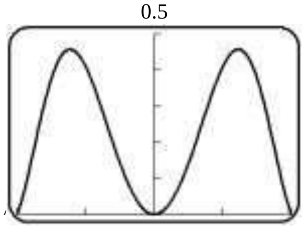


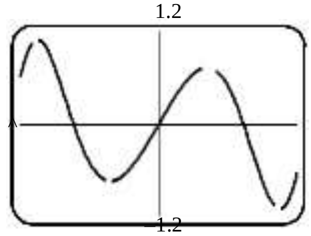
(c) $f'(x) = x\dfrac{-x}{\sqrt{4-x^2}} + \sqrt{4-x^2} = \dfrac{4-2x^2}{\sqrt{4-x^2}}$.

(d) $f(1) = \sqrt{3}$ and $f'(1) = \dfrac{2}{\sqrt{3}}$ so the tangent line has the equation $y - \sqrt{3} = \dfrac{2}{\sqrt{3}}(x-1)$.



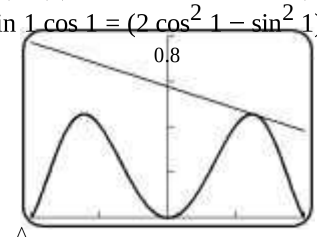
60. (a)





(c) $f'(x) = 2x\cos(x^2)\cos x - \sin x\sin(x^2)$.

(d) $f(1) = \sin 1\cos 1$ and $f'(1) = 2\cos^2 1 - \sin^2 1$, so the tangent line has the equation $y - \sin 1\cos 1 = (2\cos^2 1 - \sin^2 1)(x-1)$.



61. False. $\dfrac{d}{dx}[\sqrt{y}] = \dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx} = \dfrac{f'(x)}{2\sqrt{f(x)}}$.

62. False. $dy/dx = f'(u)g'(x) = f'(g(x))\,g'(x)$.

63. False. $dy/dx = -\sin[g(x)]\,g'(x)$.

64. True. Let $u = 3x^3$ and $v = \sin u$, so $y = v^3$. Then $\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx} = 3v^2 \cdot (\cos u) \cdot 9x^2 = 3\sin^2(3x^3) \cdot \cos(3x^3) \cdot 9x^2 = 27x^2 \sin^2(3x^3)\cos(3x^3)$.

65. (a) $dy/dt = -A\omega \sin \omega t$, $d^2y/dt^2 = -A\omega^2 \cos \omega t = -\omega^2 y$

(b) One complete oscillation occurs when ωt increases over an interval of length 2π, or if t increases over an interval of length $2\pi/\omega$.

(c) $f = 1/T$

(d) Amplitude = 0.6 cm, $T = 2\pi/15$ s/oscillation, $f = 15/(2\pi)$ oscillations/s.

66. $dy/dt = 3A\cos 3t$, $d^2y/dt^2 = -9A \sin 3t$, so $-9A\sin 3t + 2A\sin 3t = 4\sin 3t$, $-7A\sin 3t = 4\sin 3t$, $-7A = 4$, and $A = -4/7$

67. By the chain rule, $\frac{d}{dx}\left[\sqrt{x+f(x)}\right] = \frac{1+f'(x)}{2\sqrt{x+f(x)}}$. From the graph, $f(x) = \frac{4}{3}x + 5$ for $x < 0$, so $f(-1) = \frac{11}{3}$, $f'(-1) = \frac{4}{3}$, and $\frac{d}{dx}\left[\sqrt{x+f(x)}\right]_{x=-1} = \frac{7/3}{2\sqrt{8/3}} = \frac{7\sqrt{6}}{24}$.

68. $2\sin(\pi/6) = 1$, so we can assume $f(x) = -\frac{5}{2}x + 5$. Thus for sufficiently small values of $|x - \pi/6|$ we have $\frac{d}{dx}[f(2\sin x)]_{x=\pi/6} = f'(2\sin x)\frac{d}{dx}2\sin x\Big|_{x=\pi/6} = -\frac{5}{2}\,2\cos x\Big|_{x=\pi/6} = -\frac{5}{2}\,2\,\frac{\sqrt{3}}{2} = -\frac{5\sqrt{3}}{2}$.

69. (a) $p \approx 10$ lb/in^2, $dp/dh \approx -2$ lb/in^2/mi. (b) $\frac{dp}{dt} = \frac{dp}{dh}\frac{dh}{dt} \approx (-2)(0.3) = -0.6$ lb/in^2/s.

70. (a) $F = \frac{45}{\cos\theta + 0.3\sin\theta}$, $\frac{dF}{d\theta} = -\frac{45(-\sin\theta + 0.3\cos\theta)}{(\cos\theta + 0.3\sin\theta)^2}$; if $\theta = 30^\circ$, then $dF/d\theta \approx 10.5$ lb/rad ≈ 0.18 lb/deg.

(b) $\frac{dF}{dt} = \frac{dF}{d\theta}\frac{d\theta}{dt} \approx (0.18)(-0.5) = -0.09$ lb/s.

71. With $u = \sin x$, $\frac{d}{dx}(|\sin x|) = \frac{d}{dx}(|u|) = \frac{d}{du}(|u|)\frac{du}{dx} = \frac{d}{du}(|u|)\cos x = \begin{cases} \cos x, & u > 0 \\ -\cos x, & u < 0 \end{cases} = \begin{cases} \cos x, & \sin x > 0 \\ -\cos x, & \sin x < 0 \end{cases}$

$= \begin{cases} \cos x, & 0 < x < \pi \\ -\cos x, & -\pi < x < 0 \end{cases}$

72. $\frac{d}{dx}(\cos x) = \frac{d}{dx}[\sin(\pi/2 - x)] = -\cos(\pi/2 - x) = -\sin x$.

73. (a) For $x \neq 0$, $|f(x)| \le |x|$, and $\lim_{x\to 0}|x| = 0$, so by the Squeezing Theorem, $\lim_{x\to 0} f(x) = 0$.

(b) If $f'(0)$ were to exist, then the limit (as x approaches 0) $\frac{f(x) - f(0)}{x - 0} = \sin(1/x)$ would have to exist, but it doesn't.

(c) For $x = 0$, $f^0(x) = x \cos\frac{1}{x} \left(-\frac{1}{x^2}\right) + \sin\frac{1}{x} = -\frac{1}{x}\cos\frac{1}{x} + \sin\frac{1}{x}$.

(d) If $x = \frac{1}{2\pi n}$ for an integer $n = 0$, then $f^0(x) = -2\pi n \cos(2\pi n) + \sin(2\pi n) = -2\pi n$. This approaches $+\infty$ as $n \to -\infty$, so there are points x arbitrarily close to 0 where $f^0(x)$ becomes arbitrarily large. Hence $\lim_{x\to 0} f^0(x)$ does not exist.

74. (a) $-x^2 \le x^2 \sin(1/x) \le x^2$, so by the Squeezing Theorem $\lim_{x\to 0} f(x) = 0$.

(b) $f'(0) = \lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} x \sin(1/x) = 0$ by Exercise 73, part (a).

(c) For $x \neq 0$, $f'(x) = 2x \sin(1/x) + x^2 \cos(1/x)(-1/x^2) = 2x \sin(1/x) - \cos(1/x)$.

(d) If $f'(x)$ were continuous at $x = 0$ then so would $\cos(1/x) = 2x \sin(1/x) - f'(x)$ be, since $2x \sin(1/x)$ is continuous there. But $\cos(1/x)$ oscillates at $x = 0$.

75. (a) $g'(x) = 3[f(x)]^2 f'(x)$, $g'(2) = 3[f(2)]^2 f'(2) = 3(1)^2(7) = 21$.

(b) $h'(x) = f'(x^3)(3x^2)$, $h'(2) = f'(8)(12) = (-3)(12) = -36$.

76. $F'(x) = f'(g(x))g'(x) = \sqrt{3(x^2 - 1) + 4} \cdot 2x = 2x\sqrt{3x^2 + 1}$.

77. $F'(x) = f'(g(x))g'(x) = f'(\sqrt{3x - 1}) \dfrac{3}{2\sqrt{3x - 1}} = \dfrac{\sqrt{3x - 1}}{(3x - 1) + 1} \dfrac{3}{2\sqrt{3x - 1}} = \dfrac{1}{2x}$.

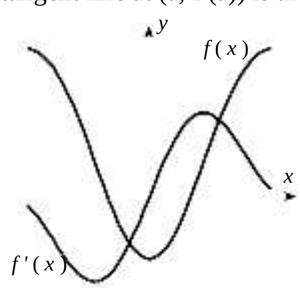
78. $\dfrac{d}{dx}[f(x^2)] = f'(x^2)(2x)$, thus $f'(x^2)(2x) = x^2$ so $f'(x^2) = x/2$ if $x \neq 0$.

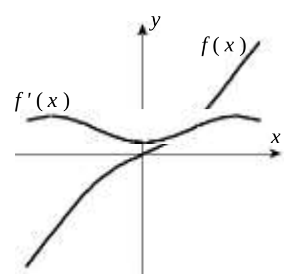
79. $\dfrac{d}{dx}[f(3x)] = f'(3x)\dfrac{d}{dx}(3x) = 3f'(3x) = 6x$, so $f'(3x) = 2x$. Let $u = 3x$ to get $f'(u) = \dfrac{2}{3}u$; $\dfrac{d}{dx}[f(x)] = f'(x) = \dfrac{2}{3}x$.

80. (a) If $f(-x) = f(x)$, then $\dfrac{d}{dx}[f(-x)] = \dfrac{d}{dx}[f(x)]$, $f'(-x)(-1) = f'(x)$, $f'(-x) = -f'(x)$ so f' is odd.

(b) If $f(-x) = -f(x)$, then $\dfrac{d}{dx}[f(-x)] = -\dfrac{d}{dx}[f(x)]$, $f'(-x)(-1) = -f'(x)$, $f'(-x) = f'(x)$ so f' is even.

81. For an even function, the graph is symmetric about the y-axis; the slope of the tangent line at $(a, f(a))$ is the negative of the slope of the tangent line at $(-a, f(-a))$. For an odd function, the graph is symmetric about the origin; the slope of the tangent line at $(a, f(a))$ is the same as the slope of the tangent line at $(-a, f(-a))$.





82. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dv}\dfrac{dv}{dw}\dfrac{dw}{dx}$.

83. $\dfrac{d}{dx}[f(g(h(x)))] = \dfrac{d}{dx}[f(g(u))]$, $u = h(x)$, $\dfrac{d}{du}[f(g(u))]\dfrac{du}{dx} = f'(g(u))g'(u)\dfrac{du}{dx} = f'(g(h(x)))g'(h(x))h'(x)$.

84. $g'(x) = f'\left(\dfrac{\pi}{2} - x\right) \cdot \dfrac{d}{dx}\left(\dfrac{\pi}{2} - x\right) = -f'\left(\dfrac{\pi}{2} - x\right)$, so g' is the negative of the co-function of f'.

The derivatives of $\sin x$, $\tan x$, and $\sec x$ are $\cos x$, $\sec^2 x$, and $\sec x \tan x$, respectively. The negatives of the co-functions of these are $-\sin x$, $-\csc^2 x$, and $-\csc x \cot x$, which are the derivatives of $\cos x$, $\cot x$, and $\csc x$, respectively.

Exercise Set 2.7

1. (a) $1 + y + x\frac{dy}{dx} - 6x^2 = 0$, $\frac{dy}{dx} = \frac{6x^2 - y - 1}{x}$.

   (b) $y = \frac{2 + 2x^3}{x} - 1 = \frac{2}{x} + 2x^2 - 1$, $\frac{dy}{dx} = -\frac{2}{x^2} + 4x$.

   (c) From part (a), $\frac{dy}{dx} = 6x - \frac{1}{x} - \frac{1}{x}y = 6x - \frac{1}{x} - \frac{1}{x}\left(\frac{2}{x} + 2x^2 - 1\right) = 4x - \frac{2}{x^2}$.

2. (a) $\frac{1}{2}y^{-1/2}\frac{dy}{dx} - \cos x = 0$ or $\frac{dy}{dx} = 2\sqrt{y}\cos x$.

   (b) $y = (2 + \sin x)^2 = 4 + 4\sin x + \sin^2 x$ so $\frac{dy}{dx} = 4\cos x + 2\sin x\cos x$.

   (c) From part (a), $\frac{dy}{dx} = 2\sqrt{y}\cos x = 2\cos x(2 + \sin x) = 4\cos x + 2\sin x\cos x$.

3. $2x + 2y\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = -\frac{x}{y}$.

4. $3x^2 + 3y^2\frac{dy}{dx} = 3y^2 + 6xy\frac{dy}{dx}$, $\frac{dy}{dx} = \frac{3y^2 - 3x^2}{3y^2 - 6xy} = \frac{y^2 - x^2}{y^2 - 2xy}$.

5. $x^2\frac{dy}{dx} + 2xy + 3x(3y^2)\frac{dy}{dx} + 3y^3 - 1 = 0$, $(x^2 + 9xy^2)\frac{dy}{dx} = 1 - 2xy - 3y^3$, so $\frac{dy}{dx} = \frac{1 - 2xy - 3y^3}{x^2 + 9xy^2}$.

6. $x^3(2y)\frac{dy}{dx} + 3x^2y^2 - 5x^2\frac{dy}{dx} - 10xy + 1 = 0$, $(2x^3y - 5x^2)\frac{dy}{dx} = 10xy - 3x^2y^2 - 1$, so $\frac{dy}{dx} = \frac{10xy - 3x^2y^2 - 1}{2x^3y - 5x^2}$.

7. $-\frac{1}{2x^{3/2}} - \frac{\frac{dy}{dx}}{2y^{3/2}} = 0$, so $\frac{dy}{dx} = -\frac{y^{3/2}}{x^{3/2}}$.

8. $2x = \frac{(x-y)(1 + dy/dx) - (x+y)(1 - dy/dx)}{(x-y)^2}$, $2x(x-y)^2 = -2y + 2x\frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{x(x-y)^2 + y}{x}$.

9. $\cos(x^2y^2)\left[x^2(2y)\frac{dy}{dx} + 2xy^2\right] = 1$, so $\frac{dy}{dx} = \frac{1 - 2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2)}$.

10. $-\sin(xy^2)\left[y^2 + 2xy\frac{dy}{dx}\right] = \frac{dy}{dx}$, so $\frac{dy}{dx} = -\frac{y^2\sin(xy^2)}{2xy\sin(xy^2) + 1}$.

$\dfrac{dy}{dx} \quad \dfrac{dy}{dx} \quad 2xy\sin(xy^2) + 1$

11. $3\tan^2(xy^2 + y)\sec^2(xy^2 + y)\left(2xy\dfrac{dy}{dx} + y^2 + \dfrac{dy}{dx}\right) = 1$, so $\dfrac{dy}{dx} = \dfrac{1 - 3y^2\tan^2(xy^2 + y)\sec^2(xy^2 + y)}{3(2xy + 1)\tan^2(xy^2 + y)\sec^2(xy^2 + y)}$.

12. $\dfrac{(1 + \sec y)[3xy^2(dy/dx) + y^3] - xy^3(\sec y \tan y)(dy/dx)}{(1 + \sec y)^2} = 4y^3\dfrac{dy}{dx}$, multiply through by $(1 + \sec y)^2$ and solve for $\dfrac{dy}{dx}$ to get $\dfrac{dy}{dx} = \dfrac{y(1 + \sec y)}{4y(1 + \sec y)^2 - 3x(1 + \sec y) + xy\sec y\tan y}$.

13. $4x - 6y\dfrac{dy}{dx} = 0$, $\dfrac{dy}{dx} = \dfrac{2x}{3y}$, $4 - 6\left(\dfrac{dy}{dx}\right)^2 - 6y\dfrac{d^2y}{dx^2} = 0$, so $\dfrac{d^2y}{dx^2} = -\dfrac{3\left(\dfrac{dy}{dx}\right)^2 - 2}{3y} = \dfrac{2(3y^2 - 2x^2)}{9y^3} = -\dfrac{8}{9y^3}$.

14. $\dfrac{dy}{dx} = -\dfrac{x^2}{y^2}$, $\dfrac{d^2y}{dx^2} = -\dfrac{y^2(2x) - x^2(2y\,dy/dx)}{y^4} = -\dfrac{2xy^2 - 2x^2y(-x^2/y^2)}{y^4} = -\dfrac{2x(y^3 + x^3)}{y^5}$, but $x^3 + y^3 = 1$, so $\dfrac{d^2y}{dx^2} = -\dfrac{2x}{y^5}$.

15. $\dfrac{dy}{dx} = -\dfrac{y}{x}$, $\dfrac{d^2y}{dx^2} = -\dfrac{x(dy/dx) - y(1)}{x^2} = -\dfrac{x(-y/x) - y}{x^2} = \dfrac{2y}{x^2}$.

16. $y + x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$, $\dfrac{dy}{dx} = -\dfrac{y}{x + 2y}$, $2\dfrac{dy}{dx} + x\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 + 2y\dfrac{d^2y}{dx^2} = 0$, $\dfrac{d^2y}{dx^2} = \dfrac{2y(x + y)}{(x + 2y)^3}$.

17. $\dfrac{dy}{dx} = (1 + \cos y)^{-1}$, $\dfrac{d^2y}{dx^2} = -(1 + \cos y)^{-2}(-\sin y)\dfrac{dy}{dx} = \dfrac{\sin y}{(1 + \cos y)^3}$.

18. $\dfrac{dy}{dx} = \dfrac{\cos y}{1 + x\sin y}$, $\dfrac{d^2y}{dx^2} = \dfrac{(1 + x\sin y)(-\sin y)(dy/dx) - (\cos y)[(x\cos y)(dy/dx) + \sin y]}{(1 + x^2\sin y)}$ $= -\dfrac{2\sin y\cos y + (x\cos y)(2\sin^2 y + \cos^2 y)}{(1 + x\sin y)^3}$, but $x\cos y = y$, $2\sin y\cos y = \sin 2y$, and $\sin^2 y + \cos^2 y = 1$, so $\dfrac{d^2y}{dx^2} = -\dfrac{\sin 2y + y(\sin^2 y + 1)}{(1 + x\sin y)^3}$.

19. By implicit differentiation, $2x + 2y(dy/dx) = 0$, $\dfrac{dy}{dx} = -\dfrac{x}{y}$; at $(1/2, \sqrt{3}/2)$, $\dfrac{dy}{dx} = -\sqrt{3}/3$; at $(1/2, -\sqrt{3}/2)$, $\dfrac{dy}{dx} = +\sqrt{3}/3$. Directly, at the upper point $y = \sqrt{1 - x^2}$, $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1 - x^2}} = -\dfrac{1/2}{\sqrt{3/4}} = -1/\sqrt{3}$ and at the lower point $y = -\sqrt{1 - x^2}$, $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1 - x^2}} = +1/\sqrt{3}$.

20. If $y^2 - x + 1 = 0$, then $y = \sqrt{x - 1}$ goes through the point $(10, 3)$ so $dy/dx = 1/(2\sqrt{x - 1})$. By implicit differentiation $dy/dx = 1/(2y)$. In both cases, $dy/dx|_{(10,3)} = 1/6$. Similarly $y = -\sqrt{x - 1}$ goes through $(10, -3)$ so $dy/dx = -1/(2\sqrt{x - 1}) = -1/6$ which yields $dy/dx = 1/(2y) = -1/6$.

21. False; $x = y^2$ defines two functions $y = \pm\sqrt{x}$. See Definition 2.7.1.

22. True.

23. False; the equation is equivalent to $x^2 = y^2$ which is satisfied by $y = |x|$.

24. True.

25. $x^m x^{-m} = 1$, $x^{-m}\dfrac{d}{dx}(x^m) - mx^{-m-1}x^m = 0$, $\dfrac{d}{dx}(x^m) = x^m(mx^{-m-1})x^m = mx^{m-1}$.

26. $x^m = (x^r)^n$, $mx^{m-1} = n(x^r)^{n-1}\dfrac{d}{dx}(x^r)$, $\dfrac{d}{dx}(x^r) = \dfrac{m}{n}x^{m-1}(x^r)^{1-n} = rx^{r-1}$.

27. $4x^3 + 4y^3\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{x^3}{y^3} = -\dfrac{1}{15^{3/4}} \approx -0.1312$.

28. $3y^2\dfrac{dy}{dx} + x^2\dfrac{dy}{dx} + 2xy + 2x - 6y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -2x\dfrac{y+1}{3y^2 + x^2 - 6y} = 0$ at $x = 0$.

29. $4(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 25\left(2x - 2y\frac{dy}{dx}\right)$, $\frac{dy}{dx} = \frac{x[25 - 4(x + y)]}{y[25 + 4(x^2 + y^2)]}$; at (3, 1) $\frac{dy}{dx} = -9/13$.

30. $\frac{2}{3}x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} = \sqrt{3}$ at $(-1, 3\sqrt{3})$.

31. $4a^3\frac{da}{dt} - 4t^3 = 6\left(a^2 + 2at\frac{da}{dt}\right)$, solve for $\frac{da}{dt}$ to get $\frac{da}{dt} = \frac{2t^3 + 3a^2}{2a^3 - 6at}$.

32. $\frac{1}{2}u^{-1/2}\frac{du}{dv} + \frac{1}{2}v^{-1/2} = 0$, so $\frac{du}{dv} = -\frac{\sqrt{u}}{\sqrt{v}}$.

33. $2a^2\omega\frac{d\omega}{d\lambda} + 2b^2\lambda = 0$, so $\frac{d\omega}{d\lambda} = -\frac{b^2\lambda}{a^2\omega}$.

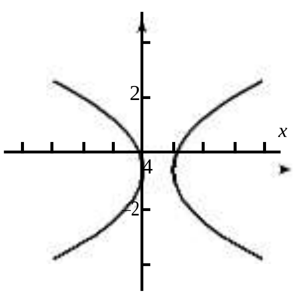
34. $1 = (\cos x)\frac{dx}{dy}$, so $\frac{dx}{dy} = \frac{1}{\cos x} = \sec x$.

35. $2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$. Substitute $y = -2x$ to obtain $-3x\frac{dy}{dx} = 0$. Since $x = \pm 1$ at the indicated points, $\frac{dy}{dx} = 0$ there.

36. (a) The equation and the point (1, 1) are both symmetric in x and y (if you interchange the two variables you get the same equation and the same point). Therefore the outcome "horizontal tangent at (1, 1)" could be replaced by "vertical tangent at (1, 1)", and these cannot both be the case.

(b) Implicit differentiation yields $\frac{dy}{dx} = \frac{2x - y}{x - 2y}$, which is zero only if $y = 2x$; coupled with the equation $x^2 - xy + y^2 = 1$ we obtain $x^2 - 2x^2 + 4x^2 = 1$, or $3x^2 = 1$, $x = (\sqrt{3}/3, 2\sqrt{3}/3)$ and $(-\sqrt{3}/3, -2\sqrt{3}/3)$.

37. (a)



(b) Implicit differentiation of the curve yields $(4y^3 + 2y)\frac{dy}{dx} = 2x - 1$, so $\frac{dy}{dx} = 0$ only if $x = 1/2$ but $y^4 + y^2 \geq 0$ so $x = 1/2$ is impossible.

(c) $x^2 - x - (y^4 + y^2) = 0$, so by the Quadratic Formula, $x = \frac{1 \pm \sqrt{(2y^2 + 1)^2}}{2} = 1 + y^2$ or $-y^2$, and we have the two parabolas $x = -y^2$, $x = 1 + y^2$.

38. By implicit differentiation, $2y(2y^2 + 1)\frac{dy}{dx} = 2x - 1$, $\frac{dx}{dy} = \frac{2y(2y^2 + 1)}{2x - 1} = 0$ only if $2y(2y^2 + 1) = 0$, which can only hold if $y = 0$. From $y^4 + y^2 = x(x - 1)$, if $y = 0$ then $x = 0$ or 1, and so (0, 0) and (1, 0) are the two points where the tangent is vertical.

39. The point (1,1) is on the graph, so 1 + a = b. The slope of the tangent line at (1,1) is −4/3; use implicit differentiation to get $\frac{dy}{dx} = -\frac{2xy}{x^2 + 2ay}$ so at (1,1), $-\frac{2}{1 + 2a} = -\frac{4}{3}$, 1 + 2a = 3/2, a = 1/4 and hence b = 1 + 1/4 = 5/4.

40. The slope of the line $x + 2y - 2 = 0$ is $m_1 = -1/2$, so the line perpendicular has slope $m = 2$ (negative reciprocal). The slope of the curve $y^3 = 2x^2$ can be obtained by implicit differentiation: $3y^2 \frac{dy}{dx} = 4x$, $\frac{dy}{dx} = \frac{4x}{3y^2}$. Set $\frac{dy}{dx} = 2$; $\frac{4x}{3y^2} = 2$, $x = (3/2)y^2$. Use this in the equation of the curve: $y^3 = 2x^2 = 2((3/2)y^2)^2 = (9/2)y^4$, $y = 2/9$, $x = \frac{3}{2}\left(\frac{2}{9}\right)^2 = \frac{2}{27}$.

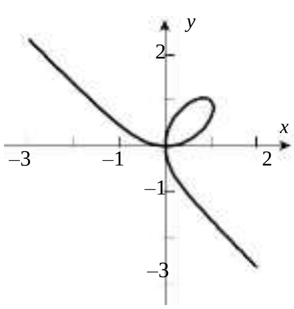
41. Solve the simultaneous equations $y = x$, $x^2 - xy + y^2 = 4$ to get $x^2 - x^2 + x^2 = 4$, $x = \pm 2$, $y = x = \pm 2$, so the points of intersection are $(2, 2)$ and $(-2, -2)$. By implicit differentiation, $\frac{dy}{dx} = \frac{y-2x}{2y-x}$. When $x = y = 2$, $\frac{dy}{dx} = -1$; when $x = y = -2$, $\frac{dy}{dx} = -1$; the slopes are equal.

42. Suppose $a^2 - 2ab + b^2 = 4$. Then $(-a)^2 - 2(-a)(-b) + (-b)^2 = a^2 - 2ab + b^2 = 4$ so if $P(a, b)$ lies on C then so does $Q(-a, -b)$. By implicit differentiation (see Exercise 41), $\frac{dy}{dx} = \frac{y-2x}{2y-x}$. When $x = a$, $y = b$ then $\frac{dy}{dx} = \frac{b-2a}{2b-a}$, and when $x = -a$, $y = -b$, then $\frac{dy}{dx} = \frac{b-2a}{2b-a}$, so the slopes at P and Q are equal.

43. We shall find when the curves intersect and check that the slopes are negative reciprocals. For the intersection solve the simultaneous equations $x^2 + (y - c)^2 = c^2$ and $(x - k)^2 + y^2 = k^2$ to obtain $cy = kx = \frac{1}{2}(x^2 + y^2)$. Thus $x^2 + y^2 = cy + kx$, or $y^2 - cy = -x^2 + kx$, and $\frac{y-c}{x} = -\frac{x-k}{y}$. Differentiating the two families yields (black) $\frac{dy}{dx} = -\frac{x}{y-c}$, and (gray) $\frac{dy}{dx} = -\frac{x-k}{y}$. But it was proven that these quantities are negative reciprocals of each other.

44. Differentiating, we get the equations (black) $x\frac{dy}{dx} + y = 0$ and (gray) $2x - 2y\frac{dy}{dx} = 0$. The first says the (black) slope is $-\frac{y}{x}$ and the second says the (gray) slope is $\frac{x}{y}$, and these are negative reciprocals of each other.

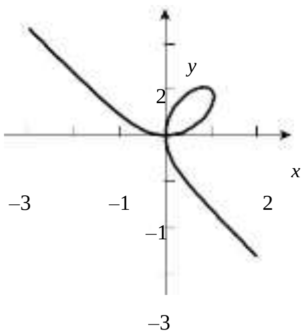
45. (a)



(b) $x \approx 0.84$

(c) Use implicit differentiation to get $dy/dx = (2y - 3x^2)/(3y^2 - 2x)$, so $dy/dx = 0$ if $y = (3/2)x^2$. Substitute this into $x^3 - 2xy + y^3 = 0$ to obtain $27x^6 - 16x^3 = 0$, $x^3 = 16/27$, $x = 2^{4/3}/3$ and hence $y = 2^{5/3}/3$.

46. (a)



(b) Evidently (by symmetry) the tangent line at the point x = 1, y = 1 has slope −1.

(c) Use implicit differentiation to get $dy/dx = (2y - 3x^2)/(3y^2 - 2x)$, so $dy/dx = -1$ if $2y - 3x^2 = -3y^2 + 2x$, $2(y - x) + 3(y - x)(y + x) = 0$. One solution is $y = x$; this together with $x^3 + y^3 = 2xy$ yields $x = y = 1$. For these values $dy/dx = -1$, so that (1, 1) is a solution. To prove that there is no other solution, suppose $y \neq x$. From $dy/dx = -1$ it follows that $2(y - x) + 3(y - x)(y + x) = 0$. But $y \neq x$, so $x + y = -2/3$, which is not true for any point in the first quadrant.

47. By the chain rule, $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$. Using implicit differentiation for $2y^3 t + t^3 y = 1$ we get $\frac{dy}{dt} = -\frac{2y^3 + 3t^2 y}{6ty^2 + t^3}$, but $\frac{dt}{dx} = \frac{1}{\cos t}$, so $\frac{dy}{dx} = -\frac{2y^3 + 3t^2 y}{(6ty^2 + t^3)\cos t}$.

48. Let $P(x_0, y_0)$ be a point where a line through the origin is tangent to the curve $2x^2 - 4x + y^2 + 1 = 0$. Implicit differentiation applied to the equation of the curve gives $dy/dx = (2-2x)/y$. At P the slope of the curve must equal the slope of the line so $(2 - 2x_0)/y_0 = y_0/x_0$, or $y_0^2 = 2x_0(1 - x_0)$. But $2x_0^2 - 4x_0 + y_0^2 + 1 = 0$ because (x_0, y_0) is on the curve, and elimination of y_0^2 in the latter two equations gives $2x_0 = 4x_0 - 1$, $x_0 = 1/2$ which when substituted into $y_0^2 = 2x_0(1 - x_0)$ yields $y_0^2 = 1/2$, so $y_0 = \pm\sqrt{2}/2$. The slopes of the lines are $(\pm\sqrt{2}/2)/(1/2) = \pm\sqrt{2}$ and their equations are $y = \sqrt{2}x$ and $y = -\sqrt{2}x$.

Exercise Set 2.8

1. $\frac{dy}{dt} = 3\frac{dx}{dt}$

(a) $\frac{dy}{dt} = 3(2) = 6$. (b) $-1 = 3\frac{dx}{dt}$, $\frac{dx}{dt} = -\frac{1}{3}$.

2. $\frac{dx}{dt} + 4\frac{dy}{dt} = 0$

(a) $1 + 4\frac{dy}{dt} = 0$ so $\frac{dy}{dt} = -\frac{1}{4}$ when x = 2. (b) $\frac{dx}{dt} + 4(4) = 0$ so $\frac{dx}{dt} = -16$ when x = 3.

3. $8x\frac{dx}{dt} + 18y\frac{dy}{dt} = 0$

(a) $8\frac{1}{2\sqrt{2}}\cdot 3 + 18\frac{1}{3\sqrt{2}}\frac{dy}{dt} = 0$, $\frac{dy}{dt} = -2$. (b) $8\frac{1}{3}\frac{dx}{dt} - 18\frac{\sqrt{5}}{9}\cdot 8 = 0$, $\frac{dx}{dt} = 6\sqrt{5}$.

4. $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2\frac{dx}{dt} + 4\frac{dy}{dt}$

(a) $2\cdot 3(-5) + 2\cdot 1\frac{dy}{dt} = 2(-5) + 4\frac{dy}{dt}$, $\frac{dy}{dt} = -10.$

(b) $2(1+\sqrt{2})\frac{dx}{dt} + 2(2+\sqrt{3})\cdot 6 = 2\frac{dx}{dt} + 4\cdot 6$, $\frac{dx}{dt} = -12\frac{\sqrt{3}}{2\sqrt{2}} = -3\sqrt{3}\sqrt{2}.$

5. (b) $A = x^2$.

(c) $\frac{dA}{dt} = 2x\frac{dx}{dt}$.

(d) Find $\left.\frac{dA}{dt}\right|_{x=3}$ given that $\left.\frac{dx}{dt}\right|_{x=3} = 2$. From part (c), $\left.\frac{dA}{dt}\right|_{x=3} = 2(3)(2) = 12$ ft^2/min.

6. (b) $A = \pi r^2$.

(c) $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$.

(d) Find $\left.\frac{dA}{dt}\right|_{r=5}$ given that $\left.\frac{dr}{dt}\right|_{r=5} = 2$. From part (c), $\left.\frac{dA}{dt}\right|_{r=5} = 2\pi(5)(2) = 20\pi$ cm^2/s.

7. (a) $V = \pi r^2 h$, so $\frac{dV}{dt} = \pi\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$.

(b) Find $\left.\frac{dV}{dt}\right|_{h=6,\, r=10}$ given that $\left.\frac{dh}{dt}\right|_{h=6,\, r=10} = 1$ and $\left.\frac{dr}{dt}\right|_{h=6,\, r=10} = -1$. From part (a), $\left.\frac{dV}{dt}\right|_{h=6,\, r=10} = \pi[10^2(1)+2(10)(6)(-1)] = -20\pi$ in^3/s; the volume is decreasing.

8. (a) $\ell^2 = x^2 + y^2$, so $\frac{d\ell}{dt} = \frac{1}{\ell}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$.

(b) Find $\left.\frac{d\ell}{dt}\right|_{x=3,\, y=4}$ given that $\frac{dx}{dt} = \frac{1}{2}$ and $\frac{dy}{dt} = -\frac{1}{4}$. From part (a) and the fact that $\ell = 5$ when $x = 3$ and $y = 4$, $\left.\frac{d\ell}{dt}\right|_{x=3,\, y=4} = \frac{1}{5}\left(3\cdot\frac{1}{2} + 4\cdot\left(-\frac{1}{4}\right)\right) = \frac{1}{10}$ ft/s; the diagonal is increasing.

9. (a) $\tan\theta = \frac{y}{x}$, so $\sec^2\theta\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$, $\frac{d\theta}{dt} = \frac{\cos^2\theta}{x^2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$.

(b) Find $\left.\frac{d\theta}{dt}\right|_{x=2,\, y=2}$ given that $\left.\frac{dx}{dt}\right|_{x=2,\, y=2} = 1$ and $\left.\frac{dy}{dt}\right|_{x=2,\, y=2} = -\frac{1}{4}$. When $x = 2$ and $y = 2$, $\tan\theta = 2/2 = 1$ so $\theta = \frac{\pi}{4}$ and $\cos\theta = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Thus from part (a), $\left.\frac{d\theta}{dt}\right|_{x=2,\, y=2} = \frac{(1/\sqrt{2})^2}{2^2}\left[2\left(-\frac{1}{4}\right) - 2(1)\right] = -\frac{5}{16}$ rad/s; θ is decreasing.

10. Find $\left.\frac{dz}{dt}\right|_{x=1,\, y=2}$ given that $\left.\frac{dx}{dt}\right|_{x=1,\, y=2} = -2$ and $\left.\frac{dy}{dt}\right|_{x=1,\, y=2} = 3$. $\frac{dz}{dt} = 2x^3y\frac{dy}{dt} + 3x^2y^2\frac{dx}{dt}$, $\left.\frac{dz}{dt}\right|_{x=1,\, y=2} = (4)(3) + (12)(-2) = -12$ units/s; z is decreasing.

11. Let A be the area swept out, and θ the angle through which the minute hand has rotated. Find $\frac{dA}{dt}$ given that $\frac{d\theta}{dt} = \frac{\pi}{30}$ rad/min; $A = \frac{1}{2}r^2\theta = 8\theta$, so $\frac{dA}{dt} = 8\frac{d\theta}{dt} = \frac{4\pi}{15}$ in^2/min.

12. Let r be the radius and A the area enclosed by the ripple. We want $\left.\frac{dA}{dt}\right|_{t=10}$ given that $\frac{dr}{dt} = 3$. We know that $A = \pi r^2$, so $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$. Because r is increasing at the constant rate of 3 ft/s, it follows that $r = 30$ ft after 10

dt

dA

2

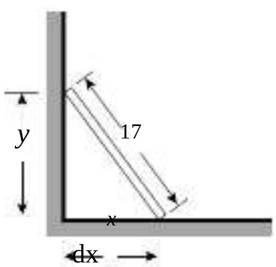
seconds so $\frac{dA}{dt}\Big|_{t=10} = 2\pi(30)(3) = 180\pi$ ft^2/s.

13. Find $\left.\frac{dr}{dt}\right|_{A=9}$ given that $\frac{dA}{dt}=6$. From $A=\pi r^2$ we get $\frac{dA}{dt}=2\pi r\frac{dr}{dt}$ so $\frac{dr}{dt}=\frac{1}{2\pi r}\frac{dA}{dt}$. If $A=9$ then $\pi r^2=9$, $r=3/\sqrt{\pi}$ so $\left.\frac{dr}{dt}\right|_{A=9}=\frac{1}{2\pi(3/\sqrt{\pi})}(6)=1/\sqrt{\pi}$ mi/h.

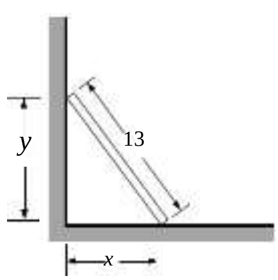
14. The volume V of a sphere of radius r is given by $V=\frac{4}{3}\pi r^3$ or, because $r=\frac{D}{2}$ where D is the diameter, $V=\frac{4}{3}\pi\left(\frac{D}{2}\right)^3=\frac{1}{6}\pi D^3$. We want $\left.\frac{dD}{dt}\right|_{r=1}$ given that $\frac{dV}{dt}=3$. From $V=\frac{1}{6}\pi D^3$ we get $\frac{dV}{dt}=\frac{1}{2}\pi D^2\frac{dD}{dt}$, $\frac{dD}{dt}=\frac{2}{\pi D^2}\frac{dV}{dt}$, so $\left.\frac{dD}{dt}\right|_{r=1}=\frac{2}{\pi(2)^2}(3)=\frac{3}{2\pi}$ ft/min.

15. Find $\left.\frac{dV}{dt}\right|_{r=9}$ given that $\frac{dr}{dt}=-15$. From $V=\frac{4}{3}\pi r^3$ we get $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$ so $\left.\frac{dV}{dt}\right|_{r=9}=4\pi(9)^2(-15)=-4860\pi$. Air must be removed at the rate of 4860π cm^3/min.

16. Let x and y be the distances shown in the diagram. We want to find $\left.\frac{dy}{dt}\right|_{y=8}$ given that $\frac{dx}{dt}=5$. From $x^2+y^2=17^2$ we get $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$, so $\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}$. When $y=8$, $x^2+8^2=17^2$, $x^2=289-64=225$, $x=15$ so $\left.\frac{dy}{dt}\right|_{y=8}=-\frac{15}{8}(5)=-\frac{75}{8}$ ft/s; the top of the ladder is moving down the wall at a rate of 75/8 ft/s.



17. Find $\left.\frac{dx}{dt}\right|_{y=5}$ given that $\frac{dy}{dt}=-2$. From $x^2+y^2=13^2$ we get $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$ so $\frac{dx}{dt}=-\frac{y}{x}\frac{dy}{dt}$. Use $x^2+y^2=169$ to find that $x=12$ when $y=5$ so $\left.\frac{dx}{dt}\right|_{y=5}=-\frac{5}{12}(-2)=\frac{5}{6}$ ft/s.



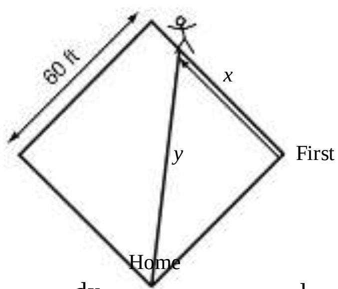
18. Let θ be the acute angle, and x the distance of the bottom of the plank from the wall. Find $\left.\frac{d\theta}{dt}\right|_{x=2}$ given that $\left.\frac{dx}{dt}\right|_{x=2}=-\frac{1}{2}$ ft/s. The variables θ and x are related by the equation $\cos\theta=\frac{x}{10}$ so $-\sin\theta\frac{d\theta}{dt}=\frac{1}{10}\frac{dx}{dt}$, $\frac{d\theta}{dt}=-$

$\frac{1}{10 \sin \theta} \frac{dx}{dt}$. When $x = 2$, the top of the plank is $\sqrt{10^2 - 2^2} = \sqrt{96}$ ft above the ground so $\sin \theta = \sqrt{96}/10$

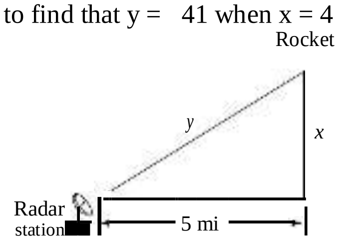
and $\left.\frac{d\theta}{dt}\right|_{x=2} = -\frac{1}{\sqrt{96}} \left(-\frac{1}{2}\right) = \frac{1}{2\sqrt{96}} \approx 0.051$ rad/s.

19. Let x denote the distance from first base and y the distance from home plate. Then $x^2 + 60^2 = y^2$ and $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$.

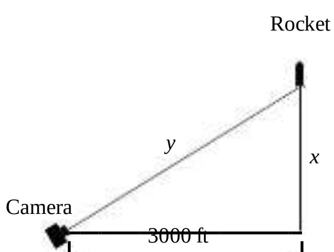
When $x = 50$ then $y = 10\sqrt{61}$ so $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} = \frac{50}{10\sqrt{61}}(25) = \frac{125}{\sqrt{61}}$ ft/s.



20. Find $\left.\frac{dx}{dt}\right|_{x=4}$ given that $\left.\frac{dy}{dt}\right|_{x=4} = 2000$. From $x^2 + 5^2 = y^2$ we get $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$ so $\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt}$. Use $x^2 + 25 = y^2$ to find that $y = \sqrt{41}$ when $x = 4$ so $\left.\frac{dx}{dt}\right|_{x=4} = \frac{\sqrt{41}}{4}(2000) = 500\sqrt{41}$ mi/h.



21. Find $\left.\frac{dy}{dt}\right|_{x=4000}$ given that $\left.\frac{dx}{dt}\right|_{x=4000} = 880$. From $y^2 = x^2 + 3000^2$ we get $2y\frac{dy}{dt} = 2x\frac{dx}{dt}$ so $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$. If $x = 4000$, then $y = 5000$ so $\left.\frac{dy}{dt}\right|_{x=4000} = \frac{4000}{5000}(880) = 704$ ft/s.



22. Find $\left.\frac{dx}{dt}\right|_{\varphi=\pi/4}$ given that $\left.\frac{d\varphi}{dt}\right|_{\varphi=\pi/4} = 0.2$. But $x = 3000\tan\varphi$ so $\frac{dx}{dt} = 3000(\sec^2\varphi)\frac{d\varphi}{dt}$, $\left.\frac{dx}{dt}\right|_{\varphi=\pi/4} = 3000\sec^2\frac{\pi}{4}(0.2) = 1200$ ft/s.

23. (a) If x denotes the altitude, then $r - x = 3960$, the radius of the Earth. $\theta = 0$ at perigee, so $r = 4995/1.12 \approx 4460$; the altitude is $x = 4460 - 3960 = 500$ miles. $\theta = \pi$ at apogee, so $r = 4995/0.88 \approx 5676$; the altitude is $x = 5676 - 3960 = 1716$ miles.

(b) If $\theta = 120^\circ$, then $r = 4995/0.94 \approx 5314$; the altitude is $5314 - 3960 = 1354$ miles. The rate of change of the altitude is given by $\frac{dx}{dt} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{4995(0.12\sin\theta)}{(1 + 0.12\cos\theta)^2}\frac{d\theta}{dt}$. Use $\theta = 120^\circ$ and $d\theta/dt = 2.7^\circ/\text{min} = (2.7)(\pi/180)$ rad/min to get $dr/dt \approx 27.7$ mi/min.

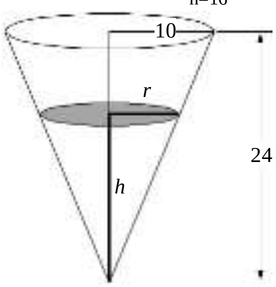
24. (a) Let x be the horizontal distance shown in the figure. Then $x = 4000 \cot \theta$ and $\frac{dx}{dt} = 4000 \csc^2 \theta \frac{d\theta}{dt}$, so

$\frac{d\theta}{dt} = -\frac{\sin^2 \theta}{4000}\frac{dx}{dt}$. Use $\theta = 30^\circ$ and $dx/dt = 300$ mi/h $= 300(5280/3600)$ ft/s $= 440$ ft/s to get $d\theta/dt =$

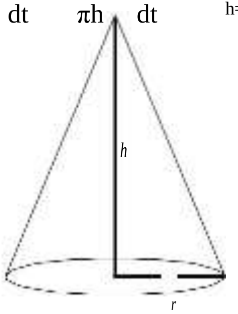
-0.0275 rad/s $\approx -1.6^\circ$/s; θ is decreasing at the rate of 1.6°/s.

(b) Let y be the distance between the observation point and the aircraft. Then $y = 4000 \csc\theta$ so $dy/dt = -4000(\csc\theta\cot\theta)(d\theta/dt)$. Use $\theta = 30^\circ$ and $d\theta/dt = -0.0275$ rad/s to get $dy/dt \approx 381$ ft/s.

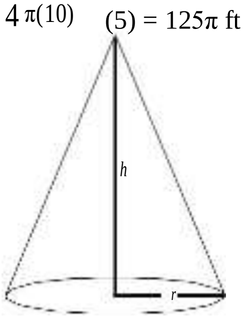
25. Find $\left.\frac{dh}{dt}\right|_{h=16}$ given that $\frac{dV}{dt} = 20$. The volume of water in the tank at a depth h is $V = \frac{1}{3}\pi r^2 h$. Use similar triangles (see figure) to get $\frac{r}{h} = \frac{10}{24}$ so $r = \frac{5}{12}h$ thus $V = \frac{1}{3}\pi\left(\frac{5}{12}h\right)^2 h = \frac{25}{432}\pi h^3$, $\frac{dV}{dt} = \frac{25}{144}\pi h^2 \frac{dh}{dt}$; $\frac{dh}{dt} = \frac{144}{25\pi h^2}\frac{dV}{dt}$, $\left.\frac{dh}{dt}\right|_{h=16} = \frac{144}{25\pi(16)^2}(20) = \frac{9}{20\pi}$ ft/min.



26. Find $\left.\frac{dh}{dt}\right|_{h=6}$ given that $\frac{dV}{dt} = 8$. $V = \frac{1}{3}\pi r^2 h$, but $r = \frac{1}{2}h$ so $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$, $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$, $\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$, $\left.\frac{dh}{dt}\right|_{h=6} = \frac{4}{\pi(6)^2}(8) = \frac{8}{9\pi}$ ft/min.



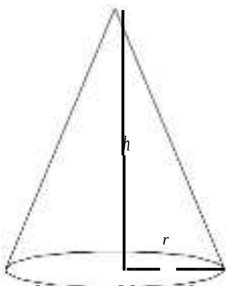
27. Find $\left.\frac{dV}{dt}\right|_{h=10}$ given that $\frac{dh}{dt} = 5$. $V = \frac{1}{3}\pi r^2 h$, but $r = \frac{1}{2}h$ so $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$, $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$, $\left.\frac{dV}{dt}\right|_{h=10} = \frac{1}{4}\pi(10)^2(5) = 125\pi$ ft^3/min.



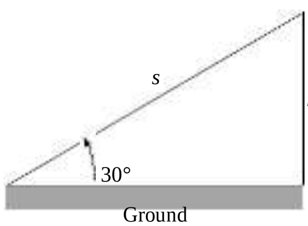
28. Let r and h be as shown in the figure. If C is the circumference of the base, then we want to find $\left.\frac{dC}{dt}\right|_{h=8}$ given that $\frac{dV}{dt} = 10$. It is given that $r = \frac{1}{2}h$, thus $C = 2\pi r = \pi h$ so $\frac{dC}{dt} = \pi\frac{dh}{dt}$. Use $V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3$ to get

$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$. Substitution of $\frac{dh}{dt}$ into $\frac{dC}{dt}$ gives $\frac{dC}{dt} = \frac{4}{h^2}\frac{dV}{dt}$ so $\left.\frac{dC}{dt}\right|_{h=8} = \frac{4}{64}(10) = \frac{5}{8}$

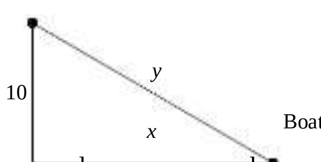
ft/min.



29. With s and h as shown in the figure, we want to find $\frac{dh}{dt}$ given that $\frac{ds}{dt} = 500$. From the figure, $h = s\sin 30^\circ = \frac{1}{2}s$ so $\frac{dh}{dt} = \frac{1}{2}\frac{ds}{dt} = \frac{1}{2}(500) = 250$ mi/h.



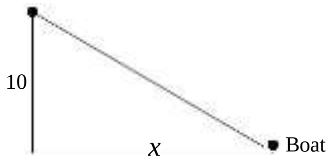
30. Find $\left.\frac{dx}{dt}\right|_{y=125}$ given that $\frac{dy}{dt} = -20$. From $x^2 + 10^2 = y^2$ we get $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$ so $\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt}$. Use $x^2 + 100 = y^2$ to find that $x = \sqrt{15,525} = 15\sqrt{69}$ when $y = 125$ so $\left.\frac{dx}{dt}\right|_{y=125} = \frac{125}{15\sqrt{69}}(-20) = -\frac{500}{3\sqrt{69}}$. The boat is approaching the dock at the rate of $\frac{500}{3\sqrt{69}}$ ft/min.



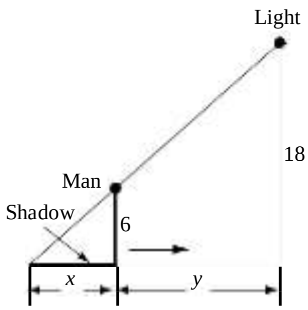
31. Find $\frac{dy}{dt}$ given that $\left.\frac{dx}{dt}\right|_{y=125} = 12$. From $x^2 + 10^2 = y^2$ we get $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$ so $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$. Use $x^2 + 100 = y^2$ to find that $x = \sqrt{15,525} = 15\sqrt{69}$ when $y = 125$ so $\frac{dy}{dt} = \frac{15\sqrt{69}}{125}(-12) = -\frac{36\sqrt{69}}{25}$. The rope must be pulled at the rate of $\frac{36\sqrt{69}}{25}$ ft/min.

Pulley

y

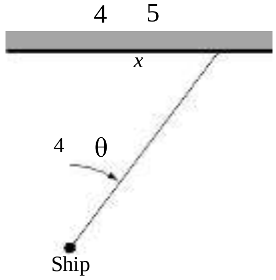


32. (a) Let x and y be as shown in the figure. It is required to find $\frac{dx}{dt}$, given that $\frac{dy}{dt} = -3$. By similar triangles, $\frac{x}{6} = \frac{x + y}{18}$, $18x = 6x + 6y$, $12x = 6y$, $x = \frac{1}{2}y$, so $\frac{dx}{dt} = \frac{1}{2}\frac{dy}{dt} = \frac{1}{2}(-3) = -\frac{3}{2}$ ft/s.



(b) The tip of the shadow is $z = x + y$ feet from the street light, thus the rate at which it is moving is given by $\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$. In part (a) we found that $\frac{dx}{dt} = -\frac{3}{2}$ when $\frac{dy}{dt} = -3$ so $\frac{dz}{dt} = (-3/2) + (-3) = -9/2$ ft/s; the tip of the shadow is moving at the rate of 9/2 ft/s toward the street light.

33. Find $\left.\frac{dx}{dt}\right|_{\theta=\pi/4}$ given that $\frac{d\theta}{dt} = \frac{2\pi}{10} = \frac{\pi}{5}$ rad/s. Then $x = 4\tan\theta$ (see figure) so $\frac{dx}{dt} = 4\sec^2\theta\,\frac{d\theta}{dt}$, $\left.\frac{dx}{dt}\right|_{\theta=\pi/4} = 4\sec^2\left(\frac{\pi}{4}\right)\left(\frac{\pi}{5}\right) = 8\pi/5$ km/s.



34. If x, y, and z are as shown in the figure, then we want $\left.\frac{dz}{dt}\right|_{x=2,\,y=4}$ given that $\frac{dx}{dt} = -600$ and $\left.\frac{dy}{dt}\right|_{x=2,\,y=4} = -1200$.

But $z^2 = x^2 + y^2$ so $2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$, $\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$. When $x = 2$ and $y = 4$, $z^2 = 2^2 + 4^2 = 20$, $z = \sqrt{20} = 2\sqrt{5}$ so $\left.\frac{dz}{dt}\right|_{x=2,\,y=4} = \frac{1}{2\sqrt{5}}[2(-600) + 4(-1200)] = -\frac{3000}{\sqrt{5}} = -600\sqrt{5}$ mi/h; the distance between missile and aircraft is decreasing at the rate of $600\sqrt{5}$ mi/h.

P x Aircraft

y
z

Missile

35. We wish to find $\left.\frac{dz}{dt}\right|_{x=2,\,y=4}$ given $\frac{dx}{dt} = -600$ and $\left.\frac{dy}{dt}\right|_{x=2,\,y=4} = -1200$ (see figure). From the law of cosines, $z^2 =$

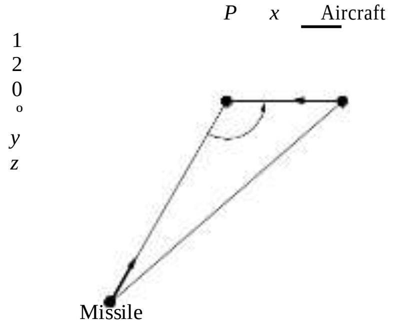
dz dx dy dy dx dz

$x^2 + y^2 - 2xy\cos 120^\circ = x^2 + y^2 - 2xy(-1/2) = x^2 + y^2 + xy$, so $2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + x\frac{dy}{dt} + y\frac{dx}{dt}$, $\frac{dz}{dt} = \frac{1}{2z}\left[(2x + y)\frac{dx}{dt} + (2y + x)\frac{dy}{dt}\right]$. When $x = 2$ and $y = 4$, $z^2 = 2^2 + 4^2 + (2)(4) = 28$, so $z = \sqrt{28} = 2\sqrt{7}$, thus $\frac{dz}{dt} = \frac{1}{4\sqrt{7}}[(2(2) + 4)(-600) + (2(4) + 2)(-1200)] = -\frac{4200}{\sqrt{7}} = -600\sqrt{7}$ mi/h; the distance between missile

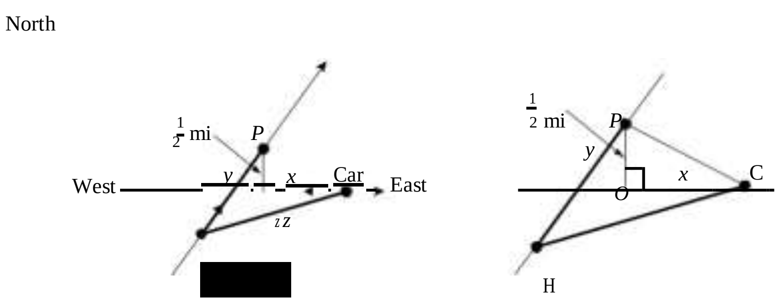
7)

y=4

and aircraft is decreasing at the rate of $600\sqrt{7}$ mi/h.



36. (a) Let P be the point on the helicopter's path that lies directly above the car's path. Let x, y, and z be the distances shown in the first figure. Find $\left.\frac{dz}{dt}\right|_{x=2,\,y=0}$ given that $\frac{dx}{dt} = -75$ and $\frac{dy}{dt} = 100$. In order to find an equation relating x, y, and z, first draw the line segment that joins the point P to the car, as shown in the second figure. Because triangle OPC is a right triangle, it follows that PC has length $\sqrt{x^2 + (1/2)^2}$; but triangle HPC is also a right triangle so $z^2 = \left(\sqrt{x^2 + (1/2)^2}\right)^2 + y^2 = x^2 + y^2 + 1/4$ and $2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 0$, $\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$. Now, when x = 2 and y = 0, $z^2 = (2)^2 + (0)^2 + 1/4 = 17/4$, $z = \sqrt{17}/2$ so $\left.\frac{dz}{dt}\right|_{x=2,\,y=0} = \frac{1}{(\sqrt{17}/2)}[2(-75) + 0(100)] = -300/\sqrt{17}$ mi/h.



(b) Decreasing, because $\frac{dz}{dt} < 0$.

37. (a) We want $\left.\frac{dy}{dt}\right|_{x=1,\,y=2}$ given that $\left.\frac{dx}{dt}\right|_{x=1,\,y=2} = 6$. For convenience, first rewrite the equation as $xy^3 = \frac{8}{5} + \frac{8}{5}y^2$ then $3xy^2\frac{dy}{dt} + y^3\frac{dx}{dt} = \frac{16}{5}y\frac{dy}{dt}$, $\frac{dy}{dt} = \frac{y^3}{\frac{16}{5}y - 3xy^2}\frac{dx}{dt}$, so $\left.\frac{dy}{dt}\right|_{x=1,\,y=2} = \frac{2^3}{\frac{16}{5}(2) - 3(1)2^2}(6) = -60/7$ units/s.

(b) Falling, because $\frac{dy}{dt} < 0$.

38. Find $\left.\frac{dx}{dt}\right|_{(2,5)}$ given that $\left.\frac{dy}{dt}\right|_{(2,5)} = 2$. Square and rearrange to get $x^3 = y^2 - 17$, so $3x^2\frac{dx}{dt} = 2y\frac{dy}{dt}$, $\frac{dx}{dt} = \frac{2y}{3x^2}\frac{dy}{dt}$, $\left.\frac{dx}{dt}\right|_{(2,5)} = \frac{5}{6}(2) = \frac{5}{3}$ units/s.

39. The coordinates of P are (x, 2x), so the distance between P and the point (3, 0) is $D = \sqrt{(x-3)^2 + (2x-0)^2} =$

$\sqrt{5x^2 - 6x + 9}$. Find $\left.\frac{dD}{dt}\right|_{x=3}$ given that $\left.\frac{dx}{dt}\right|_{x=3} = -2$. $\frac{dD}{dt} = \frac{5x - 3}{\sqrt{5x^2 - 6x + 9}}\frac{dx}{dt}$, so $\left.\frac{dD}{dt}\right|_{x=3} = \frac{12}{\sqrt{36}}(-2) = -4$ units/s.

40. (a) Let D be the distance between P and (2, 0). Find $\frac{dD}{dt}$

p

$(x-2)^2 + x^2$ $x^2 - 3x$ + 4, so $\frac{dD}{dt}$ =

given that $\frac{dx}{dt}\Big|_{x=3} = 4$. $D = \sqrt{(x-2)^2 + y^2} =$ ______

$x=3$

$\frac{dD}{}$

x=3 = 3 √ 4 = 3 units/s.
dt 2 x^2 − 3x + 4 dt dt 2 4

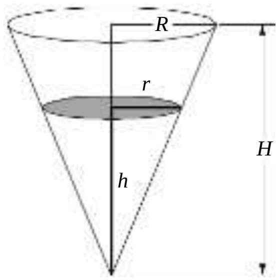
(b) Let θ be the angle of inclination. Find $\left.\frac{d\theta}{dt}\right|_{x=3}$ given that $\left.\frac{dx}{dt}\right|_{x=3} = 4$. $\tan\theta = \frac{y}{x-2} = \frac{\sqrt{x}}{x-2}$, so $\sec^2\theta \frac{d\theta}{dt} = -\frac{x+2}{2\sqrt{x}(x-2)^2}\frac{dx}{dt}$, $\frac{d\theta}{dt} = -\cos^2\theta \frac{x+2}{2\sqrt{x}(x-2)^2}\frac{dx}{dt}$. When $x = 3$, $D = 2$ so $\cos\theta = \frac{1}{2}$ and $\left.\frac{d\theta}{dt}\right|_{x=3} = -\frac{1}{4}\frac{5}{2\sqrt{3}}(4) = -\frac{5}{2\sqrt{3}}$ rad/s.

41. Solve $\frac{dx}{dt} = 3\frac{dy}{dt}$ given $y = x/(x^2+1)$. Then $y(x^2+1) = x$. Differentiating with respect to x, $(x^2+1)\frac{dy}{dx} + y(2x) = 1$. But $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{3}$ so $(x^2+1)\frac{1}{3} + 2xy = 1$, $x^2+1+6xy = 3$, $x^2+1+6x^2/(x^2+1) = 3$, $(x^2+1)^2 + 6x^2 - 3x^2 - 3 = 0$, $x^4 + 5x^2 - 2 = 0$. By the quadratic formula applied to x^2 we obtain $x^2 = (-5 \pm \sqrt{25+8})/2$. The minus sign is spurious since x^2 cannot be negative, so $x^2 = (-5+\sqrt{33})/2$, and $x = \pm\sqrt{(-5+\sqrt{33})/2}$.

42. $32x\frac{dx}{dt} + 18y\frac{dy}{dt} = 0$; if $\frac{dy}{dt} = \frac{dx}{dt} \neq 0$, then $(32x+18y)\frac{dx}{dt} = 0$, $32x + 18y = 0$, $y = -\frac{16}{9}x$, so $16x^2 + 9\frac{256}{81}x^2 = 144$, $\frac{400}{9}x^2 = 144$, $x^2 = \frac{81}{25}$, $x = \pm\frac{9}{5}$. If $x = \frac{9}{5}$, then $y = -\frac{16}{9}\frac{9}{5} = -\frac{16}{5}$. Similarly, if $x = -\frac{9}{5}$, then $y = \frac{16}{5}$. The points are $\left(\frac{9}{5}, -\frac{16}{5}\right)$ and $\left(-\frac{9}{5}, \frac{16}{5}\right)$.

43. Find $\left.\frac{dS}{dt}\right|_{s=10}$ given that $\left.\frac{ds}{dt}\right|_{s=10} = -2$. From $\frac{1}{s} + \frac{1}{S} = \frac{1}{6}$ we get $-\frac{1}{s^2}\frac{ds}{dt} - \frac{1}{S^2}\frac{dS}{dt} = 0$, so $\frac{dS}{dt} = -\frac{S^2}{s^2}\frac{ds}{dt}$. If $s = 10$, then $\frac{1}{10} + \frac{1}{S} = \frac{1}{6}$ which gives $S = 15$. So $\left.\frac{dS}{dt}\right|_{s=10} = -\frac{225}{100}(-2) = 4.5$ cm/s. The image is moving away from the lens.

44. Suppose that the reservoir has height H and that the radius at the top is R. At any instant of time let h and r be the corresponding dimensions of the cone of water (see figure). We want to show that $\frac{dh}{dt}$ is constant and independent of H and R, given that $\frac{dV}{dt} = -kA$ where V is the volume of water, A is the area of a circle of radius r, and k is a positive constant. The volume of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$. By similar triangles $\frac{r}{h} = \frac{R}{H}$, $r = \frac{R}{H}h$ thus $V = \frac{1}{3}\pi\left(\frac{R}{H}\right)^2 h^3$, so $\frac{dV}{dt} = \pi\left(\frac{R}{H}\right)^2 h^2\frac{dh}{dt}$. But it is given that $\frac{dV}{dt} = -kA$ or, because $A = \pi r^2 = \pi\left(\frac{R}{H}\right)^2 h^2$, $\frac{dV}{dt} = -k\pi\left(\frac{R}{H}\right)^2 h^2$, which when substituted into the previous equation for $\frac{dV}{dt}$ gives $-k\pi\left(\frac{R}{H}\right)^2 h^2 = \pi\left(\frac{R}{H}\right)^2 h^2\frac{dh}{dt}$, and $\frac{dh}{dt} = k$.



45. Let r be the radius, V the volume, and A the surface area of a sphere. Show that $\frac{dr}{dt}$ is a constant given that $\frac{dV}{dt} = -kA$, where k is a positive constant. Because $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. But it is given that

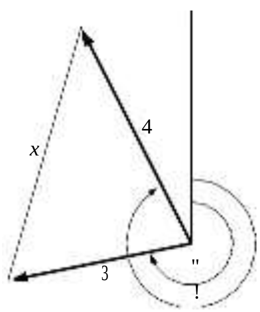
dV dV dV

$\overline{\quad}$ $= kA$ or, because $A = 4\pi r^2$, $\overline{dt}$ $=$ $4\pi r^2 k$ which when substituted into the previous equation for $\overline{\quad}$ gives
dt dt

$-4\pi r^2 k = 4\pi r^2 \dfrac{dr}{dt}$, and $\dfrac{dr}{dt} = -k$.

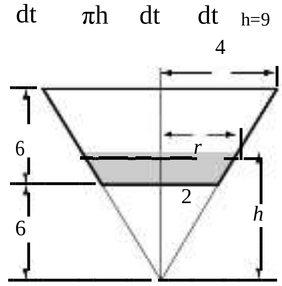
46. Let x be the distance between the tips of the minute and hour hands, and α and β the angles shown in the figure. Because the minute hand makes one revolution in 60 minutes, $\dfrac{d\alpha}{dt} = \dfrac{2\pi}{60} = \pi/30$ rad/min; the hour hand makes one revolution in 12 hours (720 minutes), thus $\dfrac{d\beta}{dt} = \dfrac{2\pi}{720} = \pi/360$ rad/min. We want to find $\left.\dfrac{dx}{dt}\right|_{\alpha=2\pi,\ \beta=3\pi/2}$ given that $\dfrac{d\alpha}{dt} = \pi/30$ and $\dfrac{d\beta}{dt} = \pi/360$. Using the law of cosines on the triangle shown in the figure, $x^2 = 3^2 + 4^2 - 2(3)(4)\cos(\alpha - \beta) = 25 - 24\cos(\alpha - \beta)$, so $2x\dfrac{dx}{dt} = 0 + 24\sin(\alpha - \beta)\left(\dfrac{d\alpha}{dt} - \dfrac{d\beta}{dt}\right)$, $\dfrac{dx}{dt} = \dfrac{12}{x}\left(\dfrac{d\alpha}{dt} - \dfrac{d\beta}{dt}\right)\sin(\alpha - \beta)$. When $\alpha = 2\pi$ and $\beta = 3\pi/2$, $x^2 = 25 - 24\cos(2\pi - 3\pi/2) = 25$, $x = 5$; so

$$\left.\frac{dx}{dt}\right|_{\alpha=2\pi,\ \beta=3\pi/2} = \frac{12}{5}(\pi/30 - \pi/360)\sin(2\pi - 3\pi/2) = \frac{11\pi}{150} \text{ in/min.}$$



47. Extend sides of cup to complete the cone and let V_0 be the volume of the portion added, then (see figure) $V = \dfrac{1}{3}\pi r^2 h - V_0$ where $\dfrac{r}{h} = \dfrac{4}{12} = \dfrac{1}{3}$ so $r = \dfrac{1}{3}h$ and $V = \dfrac{1}{3}\pi\left(\dfrac{h}{3}\right)^2 h - V_0 = \dfrac{1}{27}\pi h^3 - V_0$, $\dfrac{dV}{dt} = \dfrac{1}{9}\pi h^2 \dfrac{dh}{dt}$,

$\dfrac{dh}{dt} = \dfrac{9}{\pi h^2}\dfrac{dV}{dt}$, $\left.\dfrac{dh}{dt}\right|_{h=9} = \dfrac{9}{\pi(9)^2}(20) = \dfrac{20}{9\pi}$ cm/s.



Exercise Set 2.9

1. (a) $f(x) \approx f(1) + f'(1)(x - 1) = 1 + 3(x - 1)$.

   (b) $f(1 + \Delta x) \approx f(1) + f'(1)\Delta x = 1 + 3\Delta x$.

(c) From part (a), $(1.02)^3 \approx 1 + 3(0.02) = 1.06$. From part (b), $(1.02)^3 \approx 1 + 3(0.02) = 1.06$.

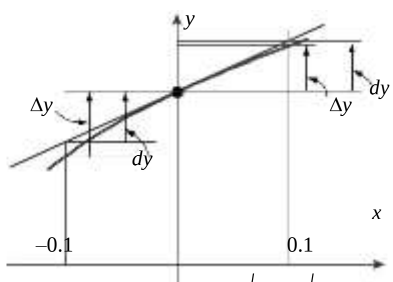
2. (a) $f(x) \approx f(2) + f'(2)(x - 2) = 1/2 + (-1/2^2)(x - 2) = (1/2) - (1/4)(x - 2)$.

   (b) $f(2 + \Delta x) \approx f(2) + f'(2)\Delta x = 1/2 - (1/4)\Delta x$.

   (c) From part (a), $1/2.05 \approx 0.5 - 0.25(0.05) = 0.4875$, and from part (b), $1/2.05 \approx 0.5 - 0.25(0.05) = 0.4875$.

3. (a) $f(x) \approx f(x_0) + f'(x_0)(x - x_0) = 1 + (1/(2\sqrt{1}))(x - 0) = 1 + (1/2)x$, so with $x_0 = 0$ and $x = -0.1$, we have $\sqrt{0.9} = f(-0.1) \approx 1 + (1/2)(-0.1) = 1 - 0.05 = 0.95$. With $x = 0.1$ we have $\sqrt{1.1} = f(0.1) \approx 1 + (1/2)(0.1) = 1.05$.

(b)



4. (b) The approximation is $\sqrt{x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}(x - x_0)$, so show that $\sqrt{x_0} + \frac{1}{2\sqrt{x_0}}(x - x_0) \geq \sqrt{x}$ which is equivalent to $g(x) = \sqrt{x} - \frac{x}{2\sqrt{x_0}} \leq \frac{\sqrt{x_0}}{2}$. But $g(x_0) = \frac{\sqrt{x_0}}{2}$, and $g'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x_0}}$ which is negative for $x > x_0$ and positive for $x < x_0$. This shows that g has a maximum value at $x = x_0$, so the student's observation is correct.

5. $f(x) = (1 + x)^{15}$ and $x_0 = 0$. Thus $(1 + x)^{15} \approx f(x_0) + f'(x_0)(x - x_0) = 1 + 15(1)^{14}(x - 0) = 1 + 15x$.

6. $f(x) = \frac{1}{\sqrt{1 - x}}$ and $x_0 = 0$, so $\frac{1}{\sqrt{1 - x}} \approx f(x_0) + f'(x_0)(x - x_0) = 1 + \frac{1}{2(1 - 0)^{3/2}}(x - 0) = 1 + x/2$.

7. $\tan x \approx \tan(0) + \sec^2(0)(x - 0) = x$.

8. $\frac{1}{1 + x} \approx 1 + \frac{-1}{(1 + 0)^2}(x - 0) = 1 - x$.

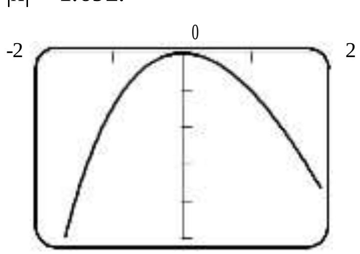
9. $x^4 \approx (1)^4 + 4(1)^3(x - 1)$. Set $\Delta x = x - 1$; then $x = \Delta x + 1$ and $(1 + \Delta x)^4 = 1 + 4\Delta x$.

10. $\sqrt{x} \approx \sqrt{1} + \frac{1}{2\sqrt{1}}(x - 1)$, and $x = 1 + \Delta x$, so $\sqrt{1 + \Delta x} \approx 1 + \Delta x/2$.

11. $\frac{1}{2 + x} \approx \frac{1}{2 + 1} - \frac{1}{(2 + 1)^2}(x - 1)$, and $2 + x = 3 + \Delta x$, so $\frac{1}{3 + \Delta x} \approx \frac{1}{3} - \frac{1}{9}\Delta x$.

12. $(4 + x)^3 \approx (4 + 1)^3 + 3(4 + 1)^2(x - 1)$ so, with $4 + x = 5 + \Delta x$ we get $(5 + \Delta x)^3 \approx 125 + 75\Delta x$.

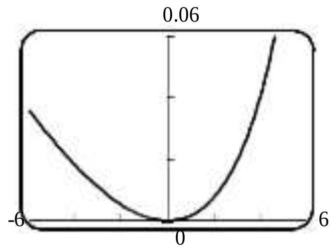
13. $f(x) = \sqrt{x + 3}$ and $x_0 = 0$, so $\sqrt{x + 3} \approx \sqrt{3} + \frac{1}{2\sqrt{3}}(x - 0) = \sqrt{3} + \frac{1}{2\sqrt{3}}x$, and $\left|f(x) - \left(\sqrt{3} + \frac{1}{2\sqrt{3}}x\right)\right| < 0.1$ if $|x| < 1.692$.



$\left| f(x) - \left(\sqrt{3} + \frac{1}{2\sqrt{3}}x\right)\right|$

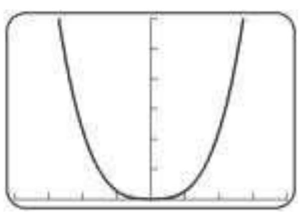
1 1 1 1 1 1 1 1

14. $f(x) = \frac{1}{\sqrt{9-x}}$ so $f(x) \approx \frac{1}{\sqrt{9}} + \frac{1}{2(9-0)^{3/2}}(x-0) = \frac{1}{3} + \frac{1}{54}x$, and $\left| f(x) - \left(\frac{1}{3} + \frac{1}{54}x\right) \right| < 0.1$ if $|x| < 5.5114$.



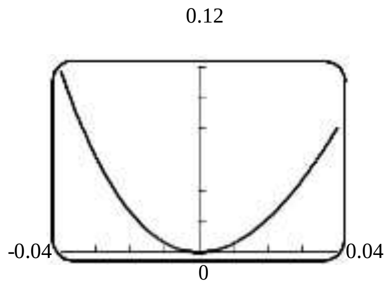
$\left| f(x) - \left(\frac{1}{3} + \frac{1}{54}x\right) \right|$

15. $\tan 2x \approx \tan 0 + (\sec^2 0)(2x - 0) = 2x$, and $|\tan 2x - 2x| < 0.1$ if $|x| < 0.3158$.



$|f(x) - 2x|$

16. $\dfrac{1}{(1+2x)^5} \approx \dfrac{1}{(1+2\cdot 0)^5} + \dfrac{-5(2)}{(1+2\cdot 0)^6}(x - 0) = 1 - 10x$, and $|f(x) - (1 - 10x)| < 0.1$.



$|f(x) - (1 - 10x)|$

17. (a) The local linear approximation $\sin x \approx x$ gives $\sin 1^\circ = \sin(\pi/180) \approx \pi/180 = 0.0174533$ and a calculator gives $\sin 1^\circ = 0.0174524$. The relative error $|\sin(\pi/180) - (\pi/180)|/(\sin \pi/180) = 0.000051$ is very small, so for such a small value of x the approximation is very good.

(b) Use $x_0 = 45^\circ$ (this assumes you know, or can approximate, $\sqrt{2}/2$).

(c) $44^\circ = \dfrac{44\pi}{180}$ radians, and $45^\circ = \dfrac{45\pi}{180} = \dfrac{\pi}{4}$ radians. With $x = \dfrac{44\pi}{180}$ and $x_0 = \dfrac{\pi}{4}$ we obtain $\sin 44^\circ = \sin\dfrac{44\pi}{180} \approx \sin\dfrac{\pi}{4} + \cos\dfrac{\pi}{4}\left(\dfrac{44\pi}{180} - \dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\left(\dfrac{-\pi}{180}\right) = 0.694765$. With a calculator, $\sin 44^\circ = 0.694658$.

18. (a) $\tan x \approx \tan 0 + \sec^2 0(x - 0) = x$, so $\tan 2^\circ = \tan(2\pi/180) \approx 2\pi/180 = 0.034907$, and with a calculator $\tan 2^\circ = 0.034921$.

(b) Use $x_0 = \pi/3$ because we know $\tan 60^\circ = \tan(\pi/3) = \sqrt{3}$.

(c) With $x_0 = \dfrac{\pi}{3} = \dfrac{60\pi}{180}$ and $x = \dfrac{61\pi}{180}$ we have $\tan 61^\circ = \tan\dfrac{61\pi}{180} \approx \tan\dfrac{\pi}{3} + \sec^2\dfrac{\pi}{3}\left(\dfrac{61\pi}{180} - \dfrac{\pi}{3}\right) = \sqrt{3} + \dfrac{4\pi}{180} = 1.8019$, and with a calculator $\tan 61^\circ = 1.8040$.

19. $f(x) = x^4$, $f'(x) = 4x^3$, $x_0 = 3$, $\Delta x = 0.02$; $(3.02)^4 \approx 3^4 + (108)(0.02) = 81 + 2.16 = 83.16$.

20. $f(x) = x^3$, $f'(x) = 3x^2$, $x_0 = 2$, $\Delta x = -0.03$; $(1.97)^3 \approx 2^3 + (12)(-0.03) = 8 - 0.36 = 7.64$.

21. $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$, $x_0 = 64$, $\Delta x = 1$; $\sqrt{65} \approx \sqrt{64} + \frac{1}{16}(1) = 8 + \frac{1}{16} = 8.0625$.

22. $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$, $x_0 = 25$, $\Delta x = -1$; $\sqrt{24} \approx \sqrt{25} + \frac{1}{10}(-1) = 5 - 0.1 = 4.9$.

23. $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$, $x_0 = 81$, $\Delta x = -0.1$; $\sqrt{80.9} \approx \sqrt{81} + \frac{1}{18}(-0.1) \approx 8.9944$.

24. $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$, $x_0 = 36$, $\Delta x = 0.03$; $\sqrt{36.03} \approx \sqrt{36} + \frac{1}{12}(0.03) = 6 + 0.0025 = 6.0025$.

25. $f(x) = \sin x$, $f'(x) = \cos x$, $x_0 = 0$, $\Delta x = 0.1$; $\sin 0.1 \approx \sin 0 + (\cos 0)(0.1) = 0.1$.

26. $f(x) = \tan x$, $f'(x) = \sec^2 x$, $x_0 = 0$, $\Delta x = 0.2$; $\tan 0.2 \approx \tan 0 + (\sec^2 0)(0.2) = 0.2$.

27. $f(x) = \cos x$, $f'(x) = -\sin x$, $x_0 = \pi/6$, $\Delta x = \pi/180$; $\cos 31^\circ \approx \cos 30^\circ + \left(-\frac{1}{2}\right)\left(\frac{\pi}{180}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{360} \approx 0.8573$.

28. (a) Let $f(x) = (1 + x)^k$ and $x_0 = 0$. Then $(1 + x)^k \approx 1^k + k(1)^{k-1}(x - 0) = 1 + kx$. Set $k = 37$ and $x = 0.001$ to obtain $(1.001)^{37} \approx 1.037$.

(b) With a calculator $(1.001)^{37} = 1.03767$.

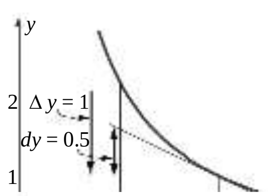
(c) It is the linear term of the expansion.

29. $\sqrt[3]{8.24} = 8^{1/3}\sqrt[3]{1.03} \approx 2(1 + \frac{1}{3}0.03) \approx 2.02$, and $4.08^{3/2} = 4^{3/2}\, 1.02^{3/2} = 8(1 + 0.02(3/2)) = 8.24$.

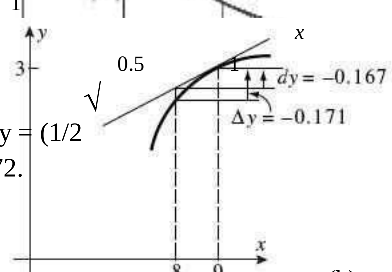
30. $6^\circ = \pi/30$ radians; $h = 500\tan(\pi/30) \approx 500[\tan 0 + (\sec^2 0)\frac{\pi}{30}] = 500\pi/30 \approx 52.36$ ft.

31. (a) $dy = (-1/x^2)dx = (-1)(-0.5) = 0.5$ and $\Delta y = 1/(x + \Delta x) - 1/x = 1/(1 - 0.5) - 1/1 = 2 - 1 = 1$.

(b)



32. (a) $dy = (1/2\sqrt{x})dx = (1/(2 \cdot 3))(-1) = -1/6 \approx -0.167$ and $\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{9 + (-1)} - \sqrt{9} = -0.172$.



(b)

$8{-}3 \approx$

33. $dy = 3x^2\,dx$; $\Delta y = (x + \Delta x)^3 - x^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$.

34. $dy = 8dx$; $\Delta y = [8(x + \Delta x) - 4] - [8x - 4] = 8\Delta x$.

35. $dy = (2x - 2)dx$; $\Delta y = [(x + \Delta x)^2 - 2(x + \Delta x) + 1] - [x^2 - 2x + 1] = x^2 + 2x\,\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1 = 2x\,\Delta x + (\Delta x)^2 - 2\Delta x$.

36. $dy = \cos x\,dx$; $\Delta y = \sin(x + \Delta x) - \sin x$.

37. (a) $dy = (12x^2 - 14x)dx$.

(b) $dy = x\,d(\cos x) + \cos x\,dx = x(-\sin x)dx + \cos x\,dx = (-x\sin x + \cos x)dx$.

38. (a) $dy = (-1/x^2)dx$.

(b) $dy = 5\sec^2 x\,dx$.

39. (a) $dy = \left(\sqrt{1-x} - \frac{x}{2\sqrt{1-x}}\right)dx = \frac{2-3x}{2\sqrt{1-x}}\,dx$.

(b) $dy = -17(1 + x)^{-18}\,dx$.

40. (a) $dy = \frac{(x^3-1)(0)-(1)(3x^2)}{(x^3-1)^2}\,dx = -\frac{3x^2}{(x^3-1)^2}\,dx$.

(b) $dy = \frac{(2-x)(-3x^2)dx - (1-x^3)(-1)dx}{(2-x)^2} = \frac{2x^3 - 6x^2 + 1}{(2-x)^2}\,dx$.

41. False; $dy = (dy/dx)dx$.

42. True.

43. False; they are equal whenever the function is linear.

44. False; if $f'(x_0) = 0$ then the approximation is constant.

45. $dy = \frac{3}{2\sqrt{3x-2}}\,dx$, $x = 2$, $dx = 0.03$; $\Delta y \approx dy = \frac{3}{4}(0.03) = 0.0225$.

46. $dy = \frac{x}{\sqrt{x^2+8}}\,dx$, $x = 1$, $dx = -0.03$; $\Delta y \approx dy = (1/3)(-0.03) = -0.01$.

47. $dy = \frac{1-x^2}{(x^2+1)^2}\,dx$, $x = 2$, $dx = -0.04$; $\Delta y \approx dy = -\frac{3}{25}(-0.04) = 0.0048$.

48. $dy = \left(\frac{4x}{\sqrt{8x+1}} + \sqrt{8x+1}\right)dx$, $x = 3$, $dx = 0.05$; $\Delta y \approx dy = (37/5)(0.05) = 0.37$.

49. (a) $A = x^2$ where x is the length of a side; $dA = 2x\,dx = 2(10)(\pm 0.1) = \pm 2\ \text{ft}^2$.

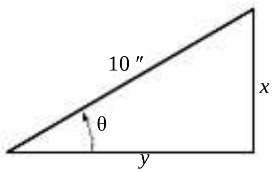
(b) Relative error in x is within $\frac{dx}{x} = \frac{\pm 0.1}{10} = \pm 0.01$ so percentage error in x is ±1%; relative error in A is within

$$\frac{dA}{A} = \frac{2x\,dx}{x^2} = 2\frac{dx}{x} = 2(\pm 0.01) = \pm 0.02$$ so percentage error in A is ±2%.

50. (a) $V = x^3$ where x is the length of a side; $dV = 3x^2\,dx = 3(25)^2(\pm1) = \pm1875$ cm^3.

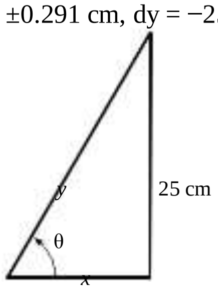
(b) Relative error in x is within $\frac{dx}{x} = \frac{\pm1}{25} = \pm0.04$ so percentage error in x is ±4%; relative error in V is within $\frac{dV}{V} = \frac{3x^2\,dx}{x^3} = 3\frac{dx}{x} = 3(\pm0.04) = \pm0.12$ so percentage error in V is ±12%.

51. (a) $x = 10\sin\theta$, $y = 10\cos\theta$ (see figure), $dx = 10\cos\theta\,d\theta = 10\cos\frac{\pi}{6}\left(\pm\frac{\pi}{180}\right) = 10\frac{\sqrt{3}}{2}\left(\pm\frac{\pi}{180}\right) \approx \pm0.151$ in, $dy = -10(\sin\theta)d\theta = -10\sin\frac{\pi}{6}\left(\pm\frac{\pi}{180}\right) = -10\frac{1}{2}\left(\pm\frac{\pi}{180}\right) \approx \pm0.087$ in.



(b) Relative error in x is within $\frac{dx}{x} = (\cot\theta)d\theta = \cot\frac{\pi}{6}\left(\pm\frac{\pi}{180}\right) = \sqrt{3}\left(\pm\frac{\pi}{180}\right) \approx \pm0.030$, so percentage error in x is ≈ ±3.0%; relative error in y is within $\frac{dy}{y} = -\tan\theta\,d\theta = -\tan\frac{\pi}{6}\left(\pm\frac{\pi}{180}\right) = -\frac{1}{\sqrt{3}}\left(\pm\frac{\pi}{180}\right) \approx \pm0.010$, so percentage error in y is ≈ ±1.0%.

52. (a) $x = 25\cot\theta$, $y = 25\csc\theta$ (see figure); $dx = -25\csc^2\theta\,d\theta = -25\csc^2\frac{\pi}{3}\left(\pm\frac{\pi}{360}\right) = -25\frac{4}{3}\left(\pm\frac{\pi}{360}\right) \approx \pm0.291$ cm, $dy = -25\csc\theta\cot\theta\,d\theta = -25\csc\frac{\pi}{3}\cot\frac{\pi}{3}\left(\pm\frac{\pi}{360}\right) = -25\frac{2}{\sqrt{3}}\frac{1}{\sqrt{3}}\left(\pm\frac{\pi}{360}\right) \approx \pm0.145$ cm.



(b) Relative error in x is within $\frac{dx}{x} = -\frac{\csc^2\theta}{\cot\theta}d\theta = -\frac{4/3}{1/\sqrt{3}}\left(\pm\frac{\pi}{360}\right) \approx \pm0.020$, so percentage error in x is ≈ ±2.0%; relative error in y is within $\frac{dy}{y} = -\cot\theta\,d\theta = -\frac{1}{\sqrt{3}}\left(\pm\frac{\pi}{360}\right) \approx \pm0.005$, so percentage error in y is ≈ ±0.5%.

53. $\frac{dR}{R} = \frac{(-2k/r^3)dr}{(k/r^2)} = -2\frac{dr}{r}$, but $\frac{dr}{r} = \pm0.05$ so $\frac{dR}{R} = -2(\pm0.05) = \pm0.10$; percentage error in R is ±10%.

54. $h = 12\sin\theta$ thus $dh = 12\cos\theta\,d\theta$ so, with $\theta = 60^\circ = \pi/3$ radians and $d\theta = -1^\circ = -\pi/180$ radians, $dh = 12\cos(\pi/3)(-\pi/180) = -\pi/30 \approx -0.105$ ft.

55. $A = \frac{1}{4}(4)^2\sin 2\theta = 4\sin 2\theta$ thus $dA = 8\cos 2\theta\,d\theta$ so, with $\theta = 30^\circ = \pi/6$ radians and $d\theta = \pm15^0 = \pm1/4^\circ = \pm\pi/720$

radians, $dA = 8\cos(\pi/3)(\pm\pi/720) = \pm\pi/180 \approx \pm0.017\ \text{cm}^2$.

56. $A = x^2$ where x is the length of a side; $\frac{dA}{A} = \frac{2x\,dx}{x^2} = 2\frac{dx}{x}$, but $\frac{dx}{x} = \pm0.01$, so $\frac{dA}{A} = 2(\pm0.01) = \pm0.02$;

percentage error in A is ±2%

57. $V = x^3$ where x is the length of a side; $\frac{dV}{V} = \frac{3x^2\,dx}{x^3} = 3\frac{dx}{x}$, but $\frac{dx}{x} = \pm 0.02$, so $\frac{dV}{V} = 3(\pm 0.02) = \pm 0.06$; percentage error in V is ±6%.

58. $\frac{dV}{V} = \frac{4\pi r^2\,dr}{4\pi r^3/3} = 3\frac{dr}{r}$, but $\frac{dV}{V} = \pm 0.03$ so $3\frac{dr}{r} = \pm 0.03$, $\frac{dr}{r} = \pm 0.01$; maximum permissible percentage error in r is ±1%.

59. $A = \frac{1}{4}\pi D^2$ where D is the diameter of the circle; $\frac{dA}{A} = \frac{(\pi D/2)dD}{\pi D^2/4} = 2\frac{dD}{D}$, but $\frac{dA}{A} = \pm 0.01$ so $2\frac{dD}{D} = \pm 0.01$, $\frac{dD}{D} = \pm 0.005$; maximum permissible percentage error in D is ±0.5%.

60. $V = x^3$ where x is the length of a side; approximate ΔV by dV if x = 1 and $dx = \Delta x = 0.02$, $dV = 3x^2\,dx = 3(1)^2(0.02) = 0.06 \text{ in}^3$.

61. V = volume of cylindrical rod $= \pi r^2 h = \pi r^2(15) = 15\pi r^2$; approximate ΔV by dV if r = 2.5 and $dr = \Delta r = 0.1$. $dV = 30\pi r\,dr = 30\pi(2.5)(0.1) \approx 23.5619 \text{ cm}^3$.

62. $P = \frac{2\pi}{\sqrt{g}}\sqrt{L}$, $dP = \frac{2\pi}{\sqrt{g}}\frac{1}{2\sqrt{L}}dL = \frac{\pi}{\sqrt{g}\sqrt{L}}dL$, $\frac{dP}{P} = \frac{1}{2}\frac{dL}{L}$ so the relative error in P ≈ $\frac{1}{2}$ the relative error in L. Thus the percentage error in P is ≈ $\frac{1}{2}$ the percentage error in L.

63. (a) $\alpha = \Delta L/(L\Delta T) = 0.006/(40 \times 10) = 1.5 \times 10^{-5}/^\circ\text{C}$.

(b) $\Delta L = 2.3 \times 10^{-5}(180)(25) \approx 0.1$ cm, so the pole is about 180.1 cm long.

64. $\Delta V = 7.5 \times 10^{-4}(4000)(-20) = -60$ gallons; the truck delivers 4000 − 60 = 3940 gallons.

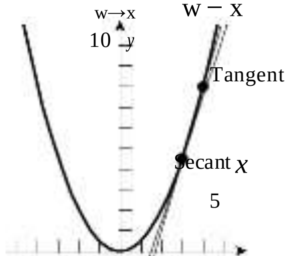
Chapter 2 Review Exercises

2. (a) $m_{sec} = \frac{f(4) - f(3)}{4-3} = \frac{(4)^2/2 - (3)^2/2}{1} = \frac{7}{2}$

(b) $m_{tan} = \lim_{w\to 3}\frac{f(w) - f(3)}{w-3} = \lim_{w\to 3}\frac{w^2/2 - 9/2}{w-3} = \lim_{w\to 3}\frac{w^2-9}{2(w-3)} = \lim_{w\to 3}\frac{(w+3)(w-3)}{2(w-3)} = \lim_{w\to 3}\frac{w+3}{2} = 3.$

(c) $m_{tan} = \lim_{w\to x}\frac{f(w) - f(x)}{w-x} = \lim_{w\to x}\frac{w^2/2 - x^2/2}{w-x} = \lim_{w\to x}\frac{w^2-x^2}{2(w-x)} = \lim_{w\to x}\frac{w+x}{2} = x.$

(d)



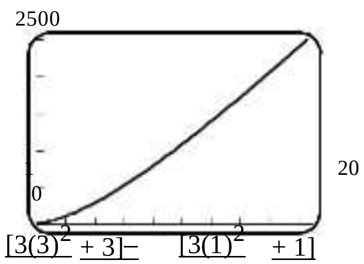
3. (a) $m_{tan} = \lim_{w\to x}\frac{f(w) - f(x)}{w-x} = \lim_{w\to x}\frac{(w^2+1) - (x^2+1)}{w-x} = \lim_{w\to x}\frac{w^2-x^2}{w-x} = \lim_{w\to x}(w+x) = 2x.$

(b) $m_{tan} = 2(2) = 4.$

4. To average 60 mi/h one would have to complete the trip in two hours. At 50 mi/h, 100 miles are completed after two hours. Thus time is up, and the speed for the remaining 20 miles would have to be infinite.

5. $v_{inst} = \lim\limits_{h\to0} \dfrac{3(h+1)^{2.5} + 580h - 3}{10h} = 58 + \dfrac{1}{10}\dfrac{d}{dx}3x^{2.5}\Big|_{x=1} = 58 + \dfrac{1}{10}(2.5)(3)(1)^{1.5} = 58.75$ ft/s.

6. 164 ft/s



7. (a) $v_{ave} = \dfrac{[3(3)^2 + 3] - [3(1)^2 + 1]}{3-1} = 13$ mi/h.

(b) $v_{inst} = \lim\limits_{t_1\to1} \dfrac{(3t_1^2 + t_1) - 4}{t_1 - 1} = \lim\limits_{t_1\to1} \dfrac{(3t_1 + 4)(t_1 - 1)}{t_1 - 1} = \lim\limits_{t_1\to1}(3t_1 + 4) = 7$ mi/h.

9. (a) $\dfrac{dy}{dx} = \lim\limits_{h\to0} \dfrac{\sqrt{9-4(x+h)} - \sqrt{9-4x}}{h} = \lim\limits_{h\to0} \dfrac{9-4(x+h) - (9-4x)}{h\left(\sqrt{9-4(x+h)} + \sqrt{9-4x}\right)} =$

$= \lim\limits_{h\to0} \dfrac{-4h}{h\left(\sqrt{9-4(x+h)} + \sqrt{9-4x}\right)} = \dfrac{-4}{2\sqrt{9-4x}} = \dfrac{-2}{\sqrt{9-4x}}$.

(b) $\dfrac{dy}{dx} = \lim\limits_{h\to0} \dfrac{\dfrac{x+h}{x+h+1} - \dfrac{x}{x+1}}{h} = \lim\limits_{h\to0} \dfrac{(x+h)(x+1) - x(x+h+1)}{h(x+h+1)(x+1)} = \lim\limits_{h\to0} \dfrac{h}{h(x+h+1)(x+1)} = \dfrac{1}{(x+1)^2}$.

10. $f(x)$ is continuous and differentiable at any $x = 1$, so we consider $x = 1$.

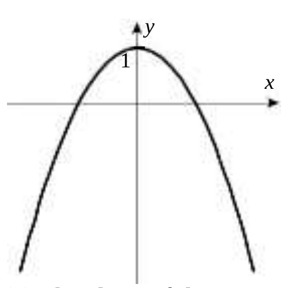
(a) $\lim\limits_{x\to1^-}(x^2 - 1) = \lim\limits_{x\to1^+} k(x-1) = 0 = f(1)$, so any value of k gives continuity at $x = 1$.

(b) $\lim\limits_{x\to1^-} f'(x) = \lim\limits_{x\to1^-} 2x = 2$, and $\lim\limits_{x\to1^+} f'(x) = \lim\limits_{x\to1^+} k = k$, so only if $k = 2$ is $f(x)$ differentiable at $x = 1$.

11. (a) $x = -2, -1, 1, 3$ (b) $(-\infty, -2), (-1, 1), (3, +\infty)$ (c) $(-2, -1), (1, 3)$

(d) $g''(x) = f''(x)\sin x + 2f'(x)\cos x - f(x)\sin x$; $g''(0) = 2f'(0)\cos 0 = 2(2)(1) = 4$

12.

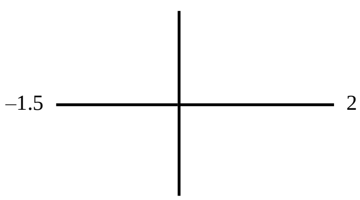


13. (a) The slope of the tangent line $\approx \dfrac{10 - 2.2}{2050 - 1950} = 0.078$ billion, so in 2000 the world population was increasing at the rate of about 78 million per year.

(b) $\frac{dN/dt}{N_6} \approx \frac{0.078}{\;} = 0.013 = 1.3$ %/year

14. When $x^4 - x - 1 > 0$, $f(x) = x^4 - 2x - 1$; when $x^4 - x - 1 < 0$, $f(x) = -x^4 + 1$, and f is differentiable in both cases. The ... $(x_2, +\infty)$, and $x^4 - x - 1 < 0$ on (x_1, x_2). Then $\lim_{x\to x_1^-} f'(x) = \lim_{x\to x_1^-} (4x^3 - 2) = 4x_1^3 - 2$ and $\lim_{x\to x_1^+} f'(x) = \lim_{x\to x_1^+} -4x^3 = -4x_1^3$ which is not equal to $4x_1^3 - 2$, so f is not differentiable at $x = x_1$; similarly f is not differentiable at $x = x_2$.

1.5



–1.5

15. (a) $f'(x) = 2x \sin x + x^2 \cos x$ (c) $f''(x) = 4x \cos x + (2 - x^2) \sin x$

16. (a) $f'(x) = \frac{1 - 2\sqrt{x} \sin 2x}{2\sqrt{x}}$ (c) $f''(x) = \frac{-1 - 8x^{3/2} \cos 2x}{4x^{3/2}}$

17. (a) $f'(x) = \frac{6x^2 + 8x - 17}{(3x + 2)^2}$ (c) $f''(x) = \frac{118}{(3x + 2)^3}$

18. (a) $f'(x) = \frac{(1 + x^2) \sec^2 x - 2x \tan x}{(1 + x^2)^2}$

(c) $f''(x) = \frac{(2 + 4x^2 + 2x^4) \sec^2 x \tan x - (4x + 4x^3) \sec^2 x + (-2 + 6x^2) \tan x}{(1 + x^2)^3}$.

19. (a) $\frac{dW}{dt} = 200(t - 15)$; at $t = 5$, $\frac{dW}{dt} = -2000$; the water is running out at the rate of 2000 gal/min.

(b) $\frac{W(5) - W(0)}{5 - 0} = \frac{10000 - 22500}{5} = -2500$; the average rate of flow out is 2500 gal/min.

20. (a) $\frac{4^3 - 2^3}{4 - 2} = \frac{56}{2} = 28$ (b) $(dV/d\ell)\big|_{\ell=5} = 3\ell^2\big|_{\ell=5} = 3(5)^2 = 75$

21. (a) $f'(x) = 2x$, $f'(1.8) = 3.6$ (b) $f'(x) = (x^2 - 4x)/(x - 2)^2$, $f'(3.5) = -7/9 \approx -0.777778$

22. (a) $f'(x) = 3x^2 - 2x$, $f'(2.3) = 11.27$ (b) $f'(x) = (1 - x^2)/(x^2 + 1)^2$, $f'(-0.5) = 0.48$

23. f is continuous at $x = 1$ because it is differentiable there, thus $\lim_{h\to 0} f(1 + h) = f(1)$ and so $f(1) = 0$ because $\lim_{h\to 0} \frac{f(1 + h)}{h}$ exists; $f'(1) = \lim_{h\to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h\to 0} \frac{f(1 + h)}{h} = 5$.

24. Multiply the given equation by $\lim_{x\to 2} (x - 2) = 0$ to get $0 = \lim_{x\to 2} (x^3 f(x) - 24)$. Since f is continuous at $x = 2$, this

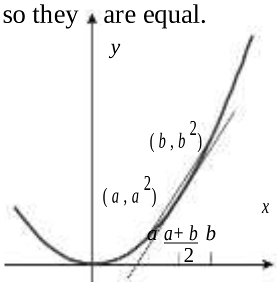
equals $2^3 f(2)$ 24, so $f(2) = 3$. Now let $g(x) = x^3 f(x)$. Then $g'(2) = \lim_{x\to 2} \frac{g(x) - g(2)}{x-2} = \lim_{x\to 2} \frac{x^3 f(x) - 2^3 f(2)}{x-2} = \lim_{x\to 2} \frac{x^3 f(x) - 24}{x-2} = 28$. But $g'(x) = x^3 f'(x) + 3x^2 f(x)$, so $28 = g'(2) = 2^3 f'(2) + 3 \cdot 2^2 f(2) = 8f'(2) + 36$, and $f'(2) = -1$.

25. The equation of such a line has the form $y = mx$. The points (x_0, y_0) which lie on both the line and the parabola and for which the slopes of both curves are equal satisfy $y_0 = mx_0 = x_0^3 - 9x_0^2 - 16x_0$, so that $m = x_0^2 - 9x_0 - 16$. By

differentiating, the slope is also given by $m = 3x_0^2 - 18x_0 - 16$. Equating, we have $x_0^2 - 9x_0 - 16 = 3x_0^2 - 18x_0 - 16$, or $2x_0^2 - 9x_0 = 0$. The root $x_0 = 0$ corresponds to $m = -16$, $y_0 = 0$ and the root $x_0 = 9/2$ corresponds to $m = -145/4$, $y_0 = -1305/8$. So the line $y = -16x$ is tangent to the curve at the point $(0, 0)$, and the line $y = -145x/4$ is tangent to the curve at the point $(9/2, -1305/8)$.

26. The slope of the line $x + 4y = 10$ is $m_1 = -1/4$, so we set the negative reciprocal $4 = m_2 = \frac{d}{dx}(2x^3 - x^2) = 6x^2 - 2x$ and obtain $6x^2 - 2x - 4 = 0$ with roots $x = \frac{1 \pm \sqrt{1+24}}{6} = 1, -2/3$.

27. The slope of the tangent line is the derivative $y' = 2x\big|_{x=\frac{1}{2}(a+b)} = a + b$. The slope of the secant is $\frac{a^2 - b^2}{a - b} = a + b$, so they are equal.



28. (a) $f'(1)g(1) + f(1)g'(1) = 3(-2) + 1(-1) = -7$ (b) $\frac{g(1)f'(1) - f(1)g'(1)}{g(1)^2} = \frac{-2(3) - 1(-1)}{(-2)^2} = -\frac{5}{4}$

(c) $\frac{1}{2\sqrt{f(1)}} f'(1) = \frac{1}{2\sqrt{1}}(3) = \frac{3}{2}$ (d) 0 (because $f(1)g'(1)$ is constant)

29. (a) $8x^7 - \frac{3}{2\sqrt{x}} - 15x^{-4}$ (b) $2 \cdot 101(2x + 1)^{100}(5x^2 - 7) + 10x(2x + 1)^{101} = (2x + 1)^{100}(1030x^2 + 10x - 1414)$

30. (a) $\cos x - 6\cos^2 x \sin x$ (b) $(1 + \sec x)(2x - \sec^2 x) + (x^2 - \tan x)\sec x \tan x$

31. (a) $2(x-1)\sqrt{3x+1} + \frac{3}{2\sqrt{3x+1}}(x-1)^2 = \frac{(x-1)(15x+1)}{2\sqrt{3x+1}}$

(b) $3\left(\frac{3x+1}{x^2}\right)^2 \frac{x^2(3) - (3x+1)(2x)}{x^4} = -\frac{3(3x+1)^2(3x+2)}{x^7}$

32. (a) $\frac{-2(x^3+5)\csc 2x\cot 2x - 3x^2\csc 2x}{(x^3+5)^2}$ (b) $-\frac{2 + 3\sin^2 x\cos x}{(2x + \sin^3 x)^2}$

33. Set $f'(x) = 0$: $f'(x) = 6(2)(2x + 7)^5(x - 2)^5 + 5(2x + 7)^6(x - 2)^4 = 0$, so $2x + 7 = 0$ or $x - 2 = 0$ or, factoring out $(2x + 7)^5(x - 2)^4$, $12(x - 2) + 5(2x + 7) = 0$. This reduces to $x = -7/2$, $x = 2$, or $22x + 11 = 0$, so the tangent line is horizontal at $x = -7/2, 2, -1/2$.

34. Set $f'(x) = 0$: $f'(x) = \frac{4(x^2 + 2x)(x - 3)^3 - (2x + 2)(x - 3)^4}{(x^2 + 2x)^2}$, and a fraction can equal zero only if its numerator equals zero. So either $x - 3 = 0$ or, after factoring out $(x - 3)^3$, $4(x^2 + 2x) - (2x + 2)(x - 3) = 0$, $2x^2 + 12x + 6 = 0$, whose roots are (by the quadratic formula) $x = \frac{-6 \pm \sqrt{36 - 4 \cdot 3}}{2} = -3 \pm \sqrt{6}$. So the tangent line is horizontal at $x = 3, -3 \pm \sqrt{6}$.

35. Suppose the line is tangent to $y = x^2 + 1$ at (x_0, y_0) and tangent to $y = -x^2 - 1$ at (x_1, y_1). Since it's tangent to $y = x^2 + 1$, its slope is $2x_0$; since it's tangent to $y = -x^2 - 1$, its slope is $-2x_1$. Hence $x_1 = -x_0$ and $y_1 = -y_0$.

Since the line passes through both points, its slope is $\frac{y_1 - y_0}{x_1 - x_0} = \frac{-2y_0}{-2x_0} = \frac{y_0}{x_0} = \frac{x_0^2 + 1}{x_0}$. Thus $2x_0 = \frac{x_0^2 + 1}{x_0}$, so $2x_0^2 = x_0^2 + 1$, $x_0^2 = 1$, and $x_0 = \pm 1$. So there are two lines which are tangent to both graphs, namely $y = 2x$ and $y = -2x$.

36. (a) Suppose $y = mx + b$ is tangent to $y = x^n + n - 1$ at (x_0, y_0) and to $y = -x^n - n + 1$ at (x_1, y_1). Then $m = nx_0^{n-1} = -nx_1^{n-1}$; since n is even this implies that $x_1 = -x_0$. Again since n is even, $y_1 = -x_1^n - n + 1 = -x_0^n - n + 1 = -(x_0^n + n - 1) = -y_0$. Thus the points (x_0, y_0) and (x_1, y_1) are symmetric with respect to the origin and both lie on the tangent line and thus $b = 0$. The slope m is given by $m = nx_0^{n-1}$ and by $m = y_0/x_0 = (x_0^n + n - 1)/x_0$, hence $nx_0^n = x_0^n + n - 1$, $(n - 1)x_0^n = n - 1$, $x_0^n = 1$. Since n is even, $x_0 = \pm 1$. One easily checks that $y = nx$ is tangent to $y = x^n + n - 1$ at $(1, n)$ and to $y = -x^n - n + 1$ at $(-1, -n)$, while $y = -nx$ is tangent to $y = x^n + n - 1$ at $(-1, n)$ and to $y = -x^n - n + 1$ at $(1, -n)$.

(b) Suppose there is such a common tangent line with slope m. The function $y = x^n + n - 1$ is always increasing, so $m \geq 0$. Moreover the function $y = -x^n - n + 1$ is always decreasing, so $m \leq 0$. Thus the tangent line has slope 0, which only occurs on the curves for $x = 0$. This would require the common tangent line to pass through $(0, n - 1)$ and $(0, -n + 1)$ and do so with slope $m = 0$, which is impossible.

37. The line $y - x = 2$ has slope $m_1 = 1$ so we set $m_2 = \frac{d}{dx}(3x - \tan x) = 3 - \sec^2 x = 1$, or $\sec^2 x = 2$, $\sec x = \pm\sqrt{2}$ so $x = n\pi \pm \pi/4$ where $n = 0, \pm 1, \pm 2, \ldots$.

38. Solve $3x^2 - \cos x = 0$ to get $x = \pm 0.535428$.

39. $3 = f(\pi/4) = (M + N)\sqrt{2}/2$ and $1 = f'(\pi/4) = (M - N)\sqrt{2}/2$. Add these two equations to get $4 = \sqrt{2}M$, $M = 2^{3/2}$.

Subtract to obtain $2 = \sqrt{2}N$, $N = \sqrt{2}$. Thus $f(x) = 2\sqrt{2}\sin x + \sqrt{2}\cos x$. $f'\left(\frac{3\pi}{4}\right) = -3$, so the tangent line is $y - 1 = -3\left(x - \frac{3\pi}{4}\right)$.

40. $f(x) = M\tan x + N\sec x$, $f'(x) = M\sec^2 x + N\sec x\tan x$. At $x = \pi/4$, $2M + \sqrt{2}N$, $0 = 2M + \sqrt{2}N$. Add to get $M = 2$, subtract to get $N = \sqrt{2} + M/\sqrt{2} = 2\sqrt{2}$, $f(x) = -2\tan x + \sqrt{2}\sec x$. $f'(0) = -2$, so the tangent line is $y - \sqrt{2} = -2x$.

41. $f'(x) = 2xf(x)$, $f(2) = 5$

(a) $g(x) = f(\sec x)$, $g'(x) = f'(\sec x)\sec x\tan x = 2 \cdot 2f(2) \cdot 2 \cdot \sqrt{3} = 40\sqrt{3}$.

(b) $h'(x) = 4\left[\frac{f(x)}{x - 1}\right]^3 \frac{(x - 1)f'(x) - f(x)}{(x - 1)^2}$, $h'(2) = 4\frac{5^3}{1}\cdot\frac{f'(2) - f(2)}{1} = 4\cdot 5^3\frac{2\cdot 2f(2) - f(2)}{1} = 4\cdot 5^3\cdot 3\cdot 5 = 7500$.

42. $\frac{dy}{dx} = \frac{1}{4}(6x - 5)^{-3/4}(6)$.

43. $\frac{dy}{dx} = \frac{1}{3}(x^2 + x)^{-2/3}(2x + 1)$.

$\frac{dy}{dx} = \frac{x^2(4/3)(3 - 2x)^{1/3}(-2) - 2x(3 - 2x)^{4/3}}{}$

44. $dx \quad = \qquad x^4$.

45. (a) $3x^2 + x\dfrac{dy}{dx} + y - 2 = 0,\quad \dfrac{dy}{dx} = \dfrac{2 - y - 3x^2}{x}$.

(b) $y = (1 + 2x - x^3)/x = 1/x + 2 - x^2$, $dy/dx = -1/x^2 - 2x$.

(c) $\dfrac{dy}{dx} = \dfrac{2 - (1/x + 2 - x^2) - 3x^2}{x} = -1/x^2 - 2x$.

46. (a) $xy = x - y$, $x\frac{dy}{dx} + y = 1 - \frac{dy}{dx}$, $\frac{dy}{dx} = \frac{1-y}{x+1}$.

(b) $y(x+1) = x$, $y = \frac{x}{x+1}$, $y' = \frac{1}{(x+1)^2}$.

(c) $\frac{dy}{dx} = \frac{1-y}{x+1} = \frac{1 - \frac{x}{1+x}}{x+1} = \frac{1}{(x+1)^2}$.

47. $-\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{x^2} = 0$ so $\frac{dy}{dx} = -\frac{y^2}{x^2}$.

48. $3x^2 - 3y^2\frac{dy}{dx} = 6\left(x\frac{dy}{dx} + y\right)$, $-(3y^2 + 6x)\frac{dy}{dx} = 6y - 3x^2$ so $\frac{dy}{dx} = \frac{x^2 - 2y}{y^2 + 2x}$.

49. $\left(x\frac{dy}{dx} + y\right)\sec(xy)\tan(xy) = \frac{dy}{dx}$, $\frac{dy}{dx} = \frac{y\sec(xy)\tan(xy)}{1 - x\sec(xy)\tan(xy)}$.

50. $2x = \frac{(1+\csc y)(-\csc^2 y)(dy/dx) - (\cot y)(-\csc y\cot y)(dy/dx)}{(1+\csc y)^2}$, $2x(1+\csc y)^2 = -\csc y(\csc y + \csc^2 y - \cot^2 y)\frac{dy}{dx}$,

but $\csc^2 y - \cot^2 y = 1$, so $\frac{dy}{dx} = -\frac{2x(1+\csc y)}{\csc y}$.

51. $\frac{dy}{dx} = \frac{3x}{4y}$, $\frac{d^2y}{dx^2} = \frac{(4y)(3) - (3x)(4dy/dx)}{16y^2} = \frac{12y - 12x(3x/(4y))}{16y^2} = \frac{12y^2 - 9x^2}{16y^3} = \frac{-3(3x^2 - 4y^2)}{16y^3}$, but $3x^2 - 4y^2 = 7$ so $\frac{d^2y}{dx^2} = \frac{-3(7)}{16y^3} = -\frac{21}{16y^3}$.

52. $\frac{dy}{dx} = \frac{y}{y-x}$, $\frac{d^2y}{dx^2} = \frac{(y-x)(dy/dx) - y(dy/dx - 1)}{(y-x)^2} = \frac{(y-x)\frac{y}{y-x} - y\left(\frac{y}{y-x} - 1\right)}{(y-x)^2} = \frac{y^2 - 2xy}{(y-x)^3}$ but $y^2 - 2xy = -3$, so $\frac{d^2y}{dx^2} = -\frac{3}{(y-x)^3}$.

53. $\frac{dy}{dx} = \tan(\pi y/2) + x(\pi/2)\frac{dy}{dx}\sec^2(\pi y/2)$, $\left.\frac{dy}{dx}\right|_{y=1/2} = 1 + (\pi/4)\left.\frac{dy}{dx}\right|_{y=1/2}(2)$, $\left.\frac{dy}{dx}\right|_{y=1/2} = \frac{2}{2-\pi}$.

54. Let $P(x_0, y_0)$ be the required point. The slope of the line $4x - 3y + 1 = 0$ is $4/3$ so the slope of the tangent to $y^2 = 2x^3$ at P must be $-3/4$. By implicit differentiation $dy/dx = 3x^2/y$, so at P, $3x_0^2/y_0 = -3/4$, or $y_0 = -4x_0^2$. But $y_0^2 = 2x_0^3$ because P is on the curve $y^2 = 2x^3$. Elimination of y_0 gives $16x_0^4 = 2x_0^3$, $x_0^3(8x_0 - 1) = 0$, so

$x_0 = 0$ or 1/8. From $y_0 = -4x_0$ it follows that $y_0 = 0$ when $x_0 = 0$, and $y_0 = -1/16$ when $x_0 = 1/8$. It does not follow, however, that (0, 0) is a solution because $dy/dx = 3x^2/y$ (the slope of the curve as determined by implicit differentiation) is valid only if $y = 0$. Further analysis shows that the curve is tangent to the x-axis at (0, 0), so the point (1/8, −1/16) is the only solution.

55. Substitute $y = mx$ into $x^2 + xy + y^2 = 4$ to get $x^2 + mx^2 + m^2x^2 = 4$, which has distinct solutions $x = \pm 2/\sqrt{m^2 + m + 1}$. They are distinct because $m^2 + m + 1 = (m + 1/2)^2 + 3/4 \geq 3/4$, so $m^2 + m + 1$ is never zero.

Note that the points of intersection occur in pairs (x_0, y_0) and $(-x_0, -y_0)$. By implicit differentiation, the slope of the tangent line to the ellipse is given by $dy/dx = -(2x + y)/(x + 2y)$. Since the slope is unchanged if we replace (x, y) with $(-x, -y)$, it follows that the slopes are equal at the two point of intersection. Finally we must examine the special case $x = 0$ which cannot be written in the form $y = mx$. If $x = 0$ then $y = \pm 2$, and the formula for dy/dx gives $dy/dx = -1/2$, so the slopes are equal.

56. By implicit differentiation, $3x^2 - y - xy^0 + 3y^2y^0 = 0$, so $y^0 = (3x^2 - y)/(x - 3y^2)$. This derivative is zero when

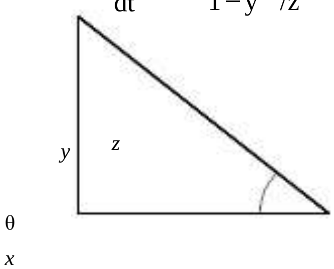
$y = 3x^2$. Substituting this into the original equation $x^3 - xy + y^3 = 0$, one has $x^3 - 3x^3 + 27x^6 = 0$, $x^3 (27x^3 - 2) = 0$. The unique solution in the first quadrant is $x = 2^{1/3}/3$, $y = 3x^2 = 2^{2/3}/3$.

57. By implicit differentiation, $3x^2 - y - xy' + 3y^2 y' = 0$, so $y' = (3x^2 - y)/(x - 3y^2)$. This derivative exists except when $x = 3y^2$. Substituting this into the original equation $x^3 - xy + y^3 = 0$, one has $27y^6 - 3y^3 + y^3 = 0$, $y^3(27y^3 - 2) = 0$. The unique solution in the first quadrant is $y = 2^{1/3}/3$, $x = 3y^2 = 2^{2/3}/3$

58. By implicit differentiation, $dy/dx = k/(2y)$ so the slope of the tangent to $y^2 = kx$ at (x_0, y_0) is $k/(2y_0)$ if $y_0 \neq 0$. The tangent line in this case is $y - y_0 = \frac{k}{2y_0}(x - x_0)$, or $2y_0 y - 2y_0^2 = kx - kx_0$. But $y_0^2 = kx_0$ because (x_0, y_0) is on the curve $y^2 = kx$, so the equation of the tangent line becomes $2y_0 y - 2kx_0 = kx - kx_0$ which gives $y_0 y = k(x + x_0)/2$. If $y_0 = 0$, then $x_0 = 0$; the graph of $y^2 = kx$ has a vertical tangent at $(0, 0)$ so its equation is $x = 0$, but $y_0 y = k(x + x_0)/2$ gives the same result when $x_0 = y_0 = 0$.

59. The boom is pulled in at the rate of 5 m/min, so the circumference $C = 2r\pi$ is changing at this rate, which means that $\frac{dr}{dt} = \frac{dC}{dt} \cdot \frac{1}{2\pi} = -5/(2\pi)$. $A = \pi r^2$ and $\frac{dr}{dt} = -5/(2\pi)$, so $\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt} = 2\pi r(-5/2\pi) = -250$, so the area is shrinking at a rate of 250 m^2/min.

60. Find $\left.\frac{d\theta}{dt}\right|_{x=1,\, y=1}$ given $\frac{dz}{dt} = a$ and $\frac{dy}{dt} = -b$. From the figure $\sin\theta = y/z$; when $x = y = 1$, $z = \sqrt{2}$. So $\theta = \sin^{-1}(y/z)$ and $\frac{d\theta}{dt} = \frac{1}{\sqrt{1 - y^2/z^2}}\left(\frac{1}{z}\frac{dy}{dt} - \frac{y}{z^2}\frac{dz}{dt}\right) = -b - \frac{a}{\sqrt{2}}$ when $x = y = 1$.



61. (a) $\Delta x = 1.5 - 2 = -0.5$; $dy = \frac{-1}{(x-1)^2}\Delta x = \frac{-1}{(2-1)^2}(-0.5) = 0.5$; and $\Delta y = \frac{1}{(1.5-1)} - \frac{1}{(2-1)} = 2 - 1 = 1$.

(b) $\Delta x = 0 - (-\pi/4) = \pi/4$; $dy = \sec^2(-\pi/4)(\pi/4) = \pi/2$; and $\Delta y = \tan 0 - \tan(-\pi/4) = 1$.

(c) $\Delta x = 3 - 0 = 3$; $dy = \frac{-x}{\sqrt{25 - x^2}} = \frac{-0}{\sqrt{25 - (0)^2}}(3) = 0$; and $\Delta y = \sqrt{25 - 3^2} - \sqrt{25 - 0^2} = 4 - 5 = -1$.

62. $\cot 46^\circ = \cot\frac{46\pi}{180}$; let $x_0 = \frac{\pi}{4}$ and $x = \frac{46\pi}{180}$. Then $\cot 46^\circ = \cot x \approx \cot\frac{\pi}{4} - \csc^2\frac{\pi}{4}\left(x - \frac{\pi}{4}\right) = 1 - 2\left(\frac{46\pi}{180} - \frac{\pi}{4}\right) = 0.9651$; with a calculator, $\cot 46^\circ = 0.9657$.

63. (a) $h = 115\tan\varphi$, $dh = 115\sec^2\varphi\, d\varphi$; with $\varphi = 51^\circ = \frac{51}{180}\pi$ radians and $d\varphi = \pm 0.5^\circ = \pm 0.5\frac{\pi}{180}$ radians, $h \pm dh = 115(1.2349) \pm 2.5340 = 142.0135 \pm 2.5340$, so the height lies between 139.48 m and 144.55 m.

(b) If $|dh| \le 5$ then $|d\varphi| \le \frac{5}{115}\cos^2\frac{51}{180}\pi \approx 0.017$ radian, or $|d\varphi| \le 0.98^\circ$.

1. (a) By property (ii), $f(0) = f(0+0) = f(0)f(0)$, so $f(0) = 0$ or 1. By property (iii), $f(0) = 0$, so $f(0) = 1$.

(b) By property (ii), $f(x) = f\left(\frac{x}{2} + \frac{x}{2}\right) = f\left(\frac{x}{2}\right)^2 \geq 0$. If $f(x) = 0$, then $1 = f(0) = f(x + (-x)) = f(x)f(-x) = 0 \cdot f(-x) = 0$, a contradiction. Hence $f(x) > 0$.

(c) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{f(x)f(h) - f(x)}{h} = \lim_{h\to 0} f(x)\dfrac{f(h)-1}{h} = f(x)\lim_{h\to 0} \dfrac{f(h)-f(0)}{h} = f(x)f'(0) = f(x)$

2. (a) By the chain rule and Exercise 1(c), $y' = f'(2x)\cdot \dfrac{d}{dx}(2x) = f(2x)\cdot 2 = 2y$.

(b) By the chain rule and Exercise 1(c), $y' = f'(kx)\cdot \dfrac{d}{dx}(kx) = kf'(kx) = kf(kx)$.

(c) By the product rule and Exercise 1(c), $y' = f(x)g'(x) + g(x)f'(x) = f(x)g(x) + g(x)f(x) = 2f(x)g(x) = 2y$, so $k = 2$.

(d) By the quotient rule and Exercise 1(c), $h'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} = \dfrac{g(x)f(x) - f(x)g(x)}{g(x)^2} = 0$. As we will see in Theorem 3.1.2(c), this implies that $h(x)$ is a constant. Since $h(0) = f(0)/g(0) = 1/1 = 1$ by Exercise 1(a), $h(x) = 1$ for all x, so $f(x) = g(x)$.

3. (a) For brevity, we omit the "(x)" throughout.

$(f\cdot g\cdot h)' = \dfrac{d}{dx}[(f\cdot g)\cdot h] = (f\cdot g)\cdot \dfrac{dh}{dx} + h\cdot \dfrac{d}{dx}(f\cdot g) = f\cdot g\cdot h' + h\cdot \left(f\cdot \dfrac{dg}{dx} + g\cdot \dfrac{df}{dx}\right)$

$= f'\cdot g\cdot h + f\cdot g'\cdot h + f\cdot g\cdot h'$

(b) $(f\cdot g\cdot h\cdot k)' = \dfrac{d}{dx}[(f\cdot g\cdot h)\cdot k] = (f\cdot g\cdot h)\cdot \dfrac{dk}{dx} + k\cdot \dfrac{d}{dx}(f\cdot g\cdot h)$

$= f\cdot g\cdot h\cdot k' + k\cdot (f'\cdot g\cdot h + f\cdot g'\cdot h + f\cdot g\cdot h') = f'\cdot g\cdot h\cdot k + f\cdot g'\cdot h\cdot k + f\cdot g\cdot h'\cdot k + f\cdot g\cdot h\cdot k'$

(c) Theorem: If $n \geq 1$ and $f_1, \cdots, f_n$ are differentiable functions of x, then

$(f_1\cdot f_2\cdot \cdots \cdot f_n)' = \sum_{i=1}^{n} f_1\cdot \cdots \cdot f_{i-1}\cdot f_i'\cdot f_{i+1}\cdot \cdots \cdot f_n$.

Proof: For $n = 1$ the statement is obviously true: $f_1' = f_1'$. If the statement is true for $n-1$, then

$(f_1\cdot f_2\cdot \cdots \cdot f_n)' = \dfrac{d}{dx}[(f_1\cdot f_2\cdot \cdots \cdot f_{n-1})\cdot f_n] = (f_1\cdot f_2\cdot \cdots \cdot f_{n-1})\cdot f_n' + f_n\cdot (f_1\cdot f_2\cdot \cdots \cdot f_{n-1})'$

$= f_1\cdot f_2 \cdot \cdots \cdot f_{n-1}\cdot f_n' + f_n\cdot \sum_{i=1}^{n-1} f_1\cdot \cdots \cdot f_{i-1}\cdot f_i'\cdot f_{i+1}\cdot \cdots \cdot f_{n-1} = \sum_{i=1}^{n} f_1\cdot \cdots \cdot f_{i-1}\cdot f_i'\cdot f_{i+1}\cdot \cdots \cdot f_n$

so the statement is true for n. By induction, it's true for all n.

4. (a) $[(f/g)/h]' = \dfrac{h\cdot (f/g)' - (f/g)\cdot h'}{h^2} = \dfrac{h\cdot \frac{g\cdot f' - f\cdot g'}{g^2} - \frac{f\cdot h'}{g}}{h^2} = \dfrac{f'\cdot g\cdot h - f\cdot g'\cdot h - f\cdot g\cdot h'}{g^2h^2}$

(b) $[(f/g)/h]' = [f/(g\cdot h)]' = \dfrac{(g\cdot h)\cdot f' - f\cdot (g\cdot h)'}{(g\cdot h)^2} = \dfrac{f'\cdot g\cdot h - f\cdot (g\cdot h' + h\cdot g')}{g^2h^2}$

$= \dfrac{f'\cdot g\cdot h - f\cdot g'\cdot h - f\cdot g\cdot h'}{g^2h^2}$

(c) $[f/(g/h)]' = \dfrac{(g/h)\cdot f' - f\cdot (g/h)'}{(g/h)^2} = \dfrac{\frac{f'\cdot g}{h} - f\cdot \frac{h\cdot g' - g\cdot h'}{h^2}}{(g/h)^2} = \dfrac{f'\cdot g\cdot h - f\cdot g'\cdot h + f\cdot g\cdot h'}{g^2}$

$[f/(g/h)]' = [(f\cdot h)/g]' = \dfrac{g\cdot (f\cdot h)' - (f\cdot h)\cdot g'}{g^2} = f'\cdot g\cdot h - f\cdot g'\cdot h + f\cdot g$

$g \cdot (f \cdot h^0 + h \cdot f^0) - f \cdot g^0 \cdot h$

$=$

$=$

$\underline{\cdot\ h^0}\ g^2$

g^2

5. (a) By the chain rule, $\frac{d}{dx}[g(x)]^{-1} = -[g(x)]^{-2} g'(x) = -\frac{g'(x)}{[g(x)]^2}$. By the product rule, $h'(x) = \frac{d}{dx}\left[f(x)[g(x)]^{-1}\right] = f(x)\left(-\frac{g'(x)}{[g(x)]^2}\right) + \frac{f'(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$.

(b) By the product rule, $f'(x) = \frac{d}{dx}[h(x)g(x)] = h(x)g'(x) + g(x)h'(x)$. So $h'(x) = \frac{1}{g(x)}[f'(x) - h(x)g'(x)] = \frac{1}{g(x)} \cdot \left[f'(x) - \frac{f(x)}{g(x)}g'(x)\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$.