Solution Manual for Friendly Introduction to Number Theory 4th Edition by Silverman ISBN 0321816196 9780321816191

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Chapter 2

Pythagorean Triples

Exercises

2.1. (a) We showed that in any primitive Pythagorean triple (a, b, c), either a or b is even. Use the same sort of argument to show that either a or b must be a multiple of 3.

(b) By examining the above list of primitive Pythagorean triples, make a guess about when a, b, or c is a multiple of 5. Try to show that your guess is correct.

Solution to Exercise 2.1.

(a) If a is not a multiple of 3, it must equal either $3x + 1$ or $3x + 2$. Similarly, if b is not a multiple of 3, it must equal $3y + 1$ or $3y + 2$. There are four possibilities for $a^2 + b^2$, namely

$$\begin{aligned}
a^2 + b^2 &= (3x + 1)^2 + (3y + 1)^2 = 9x^2 + 6x + 1 + 9y^2 + 6y + 1\\
&= 3(3x^2 + 2x + 3y^2 + 2y) + 2,\\
a^2 + b^2 &= (3x + 1)^2 + (3y + 2)^2 = 9x^2 + 6x + 1 + 9y^2 + 12y + 4\\
&= 3(3x^2 + 2x + 3y^2 + 4y + 1) + 2,\\
a^2 + b^2 &= (3x + 2)^2 + (3y + 1)^2 = 9x^2 + 12x + 4 + 9y^2 + 6y + 1\\
&= 3(3x^2 + 4x + 3y^2 + 2y + 1) + 2,\\
a^2 + b^2 &= (3x + 2)^2 + (3y + 2)^2 = 9x^2 + 12x + 4 + 9y^2 + 12y + 4\\
&= 3(3x^2 + 4x + 3y^2 + 4y + 2) + 2.
\end{aligned}$$

So if a and b are not multiples of 3, then $c^2 = a^2 + b^2$ looks like 2 more than a multiple of 3. But regardless of whether c is $3z$ or $3z + 1$ or $3z + 2$, the numbers c^2 cannot be 2 more than a multiple of 3. This is true because

$$(3z)^2 = 3 \cdot 3z,$$

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$$(3z + 1)^2 = 3(3z^2 + 2z) + 1,$$
$$(3z + 2)^2 = 3(3z^2 + 4z + 1) + 1.$$

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(b) The table suggests that in every primitive Pythagorean triple, exactly one of a, b, or c is a multiple of 5. To verify this, we use the Pythagorean Triples Theorem to write a and b as $a = st$ and $b = \frac{1}{2}(s^2 - t^2)$. If either s or t is a multiple of 5, then a is a multiple of 5 and we're done. Otherwise s looks like $s = 5S + i$ and t looks like $5T + j$ with i and j being integers in the set $\{1, 2, 3, 4\}$. Next we observe that

$$2b = s^2 - t^2 = (5S + i)^2 - (5T + j)^2 = 25(S^2 - T^2) + 10(Si - Tj) + i^2 - j^2.$$

If $i^2 - j^2$ is a multiple of 5, then b is a multiple of 5, and again we're done. Looking atthe 16 possibilities for the pair (i, j), we see that this accounts for 8 of them, leaving the possibilities

$$(i, j) = (1, 2),\ (1, 3),\ (2, 1),\ (2, 4),\ (3, 1),\ (3, 4),\ (4, 2),\ \text{or } (4, 3).$$

Now for each of these remaining possibilities, we need to check that

$$2c = s^2 + t^2 = (5S + i)^2 + (5T + j)^2 = 25(S^2 + T^2) + 10(Si + Tj) + i^2 + j^2$$

is a multiple of 5, which means checking that $i^2 + j^2$ is a multiple of 5. This is easily accomplished:

$$1^2 + 2^2 = 5 \quad 1^2 + 3^2 = 10 \quad 2^1 + 1^2 = 5 \quad 2^2 + 4^2 = 20 \tag{2.1}$$

$$3^1 + 1^2 = 10 \quad 3^2 + 4^2 = 25 \quad 4^2 + 2^2 = 20 \quad 4^2 + 3^2 = 25. \tag{2.2}$$

.

2.2. A nonzero integer d is said to *divide* an integer m if $m = dk$ for some number k. Show that if d divides both m and n, then d also divides $m - n$ and $m + n$.

Solution to Exercise 2.2.
Both m and n are divisible by d, so $m = dk$ and $n = dk'$. Thus $m \pm n = dk \pm dk' = d(k \pm k')$, so $m + n$ and $m - n$ are divisible by d.

2.3. For each of the following questions, begin by compiling some data; next examine the data and formulate a conjecture; and finally try to prove that your conjecture is correct. (But don't worry if you can't solve every part of this problem; some parts are quite difficult.)

(a) Which odd numbers a can appear in a primitive Pythagorean triple (a, b, c)?

(b) Which even numbers b can appear in a primitive Pythagorean triple (a, b, c)?

(c) Which numbers c can appear in a primitive Pythagorean triple (a, b, c)?

Solution to Exercise 2.3.
(a) Any odd number can appear as the a in a primitive Pythagorean triple. To find such a triple, we can just take $t = a$ and $s = 1$ in the Pythagorean Triples Theorem. This gives the primitive Pythagorean triple $(a, (a^2 - 1)/2, (a^2 + 1)/2)$.

(b) Looking at the table, it seems first that b must be a multiple of 4, and second that every multiple of 4 seems to be possible. We know that b looks like $b = (s^2 - t^2)/2$ with

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s and t odd. This means we can write $s = 2m+1$ and $t = 2n+1$. Multiplying things out gives

$$b = \frac{(2m+1)^2 - (2n+1)^2}{2} = 2m^2 + 2m - 2n^2 - 2n$$
$$= 2m(m+1) - 2n(n+1).$$

Can you see that $m(m+1)$ and $n(n+1)$ must both be even, regardless of the value of m and n? So b must be divisible by 4.

On the other hand, if b is divisible by 4, then we can write it as $b = 2^r B$ for some odd number B and some $r \geq 2$. Then we can try to find values of s and t such that $(s^2 - t^2)/2 = b$. We factor this as

$$(s-t)(s+t) = 2b = 2^{r+1}B.$$

Now both $s - t$ and $s + t$ must be even (since s and t are odd), so we might try

$$s - t = 2^r \qquad \text{and} \qquad s + t = 2B.$$

Solving for s and t gives $s = 2^{r-1} + B$ and $t = -2^{r-1} + B$. Notice that s and t are odd, since B is odd and $r \geq 2$. Then

$$a = st = B^2 - 2^{2r-2},$$
$$b = \frac{s^2 - t^2}{2} = 2^r B,$$
$$c = \frac{s^2 + t^2}{2} = B^2 + 2^{2r-2}.$$

This gives a primitive Pythagorean triple with the right value of b provided that $B > 2^{r-1}$. On the other hand, if $B < 2^{r-1}$, then we can just take $a = 2^{2r-2} - B^2$ instead.

(c) This part is quite difficult to prove, and it's not even that easy to make the correct conjecture. It turns out that an odd number c appears as the hypotenuse of a primitive Pythagorean triple if and only if every prime dividing c leaves a remainder of 1 when divided by 4. Thus c appears if it is divisible by the primes $5, 13, 17, 29, 37, \ldots$, but it does not appear if it is divisible by any of the primes $3, 7, 11, 19, 23,$. We will prove this in Chapter 25. Note that it is not enough that c itself leave a remainder of 1 when divided by 4. For example, neither 9 nor 21 can appear as the hypotenuse of a primitive Pythagorean triple.

2.4. In our list of examples are the two primitive Pythagorean triples

$$33^2 + 56^2 = 65^2 \qquad \text{and} \qquad 16^2 + 63^2 = 65^2.$$

Find at least one more example of two primitive Pythagorean triples with the same value of c. Can you find three primitive Pythagorean triples with the same c? Can you find more than three?

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Solution to Exercise 2.4.
The next example is $c = 5 \cdot 17 = 85$. Thus

$$85^2 = 13^2 + 84^2 = 36^2 + 77^2.$$

A general rule is that if $c = p_1 p_2 \cdots p_r$ is a product of r distinct odd primes which all leave a remainder of 1 when divided by 4, then c appears as the hypotenuse in 2^{r-1} primitive Pythagorean triples. (This is counting (a, b, c) and (b, a, c) as the same triple.) So for example, $c = 5 \cdot 13 \cdot 17 = 1105$ appears in 4 triples,

$$1105^2 = 576^2 + 943^2 = 744^2 + 817^2 = 264^2 + 1073^2 = 47^2 + 1104^2.$$

But it would be difficult to prove the general rule using only the material we have developed so far.

2.5. In Chapter 1 we saw that the n^{th} triangular number T_n is given by the formula

$$T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

The first few triangular numbers are 1, 3, 6, and 10. In the list of the first few Pythagorean triples (a, b, c), we find (3, 4, 5), (5, 12, 13), (7, 24, 25), and (9, 40, 41). Notice that in each case, the value of b is four times a triangular number.

(a) Find a primitive Pythagorean triple (a, b, c) with $b = 4T_5$. Do the same for $b = 4T_6$ and for $b = 4T_7$.

(b) Do you think that for every triangular number T_n, there is a primitive Pythagorean triple (a, b, c) with $b = 4T_n$? If you believe that this is true, then prove it. Otherwise, find some triangular number for which it is not true.

Solution to Exercise 2.5.
(a) $T_5 = 15$ and (11, 60, 61). $T_6 = 21$ and (13, 84, 85). $T_7 = 28$ and (15, 112, 113).
(b) The primitive Pythagorean triples with b even are given by $b = (s^2 - t^2)/2$, $s > t \geq 1$, s and t odd integers, and $\gcd(s, t) = 1$. Since s is odd, we can write it as $s = 2n + 1$, and we can take $t = 1$. (The examples suggest that we want $c = b + 1$, which means we need to take $t = 1$.) Then

$$b = \frac{s^2 - t^2}{2} = \frac{(2n+1)^2 - 1}{2} = 2n^2 + 2n = 4\frac{n^2 + n}{2} = 4T_n.$$

So for every triangular number T_n, there is a Pythagorean triple

$$(2n + 1, 4T_n, 4T_n + 1).$$

(Thanks to Mike McConnell and his class for suggesting this problem.)

2.6. If you look at the table of primitive Pythagorean triples in this chapter, you will see many triples in which c is 2 greater than a. For example, the triples (3, 4, 5), (15, 8, 17), (35, 12, 37), and (63, 16, 65) all have this property.

(a) Find two more primitive Pythagorean triples (a, b, c) having $c = a + 2$.

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(b) Find a primitive Pythagorean triple (a, b, c) having $c = a + 2$ and $c > 1000$.

(c) Try to find a formula that describes all primitive Pythagorean triples (a, b, c) having $c = a + 2$.

Solution to Exercise 2.6.
The next few primitive Pythagorean triples with $c = a + 2$ are

$$(99, 20, 101), \quad (143, 24, 145), \quad (195, 28, 197),$$
$$(255, 32, 257), \quad (323, 36, 325), \quad (399, 40, 401).$$

One way to find them is to notice that the b values are going up by 4 each time. An even better way is to use the Pythagorean Triples Theorem. This says that $a = st$ and $c = (s^2 + t^2)/2$. We want $c - a = 2$, so we set

$$\frac{s^2 + t^2}{2} - st = 2$$

and try to solve for s and t. Multiplying by 2 gives

$$\begin{aligned} s^2 + t^2 - 2st &= 4, \\ (s - t)^2 &= 4, \\ s - t &= \pm 2. \end{aligned}$$

The Pythagorean Triples Theorem also says to take $s > t$, so we need to have $s - t = 2$. Further, s and t are supposed to be odd. If we substitute $s = t + 2$ into the formulas for a, b, c, we get a general formula for all primitive Pythagorean triples with $c = a + 2$. Thus

$$\begin{aligned} a &= st = (t + 2)t = t^2 + 2t, \\ b &= \frac{s^2 - t^2}{2} = \frac{(t + 2)^2 - t^2}{2} = 2t + 2, \\ c &= \frac{s^2 + t^2}{2} = \frac{(t + 2)^2 + t^2}{2} = t^2 + 2t + 2. \end{aligned}$$

We will get all PPT's with $c = a + 2$ by taking $t = 1, 3, 5, 7, \ldots$ in these formulas. For example, to get one with $c > 1000$, we just need to choose t large enough to make $t^2 + 2t + 2 > 1000$. The least t which will work is $t = 31$, which gives the PPT $(1023, 64, 1025)$. The next few with $c > 1000$ are $(1155, 68, 1157)$, $(1295, 72, 1297)$, $(1443, 76, 1445)$, obtained by setting $t = 33$, 35, and 37 respectively.

2.7. For each primitive Pythagorean triple (a, b, c) in the table in this chapter, compute the quantity $2c - 2a$. Do these values seem to have some special form? Try to prove that your observation is true for all primitive Pythagorean triples.

Solution to Exercise 2.7.
First we compute $2c - 2a$ for the PPT's in the Chapter 2 table.

a	3	5	7	9	15	21	35	45	63
b	4	12	24	40	8	20	12	28	16
c	5	13	25	41	17	29	37	53	65
$2c - 2a$	4	16	36	64	4	16	4	16	4

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all the differences $2c-2a$ seem to be perfect squares. We can show that this is always the case by using the Pythagorean Triples Theorem, which says that $a = st$ and $c = (s^2 + t^2)/2$. Then

$$2c - 2a = (s^2 + t^2) - 2st = (s - t)^2,$$

so $2c - 2a$ is always a perfect square.

2.8. Let m and n be numbers that differ by 2, and write the sum $\frac{1}{m} + \frac{1}{n}$ as a fraction in lowest terms. For example, $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ and $\frac{1}{3} + \frac{1}{5} = \frac{8}{15}$.

- **(a)** Compute the next three examples.
- **(b)** Examine the numerators and denominators of the fractions in (a) and compare them with the table of Pythagorean triples on page 18. Formulate a conjecture about such fractions.
- **(c)** Prove that your conjecture is correct.

Solution to Exercise 2.8.
(a)

$$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}, \quad \frac{1}{5} + \frac{1}{7} = \frac{12}{35}, \quad \frac{1}{6} + \frac{1}{8} = \frac{7}{24}.$$

(b) It appears that the numerator and denominator are always the sides of a (primitive) Pythagorean triple.
(c) This is easy to prove. Thus

$$\frac{1}{N} + \frac{1}{N+2} = \frac{2N+2}{N^2+2N}.$$

The fraction is in lowest terms if N is odd, otherwise we need to divide numerator and denominator by 2. But in any case, the numerator and denominator are part of a Pythagorean triple, since

$$(2N+2)^2 + (N^2+2N)^2 = N^4 + 4N^3 + 8N^2 + 8N + 4 = (N^2+2N+2)^2.$$

Once one suspects that $N^4 + 4N^3 + 8N^2 + 8N + 4$ should be a square, it's not hard to factor it. Thus if it's a square, it must look like $(N^2 + AN \pm 2)^2$ for some value of A. Now just multiply out and solve for A, then check that your answer works.

2.9. **(a)** Read about the Babylonian number system and write a short description, including the symbols for the numbers 1 to 10 and the multiples of 10 from 20 to 50.
- **(b)** Read about the Babylonian tablet called Plimpton 322 and write a brief report, including its approximate date of origin.
- **(c)** The second and third columns of Plimpton 322 give pairs of integers (a, c) having the property that $c^2 - a^2$ is a perfect square. Convert some of these pairs from Baby-lonian numbers to decimal numbers and compute the value of b so that (a, b, c) is a Pythagorean triple.

Solution to Exercise 2.9.
There is a good article in wikipedia on Plimpton 322. Another nice source for this material is
`www.math.ubc.ca/~cass/courses/m446-03/pl322/pl322.html`

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