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Chapter 2

WALKING

2.1. Weighted Voting

1. (a) A generic weighted voting system with $N = 5$ players is described using notation $[q : w_1, w_2, w_3, w_4, w_5]$ where q represents the value of the quota and w_i represents the weight of player P_i. In this case, the players are the partners, $w_1 = 15$, $w_2 = 12$, $w_3 = w_4 = 10$, and $w_5 = 3$. Since the total number of votes is

15 + 12 + 10 + 10 + 3 = 50 and the quota is determined by a simple majority (more than 50% of the total number of votes), $q = 26$. That is, the partnership can be described by $[26 : 15, 12, 10, 10, 3]$.

(b) Since $\left(\frac{2}{3}\right)50 = 33\frac{1}{3}$, we choose the quota q as the smallest integer larger than this value which is 34. The partnership can thus be described by $[34 : 15, 12, 10, 10, 3]$.

2. (a) There are 100 votes (each one representing 1% ownership). A simple majority requires *more* than half of that or $q = 51$. That is, the partnership can be described by $[51 : 30, 25, 20, 16, 9]$.

(b) Since $\left(\frac{2}{3}\right)100 = 66\frac{2}{3}$, we choose the quota q as the smallest integer larger than this value which is 67. The partnership can thus be described by $[67 : 30, 25, 20, 16, 9]$.

3. (a) The quota must be *more than* half of the total number of votes. This system has 6 + 4 + 3 + 3 + 2 + 2 = 20 total votes. Since 50% of 20 is 10, the smallest possible quota would be 11. Note: $q = 10$ is *not* sufficient. If 10 votes are cast in favor and 10 cast against a motion, that motion should not pass.

(b) The largest value q can take is 20, the total number of votes.

(c) $\frac{3}{4} \times 20 = 15$, so the value of the quota q would be 15.

(d) The value of the quota q would be *strictly larger than* 15. That is, 16.

4. (a) The quota must be more than half of the total votes. This system has 10 + 6 + 5 + 4 + 2 = 27 total votes. $\frac{1}{2} \times 27 = 13.5$, so the smallest value q can take is 14.

(b) The largest value q can take is 27, the total number of votes.

(c) $\frac{2}{3} \times 27 = 18$

(d) 19

5. (a) P_1, the player with the most votes, does not have veto power since the other players combined have 3 + 3 + 2 = 8 votes and can successfully pass a motion without him. The other players can't have veto power either then as they have fewer votes and hence less (or equal) power. So no players have veto power in this system.

(b) P_1 does have veto power here since the other players combined have only 3 + 3 + 2 = 8 votes and cannot successfully pass a motion without him. P_2, on the other hand, does not have veto power since the other players combined have the 4 + 3 + 2 = 9 votes necessary to meet the quota. Since P_3 and P_4 have the same or fewer votes than P_2, it follows that only P_1 has veto power.

(c) P_4, the player with the fewest votes, does not have veto power here since the other players combined have 4 + 3 + 3 = 10 votes and can pass a motion without him. P_3, on the other hand, does have veto power since the other players combined have only 4 + 3 + 2 = 9 votes. Since P_1 and P_2 have the same or more votes than P_3, it follows that P_1, P_2, and P_3 all have veto power.

(d) P_4, the player with the fewest votes, has veto power since the other players combined have only 4 + 3 + 3 = 10 votes and cannot pass a motion without him. Since the other players all have the same or more votes than P_4, it follows that all players have veto power in this system.

6. (a) P_1 does have veto power here since the other players combined have only 4 + 2 + 1 = 7 votes and cannot pass a motion without him. P_2, on the other hand, does not have veto power since the other players combined have the 8 + 2 + 1 = 11 votes necessary to meet the quota. Since P_3 and P_4 have the same or fewer votes than P_2, it follows that only P_1 has veto power.

(b) Based on (a), we know that P_1 still has veto power since only the quota has changed (increased). It would now be even more difficult for the other players to pass a motion without P_1. P_2 also has veto power here since the other players combined have 8 + 2 + 1 = 11 votes and cannot pass a motion without him. P_3, on the other hand, does not have veto power since the other players combined have 8 + 4 + 1 = 13 votes. Since P_4 has fewer votes than P_3, it follows that only P_1 and P_2 have veto power.

(c) Based on (a) and (b), we know that P_1 and P_2 still have veto power since only the quota has changed (increased). P_3 now also has veto power here since the other players combined have only 8 + 4 + 1 = 13 votes and cannot pass a motion without him. P_4, on the other hand, does not have veto power since the other players combined have 8 + 4 + 2 = 14 votes.

(d) P_4, the player with the fewest votes, has veto power since the other players combined have only 8 + 4 + 2 = 14 votes and cannot pass a motion without him. Since the other players all have the same or more votes than P_4, it follows that all four players have veto power in this system.

7. (a) In order for all three players to have veto power, the player having the fewest votes (the weakest) must have veto power. In order for that to happen, the quota q must be *strictly* larger than 7 + 5 = 12. The smallest value of q for which this is true is $q = 13$.

(b) In order for P_2 to have veto power, the quota q must be strictly larger than 7 + 3 = 10. The smallest value of q for which this is true is $q = 11$. [Note: When $q = 11$, we note that P_3 does not have veto power since the other two players have 7 + 5 = 12 votes.]

8. (a) In order for all five players to have veto power, the player having the fewest votes (the weakest) must have veto power. In order for that to happen, the quota q must be *strictly* larger than 10 + 8 + 6 + 4 = 28. The smallest value of q for which this is true is $q = 29$.

(b) In order for P_3 to have veto power, the quota q must be strictly larger than 10 + 8 + 4 + 2 = 24. The smallest value of q for which this is true is q = 25. [Note: When q = 25, we note that P_4 does not have veto power since the other four players have 10 + 8 + 6 + 2 = 26 votes.]

9. To determine the number of votes each player has, write the weighted voting system as [49: 4*x*, 2*x*, *x*, *x*].

(a) If the quota is defined as a simple majority of the votes, then x is the largest integer satisfying $49\ !\ \frac{4x\ \ 2x\ x\ x}{2}$ which means that $8x\ 98$ or $x\ 12.25$. So, $x = 12$ and the system can be described as [49: 48, 24, 12, 12].

(b) If the quota is defined as more than two-thirds of the votes, then x is the largest integer satisfying $49\ !\ \frac{2\ 4x\ 2x\ x\ \ x}{3}$ which means that $16x\ 147$ or $x\ 9.1875$. So, $x = 9$ and the system can be described as [49: 36, 18, 9, 9].

(c) If the quota is defined as more than three-fourths of the votes, then x is the largest integer satisfying $49\ !\ \frac{3\ 4x\ 2x\ x\ \ x}{4}$ which means that $24x\ 196$ or $x\ 8.167$. So, $x = 8$ and the system can be described as [49: 32, 16, 8, 8].

10. **(a)** [121: 96, 48, 48, 24, 12, 12]; if the quota is defined as a simple majority of the votes, then x is the largest integer satisfying $121\ !\ \frac{8x\ \ 4x\ \ 4x\ \ 2x\ \ x\ \ x}{2}$.

(b) [121: 72, 36, 36, 18, 9, 9]; if the quota is defined as more than two-thirds of the votes, then x is the largest integer satisfying $121\ !\ \frac{28x\ 4x\ 4x\ 2x\ \ x\ \ x}{3}$.

(c) [121: 64, 32, 32, 16, 8, 8]; if the quota is defined as more than three-fourths of the votes, then x is the largest integer satisfying $121\ !\ \frac{38x\ 4x\ 4x\ 2x\ \ x\ \ x}{4}$.

2.2. Banzhaf Power

11. **(a)** $w_1\ \ w_3$ 7 + 3 = 10

(b) Since (7 + 5 + 3) / 2 = 7.5, the smallest allowed value of the quota q in this system is 8. For $\{P_1, P_3\}$ to be a winning coalition, the quota q could be at most 10. So, the values of the quota for which $\{P_1, P_3\}$ is winning are 8, 9, and 10.

(c) Since 7 + 5 + 3 = 15, the largest allowed value of the quota q in this system is 15. For $\{P_1, P_3\}$ to be a losing coalition, the quota q must be strictly greater than its weight, 10. So, the values of the quota for which $\{P_1, P_3\}$ is losing are 11, 12, 13, 14, and 15.

12. **(a)** 8 + 6 + 4 = 18

(b) Since (10 + 8 + 6 + 4 + 2) / 2 = 15, the smallest allowed value of the quota q in this system is 16. For $\{P_2, P_3, P_4\}$ to be a winning coalition, the quota q could be at most 18. So, the values of the quota for which $\{P_2, P_3, P_4\}$ is winning are 16, 17, and 18.

(c) Since 10 + 8 + 6 + 4 + 2 = 30, the largest allowed value of the quota q in this system is 30. For $\{P_2, P_3, P_4\}$ to be a losing coalition, the quota q must be strictly greater than its weight, 18. So, the values of the quota for which $\{P_2, P_3, P_4\}$ is losing are integer values from 19 to 30.

13. P_1 is critical (underlined) three times; P_2 is critical three times; P_3 is critical once ; P_4 is critical once. The total number of times the players are critical is 8 (number of underlines). The Banzhaf power distribution is $E_1 \quad 3/8;\ E_2 \quad 3/8;\ E_3 \quad 1/8;\ E_4 \quad 1/8$.

14. P_1 is critical (underlined) seven times; P_2 is critical five times; P_3 is critical three times; P_4 is critical three times; P_5 is critical one time. The total number of times the players are critical (all underlines) is 19. The Banzhaf power distribution is $E_1 \quad 7/19;\ E_2 \quad 5/19;\ E_3 \quad 3/19;\ E_4 \quad 3/19;\ E_5 \quad 1/19$.

15. (a) P_1 is critical since the other players only have 5 + 2 = 7 votes. P_2 is also critical since the other players only have 6 + 2 = 8 votes. However, P_4 is not critical since the other two players have 6 + 5 = 11 (more than q = 10).

(b) The winning coalitions are those whose weights are 10 or more. These are: $\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_3, P_4\}$, $\{P_2, P_3, P_4\}$, $\{P_1, P_2, P_3, P_4\}$.

(c) We underline the critical players in each winning coalition: $\{\underline{P_1}, \underline{P_2}\}$, $\{\underline{P_1}, \underline{P_3}\}$, $\{\underline{P_1}, P_2, P_3\}$, $\{\underline{P_1}, \underline{P_2}, P_4\}$, $\{\underline{P_1}, \underline{P_3}, P_4\}$, $\{\underline{P_2}, \underline{P_3}, \underline{P_4}\}$, $\{P_1, P_2, P_3, P_4\}$.

Then, it follows that $E_1 \quad \frac{5}{12};\ E_2 \quad E_3 \quad \frac{3}{12};\ E_4 \quad \frac{1}{12}$.

16. (a) All the players are critical in this coalition since the total weight of the coalition (3 + 1 + 1 = 5) is exactly the same as the quota. If any one player were to leave the coalition, the remaining players would not have enough votes to meet the quota.

(b) The winning coalitions are those whose weights are 5 or more. These are: $\{P_1, P_2\}$, $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_3, P_4\}$, $\{P_1, P_2, P_3, P_4\}$.

(c) We underline the critical players in each winning coalition: $\{\underline{P_1}, \underline{P_2}\}$, $\{\underline{P_1}, \underline{P_2}, P_3\}$, $\{\underline{P_1}, \underline{P_2}, P_4\}$, $\{\underline{P_1}, \underline{P_3}, \underline{P_4}\}$, $\{\underline{P_1}, P_2, P_3, P_4\}$.

Then, it follows that $E_1 \quad \frac{5}{10};\ E_2 \quad \frac{3}{10};\ E_3 \quad \frac{1}{10};\ E_4 \quad \frac{1}{10}$.

17. (a) The winning coalitions (with critical players underlined) are: $\{\underline{P_1}, \underline{P_2}\}$, $\{\underline{P_1}, \underline{P_3}\}$, and $\{\underline{P_1}, P_2, P_3\}$. P_1 is critical three times; P_2 is critical one time; P_3 is critical one time. The total number of times the players are critical is 5. The Banzhaf power distribution is $E_1 \quad 3/5;\ E_2 \quad 1/5;\ E_3 \quad 1/5$.

(b) The winning coalitions (with critical players underlined) are: $\{\underline{P_1}, \underline{P_2}\}$, $\{\underline{P_1}, \underline{P_3}\}$, and $\{\underline{P_1}, P_2, P_3\}$. P_1 is critical three times; P_2 is critical one time; P_3 is critical one time. The total number of times the players are critical is 5. The Banzhaf power distribution is $E_1 \quad 3/5$; $E_2 \quad 1/5$; $E_3 \quad 1/5$. The distributions in (a) and (b) are the same.

18. (a) The winning coalitions (with critical players underlined) are: $\{\underline{P_1}, \underline{P_2}\}$ and $\{\underline{P_1}, \underline{P_2}, P_3\}$. P_1 is critical twice; P_2 is critical twice; P_3 is never critical. The total number of times the players are critical is 4. The Banzhaf power distribution is $E_1 \quad 2/4$; $E_2 \quad 2/4$; $E_3 \quad 0/4$.

(b) The winning coalitions (with critical players underlined) are: $\{\underline{P_1}, \underline{P_2}\}$ and $\{\underline{P_1}, \underline{P_2}, P_3\}$. P_1 is critical twice; P_2 is critical twice; P_3 is never critical. The total number of times the players are critical is 4. The Banzhaf power distribution is $\text{E}_1 \;\; 2/4$; $\text{E}_2 \;\; 2/4$; $\text{E}_3 \;\; 0/4$. The distributions in (a) and (b) are the same.

19. (a) The winning coalitions (with critical players underlined) are: $\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_5}\}$, $\{\underline{P_1}, \underline{P_3}, \underline{P_4}\}$, $\{\underline{P_1}, P_2, P_3, P_4\}$, $\{\underline{P_1}, \underline{P_2}, P_3, P_5\}$, $\{\underline{P_1}, \underline{P_2}, P_4, P_5\}$, $\{\underline{P_1}, \underline{P_3}, \underline{P_4}, P_5\}$, $\{\underline{P_2}, \underline{P_3}, \underline{P_4}, \underline{P_5}\}$, $\{P_1, P_2, P_3, P_4, P_5\}$. The Banzhaf power distribution is $\text{E}_1 \;\; 8/24$; $\text{E}_2 \;\; 6/24$; $\text{E}_3 \;\; 4/24$; $\text{E}_4 \;\; 4/24$; $\text{E}_5 \;\; 2/24$.

(b) The situation is like (a) except that $\{P_1, P_2, P_5\}$, $\{P_1, P_3, P_4\}$ and $\{P_2, P_3, P_4, P_5\}$ are now losing coalitions. In addition, P_2 is now critical in $\{P_1, P_2, P_3, P_4\}$, P_3 is now critical in $\{P_1, P_2, P_3, P_5\}$, P_4 is now critical in $\{P_1, P_2, P_4, P_5\}$, P_5 is now critical in $\{P_1, P_3, P_4, P_5\}$, and P_1 is now critical in the grand coalition $\{P_1, P_2, P_3, P_4, P_5\}$. The winning coalitions (with critical players underlined) are now: $\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}\}$, $\{\underline{P_1}, \underline{P_2}, P_3, P_4\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, P_5\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}, P_5\}$, $\{\underline{P_1}, \underline{P_3}, \underline{P_4}, \underline{P_5}\}$, $\{\underline{P_1}, P_2, P_3, P_4, P_5\}$. The Banzhaf power distribution is $\text{E}_1 \;\; 7/19$; $\text{E}_2 \;\; 5/19$; $\text{E}_3 \;\; 3/19$; $\text{E}_4 \;\; 3/19$; $\text{E}_5 \;\; 1/19$.

(c) This situation is like (b) with the following exceptions: $\{P_1, P_2, P_4\}$ and $\{P_1, P_3, P_4, P_5\}$ are now losing coalitions; P_3 is critical in $\{P_1, P_2, P_3, P_4\}$; P_5 is critical in $\{P_1, P_2, P_4, P_5\}$; and P_2 is critical in

$\{P_1, P_2, P_3, P_4, P_5\}$. The winning coalitions (with critical players underlined) are now: $\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, P_4\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, P_5\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}, \underline{P_5}\}$, $\{\underline{P_1}, \underline{P_2}, P_3, P_4, P_5\}$. The Banzhaf power distribution is $\text{E}_1 \;\; 5/15$; $\text{E}_2 \;\; 5/15$; $\text{E}_3 \;\; 3/15$; $\text{E}_4 \;\; 1/15$; $\text{E}_5 \;\; 1/15$.

(d) Since the quota equals the total number of votes in the system, the only winning coalition is the grand coalition and every player is critical in that coalition. The Banzhaf power distribution is easy to calculate in the case since all players share power equally. It is $\text{E}_1 \;\; 1/5$; $\text{E}_2 \;\; 1/5$; $\text{E}_3 \;\; 1/5$; $\text{E}_4 \;\; 1/5$; $\text{E}_5 \;\; 1/5$.

20. (a) P_1 is a dictator and the other players are dummies. Thus P_1 is the only critical player in each winning coalition. The Banzhaf power distribution is $\text{E}_1 \;\; 1$; $\text{E}_2 \;\; \text{E}_3 \;\; \text{E}_4 \;\; 0$.

(b) The winning coalitions (with critical players underlined) are: $\{\underline{P_1}, \underline{P_2}\}$, $\{\underline{P_1}, \underline{P_3}\}$, $\{\underline{P_1}, \underline{P_4}\}$, $\{\underline{P_1}, P_2, P_3\}$, $\{\underline{P_1}, P_2, P_4\}$, $\{\underline{P_1}, P_3, P_4\}$, $\{\underline{P_1}, P_2, P_3, P_4\}$. The Banzhaf power distribution is $\text{E}_1 \;\; \frac{7}{10} \;\; 70\%$; $\text{E}_2 \;\; \frac{1}{10} \;\; 10\%$; $\text{E}_3 \;\; \frac{1}{10} \;\; 10\%$; $\text{E}_4 \;\; \frac{1}{10} \;\; 10\%$.

(c) This situation is like (b) with the following exceptions: $\{P_1, P_4\}$ is now a losing coalition; P_1 and P_2 are both critical in $\{P_1, P_2, P_4\}$; P_1 and P_3 are both critical in $\{P_1, P_3, P_4\}$. The winning coalitions (with critical players underlined) are now: $\{\underline{P_1}, \underline{P_2}\}$, $\{\underline{P_1}, \underline{P_3}\}$, $\{\underline{P_1}, P_2, P_3\}$, $\{\underline{P_1}, \underline{P_2}, P_4\}$, $\{\underline{P_1}, \underline{P_3}, P_4\}$, $\{\underline{P_1}, P_2, P_3, P_4\}$. The Banzhaf power distribution is $\mathrm{E}_1 \ \frac{6}{10} \ 60\%$; $\mathrm{E}_2 \ \frac{2}{10} \ 20\%$; $\mathrm{E}_3 \ \frac{2}{10} \ 20\%$; $\mathrm{E}_4 \ 0$.

(d) The winning coalitions (with critical players underlined) are now: $\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, P_4\}$. The Banzhaf power distribution is $\text{E}_1 \quad \frac{2}{6};\ \text{E}_2 \quad \frac{2}{6};\ \text{E}_3 \quad \frac{2}{6};\ \text{E}_4 \quad 0.$

21. (a) A player is critical in a coalition if that coalition without the player is not on the list of winning coalitions. So, in this case, the critical players are underlined below.
$\{\underline{P_1}, \underline{P_2}\}$, $\{\underline{P_1}, \underline{P_3}\}$, $\{\underline{P_1}, P_2, P_3\}$

(b) P_1 is critical three times; P_2 is critical one time; P_3 is critical one time. The total number of times all players are critical is 3 + 1 + 1 = 5. The Banzhaf power distribution is $\text{E}_1 \quad 3/5;\ \text{E}_2 \quad 1/5;\ \text{E}_3 \quad 1/5.$

22. (a) A player is critical in a coalition if that coalition without the player is not on the list of winning coalitions. $\{\underline{P_1}, \underline{P_2}\}$, $\{\underline{P_1}, \underline{P_2}, P_3\}$, $\{\underline{P_1}, \underline{P_2}, P_4\}$, $\{\underline{P_1}, \underline{P_2}, P_3, P_4\}$

(b) $\text{E}_1 \quad 4/8,\ \text{E}_2 \quad 4/8,\ \text{E}_3 \quad \text{E}_4 \quad 0$

23. (a) The winning coalitions (with critical players underlined) are $\{\underline{P_1}, \underline{P_2}\}$, $\{\underline{P_1}, \underline{P_3}\}$, $\{\underline{P_2}, \underline{P_3}\}$, $\{P_1, P_2, P_3\}$, $\{\underline{P_1}, \underline{P_2}, P_4\}$, $\{\underline{P_1}, \underline{P_2}, P_5\}$, $\{\underline{P_1}, \underline{P_2}, P_6\}$, $\{\underline{P_1}, \underline{P_3}, P_4\}$, $\{\underline{P_1}, \underline{P_3}, P_5\}$, $\{\underline{P_1}, \underline{P_3}, P_6\}$, $\{\underline{P_2}, \underline{P_3}, P_4\}$, $\{\underline{P_2}, \underline{P_3}, P_5\}$, $\{\underline{P_2}, \underline{P_3}, P_6\}$.

(b) These winning coalitions (with critical players underlined) are $\{\underline{P_1}, \underline{P_2}, P_4\}$, $\{\underline{P_1}, \underline{P_3}, P_4\}$, $\{\underline{P_2}, \underline{P_3}, P_4\}$, $\{P_1, P_2, P_3, P_4\}$, $\{\underline{P_1}, \underline{P_2}, P_4, P_5\}$, $\{\underline{P_1}, \underline{P_2}, P_4, P_6\}$, $\{\underline{P_1}, \underline{P_3}, P_4, P_5\}$, $\{\underline{P_1}, \underline{P_3}, P_4, P_6\}$, $\{\underline{P_2}, \underline{P_3}, P_4, P_5\}$, $\{\underline{P_2}, \underline{P_3}, P_4, P_6\}$, $\{P_1, P_2, P_3, P_4, P_5\}$, $\{P_1, P_2, P_3, P_4, P_6\}$, $\{\underline{P_1}, \underline{P_2}, P_4, P_5, P_6\}$, $\{\underline{P_1}, \underline{P_3}, P_4, P_5, P_6\}$, $\{\underline{P_2}, \underline{P_3}, P_4, P_5, P_6\}$, $\{P_1, P_2, P_3, P_4, P_5, P_6\}$.

(c) P_4 is never a critical player since every time it is part of a winning coalition, that coalition is a winning coalition without P_4 as well. So, $\text{E}_4 \quad 0.$

(d) A similar argument to that used in part (c) shows that P_5 and P_6 are also dummies. One could also argue that any player with fewer votes than P_4, a dummy, will also be a dummy. So, P_4, P_5, and P_6 will never be critical -- they all have zero power.

The only winning coalitions with only two players are $\{P_1, P_2\}$, $\{P_1, P_3\}$, and $\{P_2, P_3\}$; and both players are critical in each of those coalitions. All other winning coalitions consist of one of these coalitions plus additional players, and the only critical players will be the ones from the two-player coalition. So P_1, P_2, and P_3 will be critical in every winning coalition they are in, and they will all be in the same number of winning coalitions, so they all have the same power. Thus, the Banzhaf power distribution is $\text{E}_1 \quad 1/3;\ \text{E}_2 \quad 1/3;\ \text{E}_3 \quad 1/3;\ \text{E}_4 \quad 0;\ \text{E}_5 \quad 0;\ \text{E}_6 \quad 0.$

24. (a) Three-player winning coalitions (with critical players underlined): $\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_5}\}$, $\{\underline{P_1}, \underline{P_3}, \underline{P_4}\}$, $\{\underline{P_2}, \underline{P_3}, \underline{P_4}\}$.

(b) Four-player winning coalitions (with critical players underlined): $\{P_1, P_2, P_3, P_4\}$, $\{\underline{P_1}, \underline{P_2}, P_3, P_5\}$,

$\{\underline{P_1}, \underline{P_2}, \underline{P_3}, P_6\}$, $\{\underline{P_1}, \underline{P_2}, P_4, P_5\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}, P_6\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_5}, P_6\}$, $\{\underline{P_1}, \underline{P_3}, \underline{P_4}, P_5\}$, $\{\underline{P_1}, \underline{P_3}, \underline{P_4}, P_6\}$, $\{\underline{P_1}, \underline{P_3}, \underline{P_5}, \underline{P_6}\}$, $\{\underline{P_2}, \underline{P_3}, \underline{P_4}, P_5\}$, $\{\underline{P_2}, \underline{P_3}, \underline{P_4}, P_6\}$.

(c) Five-player winning coalitions (with critical players underlined): $\{P_1, P_2, P_3, P_4, P_5\}$,
$\{P_1, P_2, P_3, P_4, P_6\}$, $\{\underline{P_1}, P_2, P_3, P_5, P_6\}$, $\{\underline{P_1}, \underline{P_2}, P_4, P_5, P_6\}$, $\{\underline{P_1}, \underline{P_3}, P_4, P_5, P_6\}$, $\{\underline{P_2}, \underline{P_3}, \underline{P_4}, P_5, P_6\}$.

(d) $E_1 \quad \frac{15}{52}$; $E_2 \quad \frac{13}{52}$; $E_3 \quad \frac{11}{52}$; $E_4 \quad \frac{9}{52}$; $E_5 \quad \frac{3}{52}$; $E_6 \quad \frac{1}{52}$. Note: The grand coalition is also a winning coalition. However, it has no critical players.

25. **(a)** $\{\underline{A}, \underline{B}\}, \{\underline{A}, \underline{C}\}, \{\underline{B}, \underline{C}\}, \{A, B, C\}, \{\underline{A}, \underline{B}, D\}, \{\underline{A}, \underline{C}, D\}, \{\underline{B}, \underline{C}, D\}, \{A, B, C, D\}$

(b) *A*, *B*, and *C* have Banzhaf power index of 4/12 each; *D* is a dummy. (*D* is never a critical player, and the other three clearly have equal power.)

26. **(a)** $\{\underline{A}, \underline{B}\}, \{\underline{A}, \underline{C}\}, \{\underline{A}, \underline{D}\}, \{\underline{A}, B, C\}, \{\underline{A}, B, D\}, \{\underline{A}, C, D\}, \{\underline{B}, \underline{C}, \underline{D}\}, \{A, B, C, D\}$

(b) *A* has Banzhaf power index of 6/12; *B*, *C*, and *D* have Banzhaf power index of 2/12 each.

2.3. Shapley-Shubik Power

27. P_1 is pivotal (underlined) ten times; P_2 is pivotal (again, underlined) ten times; P_3 is pivotal twice ; P_4 is pivotal twice. The total number of times the players are pivotal is 4! = 24 (number of underlines). The Shapley-Shubik power distribution is $\varsigma_1 \quad 10/24$; $\varsigma_2 \quad 10/24$; $\varsigma_3 \quad 2/24$; $\varsigma_4 \quad 2/24$.

28. P_1 is pivotal (underlined) ten times; P_2 is pivotal six times; P_3 is pivotal six times ; P_4 is pivotal twice. The total number of times the players are pivotal is 4! = 24. The Shapley-Shubik power distribution is $\varsigma_1 \quad 10/24$; $\varsigma_2 \quad 6/24$; $\varsigma_3 \quad 6/24$; $\varsigma_4 \quad 2/24$.

29. **(a)** There are 3! = 6 sequential coalitions of the three players. Each pivotal player is underlined.
$P_1, \underline{P_2}, P_3$!, $P_1, \underline{P_3}, P_2$!, $P_2, \underline{P_1}, P_3$!, $P_2, P_3, \underline{P_1}$!, $P_3, \underline{P_1}, P_2$!, $P_3, P_2, \underline{P_1}$!

(b) P_1 is pivotal four times; P_2 is pivotal one time; P_3 is pivotal one time. The Shapley-Shubik power distribution is $\varsigma_1 \quad 4/6$; $\varsigma_2 \quad 1/6$; $\varsigma_3 \quad 1/6$.

30. **(a)** There are 3! = 6 sequential coalitions of the three players. Each pivotal player is underlined.
$P_1, \underline{P_2}, P_3$!, $P_1, \underline{P_3}, P_2$!, $P_2, \underline{P_1}, P_3$!, $P_2, \underline{P_3}, P_1$!, $P_3, \underline{P_1}, P_2$!, $P_3, \underline{P_2}, P_1$!

(b) $\varsigma_1 \quad 2/6$; $\varsigma_2 \quad 2/6$; $\varsigma_3 \quad 2/6$.

31. **(a)** Since P_1 is a dictator, $\varsigma_1 \quad 1$, $\varsigma_2 \quad 0$, $\varsigma_3 \quad 0$, and $\varsigma_4 \quad 0$.

(b) There are 4! = 24 sequential coalitions of the four players. Each pivotal player is underlined.
$P_1, \underline{P_2}, P_3, P_4$!, $P_1, \underline{P_2}, P_4, P_3$!, $P_1, \underline{P_3}, P_2, P_4$!, $P_1, \underline{P_3}, P_4, P_2$!,
$P_1, P_4, \underline{P_2}, P_3$!, $P_1, P_4, \underline{P_3}, P_2$!, $P_2, \underline{P_1}, P_3, P_4$!, $P_2, \underline{P_1}, P_4, P_3$!,
$P_2, P_3, \underline{P_1}, P_4$!, $P_2, P_3, P_4, \underline{P_1}$!, $P_2, P_4, \underline{P_1}, P_3$!, $P_2, P_4, P_3, \underline{P_1}$!,
$P_3, \underline{P_1}, P_2, P_4$!, $P_3, \underline{P_1}, P_4, P_2$!, $P_3, P_2, \underline{P_1}, P_4$!, $P_3, P_2, P_4, \underline{P_1}$!,
$P_3, P_4, \underline{P_1}, P_2$!, $P_3, P_4, P_2, \underline{P_1}$!, $P_4, P_1, \underline{P_2}, P_3$!, $P_4, P_1, \underline{P_3}, P_2$!,

$P_4, P_2, \underline{P_1}, P_3$!, $P_4, P_2, P_3, \underline{P_1}$!, $P_4, P_3, \underline{P_1}, P_2$!, $P_4, P_3, P_2, \underline{P_1}$!

P_1 is pivotal 16 times; P_2 is pivotal 4 times; P_3 is pivotal 4 times; P_4 is pivotal 0 times. The Shapley-Shubik power distribution is $\varsigma_1 \quad 16/24$; $\varsigma_2 \quad 4/24$; $\varsigma_3 \quad 4/24$; $\varsigma_4 \quad 0$.

(c) The only way a motion will pass is if P_1 and P_2 both support it. In fact, the second of these players that appears in a sequential coalition will be the pivotal player in that coalition. It follows that $\varsigma_1 \quad 12/24 \quad 1/2$, $\varsigma_2 \quad 12/24 \quad 1/2$, $\varsigma_3 \quad 0$, and $\varsigma_4 \quad 0$.

(d) Because the quota is so high, the only way a motion will pass is if P_1, P_2, and P_3 all support it. In fact, the third of these players that appears in a sequential coalition will always be the pivotal player in that coalition. It follows that $\varsigma_1 \quad 8/24 \quad 1/3$, $\varsigma_2 \quad 8/24 \quad 1/3$, $\varsigma_3 \quad 8/24 \quad 1/3$, and $\varsigma_4 \quad 0$.

32. (a) Since P_1 is a dictator, $\varsigma_1 \quad 1$, $\varsigma_2 \quad 0$, $\varsigma_3 \quad 0$, $\varsigma_4 \quad 0$.

(b)
$P_1, \underline{P_2}, P_3, P_4$!, $P_1, \underline{P_2}, P_4, P_3$!, $P_1, \underline{P_3}, P_2, P_4$!, $P_1, \underline{P_3}, P_4, P_2$!,
$P_1, \underline{P_4}, P_2, P_3$!, $P_1, \underline{P_4}, P_3, P_2$!, $P_2, \underline{P_1}, P_3, P_4$!, $P_2, \underline{P_1}, P_4, P_3$!,
$P_2, P_3, \underline{P_1}, P_4$!, $P_2, P_3, P_4, \underline{P_1}$!, $P_2, P_4, \underline{P_1}, P_3$!, $P_2, P_4, P_3, \underline{P_1}$!,
$P_3, \underline{P_1}, P_2, P_4$!, $P_3, \underline{P_1}, P_4, P_2$!, $P_3, P_2, \underline{P_1}, P_4$!, $P_3, P_2, P_4, \underline{P_1}$!,
$P_3, P_4, \underline{P_1}, P_2$!, $P_3, P_4, P_2, \underline{P_1}$!, $P_4, \underline{P_1}, P_2, P_3$!, $P_4, \underline{P_1}, P_3, P_2$!,
$P_4, P_2, \underline{P_1}, P_3$!, $P_4, P_2, P_3, \underline{P_1}$!, $P_4, P_3, \underline{P_1}, P_2$!, $P_4, P_3, P_2, \underline{P_1}$!

P_1 is pivotal 18 times; P_2 is pivotal 2 times; P_3 is pivotal 2 times; P_4 is pivotal 2 times.
The Shapley-Shubik power distribution is $\varsigma_1 \quad 9/12$, $\varsigma_2 \quad 1/12$, $\varsigma_3 \quad 1/12$, $\varsigma_4 \quad 1/12$.

(c) $\varsigma_1 \quad 1/2$, $\varsigma_2 \quad 1/2$, $\varsigma_3 \quad 0$, $\varsigma_4 \quad 0$ (The second of P_1 or P_2 to vote is always the pivotal player in any of the 24 sequential coalitions. This happens 12 times in each case.)

(d) $\varsigma_1 \quad 1/3$, $\varsigma_2 \quad 1/3$, $\varsigma_3 \quad 1/3$, $\varsigma_4 \quad 0$ (The third of P_1, P_2, or P_3 to vote is always the pivotal player in any of the 24 sequential coalitions. This happens 8 times in each case.)

33. (a) There are 4! = 24 sequential coalitions of the four players. Each pivotal player is underlined.
$P_1, \underline{P_2}, P_3, P_4$!, $P_1, \underline{P_2}, P_4, P_3$!, $P_1, \underline{P_3}, P_2, P_4$!, $P_1, \underline{P_3}, P_4, P_2$!,
$P_1, P_4, \underline{P_2}, P_3$!, $P_1, P_4, \underline{P_3}, P_2$!, $P_2, \underline{P_1}, P_3, P_4$!, $P_2, \underline{P_1}, P_4, P_3$!,
$P_2, P_3, \underline{P_1}, P_4$!, $P_2, P_3, \underline{P_4}, P_1$!, $P_2, P_4, \underline{P_1}, P_3$!, $P_2, P_4, \underline{P_3}, P_1$!,
$P_3, \underline{P_1}, P_2, P_4$!, $P_3, \underline{P_1}, P_4, P_2$!, $P_3, P_2, \underline{P_1}, P_4$!, $P_3, P_2, \underline{P_4}, P_1$!,
$P_3, P_4, \underline{P_1}, P_2$!, $P_3, P_4, \underline{P_2}, P_1$!, $P_4, P_1, \underline{P_2}, P_3$!, $P_4, P_1, \underline{P_3}, P_2$!,
$P_4, P_2, \underline{P_1}, P_3$!, $P_4, P_2, \underline{P_3}, P_1$!, $P_4, P_3, \underline{P_1}, P_2$!, $P_4, P_3, \underline{P_2}, P_1$!

P_1 is pivotal in 10 coalitions; P_2 is pivotal in six coalitions; P_3 is pivotal in six coalitions; P_4 is pivotal in two coalitions. The Shapley-Shubik power distribution is $\varsigma_1 \quad 10/24$; $\varsigma_2 \quad 6/24$; $\varsigma_3 \quad 6/24$; $\varsigma_4 \quad 2/24$.

(b) This is the same situation as in (a) – there is essentially no difference between 51 and 59 because the players' votes are all multiples of 10. The Shapley-Shubik power distribution is thus still $\varsigma_1 \quad 10/24$; $\varsigma_2 \quad 6/24$; $\varsigma_3 \quad 6/24$; $\varsigma_4 \quad 2/24$.

(c) This is also the same situation as in (a) – any time a group of players has 51 votes, they must have 60 votes. The Shapley-Shubik power distribution is thus still $\varsigma_1 \quad 10/24$; $\varsigma_2 \quad 6/24$; $\varsigma_3 \quad 6/24$; $\varsigma_4 \quad 2/24$.

34. (a) $\varsigma_1 \quad 18/24; \varsigma_2 \quad 2/24; \varsigma_3 \quad 2/24; \varsigma_4 \quad 2/24.$

The sequential coalitions (with pivotal player in each coalition underlined) are:

$P_1, \underline{P_2}, P_3, P_4$!, $P_1, \underline{P_2}, P_4, P_3$!, $P_1, \underline{P_3}, P_2, P_4$!, $P_1, \underline{P_3}, P_4, P_2$!,

$P_1, \underline{P_4}, P_2, P_3$!, $P_1, \underline{P_4}, P_3, P_2$!, $P_2, \underline{P_1}, P_3, P_4$!, $P_2, \underline{P_1}, P_4, P_3$!,

$P_2, P_3, \underline{P_1}, P_4$!, $P_2, P_3, P_4, \underline{P_1}$!, $P_2, P_4, \underline{P_1}, P_3$!, $P_2, P_4, P_3, \underline{P_1}$!,

$P_3, \underline{P_1}, P_2, P_4$!, $P_3, \underline{P_1}, P_4, P_2$!, $P_3, P_2, \underline{P_1}, P_4$!, $P_3, P_2, P_4, \underline{P_1}$!,

$P_3, P_4, \underline{P_1}, P_2$!, $P_3, P_4, P_2, \underline{P_1}$!, $P_4, \underline{P_1}, P_2, P_3$!, $P_4, \underline{P_1}, P_3, P_2$!,

$P_4, P_2, \underline{P_1}, P_3$!, $P_4, P_2, P_3, \underline{P_1}$!, $P_4, P_3, \underline{P_1}, P_2$!, $P_4, P_3, P_2, \underline{P_1}$!

(b) $\varsigma_1 \quad 18/24; \varsigma_2 \quad 2/24; \varsigma_3 \quad 2/24; \varsigma_4 \quad 2/24.$

This is the same situation as in (a).

(c) This is also the same situation as in (a) – any time a group of players has 41 votes, they must have 50 votes. The Shapley-Shubik power distribution is thus still $\varsigma_1 \quad 12/24; \quad \varsigma_2 \quad 4/24; \quad \varsigma_3 \quad 4/24;$ $\varsigma_4 \quad 4/24.$

35. (a) There are 3! = 6 sequential coalitions of the three players.

$P_1, \underline{P_2}, P_3$!, $P_1, \underline{P_3}, P_2$!, $P_2, \underline{P_1}, P_3$!, $P_2, P_3, \underline{P_1}$!, $P_3, \underline{P_1}, P_2$!, $P_3, P_2, \underline{P_1}$!

(The second player listed will always be pivotal unless the first two players listed are P_2 and P_3 since that is not a winning coalition. In that case, P_1 is pivotal.)

(b) P_1 is pivotal four times; P_2 is pivotal one time; P_3 is pivotal one time. The Shapley-Shubik power distribution is $\varsigma_1 \quad 4/6; \quad \varsigma_2 \quad 1/6; \quad \varsigma_3 \quad 1/6.$

36. (a) $P_1, \underline{P_2}, P_3$!, $P_1, P_3, \underline{P_2}$!, $P_2, \underline{P_1}, P_3$!, $P_2, P_3, \underline{P_1}$!, $P_3, P_1, \underline{P_2}$!, $P_3, P_2, \underline{P_1}$!

(The third player listed will always be pivotal unless the first two players listed are P_1 and P_2 since that is a winning coalition. In that case, the second of these players is pivotal.)

(b) $\varsigma_1 \quad 1/2, \varsigma_2 \quad 1/2, \varsigma_3 \quad 0$

37. We proceed to identify pivotal players by moving down each column. We start with the first column. Since P_2 is pivotal in the sequential coalition $P_1, \underline{P_2}, P_3, P_4$! , we know that $\perp P_1, P_2 \overline{}$ form a winning coalition. This tells us that P_2 is also pivotal in the second sequential coalition $P_1, \underline{P_2}, P_4, P_3$! listed in the first column. A similar argument tells us that P_3 is pivotal in the fourth sequential coalition $P_1, \underline{P_3}, P_4, P_2$! listed in the first column.

In the second column, P_1 is clearly pivotal in the sequential coalition $P_2, \underline{P_1}, P_3, P_4$! since we know that $\perp P_1, P_2 \overline{}$ form a winning coalition. Similarly, P_1 is pivotal in the sequential coalition $P_2, \underline{P_1}, P_4, P_3$! .

In the third column, P_1 is pivotal in the sequential coalition $P_3, \underline{P_1}, P_2, P_4$! since the third sequential coalition in the first column ($P_1, \underline{P_3}, P_2, P_4$!) identified $\perp P_1, P_3 \overline{}$ as a winning coalition. Similarly, P_1 is pivotal in the sequential coalition $P_3, \underline{P_1}, P_4, P_2$! . Since P_1 was pivotal in the sequential coalition

$P_2, P_3, \underline{P_1}, P_4$! back in the second column, it must also be the case that P_1 is pivotal in the sequential coalition $P_3, P_2, \underline{P_1}, P_4$! . Similarly, P_4 is pivotal in the sequential coalition $P_3, P_2, \underline{P_4}, P_1$! since P_4 was also pivotal in the sequential coalition $P_2,\ P_3,\ \underline{P_4},\ P_1\ !$ back in the second column.

In the fourth column, the third player listed will always be pivotal since the first two players when listed first in the opposite order (in an earlier column) never contain a pivotal player and the fourth player was never pivotal in the first three columns. The final listing of pivotal players is found below.

$P_1,\ \underline{P_2},\ P_3,\ P_4\ !,\ P_2,\ \underline{P_1},\ P_3,\ P_4\ !,\ P_3,\ \underline{P_1},\ P_2,\ P_4\ !,\ P_4,\ P_1,\ \underline{P_2},\ P_3\ !,$

$P_1,\ \underline{P_2},\ P_4,\ P_3\ !,\ P_2,\ \underline{P_1},\ P_4,\ P_3\ !,\ P_3,\ \underline{P_1},\ P_4,\ P_2\ !,\ P_4,\ P_1,\ \underline{P_3},\ P_2\ !,$

$P_1,\ \underline{P_3},\ P_2,\ P_4\ !,\ P_2,\ P_3,\ \underline{P_1},\ P_4\ !,\ P_3,\ P_2,\ \underline{P_1},\ P_4\ !,\ P_4,\ P_2,\ \underline{P_1},\ P_3\ !,$

$P_1,\ \underline{P_3},\ P_4,\ P_2\ !,\ P_2,\ P_3,\ \underline{P_4},\ P_1\ !,\ P_3,\ P_2,\ \underline{P_4},\ P_1\ !,\ P_4,\ P_2,\ \underline{P_3},\ P_1\ !,$

$P_1,\ P_4,\ \underline{P_2},\ P_3\ !,\ P_2,\ P_4,\ \underline{P_1},\ P_3\ !,\ P_3,\ P_4,\ \underline{P_1},\ P_2\ !,\ P_4,\ P_3,\ \underline{P_1},\ P_2\ !,$

$P_1,\ P_4,\ \underline{P_3},\ P_2\ !,\ P_2,\ P_4,\ \underline{P_3},\ P_1\ !,\ P_3,\ P_4,\ \underline{P_2},\ P_1\ !,\ P_4,\ P_3,\ \underline{P_2},\ P_1\ !$

The Shapley-Shubik distribution is then $\varsigma_1 \quad 10/24;\ \varsigma_2 \quad 6/24;\ \varsigma_3 \quad 6/24;\ \varsigma_4 \quad 2/24.$

38. We identify pivotal players by moving down each column. Since P_2 is pivotal in the sequential coalition $P_1,\ \underline{P_2},\ P_3,\ P_4\ !$, we know that $\perp P_1, P_2 \overline{\ }$ form a winning coalition. This tells us that P_2 is also pivotal in the second sequential coalition $P_1,\ \underline{P_2},\ P_4,\ P_3\ !$.

In the second column, P_1 is clearly pivotal in the sequential coalition $P_2,\ \underline{P_1},\ P_3,\ P_4\ !$ since we know that $\perp P_1, P_2 \overline{\ }$ form a winning coalition. Similarly, P_1 is pivotal in the sequential coalition $P_2,\ \underline{P_1},\ P_4,\ P_3\ !$. Since P_1 is pivotal in $P_2,\ P_3,\ P_4,\ \underline{P_1}\ !$, we know that P_1 must be pivotal in $P_2,\ P_4,\ P_3,\ \underline{P_1}\ !$ too.

In the third column, P_2 is pivotal in the sequential coalition $P_3,\ P_1,\ \underline{P_2},\ P_4\ !$ according to the third sequential coalition in the first column ($P_1,\ P_3,\ \underline{P_2},\ P_4\ !$). Similarly, P_4 is pivotal in the sequential coalition $P_3,\ P_1,\ \underline{P_4},\ P_2\ !$ according to the fourth sequential coalition in the first column ($P_1,\ P_3,\ \underline{P_4},\ P_2\ !$). P_1 is pivotal in the sequential coalition $P_3,\ P_2,\ \underline{P_1},\ P_4\ !$ according to the third sequential coalition in the second column ($P_2,\ P_3,\ \underline{P_1},\ P_4\ !$). P_1 is pivotal in the sequential coalitions $P_3,\ P_2,\ P_4,\ \underline{P_1}\ !$ and $P_3,\ P_4,\ P_2,\ \underline{P_1}\ !$ according to the fourth sequential coalition in the second column ($P_2,\ P_3,\ P_4,\ \underline{P_1}\ !$).

In the fourth column, P_2 is pivotal in the sequential coalition $P_4,\ P_1,\ \underline{P_2},\ P_3\ !$ according to the fifth sequential coalition in the first column ($P_1,\ P_4,\ \underline{P_2},\ P_3\ !$). Also, P_3 is pivotal in the sequential coalition $P_4,\ P_1,\ \underline{P_3},\ P_2\ !$ according to the sixth sequential coalition in the first column ($P_1,\ P_4,\ \underline{P_3},\ P_2\ !$). We have P_1 pivotal in the sequential coalition $P_4,\ P_2,\ \underline{P_1},\ P_3\ !$ according to the fifth sequential coalition in the second column ($P_2,\ P_4,\ \underline{P_1},\ P_3\ !$). We also have P_1 pivotal in the sequential coalitions $P_4,\ P_2,\ P_3,\ \underline{P_1}\ !$ and $P_4,\ P_3,\ P_2,\ \underline{P_1}\ !$ according to the fourth sequential coalition in the second column ($P_2,\ P_3,\ P_4,\ \underline{P_1}\ !$).

Lastly, we see that P_1 pivotal in the sequential coalition $P_4, P_3, \underline{P_1}, P_2$! according to the fifth sequential coalition in the third column ($P_3, P_4, \underline{P_1}, P_2$!). The final listing of pivotal players is found below.

$P_1, \underline{P_2}, P_3, P_4$!, $P_2, \underline{P_1}, P_3, P_4$!, $P_3, P_1, \underline{P_2}, P_4$!, $P_4, P_1, \underline{P_2}, P_3$!,

$P_1, \underline{P_2}, P_4, P_3$!, $P_2, \underline{P_1}, P_4, P_3$!, $P_3, P_1, \underline{P_4}, P_2$!, $P_4, P_1, \underline{P_3}, P_2$!,

$P_1, P_3, \underline{P_2}, P_4$!, $P_2, P_3, \underline{P_1}, P_4$!, $P_3, P_2, \underline{P_1}, P_4$!, $P_4, P_2, \underline{P_1}, P_3$!,

$P_1, P_3, \underline{P_4}, P_2$!, $P_2, P_3, P_4, \underline{P_1}$!, $P_3, P_2, P_4, \underline{P_1}$!, $P_4, P_2, P_3, \underline{P_1}$!,

$P_1, P_4, \underline{P_2}, P_3$!, $P_2, P_4, \underline{P_1}, P_3$!, $P_3, P_4, \underline{P_1}, P_2$!, $P_4, P_3, \underline{P_1}, P_2$!,

$P_1, P_4, \underline{P_3}, P_2$!, $P_2, P_4, P_3, \underline{P_1}$!, $P_3, P_4, P_2, \underline{P_1}$!, $P_4, P_3, P_2, \underline{P_1}$!

The Shapley-Shubik distribution is then $\varsigma_1 \quad 14/24; \varsigma_2 \quad 6/24; \varsigma_3 \quad 2/24; \varsigma_4 \quad 2/24.$

2.4. Subsets and Permutations

39. **(a)** A set with *N* elements has 2^N subsets. So, *A* has $2^{10} \quad 1024$ subsets.

(b) There is only one subset, namely the empty set { }, having less than one element. So, using part (a), the number of subsets of *A* having one or more elements is 1024 – 1 = 1023.

(c) Each element of *A* can be used to form a subset containing exactly one element. So, there are exactly 10 such subsets.

(d) The number of subsets of *A* having two or more elements is the number of subsets of *A* not having either 0 or 1 element. From parts (a), (b) and (c), we calculate this number as 1024 – 1 – 10 = 1013.

40. **(a)** A set with *N* elements has 2^N subsets. So, *A* has $2^{12} \quad 4096$ subsets.

(b) There is only one subset, namely the empty set { }, having less than one element. So, using part (a), the number of subsets of *A* having one or more elements is 4096 – 1 = 4095.

(c) Each element of *A* can be used to form a subset containing exactly one element. So, there are exactly 12 such subsets.

(d) The number of subsets of *A* having two or more elements is the number of subsets of *A* not having either 0 or 1 element. From parts (a), (b) and (c), we calculate this number as 4096 – 1 – 12= 4083.

41. **(a)** A weighted voting system with *N* players has $2^N \ 1$ coalitions. So, a system with 10 players has $2^{10} \ 1 \quad 1023$ coalitions.

(b) The number of coalitions with two or more players is all of the coalitions minus the number of those having at most one player. But we know that there are exactly 10 coalitions consisting of exactly one player. So, the number of coalitions with two or more players is 1023 – 10 = 1013. See also Exercise 39(d).

42. **(a)** A weighted voting system with *N* players has $2^N \ 1$ coalitions. So, a system with 12 players has $2^{12} \ 1 \quad 4095$ coalitions.

(b) 4095 – 12 = 4083

43. **(a)** $2^6 \ 1 \quad 63$ coalitions

(b) There are $2^5 \ 1 \quad 31$ coalitions of the remaining five players P_2, P_3, P_4, P_5 and P_6. These are exactly those coalitions that do not include P_1 .

(c) As in (b), there are $2^5 - 1 = 31$ coalitions of the remaining five players P_1, P_2, P_4, P_5, and P_6. These are exactly those coalitions that do not include P_3.

(d) There are $2^4 - 1 = 15$ coalitions of the remaining four players P_2, P_4, P_5, P_6. These are exactly those coalitions that do not include either P_1 or P_3.

(e) 16 coalitions include P_1 and P_3 (all $2^4 - 1 = 15$ coalitions of the remaining four players P_2, P_4, P_5, P_6 together with the empty coalition could be combined with these two players to form such a coalition).

44. (a) $2^5 - 1 = 31$ coalitions

(b) There are $2^4 - 1 = 15$ coalitions of the remaining four players P_2, P_3, P_4, and P_5. These are exactly those coalitions that do not include P_1.

(c) As in (b), there are $2^4 - 1 = 15$ coalitions of the remaining four players P_1, P_2, P_3, and P_4. These are exactly those coalitions that do not include P_5.

(d) There are $2^3 - 1 = 7$ coalitions of the remaining three players P_2, P_3, P_4. These are exactly those coalitions that do not include P_1 or P_5.

(e) 8 coalitions include P_1 and P_5 (all $2^3 - 1 = 7$ coalitions of the remaining three players P_2, P_3, P_4 together with the empty coalition could be combined with these two players to form such a coalition).

45. (a) 13! = 6,227,020,800

(b) $18! = 6{,}402{,}373{,}705{,}728{,}000 \approx 6.402374 \times 10^{15}$

(c) $25! = 15{,}511{,}210{,}043{,}330{,}985{,}984{,}000{,}000 \approx 1.551121 \times 10^{25}$

(d) There are 25! sequential coalitions of 25 players.

$$25! \text{ sequential coaltions} \times \frac{1 \text{ second}}{1{,}000{,}000{,}000{,}000 \text{ sequential coalitions}} \times \frac{1 \text{ hour}}{3600 \text{ seconds}} \times \frac{1 \text{ day}}{24 \text{ hours}} \times \frac{1 \text{ year}}{365 \text{ days}}$$

$\approx 491{,}857$ years

46. (a) 12! = 479,001,600

(b) $15! = 1{,}307{,}674{,}368{,}000 \approx 1.307674 \times 10^{12}$

(c) $20! = 2{,}432{,}902{,}008{,}176{,}640{,}000 \approx 2.432902 \times 10^{18}$

(d) There are 20! sequential coalitions of 20 players.

$$20! \text{ sequential coalitions} \times \frac{1 \text{ second}}{1{,}000{,}000{,}000 \text{ sequential coalitions}} \times \frac{1 \text{ hour}}{3600 \text{ seconds}} \times \frac{1 \text{ day}}{24 \text{ hours}} \times \frac{1 \text{ year}}{365 \text{ days}}$$

≈ 77 years

47. (a) $\frac{13!}{3!} = \frac{6{,}227{,}020{,}800}{6} = 1{,}037{,}836{,}800$

(b) $\frac{13!}{3!10!} \quad \frac{6,227,020,800}{6 \upsilon 3,628,800} \quad 286$

(c) $\frac{13!}{4!9!} \quad \frac{6,227,020,800}{24 \times 362,880} \quad 715$

(d) $\frac{13!}{5!8!} \quad \frac{6,227,020,800}{120 \times 40,320} \quad 1287$

48. (a) $\frac{12!}{2!} \quad \frac{479,001,600}{2} \quad 239,500,800$

(b) $\frac{12!}{2!10!} \quad \frac{479,001,600}{2 \times 3,628,800} \quad 66$

(c) $\frac{12!}{3!9!} \quad \frac{479,001,600}{6 \times 362,880} \quad 220$

(d) $\frac{12!}{4!8!} \quad \frac{479,001,600}{24 \times 40,320} \quad 495$

49. (a) $10! \quad 10 \times 9 \times 8 \times \quad \times 3 \times 2 \times 1$

$10 \times 9!$

So, $9! \quad \frac{10!}{10} \quad \frac{3,628,800}{10} \quad 362,880$.

(b) $11! \quad 11 \times 10 \times 9 \times \quad \times 3 \times 2 \times 1 \quad 11 \times 10!$

So, $\frac{11!}{10!} \quad \frac{11 \times 10!}{10!} \quad 11$.

(c) $11! \quad 11 \times 10 \times 9 \times \quad \times 3 \times 2 \times 1 \quad 11 \times 10 \times 9!$

So, $\frac{11!}{9!} \quad \frac{11 \times 10 \times 9!}{9!} \quad 11 \times 10 \quad 110$.

(d) $\frac{9!}{6!} \quad \frac{9 \times 8 \times 7 \times 6!}{6!} \quad 9 \times 8 \times 7 \quad 504$

(e) $\frac{101!}{99!} \quad \frac{101 \times 100 \times 99!}{99!} \quad 101 \times 100 \quad 10,100$

50. (a) $20! \quad 20 \times 19!$ so $19! \quad \frac{20!}{20} \quad \frac{2,432,902,008,176,640,000}{20} \quad 121,645,100,408,832,000$

(b) 20

(c) $\frac{201!}{199!} \quad \frac{201 \times 200 \times 199!}{199!} \quad 201 \times 200 \quad 40,200$

(d) 990

$11!/8! \quad (11 \times 10 \times 9 \times 8!)/8! \quad 11 \times 10 \times 9 \quad 990$

51. (a) $7! \quad 5040$ sequential coalitions

(b) There are $6! \quad 720$ sequential coalitions of the remaining six players P_1, P_2, P_3, P_4, P_5 and P_6. These correspond exactly to those seven-player coalitions that have P_7 as the first player.

(c) As in (b), there are $6! \quad 720$ sequential coalitions of the remaining six players P_1, P_2, P_3, P_4, P_5 and P_6. These correspond exactly to those seven-player coalitions that have P_7 as the last player.

(d) 5040 – 720 = 4320 sequential coalitions do not have P_1 listed as the first player.

52. (a) 6! = 720

(b) There are $5! \quad 120$ sequential coalitions of the remaining five players P_1, P_2, P_3, P_5 and P_6. These correspond exactly to those six-player coalitions that have P_4 as the last player.

(c) 5! = 120; see (b)

(d) 720 – 120 = 600

53. (a) First, note that P_1 is pivotal in all sequential coalitions except when it is the first player. By 51(d), P_1 is pivotal in 4320 of the 5040 sequential coalitions.

(b) Based on the results of (a), $\varsigma_1 \quad 4320/5040 \quad 6/7$.

(c) Since players P_2, P_3, P_4, P_5, P_6 and P_7 share the remaining power equally, $\varsigma_1 \quad 4320/5040 \quad 6/7$, $\varsigma_2 \quad \varsigma_3 \quad \varsigma_4 \quad \varsigma_5 \quad \varsigma_6 \quad \varsigma_7 \quad 120/5040 \quad 1/42$. [Note also that 1/42 is 1/7 divided into 6 equal parts.]

54. (a) 600 (see Exercise 52(d))

(b) 600/720 = 5/6

(c) $\varsigma_1 \quad 5/6$, $\varsigma_2 \quad \varsigma_3 \quad \varsigma_4 \quad \varsigma_5 \quad \varsigma_6 \quad 1/30$ (the other players share power equally)

JOGGING

55. (a) Suppose that a winning coalition that contains *P* is not a winning coalition without *P*. Then *P* would be a critical player in that coalition, contradicting the fact that *P* is a dummy.

(b) *P* is a dummy □ *P* is never critical □ the numerator of its Banzhaf power index is 0 □ it's Banzhaf power index is 0.

(c) Suppose *P* is not a dummy. Then, *P* is critical in some winning coalition. Let *S* denote the other players in that winning coalition. The sequential coalition with the players in *S* first (in any order), followed by *P* and then followed by the remaining players has *P* as its pivotal player. Thus, *P*'s Shapley-Shubik power index is not 0. Conversely, if *P*'s Shapley-Shubik power index is not 0, then *P* is pivotal in some sequential coalition. A coalition consisting of *P* together with the players preceding *P* in that sequential coalition is a winning coalition and *P* is a critical player in it. Thus, *P* is not a dummy.

56. (a) P_5 is a dummy. It takes (at least) three of the first four players to pass a motion. P_5*'s* vote doesn't make any difference.

(b) $E_1 \quad \varsigma_1 \quad 1/4$; $E_2 \quad \varsigma_2 \quad 1/4$; $E_3 \quad \varsigma_3 \quad 1/4$; $E_4 \quad \varsigma_4 \quad 1/4$; $E_5 \quad \varsigma_5 \quad 0$

(c) $q = 21$, $q = 31$, $q = 41$.

(d) Since we assume that the weights are listed in non-increasing order, P_5 is a dummy if $w = 1, 2, 3$.

57. **(a)** The quota must be at least half of the total number of votes and not more than the total number of votes. $7 \delta q \delta 13$.

(b) For $q = 7$ or $q = 8$, P_1 is a dictator because $\{P_1\}$ is a winning coalition.

(c) For $q = 9$, only P_1 has veto power since P_2 and P_3 together have just 5 votes.

(d) For $10 \delta q \delta 12$, both P_1 and P_2 have veto power since no motion can pass without both of their votes. For $q = 13$, all three players have veto power.

(e) For $q = 7$ or $q = 8$, both P_2 and P_3 are dummies because P_1 is a dictator. For $10 \delta q \delta 12$, P_3 is a dummy since all winning coalitions contain $\{P_1, P_2\}$ which is itself a winning coalition.

58. **(a)** $5 \delta w \delta 9$; Since we assume that the weights are listed in non-increasing order, we have $5 \delta w$. In addition, we have $\frac{w \;\, 5 \;\, 2\, 1}{2} \, 9 \delta w \, 5 \, 2 \, 1$, i.e., $w + 8 < 18$ and $9 \delta w \, 8$. Since w is a whole number, this gives $5 \delta w \delta 9$.

(b) P_1 is a dictator if $w = 9$.

(c) For $w = 5$, P_1 and P_2 have veto power. If $6 \delta w \delta 8$, then P_1 alone has veto power.

(d) If $w = 5$, all winning coalitions contain $\{P_1, P_2\}$ which is itself a winning coalition and so both P_3 and P_4 are dummies. If $w = 7$, then P_4 is a dummy. If $w = 9$, then P_2, P_3, and P_4 are all dummies since P_1 is a dictator.

59. **(a)** The winning coalitions for both weighted voting systems [8: 5, 3, 2] and [2: 1, 1, 0] are $\{P_1, P_2\}$ and $\{P_1, P_2, P_3\}$.

(b) The winning coalitions for both weighted voting systems [7: 4, 3, 2, 1] and [5: 3, 2, 1, 1] are $\{P_1, P_2\}$, $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_3, P_4\}$, and $\{P_1, P_2, P_3, P_4\}$.

(c) The winning coalitions for both weighted voting systems are those consisting of any three of the five players, any four of the five players, and the grand coalition.

(d) If a player is critical in a winning coalition, then that coalition is no longer winning if the player is removed. An equivalent system (with the same winning coalitions) will find that player critical in the same coalitions. When calculating the Banzhaf power indexes in two equivalent systems, the same players will be critical in the same coalitions. Thus, the numerators and denominators in each player's Banzhaf power index will be the same.

(e) If a player, P, is pivotal in a sequential coalition, then the players to P's left in the sequential coalition do not form a winning coalition, but including P does make it a winning coalition. An equivalent system (with the same winning coalitions) will find the player P pivotal in that same sequential coalition. When calculating the Shapley-Shubik indexes in two equivalent systems, the same players will be pivotal in the same sequential coalitions. Thus, the numerators and denominators in each Shapley-Shubik index will be the same.

60. **(a)** P has veto power □ the coalition formed by all players other than P is a losing coalition □ P must be a member of any winning coalition.

(b) *P* has veto power $\Leftrightarrow$ the coalition formed by all players other than *P* is a losing coalition $\Leftrightarrow$ *P* is a critical player in the grand coalition.

61. **(a)** $V / 2 \; q \; \delta \; V \; w_1$ (where V denotes the sum of all the weights). The idea here is that if P_1 does not have veto power, neither will any of the other players (since weights decrease in value).

(b) $V \; w_N \; q \; \delta \; V$; The idea here is that if P_N has veto power, so will all the other players.

(c) $V \; w_i \; q \; \delta \; V \; w_{i\;1}$

62. **(a)** [7: 3, 3, 2, 2, 2] or any answer of the form [7*k*: 3*k*, 3*k*, 2*k*, 2*k*, 2*k*] for positive integer *k* is correct. One process to solve this problem is start with small values of *c* and experiment. It is important that the sum of the children's votes (3*c*) not meet the quota and that sum of the parent's votes (2*p*) also not meet the quota.

(b) The winning coalitions in this voting system (with critical players underlined) are $\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_5}\}$, $\{\underline{P_1}, \underline{P_3}, \underline{P_4}\}$, $\{\underline{P_1}, \underline{P_3}, \underline{P_5}\}$, $\{\underline{P_1}, \underline{P_4}, \underline{P_5}\}$, $\{\underline{P_2}, \underline{P_3}, \underline{P_4}\}$, $\{\underline{P_2}, \underline{P_3}, \underline{P_5}\}$, $\{\underline{P_2}, \underline{P_4}, \underline{P_5}\}$, $\{P_1, P_2, P_3, P_4\}$, $\{P_1, P_2, P_3, P_5\}$, $\{P_1, P_2, P_4, P_5\}$, $\{\underline{P_1}, P_3, P_4, P_5\}$, $\{\underline{P_2}, P_3, P_4, P_5\}$, $\{P_1, P_2, P_3, P_4, P_5\}$. So, the Banzhaf power distribution is $E_1 \; E_2 \; 7/29$, $E_3 \; E_4 \; E_5 \; 5/29$.

63. You should buy your vote from P_3. The following table explains why.

Buying a vote from	Resulting weighted voting system	Resulting Banzhaf power distribution	Your power
P_1	[8: 5, 4, 2, 2]	$E_1 \; \frac{4}{12}; E_2 \; \frac{4}{12}; E_3 \; \frac{2}{12}; E_4 \; \frac{2}{12}$	$\frac{2}{12}$
P_2	[8: 6, 3, 2, 2]	$E_1 \; \frac{7}{10}; E_2 \; \frac{1}{10}; E_3 \; \frac{1}{10}; E_4 \; \frac{1}{10}$	$\frac{1}{10}$
P_3	[8: 6, 4, 1, 2]	$E_1 \; \frac{6}{10}; E_2 \; \frac{2}{10}; E_3 \; 0; E_4 \; \frac{2}{10}$	$\frac{2}{10}$

64. You should buy your vote from P_4. The following table explains why.

Buying a vote from	Resulting weighted voting system	Resulting Banzhaf power distribution	Your power
P_1	[27: 9, 8, 6, 4, 3]	E 1; E 1; E 1; E 1; E 0	0
P_2	[27: 10, 7, 6, 4, 3]	$E_1 \; \frac{1}{4}; E_2 \; \frac{1}{4}; E_3 \; \frac{1}{4}; E_4 \; \frac{1}{4}; E_5 \; 0$	0
P_3	[27: 10, 8, 5, 4, 3]	$E_1 \; \frac{1}{4}; E_2 \; \frac{1}{4}; E_3 \; \frac{1}{4}; E_4 \; \frac{1}{4}; E_5 \; 0$	0
P_4	[27: 10, 8, 6, 3, 3]	$E_1 \; \frac{3}{11}; E_2 \; \frac{3}{11}; E_3 \; \frac{3}{11}; E_4 \; \frac{1}{11}; E_5 \; \frac{1}{11}$	$\frac{1}{11}$

65. **(a)** You should buy your vote from P_2. The following table explains why.

Buying a vote from	Resulting weighted voting system	Resulting Banzhaf power distribution	Your power
P_1	[18: 9, 8, 6, 4, 3]	$E_1\ \frac{4}{13}$; $E_2\ \frac{3}{13}$; $E_3\ \frac{3}{13}$; $E_4\ \frac{2}{13}$; $E_5\ \frac{1}{13}$	$\frac{1}{13}$
P_2	[18: 10, 7, 6, 4, 3]	$E_1\ \frac{9}{25}$; $E_2\ \frac{1}{5}$; $E_3\ \frac{1}{5}$; $E_4\ \frac{3}{25}$; $E_5\ \frac{3}{25}$	$\frac{3}{25}$
P_3	[18: 10, 8, 5, 4, 3]	$E_1\ \frac{5}{12}$; $E_2\ \frac{1}{4}$; $E_3\ \frac{1}{6}$; $E_4\ \frac{1}{12}$; $E_5\ \frac{1}{12}$	$\frac{1}{12}$
P_4	[18: 10, 8, 6, 3, 3]	$E_1\ \frac{5}{12}$; $E_2\ \frac{1}{4}$; $E_3\ \frac{1}{6}$; $E_4\ \frac{1}{12}$; $E_5\ \frac{1}{12}$	$\frac{1}{12}$

(b) You should buy 2 votes from P_2. The following table explains why.

Buying a vote from	Resulting weighted voting system	Resulting Banzhaf power distribution	Your power
P_1	[18: 8, 8, 6, 4, 4]	$E_1\ \frac{7}{27}$; $E_2\ \frac{7}{27}$; $E_3\ \frac{7}{27}$; $E_4\ \frac{1}{9}$; $E_5\ \frac{1}{9}$	$\frac{1}{9}$
P_2	[18: 10, 6, 6, 4, 4]	$E_1\ \frac{5}{13}$; $E_2\ \frac{2}{13}$; $E_3\ \frac{2}{13}$; $\frac{2}{13}$; $E_5\ \frac{2}{13}$	$\frac{2}{13}$
P_3	[18: 10, 8, 4, 4, 4]	$E_1\ \frac{11}{25}$; $E_2\ \frac{1}{5}$; $E_3\ \frac{3}{25}$; $E_4\ \frac{3}{25}$; $E_5\ \frac{3}{25}$	$\frac{3}{25}$
P_4	[18: 10, 8, 6, 2, 4]	$E_1\ \frac{9}{25}$; $E_2\ \frac{7}{25}$; $E_3\ \frac{1}{5}$; $E_4\ \frac{1}{25}$; $E_5\ \frac{3}{25}$	$\frac{3}{25}$

(c) Buying a single vote from P_2 raises your power from $\frac{1}{25}$ 4% to $\frac{3}{25}$ 12%. Buying a second vote from P_2 raises your power to $\frac{2}{13}$ | 15.4%. The increase in power is less with the second vote, but if you value power over money, it might still be worth it to you to buy that second vote.

66. **(a)** Before the merger: $E_1\ \frac{3}{5}$; $E_2\ \frac{1}{5}$; $E_3\ \frac{1}{5}$; after the merger $E_1 : \frac{1}{2}$; $E^* : \frac{1}{2}$.

(b) Before the merger: $E_1\ \frac{1}{2}$; $E_2\ \frac{1}{2}$; E_3 0; after the merger $E_1\ \frac{1}{2}$; $E^*\ \frac{1}{2}$.

(c) Before the merger: $E_1\ \frac{1}{3}$; $E_2\ \frac{1}{3}$; $E_3\ \frac{1}{3}$; after the merger $E_1\ \frac{1}{2}$; $E^*\ \frac{1}{2}$.

(d) The Banzhaf power index of the merger of two players can be greater than, equal to, or less than the sum of Banzhaf power indices of the individual players.

67. **(a)** [24: 14, 8, 6, 4] is just [12: 7, 4, 3, 2] with each value multiplied by 2. Both have Banzhaf power distribution E_1 2/5; E_2 1/5; E_3 1/5; E_4 1/5.

(b) In the weighted voting system $[q : w_1, w_2, \quad , w_N]$, if P_k is critical in a coalition then the sum of the weights of all the players in that coalition (including P_k) is at least q, but the sum of the weights of all the players in the coalition except P_k is less than q. Consequently, if the weights of all the players in that coalition are multiplied by $c > 0$ ($c = 0$ would make no sense), then the sum of the weights of all the players in the coalition (including P_k) is at least cq but the sum of the weights of all the players in the coalition except P_k is less than cq. Therefore P_k is critical in the same coalition in the weighted voting system $[cq : cw_1, cw_2, \quad , cw_N]$. Since the critical players are the same in both weighted voting systems, the Banzhaf power distributions will be the same.

68. (a) Both have Shapley-Shubik power distribution $\sigma_1 = 1/2;\ \sigma_2 = 1/6;\ \sigma_3 = 1/6;\ \sigma_4 = 1/6$.

(b) In the weighted voting system $[q: w_1, w_2, \ldots, w_N]$, P_k is pivotal in the sequential coalition $\langle P_1, P_2, \ldots, P_k, \ldots, P_N \rangle$ means $w_1 + w_2 + \cdots + w_k \geq q$ but $w_1 + w_2 + \cdots + w_{k-1} < q$. In the weighted voting system $[cq: cw_1, cw_2, \ldots, cw_N]$, P_k is pivotal in the sequential coalition $\langle P_1, P_2, \ldots, P_k, \ldots, P_N \rangle$ means $cw_1 + cw_2 + \cdots + cw_k \geq cq$ but $cw_1 + cw_2 + \cdots + cw_{k-1} < cq$. These two statements are equivalent since $cw_1 + cw_2 + \cdots + cw_k = c(w_1 + w_2 + \cdots + w_k) \geq cq$ if and only if $w_1 + w_2 + \cdots + w_k \geq q$, and $cw_1 + cw_2 + \cdots + cw_{k-1} = c(w_1 + w_2 + \cdots + w_{k-1}) < cq$ if and only if $w_1 + w_2 + \cdots + w_{k-1} < q$. This same reasoning applies to any sequential coalition and so the pivotal players are exactly the same.

69. (a) A player is critical in a coalition if that coalition without the player is not on the list of winning coalitions. The critical players in the winning coalitions are underlined: $\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}\}$, $\{\underline{P_1}, \underline{P_3}, \underline{P_4}\}$, $\{\underline{P_1}, P_2, P_3, P_4\}$. So the Banzhaf power distribution is $\beta_1 = 4/10$, $\beta_2 = \beta_3 = \beta_4 = 2/10$.

(b) A player P is pivotal in a sequential coalition if the players that appear before P in that sequential coalition do not, as a group, appear on the list of winning coalitions, but would appear on the list of winning coalitions if P were included. For example, P_3 is pivotal in $\langle P_1, P_2, P_3, P_4 \rangle$ since $\{P_1, P_2\}$ is not on the list of winning coalitions but $\{P_1, P_2, P_3\}$ is on that list. The pivotal players in each sequential coalition is underlined:

$\langle P_1, P_2, \underline{P_3}, P_4\rangle, \langle P_1, P_2, \underline{P_4}, P_3\rangle, \langle P_1, P_3, \underline{P_2}, P_4\rangle, \langle P_1, P_3, \underline{P_4}, P_2\rangle,$
$\langle P_1, P_4, \underline{P_2}, P_3\rangle, \langle P_1, P_4, \underline{P_3}, P_2\rangle, \langle P_2, P_1, \underline{P_3}, P_4\rangle, \langle P_2, P_1, \underline{P_4}, P_3\rangle,$
$\langle P_2, P_3, \underline{P_1}, P_4\rangle, \langle P_2, P_3, P_4, \underline{P_1}\rangle, \langle P_2, P_4, \underline{P_1}, P_3\rangle, \langle P_2, P_4, P_3, \underline{P_1}\rangle,$
$\langle P_3, P_1, \underline{P_2}, P_4\rangle, \langle P_3, P_1, \underline{P_4}, P_2\rangle, \langle P_3, P_2, \underline{P_1}, P_4\rangle, \langle P_3, P_2, P_4, \underline{P_1}\rangle,$
$\langle P_3, P_4, \underline{P_1}, P_2\rangle, \langle P_3, P_4, P_2, \underline{P_1}\rangle, \langle P_4, P_1, \underline{P_2}, P_3\rangle, \langle P_4, P_1, \underline{P_3}, P_2\rangle,$
$\langle P_4, P_2, \underline{P_1}, P_3\rangle, \langle P_4, P_2, P_3, \underline{P_1}\rangle, \langle P_4, P_3, \underline{P_1}, P_2\rangle, \langle P_4, P_3, P_2, \underline{P_1}\rangle$

So, the Shapley-Shubik power indices are $\sigma_1 = 12/24 = 1/2$, $\sigma_2 = \sigma_3 = \sigma_4 = 4/24 = 1/6$.

70. (a) [5: 2,1,1,1,1,1,1]

(b) The mayor has a Shapley-Shubik power index of 2/7. (Of the 7! sequential coalitions, the mayor is pivotal anytime he/she is in the 4th slot or in the 5th slot, and there are 6! of each kind.) Each of the council members has a Shapley-Shubik power index of $\frac{1}{6}\left(1 - \frac{2}{7}\right) = \frac{5}{42}$.

RUNNING

71. Write the weighted voting system as [q: $8x$, $4x$, $2x$, x]. The total number of votes is $15x$. If x is even, then so is $15x$. Since the quota is a simple majority, $q = \frac{15x}{2} + 1$. But, $\frac{15x}{2} + 1 = 7.5x + 1 \le 8x$ since $x \ge 2$. So, P_1 (having $8x$ votes) is a dictator. If x is odd, then so is $15x$. Since the quota is a simple majority, $q = \frac{15x + 1}{2}$.

But, $\frac{15x + 1}{2} = 7.5x + \frac{1}{2} \le 8x$ since $x \ge 1$. So, again, P_1 (having $8x$ votes) is a dictator.

72. Since P_2 is a pivotal player in the sequential coalition $\langle P_1, P_2, P_3 \rangle$, it is clear that $\{P_1, P_2\}$ and $\{P_1, P_2, P_3\}$ are winning coalitions, but $\{P_1\}$ is a losing coalition. Similarly, we construct the following chart.

Pivotal Player and Sequential Coalition	**Winning Coalitions**	**Losing Coalitions**
$\langle P_1, \underline{P_2}, P_3 \rangle$	$\{P_1, P_2\}$, $\{P_1, P_2, P_3\}$	$\{P_1\}$
$\langle P_1, \underline{P_3}, P_2 \rangle$	$\{P_1, P_3\}$, $\{P_1, P_2, P_3\}$	$\{P_1\}$
$\langle P_2, \underline{P_1}, P_3 \rangle$	$\{P_1, P_2\}$, $\{P_1, P_2, P_3\}$	$\{P_2\}$
$\langle P_2, P_3, \underline{P_1} \rangle$	$\{P_1, P_2, P_3\}$	$\{P_2\}$, $\{P_3\}$, $\{P_2, P_3\}$
$\langle P_3, \underline{P_1}, P_2 \rangle$	$\{P_1, P_3\}$, $\{P_1, P_2, P_3\}$	$\{P_3\}$
$\langle P_3, P_2, \underline{P_1} \rangle$	$\{P_1, P_2, P_3\}$	$\{P_2\}$, $\{P_3\}$, $\{P_2, P_3\}$

Based on this, the winning coalitions with critical players underlined are $\{\underline{P_1}, \underline{P_2}\}$, $\{\underline{P_1}, \underline{P_3}\}$, and $\{\underline{P_1}, P_2, P_3\}$.

It follows that the Banzhaf power distribution is $\beta_1 = 3/5;\ \beta_2 = 1/5;\ \beta_3 = 1/5$.

73. (a) [4: 2, 1, 1, 1] or [9: 5, 2, 2, 2] are among the possible answers.

(b) The sequential coalitions (with pivotal players underlined) are:

$\langle H, A_1, \underline{A_2}, A_3 \rangle, \langle H, A_1, \underline{A_3}, A_2 \rangle, \langle H, A_2, \underline{A_1}, A_3 \rangle, \langle H, A_2, \underline{A_3}, A_1 \rangle,$

$\langle H, A_3, \underline{A_1}, A_2 \rangle, \langle H, A_3, \underline{A_2}, A_1 \rangle, \langle A_1, H, \underline{A_2}, A_3 \rangle, \langle A_1, H, \underline{A_3}, A_2 \rangle,$

$\langle A_1, A_2, \underline{H}, A_3 \rangle, \langle A_1, A_2, A_3, \underline{H} \rangle, \langle A_1, A_3, \underline{H}, A_2 \rangle, \langle A_1, A_3, A_2, \underline{H} \rangle,$

$\langle A_2, H, \underline{A_1}, A_3 \rangle, \langle A_2, H, \underline{A_3}, A_1 \rangle, \langle A_2, A_1, \underline{H}, A_3 \rangle, \langle A_2, A_1, A_3, \underline{H} \rangle,$

$\langle A_2, A_3, \underline{H}, A_1 \rangle, \langle A_2, A_3, A_1, \underline{H} \rangle, \langle A_3, H, \underline{A_1}, A_2 \rangle, \langle A_3, H, \underline{A_2}, A_1 \rangle,$

$\langle A_3, A_1, \underline{H}, A_2 \rangle, \langle A_3, A_1, A_2, \underline{H} \rangle, \langle A_3, A_2, \underline{H}, A_1 \rangle, \langle A_3, A_2, A_1, \underline{H} \rangle.$

H is pivotal in 12 coalitions; A_1 is pivotal in four coalitions; A_2 is pivotal in four coalitions; A_3 is pivotal in four coalitions. The Shapley-Shubik power distribution is $\sigma_H = 12/24 = 1/2;$

$\sigma_{A_1} = \sigma_{A_2} = \sigma_{A_3} = 4/24 = 1/6$.

74. There are $N!$ ways that P_1 (the senior partner) can be the first player in a sequential coalition (this is the number of sequential coalitions consisting of the other N players). When this happens, the senior partner is not pivotal (the second player listed in the sequential coalition is instead). There are $(N+1)! - N! = N!(N+1-1) = N \cdot N!$ ways that P_1 is not the first player in a sequential coalition. In each of these cases, P_1 is the pivotal player. It follows that the Shapley-Shubik power index of P_1 (the senior partner) is $\frac{N \cdot N!}{(N+1)!} = \frac{N \cdot N!}{(N+1) \cdot N!} = \frac{N}{N+1}$. Since the other N players divide the remaining power equally, the junior partners each have a Shapley-Shubik power index of $\frac{1 - \frac{N}{N+1}}{N} = \frac{\frac{N+1}{N+1} - \frac{N}{N+1}}{N} = \frac{1}{N(N+1)}$.

75. **(a)** The losing coalitions are $\{P_1\}, \{P_2\},$ and $\{P_3\}$. The complements of these coalitions are $\{P_2, P_3\}$, $\{P_1, P_3\},$ and $\{P_1, P_2\}$ respectively, all of which are winning coalitions.

(b) The losing coalitions are $\{P_1\}, \{P_2\}, \{P_3\}, \{P_4\}, \{P_2, P_3\}, \{P_2, P_4\},$ and $\{P_3, P_4\}$. The complements of these coalitions are $\{P_2, P_3, P_4\}, \{P_1, P_3, P_4\}, \{P_1, P_2, P_4\}, \{P_1, P_2, P_3\}, \{P_1, P_4\}, \{P_1, P_3\},$ and $\{P_1, P_2\}$ respectively, all of which are winning coalitions.

(c) If *P* is a dictator, the losing coalitions are all the coalitions without *P*; the winning coalitions are all the coalitions that include *P*. The complement of any coalition without *P* (losing) is a coalition with *P* (winning).

(d) Take the grand coalition out of the picture for a moment. Of the remaining 2^N 2 coalitions, half are losing coalitions and half are winning coalitions, since each losing coalition pairs up with a winning coalition (its complement). Half of 2^N 2 is $2^{N 1}$ 1. In addition, we have the grand coalition (always a winning coalition). Thus, the total number of winning coalitions is $2^{N 1}$.

76. The mayor has power index 5/13 and each of the four other council members has a power index of 2/13. The winning coalitions (with critical players underlined) are: $\{\underline{M}, \underline{P_2}, \underline{P_3}\}$, $\{\underline{M}, \underline{P_2}, \underline{P_4}\}$, $\{\underline{M}, \underline{P_2}, \underline{P_5}\}$, $\{\underline{M}, \underline{P_3}, \underline{P_4}\}$, $\{\underline{M}, \underline{P_3}, \underline{P_5}\}$, $\{\underline{M}, \underline{P_4}, \underline{P_5}\}$, $\{\underline{M}, P_2, P_3, P_4\}$, $\{\underline{M}, P_2, P_3, P_5\}$, $\{\underline{M}, P_2, P_4, P_5\}$, $\{\underline{M}, P_3, P_4, P_5\}$, $\{\underline{P_2}, \underline{P_3}, \underline{P_4}, \underline{P_5}\}$, $\{M, P_2, P_3, P_4, P_5\}$.

77. (a) According to Table 2-11, the Banzhaf index of California in the Electoral College is $E_{California}$ 0.114.

The relative voting weight of California is $w_{California}$ $\frac{55}{538}$ | 0.1022. The relative Banzhaf voting power for California is then $\Sigma_{California}$ $\frac{0.114}{0.1022}$ | 1.115.

(b) Σ_{H1} $\frac{1/3}{31/115}$ $\frac{115}{93}$, Σ_{H2} $\frac{1/3}{31/115}$ $\frac{115}{93}$, Σ_{OB} $\frac{1/3}{28/115}$ $\frac{115}{84}$, Σ_{NH} $\frac{0}{21/115}$ 0,

Σ_{LB} $\frac{0}{2/115}$ 0, Σ_{GC} $\frac{0}{2/115}$ 0.

78. (a) The possible coalitions are all coalitions with *A* but not *P* in them or with *P* but not *A* in them. If we call *B* and *C* the players with 4 and 3 votes respectively, the possible coalitions are: $\{A\}, \{A, B\}, \{A, C\}, \{A, B, C\}$ and $\{P\}, \{P, B\}, \{P, C\}, \{P, B, C\}$. The winning coalitions (with critical players underlined) are: $\{\underline{A}, \underline{B}\}, \{\underline{A}, \underline{C}\}, \{\underline{A}, B, C\}, \{\underline{P}, \underline{B}, \underline{C}\}$. The Banzhaf power distribution in this case is E_A 3/8; E_B 2/8; E_C 2/8; E_D 1/8.

(b) The possible coalitions under these circumstances are all subsets of the players that contain either *A* or *P* but not both. There are $2^{N 2}$ subsets that contain neither *A* nor *P* (all possible subsets of the *N* – 2 other players). If we throw *A* into each of these subsets, we get all the coalitions that have *A* but not *P* in them—a total of 2^{N-2} coalitions. If we throw *P* into each of the 2^{N-2} subsets we get all the coalitions that have *P* but not *A* in them. Adding these two lists gives $2 \cdot 2^{N-2} = 2^{N-1}$ coalitions.

(c) Consider the same weighted voting system discussed in part (a) but without restrictions. The Banzhaf power distribution in this case is $E_A = \frac{5}{12}; E_B = \frac{3}{12}; E_C = \frac{3}{12}; E_D = \frac{1}{12}$. When *A* becomes the antagonist of *P*, *A*'s Banzhaf power index goes down (to 3/8) and *P*'s Banzhaf power index goes up (to 1/8). If we reverse the roles of *A* and *P* (*A* becomes the "player" and *P* his "antagonist", the Banzhaf power index calculations remain the same, and in this case it is the antagonist's (*P*) power index that goes up and the player's (*A*) power that goes down.

(d) Once *P* realizes that *A* will always vote against him, he can vote exactly the opposite way of his true opinion. This puts *A*'s votes behind *P*'s true opinion. If *A* has more votes than *P*, this strategy essentially increases *P*'s power at the expense of *A*.

79. (a) [4: 3,1,1,1]; P_1 is pivotal in every sequential coalition except the six sequential coalitions in which P_1 is the first member. So, the Shapley-Shubik power index of P_1 is (24 – 6)/24 = 18/24 = 3/4. P_2, P_3, and P_4 share power equally and have Shapley-Shubik power index of 1/12.

(b) In a weighted voting system with 4 players there are 24 sequential coalitions — each player is the first member in exactly 6 sequential coalitions. The only way the first member of a coalition can be pivotal is if she is a dictator. Consequently, if a player is not a dictator, she can be pivotal in at most 24 – 6 = 18 sequential coalitions and so that player's Shapley-Shubik power index can be at most 18/24 = 3/4.

(c) In a weighted voting system with *N* players there are *N*! sequential coalitions — each player is the first member in exactly (*N*–1)! sequential coalitions. The only way the first member of a coalition can be pivotal is if she is a dictator. Consequently, if a player is not a dictator, she can be pivotal in at most *N*! – (*N*–1)! = (*N*–1)!(*N*–1) sequential coalitions and so that player's Shapley-Shubik power index can be at most (*N*–1)!(*N*–1)/*N*! = (*N*–1)/*N*.

(d) [*N*: *N*–1, 1, 1, 1,…, 1]; Here *N* is the number of players as well as the quota. P_1 is pivotal in every sequential coalition except for the (*N*–1)! sequential coalitions in which P_1 is the first member. So the Shapley-Shubik power index of P_1 is [*N*! – (*N*–1)!]/*N*! = (*N*–1)/*N*.

80. (a) A player with veto power is critical in *every* winning coalition. Therefore that player must have Banzhaf power index of *at least* as much as any other player. Since the total Banzhaf power indexes of all *N* players is 1, they cannot all be less than 1/*N*.

(b) A player with veto power is pivotal in every sequential coalition in which that player is the last player. There are (*N* – 1)! such sequential coalitions. Consequently, the player must have Shapley-Shubik power index of at least (*N* – 1)!/*N*! = 1/*N*.

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81. Power in the first Electoral College is summarized in the table below. A quota of $q = 67$ is used.

State	Number of Electors	Banzhaf Power	Shapley-Shubik Power
Virginia	21	17.2034%	17.3049%
Massachusetts	16	12.2842%	12.4892%
Pennsylvania	15	11.4728%	11.6567%
New York	12	8.9970%	9.0568%
North Carolina	12	8.9970%	9.0568%
Connecticut	9	6.6458%	6.5945%
Maryland	8	5.9173%	5.8680%
South Carolina	8	5.9173%	5.8680%
New Jersey	7	5.1474%	5.0971%
New Hampshire	6	4.2761%	4.2236%
Georgia	4	2.9160%	2.8355%
Kentucky	4	2.9160%	2.8355%
Rhode Island	4	2.9160%	2.8355%
Vermont	3	2.1968%	2.1390%
Delaware	3	2.1968%	2.1390%

82. Slightly increasing the quota from 58 produces more desirable results (in terms of reducing the gap). One quota to be considered is 65 (Note: This is slightly larger than 31 + 31 + 2 = 64). $q = 65$ produces the following gaps which are within the 4% threshold. Another interesting quota, though not one meeting the 4% threshold for gaps, is $q = 67$.

Player	Weight	Weight as a %	Banzhaf Power	Gap
Hempstead 1	31	26.96%	26%	0.96%
Hempstead 2	31	26.96%	26%	0.96%
Oyster Bay	28	24.35%	22%	2.35%
N. Hempstead	21	18.26%	22%	3.74%
Long Beach	2	1.74%	2%	0.26%
Glen Cove	2	1.74%	2%	0.26%
Quota	**65**			

83. (a) The following table describes Banzhaf power vs. the weight of each player (as a percentage) in the weighted voting system [65: 30, 28, 22, 15, 7, 6].

Player	**Weight**	**Weight as a %**	**Banzhaf Power**	**Gap**
Hempstead 1	30	27.7778%	28.8462%	1.0684%
Hempstead 2	28	25.9259%	25.0000%	0.9259%
Oyster Bay	22	20.3704%	21.1538%	0.7834%
N. Hempstead	15	13.8889%	17.3077%	3.4188%
Long Beach	7	6.4815%	5.7692%	0.7123%
Glen Cove	6	5.5556%	1.9231%	3.6325%
Quota	**65**			

(b) The largest gap for $q = 65$ is 3.6325%. The average value of the gaps for $q = 65$ is 1.7569%.

(c) Experimentation with various quotas produce the results given in the table below. The best largest and average gaps occur for any value of the quota between 60 and 63. Quotas below $q = 59$ or above $q = 65$ produce even larger gaps.

Weight	**Weight as a %**	**Banzhaf Power (q=65)**	**Gap**	**Banzhaf Power (q=64)**	**Gap**	**Banzhaf Power (q=60-63)**	**Gap**	**Banzhaf Power (q=59)**	**Gap**
30	27.78%	28.85%	1.07%	30.77%	2.99%	27.78%	0.00%	29.63%	1.85%
28	25.93%	25.00%	0.93%	26.92%	1.00%	27.78%	1.85%	25.93%	0.00%
22	20.37%	21.15%	0.78%	19.23%	1.14%	20.37%	0.00%	22.22%	1.85%
15	13.89%	17.31%	3.42%	15.38%	1.49%	12.96%	0.93%	11.11%	2.78%
7	6.48%	5.77%	0.71%	3.85%	2.64%	5.56%	0.92%	7.41%	0.93%
6	5.56%	1.92%	3.64%	3.85%	1.71%	5.56%	0.00%	3.70%	1.86%
MAX			3.64%		2.99%		1.85%		2.78%
AVG			1.76%		1.83%		0.62%		1.54%