CHAPTER 1
FUNDAMENTAL PRINCIPLES OF COUNTING

Sections 1.1 and 1.2

1. (a) By the rule of sum, there are $8 + 5 = 13$ possibilities for the eventual winner.
(b) Since there are eight Republicans and five Democrats, by the rule of product we have $8 \times 5 = 40$ possible pairs of opposing candidates.
(c) The rule of sum in part (a); the rule of product in part (b).

2. By the rule of product there are $5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^6$ license plates where the first two symbols are vowels and the last four are even digits.

3. By the rule of product there are (a) $4 \times 12 \times 3 \times 2 = 288$ distinct Buicks that can be manufactured. Of these, (b) $4 \times 1 \times 3 \times 2 = 24$ are blue.

4. (a) From the rule of product there are $10 \times 9 \times 8 \times 7 = P(10, 4) = 5040$ possible slates.
(b) (i) There are $3 \times 9 \times 8 \times 7 = 1512$ slates where a physician is nominated for president.
(ii) The number of slates with exactly one physician appearing is $4 \times [3 \times 7 \times 6 \times 5] = 2520$.
(iii) There are $7 \times 6 \times 5 \times 4 = 840$ slates where no physician is nominated for any of the four offices. Consequently, $5040 - 840 = 4200$ slates include at least one physician.

5. Based on the evidence supplied by Jennifer and Tiffany, from the rule of product we find that there are $2 \times 2 \times 1 \times 10 \times 10 \times 2 = 800$ different license plates.

6. (a) Here we are dealing with the permutations of 30 objects (the runners) taken 8 (the first eight finishing positions) at a time.. So the trophies can be awarded in $P(30, 8) = 30!/22!$ ways.
(b) Roberta and Candice can finish among the top three runners in 6 ways. For each of these 6 ways, there are $P(28, 6)$ ways for the other 6 finishers (in the top 8) to finish the race. By the rule of product there are $6 \cdot P(28, 6)$ ways to award the trophies with these two runners among the top three.

7. By the rule of product there are 2^9 possibilities.

8. By the rule of product there are (a) $12!$ ways to process the programs if there are no restrictions; (b) $(4!)(8!)$ ways so that the four higher priority programs are processed first; and (c) $(4!)(5!)(3!)$ ways where the four top priority programs are processed first and the three programs of least priority are processed last.

9. (a) $(14)(12) = 168$
(b) $(14)(12)(6)(18) = 18{,}144$
(c) $(8)(18)(6)(3)(14)(12)(14)(12) = 73{,}156{,}608$

10. Consider one such arrangement – say we have three books on one shelf and 12 on the other. This can be accomplished in $15!$ ways. In fact for any subdivision (resulting in two nonempty shelves) of the 15 books we get $15!$ ways to arrange the books on the two shelves. Since there are 14 ways to subdivide the books so that each shelf has at least one book, the total number of ways in which Pamela can arrange her books in this manner is $(14)(15!)$.

11. (a) There are four roads from town A to town B and three roads from town B to town C, so by the rule of product there are $4 \times 3 = 12$ roads from A to C that pass through B. Since there are two roads from A to C directly, there are $12 + 2 = 14$ ways in which Linda can make the trip from A to C.
(b) Using the result from part (a), together with the rule of product, we find that there are $14 \times 14 = 196$ different round trips (from A to C and back to A).
(c) Here there are $14 \times 13 = 182$ round trips.

12. (1) a,c,t (2) a,t,c (3) c,a,t (4) c,t,a (5) t,a,c (6) t,c,a

13. (a) $8! = P(8,8)$ (b) $7!$ $6!$

14. (a) $P(7,2) = 7!/(7-2)! = 7!/5! = (7)(6) = 42$
(b) $P(8,4) = 8!/(8-4)! = 8!/4! = (8)(7)(6)(5) = 1680$
(c) $P(10,7) = 10!/(10-7)! = 10!/3! = (10)(9)(8)(7)(6)(5)(4) = 604,800$
(d) $P(12,3) = 12!/(12-3)! = 12!/9! = (12)(11)(10) = 1320$

15. Here we must place a,b,c,d in the positions denoted by x: e x e x e x e x e. By the rule of product there are $4!$ ways to do this.

16. (a) With repetitions allowed there are 40^{25} distinct messages.
(b) By the rule of product there are $40 \times 30 \times 30 \times \ldots \times 30 \times 30 \times 40 = (40^2)(30^{23})$ messages.

17. Class A: $(2^7 - 2)(2^{24} - 2) = 2,113,928,964$
Class B: $2^{14}(2^{16} - 2) = 1,073,709,056$
Class C: $2^{21}(2^8 - 2) = 532,676,608$

18. From the rule of product we find that there are $(7)(4)(3)(6) = 504$ ways for Morgan to configure her low-end computer system.

19. (a) $7! = 5040$ (b) $4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = (4!)(3!) = 144$
(c) $(3!)(5)(4!) = 720$ (d) $(3!)(4!)(2) = 288$

20. (a) Since there are three A's, there are $8!/3! = 6720$ arrangements.

(b) Here we arrange the six symbols D,T,G,R,M, AAA in $6! = 720$ ways.

21. (a) $12!/(3!2!2!2!)$
(b) $[11!/(3!2!2!2!)]$ (for AG) + $[11!/(3!2!2!2!)]$ (for GA)
(c) Consider one case where all the vowels are adjacent: S,C,L,G,C,L, OIOOIA. These seven symbols can be arranged in $(7!)/(2!2!)$ ways. Since O,O,O,I,I,A can be arranged in $(6!)/(3!2!)$ ways, the number of arrangements with all the vowels adjacent is $[7!/(2!2!)][6!/(3!2!)]$.

22. (Case 1: The leading digit is 5) $(6!)/(2!)$
(Case 2: The leading digit is 6) $(6!)/(2!)^2$
(Case 3: The leading digit is 7) $(6!)/(2!)^2$
In total there are $[(6!)/(2!)][1 + (1/2) + (1/2)] = 6! = 720$ such positive integers n.

23. Here the solution is the number of ways we can arrange 12 objects — 4 of the first type, 3 of the second, 2 of the third, and 3 of the fourth. There are $12!/(4!3!2!3!) = 277,200$ ways.

24. $P(n+1,r) = (n+1)!/(n+1-r)! = [(n+1)/(n+1-r)] \cdot [n!/(n-r)!] = [(n+1)/(n+1-r)]P(n,r)$.

25. (a) $n = 10$ (b) $n = 5$
(c) $2n!/(n-2)! + 50 = (2n)!/(2n-2)! \Longrightarrow 2n(n-1) + 50 = (2n)(2n-1) \Longrightarrow n^2 = 25 \Longrightarrow n = 5$.

26. Any such path from (0,0) to (7,7) or from (2,7) to (9,14) is an arrangement of 7 R's and 7 U's. There are $(14!)/(7!7!)$ such arrangements.
In general, for m, n nonnegative integers, and any real numbers a, b, the number of such paths from (a,b) to $(a+m, b+n)$ is $(m+n)!/(m!n!)$.

27. (a) Each path consists of 2 H's, 1 V, and 7 A's. There are $10!/(2!1!7!)$ ways to arrange these 10 letters and this is the number of paths.
(b) $10!/(2!1!7!)$
(c) If a, b, and c are any real numbers and m, n, and p are nonnegative integers, then the number of paths from (a,b,c) to $(a+m, b+n, c+p)$ is $(m+n+p)!/(m!n!p!)$.

28. (a) The for loop for i is executed 12 times, while those for j and k are executed $10-5+1 = 6$ and $15-8+1 = 8$ times, respectively. Consequently, following the execution of the given program segment, the value of *counter* is

$$0 + 12(1) + 6(2) + 8(3) = 48.$$

(b) Here we have three tasks — T_1, T_2, and T_3. Task T_1 takes place each time we traverse the instructions in the i loop. Similarly, tasks T_2 and T_3 take place during each iteration of the j and k loops, respectively. The final value for the integer variable counter follows by the rule of sum.

29. (a) & (b) By the rule of product the **print** statement is executed $12 \times 6 \times 8 = 576$ times.

30. (a) For five letters there are $26 \times 26 \times 26 \times 1 \times 1 = 26^3$ palindromes. There are $26 \times 26 \times 26 \times 1 \times 1 \times 1 = 26^3$ palindromes for six letters.
(b) When letters may not appear more than two times, there are $26 \times 25 \times 24 = 15,600$ palindromes for either five or six letters.

31. By the rule of product there are (a) $9 \times 9 \times 8 \times 7 \times 6 \times 5 = 136,080$ six-digit integers with no leading zeros and no repeated digit. (b) When digits may be repeated there are 9×10^5 such six-digit integers.
(i) (a) $(9 \times 8 \times 7 \times 6 \times 5 \times 1)$ (for the integers ending in 0) + $(8 \times 8 \times 7 \times 6 \times 5 \times 4)$ (for the integers ending in 2,4,6, or 8) = 68,800. (b) When the digits may be repeated there are $9 \times 10 \times 10 \times 10 \times 10 \times 5 = 450,000$ six-digit even integers.
(ii) (a) $(9 \times 8 \times 7 \times 6 \times 5 \times 1)$ (for the integers ending in 0) + $(8 \times 8 \times 7 \times 6 \times 5 \times 1)$ (for the integers ending in 5) = 28,560. (b) $9 \times 10 \times 10 \times 10 \times 10 \times 2 = 180,000$.
(iii) We use the fact that an integer is divisible by 4 if and only if the integer formed by the last two digits is divisible by 4. (a) $(8 \times 7 \times 6 \times 5 \times 6)$ (last two digits are 04, 08, 20, 40, 60, or 80) + $(7 \times 7 \times 6 \times 5 \times 16)$ (last two digits are 12, 16, 24, 28, 32, 36, 48, 52, 56, 64, 68, 72, 76, 84, 92, or 96) = 33,600. (b) $9 \times 10 \times 10 \times 10 \times 25 = 225,000$.

32. (a) For positive integers n, k, where $n = 3k$, $n!/(3!)^k$ is the number of ways to arrange the n objects $x_1, x_1, x_1, x_2, x_2, x_2, \ldots, x_k, x_k, x_k$. This must be an integer.
(b) If n, k are positive integers with $n = mk$, then $n!/(m!)^k$ is an integer.

33. (a) With 2 choices per question there are $2^{10} = 1024$ ways to answer the examination.
(b) Now there are 3 choices per question and 3^{10} ways.

34. (4!/2!) (No 7's) + (4!) (One 7 and one 3) + (2)(4!/2!) (One 7 and two 3's) + (4!/2!) (Two 7's and no 3's) + (2)(4!/2!) (Two 7's and one 3) + (4!/(2!2!)) (Two 7's and two 3's). The total gives us 102 such four-digit integers.

35. (a) 6! (b) Let A,B denote the two people who insist on sitting next to each other. Then there are 5! (A to the right of B) + 5! (B to the right of A) = 2(5!) seating arrangements.

36. (a) Locate A. There are two cases to consider. (1) There is a person to the left of A on the same side of the table. There are 7! such seating arrangements. (2) There is a person to the right of A on the same side of the table. This gives 7! more arrangements. So there are 2(7!) possibilities. (b) 7200

37. We can select the 10 people to be seated at the table for 10 in $\binom{16}{10}$ ways. For each such selection there are 9! ways of arranging the 10 people around the table. The remaining six people can be seated around the other table in 5! ways. Consequently, there are $\binom{16}{10}9!5!$ ways to seat the 16 people around the two given tables.

38. The nine women can be situated around the table in 8! ways. Each such arrangement provides nine spaces (between women) where a man can be placed. We can select six of these places and situate a man in each of them in $\binom{9}{6}6! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ ways. Consequently, the number of seating arrangements under the given conditions is $(8!)\binom{9}{6}6! = 2,438,553,600$.

39.

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procedure SumOfFact(i, sum: positive integers; j,k: nonnegative integers;
                    factorial: array [0..9] of ten positive integers)
begin
   factorial [0] := 1
   for i := 1 to 9 do
      factorial [i] := i * factorial [i - 1]

   for i := 1 to 9 do
      for j := 0 to 9 do
         for k := 0 to 9 do
            begin
               sum := factorial [i] + factorial [j] + factorial [k]
               if (100 * i + 10 * j + k) = sum then
                  print (100 * i + 10 * j + k)
            end
end
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The unique answer is 145 since $(1!) + (4!) + (5!) = 1 + 24 + 120 = 145$.

Section 1.3

1. $\binom{6}{2} = 6!/[2!(6\text{-}2)!] = 6!/(2!4!) = (6)(5)/2 = 15$

a	b	b	c	c	e
a	c	b	d	c	f
a	d	b	e	d	e
a	e	b	f	d	f
a	f	c	d	e	f

2. Order is not relevant here and Diane can make her selection in $\binom{12}{5} = 792$ ways.

3. (a) $C(10,4) = 10!/(4!6!) = (10)(9)(8)(7)/(4)(3)(2)(1) = 210$
(b) $\binom{12}{7} = 12!/(7!5!) = (12)(11)(10)(9)(8)/(5)(4)(3)(2)(1) = 792$

(c) $C(14,12) = 14!/(12!2!) = (14)(13)/(2)(1) = 91$
(d) $\binom{15}{10} = 15!/(10!5!) = (15)(14)(13)(12)(11)/(5)(4)(3)(2)(1) = 3003$

4. (a) $2^6 - 1 = 63$ (b) $\binom{6}{3} = 20$ (c) $\binom{6}{2} + \binom{6}{4} + \binom{6}{6} = 31$

5. (a) There are $P(5,3) = 5!/(5-3)! = 5!/2! = (5)(4)(3) = 60$ permutations of size 3 for the five letters m, r, a, f, and t.
(b) There are $C(5,3) = 5!/[3!(5-3)!] = 5!/(3!2!) = 10$ combinations of size 3 for the five letters m, r, a, f, and t. They are

a,f,m	a,f,r	a,f,t	a,m,r	a,m,t
a,r,t	f,m,r	f,m,t	f,r,t	m,r,t

6.
$$\binom{n}{2} + \binom{n-1}{2} = (\frac{1}{2})(n)(n-1) + (\frac{1}{2})(n-1)(n-2) = (\frac{1}{2})(n-1)[n+(n-2)] =$$
$(\frac{1}{2})(n-1)(2n-2) = (n-1)^2$.

7. (a) $\binom{20}{12}$ (b) $\binom{10}{6}\binom{10}{6}$
(c) $\binom{10}{2}\binom{10}{10}$(2 women)$+\binom{10}{4}\binom{10}{8}$ (4 women)$+\ldots+\binom{10}{10}\binom{10}{2}$ (10 women) $= \sum_{i=1}^{5}\binom{10}{2i}\binom{10}{12-2i}$
(d) $\binom{10}{7}\binom{10}{5}$ (7 women) $+ \binom{10}{8}\binom{10}{4}$ (8 women) $+ \binom{10}{9}\binom{10}{3}$ (9 women) $+$
$\binom{10}{10}\binom{10}{2}$ (10 women) $= \sum_{i=7}^{10}\binom{10}{i}\binom{10}{12-i}$.
(e) $\sum_{i=8}^{10}\binom{10}{i}\binom{10}{12-i}$

8. (a) $\binom{4}{1}\binom{13}{5}$ (b) $\binom{4}{4}\binom{48}{1}$ (c) $\binom{13}{1}\binom{4}{4}\binom{48}{1}$ (d) $\binom{4}{3}\binom{4}{2}$
(e) $\binom{4}{3}\binom{12}{1}\binom{4}{2}$ (f) $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3744$
(g) $\binom{13}{1}\binom{4}{3}\binom{48}{1}\binom{44}{1}/2$ (Division by 2 is needed since no distinction is made for the order in which the other two cards are drawn.) This result equals $54,912 = \binom{13}{1}\binom{4}{3}\binom{48}{2} - 3744 =$
$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}$.
(h) $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}$.

9. (a) $\binom{8}{2}$ (b) $\binom{8}{4}$ (c) $\binom{8}{6}$ (d) $\binom{8}{6} + \binom{8}{7} + \binom{8}{8}$.

10. $\binom{12}{5}$; $\binom{10}{3}$.

11. (a) $\binom{10}{7} = 120$ (b) $\binom{8}{5} = 56$ (c) $\binom{6}{4}\binom{4}{3}$ (four of the first six) $+\binom{6}{5}\binom{4}{2}$ (five of the first six) $+\binom{6}{6}\binom{4}{1}$ (all of the first six) $= (15)(4) + (6)(6) + (1)(4) = 100$.

12. (a) The first three books can be selected in $\binom{12}{3}$ ways. The next three in $\binom{9}{3}$ ways. The third set of three in $\binom{6}{3}$ ways and the fourth set in $\binom{3}{3}$ ways. Consequently, the 12 books can be distributed in $\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = (12!)/[(3!)^4]$ ways.

(b) $\binom{12}{4}\binom{8}{4}\binom{4}{2}\binom{2}{2} = (12!)/[(4!)^2(2!)^2]$.

13. The letters M,I,I,I,P,P,I can be arranged in $[7!/(4!)(2!)]$ ways. Each arrangement provides eight locations (one at the start of the arrangement, one at the finish, and six between letters) for placing the four nonconsecutive S's. Four of these locations can be selected in $\binom{8}{4}$ ways. Hence, the total number of these arrangements is $\binom{8}{4}$ $[7!/(4!)(2!)]$.

14. $\binom{n}{11} = 12,376$ when $n = 17$.

15. (a) Two distinct points determine a line. With 15 points, no three collinear, there are $\binom{15}{2}$ possible lines.

(b) There are $\binom{25}{3}$ possible triangles or planes, and $\binom{25}{4}$ possible tetrahedra.

16. (a) $\sum_{i=1}^{6}(i^2+1) = (1^2+1)+(2^2+1)+(3^2+1)+(4^2+1)+(5^2+1)+(6^2+1) = 2+5+10+17+26+37$ $= 97$

(b) $\sum_{j=-2}^{2}(j^3-1) = [(-2)^3-1]+[(-1)^3-1]+(0^3-1)+(1^3-1)+(2^3-1) = -9-2-1+0+7$ $= -5$

(c) $\sum_{i=0}^{10}[1+(-1)^i] = 2+0+2+0+2+0+2+0+2+0+2 = 12$

(d) $\sum_{k=n}^{2n}(-1)^k = [(-1)^n+(-1)^{n+1}]+[(-1)^{n+2}+(-1)^{n+3}]+\ldots+[(-1)^{2n-1}+(-1)^{2n}]$ $= 0+0+\ldots+0 = 0$

(e) $\sum_{i=1}^{6} i(-1)^i = -1+2-3+4-5+6 = 3$

17. (a) $\sum_{k=2}^{n}\frac{1}{k!}$ (b) $\sum_{i=1}^{7} i^2$ (c) $\sum_{j=1}^{7}(-1)^{j-1}j^3 = \sum_{k=1}^{7}(-1)^{k+1}k^3$

(d) $\sum_{i=0}^{n}\frac{i+1}{n+i}$ (e) $\sum_{i=0}^{n}(-1)^i\left[\frac{n+i}{(2i)!}\right]$

18. (a) $10!/(4!3!3!)$ (b) $\binom{10}{8}2^2+\binom{10}{9}2+\binom{10}{10}$

(c) $\binom{10}{4}$ (four 1's, six 0's) + $\binom{10}{2}\binom{8}{1}$ (two 1's, one 2, seven 0's) + $\binom{10}{2}$ (two 2's, eight 0's)

19. $\binom{10}{3}$ (three 1's, seven 0's) + $\binom{10}{1}\binom{9}{1}$ (one 1, one 2, eight 0's) + $\binom{10}{1}$ (one 3, nine 0's) = 220

$\binom{10}{4}+\binom{10}{2}+\binom{10}{1}\binom{9}{2}+\binom{10}{1}\binom{9}{1} = 705$