Solution Manual for Fundamentals of Communication Systems 2nd Edition by Proakis Salehi ISBN 0133354857 9780133354850

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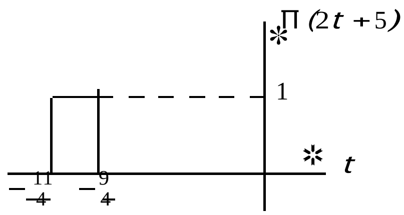
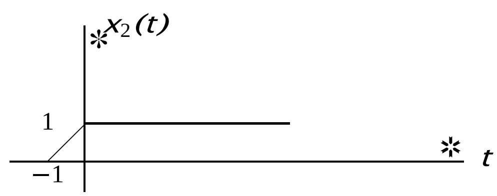
Chapter 2

Problem 2.1

1. $\Pi(2t+5) = \Pi\left(2\left(t+\frac{5}{2}\right)\right)$. This indicates first we have to plot $\Pi(2t)$ and then shift it to left by $\frac{5}{2}$. A plot is shown below:

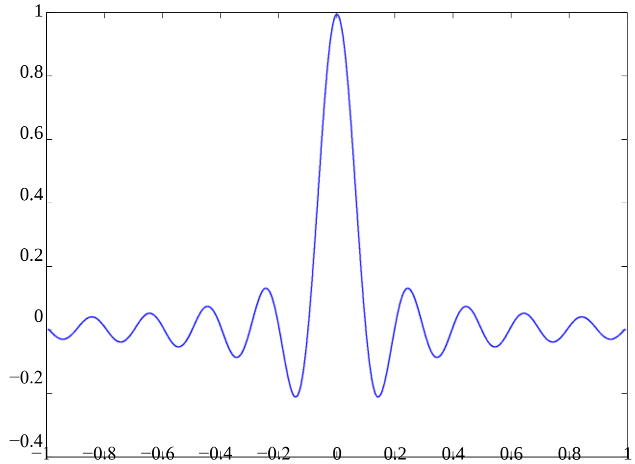


2. $\sum_{n=0}^{\infty} \Lambda(t-n)$ is a sum of shifted triangular pulses. Note that the sum of the left and right side of triangular pulses that are displaced by one unit of time is equal to 1, The plot is given below



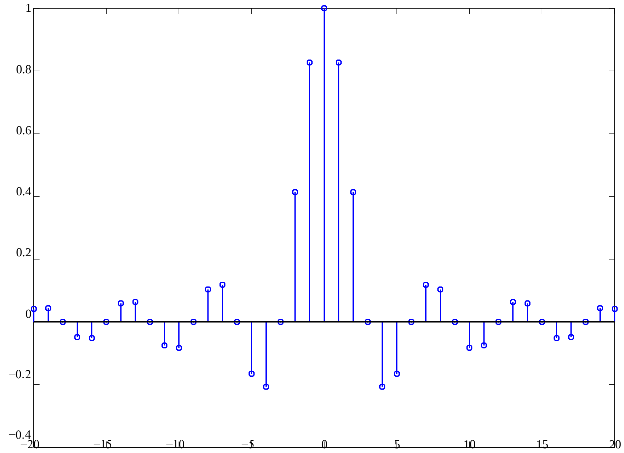
3. It is obvious from the definition of sgn(t) that sgn$(2t)$ = sgn(t). Therefore $x_3(t) = 0$.

4. $x_4(t)$ is sinc(t) contracted by a factor of 10.

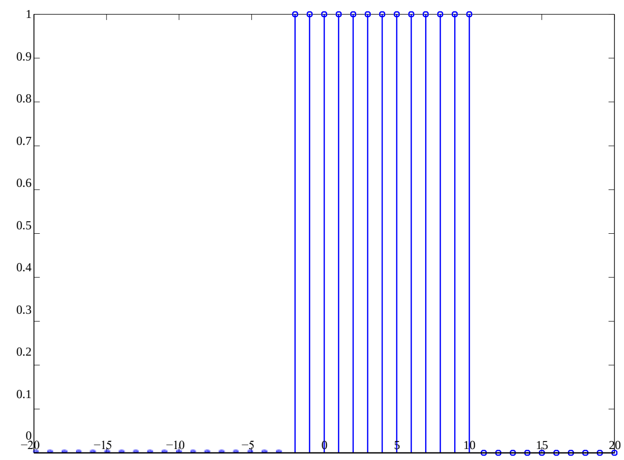


Problem 2.2

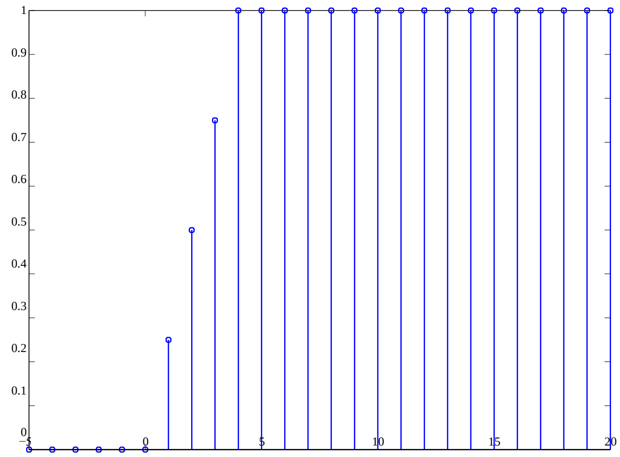
1. $x[n] = \text{sinc}(3n/9) = \text{sinc}(n/3)$.



2. $x[n] = \Pi\left(\frac{\frac{n}{4}-1}{3}\right)$. If $-\frac{1}{2} \le \frac{\frac{n}{4}-1}{3} \le \frac{1}{2}$, i.e., $-2 \le n \le 10$, we have $x[n] = 1$.



3. $x[n] = \frac{n}{4} u_{-1}(n/4) - (\frac{n}{4} - 1) u_{-1}(n/4 - 1)$. For $n < 0$, $x[n] = 0$, for $0 \le n \le 3$, $x[n] = \frac{n}{4}$ and for $n \ge 4$, $x[n] = \frac{n}{4} - \frac{n}{4} + 1 = 1$.



Problem 2.3

$x_1[n] = 1$ and $x_2[n] = \cos(2\pi n) = 1$, for all n. This shows that two signals can be different but their sampled versions be the same.

Problem 2.4

Let $x_1[n]$ and $x_2[n]$ be two periodic signals with periods N_1 and N_2, respectively, and let $N = \text{LCM}(N_1, N_2)$, and define $x[n] = x_1[n] + x_2[n]$. Then obviously $x_1[n+N] = x_1[n]$ and $x_2[n+N] = x_2[n]$, and hence $x[n] = x[n+N]$, i.e., $x[n]$ is periodic with period N.

For continuous-time signals $x_1(t)$ and $x_2(t)$ with periods T_1 and T_2 respectively, in general we cannot find a T such that $T = k_1 T_1 = k_2 T_2$ for integers k_1 and k_2. This is obvious for instance if $T_1 = 1$ and $T_2 = \pi$. The necessary and sufficient condition for the sum to be periodic is that $\frac{T_1}{T_2}$ be a rational number.

Problem 2.5

Using the result of problem 2.4 we have:

1. The frequencies are 2000 and 5500, their ratio (and therefore the ratio of the periods) is rational, hence the sum is periodic.

2. The frequencies are 2000 and $\frac{5500}{\pi}$. Their ratio is not rational, hence the sum is not periodic.

3. The sum of two periodic discrete-time signal is periodic.

4. The fist signal is periodic but $\cos[11000n]$ is *not* periodic, since there is no N such that $\cos[11000(n+N)] = \cos(11000n)$ for all n. Therefore the sum cannot be periodic.

Problem 2.6

1)

$$x_1(t) = \begin{cases} e^{-t} & t > 0 \\ -e^{t} & t < 0 \\ 0 & t = 0 \end{cases} \Rightarrow x_1(-t) = \begin{cases} -e^{-t} & t > 0 \\ e^{t} & t < 0 \\ 0 & t = 0 \end{cases} = -x_1(t)$$

Thus, $x_1(t)$ is an odd signal

2) $x_2(t) = \cos\left(120\pi t + \frac{\pi}{3}\right)$ is neither even nor odd. We have $\cos\left(120\pi t + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)\cos(120\pi t) - \sin\left(\frac{\pi}{3}\right)\sin(120\pi t)$. Therefore $x_{2e}(t) = \cos\left(\frac{\pi}{3}\right)\cos(120\pi t)$ and $x_{2o}(t) = -\sin\left(\frac{\pi}{3}\right)\sin(120\pi t)$.

(Note: This part can also be considered as a special case of part 7 of this problem)

3)

$$x_3(t) = e^{-|t|} \Rightarrow x_3(-t) = e^{-|(-t)|} = e^{-|t|} = x_3(t)$$

Hence, the signal $x_3(t)$ is even.

4)

$$x_4(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases} \Rightarrow x_4(-t) = \begin{cases} 0 & t \geq 0 \\ -t & t < 0 \end{cases}$$

The signal $x_4(t)$ is neither even nor odd. The even part of the signal is

$$x_{4,e}(t) = \frac{x_4(t) + x_4(-t)}{2} = \begin{cases} \frac{t}{2} & t \geq 0 \\ \frac{-t}{2} & t < 0 \end{cases} = \frac{|t|}{2}$$

The odd part is

$$x_{4,o}(t) = \frac{x_4(t) - x_4(-t)}{2} = \begin{cases} \frac{t}{2} & t \geq 0 \\ \frac{t}{2} & t < 0 \end{cases} = \frac{t}{2}$$

5)

$$x_5(t) = x_1(t) - x_2(t) \Rightarrow x_5(-t) = x_1(-t) - x_2(-t) = x_1(t) + x_2(t)$$

Clearly $x_5(-t) \neq x_5(t)$ since otherwise $x_2(t) = 0 \ \forall t$. Similarly $x_5(-t) \neq -x_5(t)$ since otherwise $x_1(t) = 0 \ \forall t$. The even and the odd parts of $x_5(t)$ are given by

$$x_{5,e}(t) = \frac{x_5(t) + x_5(-t)}{2} = x_1(t)$$

$$x_{5,o}(t) = \frac{x_5(t) - x_5(-t)}{2} = -x_2(t)$$

Problem 2.7

For the first two questions we will need the integral $I = \int e^{ax}\cos^2 x\,dx$.

$$
\begin{aligned}
I &= \frac{1}{a}\int \cos^2 x\, de^{ax} = \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a}\int e^{ax}\sin 2x\,dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}\int \sin 2x\, de^{ax} \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax}\cos 2x\,dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax}(2\cos^2 x - 1)\,dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax}\,dx - \frac{4}{a^2}I
\end{aligned}
$$

Thus,

$$
I = \frac{1}{4+a^2}\left[(a\cos^2 x + \sin 2x) + \frac{2}{a}\right]e^{ax}
$$

1)

$$
\begin{aligned}
E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_1^2(t)\,dt = \lim_{T\to\infty}\int_0^{\frac{T}{2}} e^{-2t}\cos^2 t\,dt \\
&= \lim_{T\to\infty}\frac{1}{8}\left[(-2\cos^2 t + \sin 2t) - 1\right]e^{-2t}\Big|_0^{\frac{T}{2}} \\
&= \lim_{T\to\infty}\frac{1}{8}\left[(-2\cos^2\frac{T}{2} + \sin T - 1)e^{-T} + 3\right] = \frac{3}{8}
\end{aligned}
$$

Thus $x_1(t)$ is an energy-type signal and the energy content is $3/8$

2)

$$
\begin{aligned}
E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_2^2(t)\,dt = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-2t}\cos^2 t\,dt \\
&= \lim_{T\to\infty}\left[\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 t\,dt + \int_0^{\frac{T}{2}} e^{-2t}\cos^2 t\,dt\right]
\end{aligned}
$$

But,

$$
\lim_{T\to\infty}\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 t\,dt = \lim_{T\to\infty}\frac{1}{8}\left[(-2\cos^2 t + \sin 2t) - 1\right]e^{-2t}\Big|_{-\frac{T}{2}}^{0}
$$

$$= \lim_{T\to\infty} \frac{1}{8} \left[-3 + (2\cos^2 \frac{T}{2} + 1 + \sin T)e^T \right] = \infty$$

since $2 + \cos\theta + \sin\theta > 0$. Thus, $E_x = \infty$ since as we have seen from the first question the second integral is bounded. Hence, the signal $x_2(t)$ is not an energy-type signal. To test if $x_2(t)$ is a power-type signal we find P_x.

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{0} e^{-2t} \cos^2 dt + \lim_{T\to\infty} \frac{1}{T} \int_{0}^{\frac{T}{2}} e^{-2t} \cos^2 dt$$

But $\lim_{T\to\infty} \frac{1}{T}\int_0^{\frac{T}{2}} e^{-2t}\cos^2 dt$ is zero and

$$\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 dt = \lim_{T\to\infty}\frac{1}{8T}\left(2\cos^2\frac{T}{2}+1+\sin T\right)e^{T}$$

$$> \lim_{T\to\infty}\frac{1}{T}e^{T} > \lim_{T\to\infty}\frac{1}{T}(1+T+T^2) > \lim_{T\to\infty} T = \infty$$

Thus the signal $x_2(t)$ is not a power-type signal.

3)

$$E_x = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_3^2(t)dx = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} \text{sgn}^2(t)dt = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \lim_{T\to\infty} T = \infty$$

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \text{sgn}^2(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \lim_{T\to\infty}\frac{1}{T}T = 1$$

The signal $x_3(t)$ is of the power-type and the power content is 1.

4)

First note that

$$\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} A\cos(2\pi f t)dt = \sum_{k=-\infty}^{\infty} A\int_{k-\frac{1}{2f}}^{k+\frac{1}{2f}} \cos(2\pi f t)dt = 0$$

so that

$$\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2\cos^2(2\pi f t)dt = \lim_{T\to\infty}\frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} (A^2 + A^2\cos(2\pi 2 f t))dt$$

$$= \lim_{T\to\infty}\frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 dt = \lim_{T\to\infty}\frac{1}{2}A^2 T = \infty$$

$$E_x = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} (A^2\cos^2(2\pi f_1 t) + B^2\cos^2(2\pi f_2 t) + 2AB\cos(2\pi f_1 t)\cos(2\pi f_2 t))dt$$

$$= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2\cos^2(2\pi f_1 t)dt + \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} B^2\cos^2(2\pi f_2 t)dt +$$

$$AB\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} [\cos^2(2\pi(f_1+f_2)+\cos^2(2\pi(f_1-f_2)]dt$$

$$= \infty + \infty + 0 = \infty$$

Thus the signal is not of the energy-type. To test if the signal is of the power-type we consider two cases $f_1 = f_2$ and $f_1 \neq f_2$. In the first case

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} (A+B)^2 \cos^2(2\pi f_1)dt \\ &= \lim_{T\to\infty} \frac{1}{2T}(A+B)^2 \int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \frac{1}{2}(A+B)^2 \end{aligned}$$

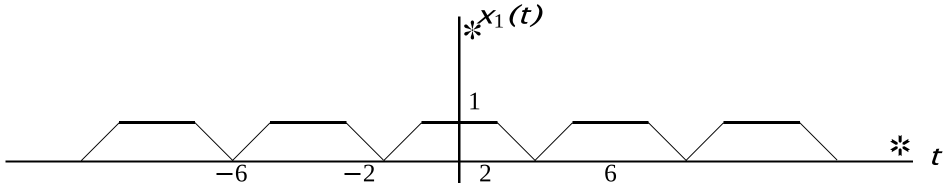
If $f_1 \neq f_2$ then

$$
\begin{aligned}
P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(A^2 \cos^2(2\pi f_1 t) + B^2 \cos^2(2\pi f_2 t) + 2AB \cos(2\pi f_1 t)\cos(2\pi f_2 t)\right) dt \\
&= \lim_{T\to\infty} \frac{1}{T} \left[\frac{A^2 T}{2} + \frac{B^2 T}{2} \right] = \frac{A^2}{2} + \frac{B^2}{2}
\end{aligned}
$$

Thus the signal is of the power-type and if $f_1 = f_2$ the power content is $(A+B)^2/2$ whereas if $f_1 \neq f_2$ the power content is $\frac{1}{2}(A^2 + B^2)$

Problem 2.8

1. Let $x(t) = 2\Lambda\left(\frac{t}{2}\right) - \Lambda(t)$, then $x_1(t) = \sum_{n=-\infty}^{\infty} x(t-4n)$. First we plot $x(t)$ then by shifting it by multiples of 4 we can plot $x_1(t)$. $x(t)$ is a triangular pulse of width 4 and height 2 from which a standard triangular pulse of width 1 and height 1 is subtracted. The result is a trapezoidal pulse, which when replicated at intervals of 4 gives the plot of $x_1(t)$.



2. This is the sum of two periodic signals with periods 2π and 1. Since the ratio of the two periods is not rational the sum is not periodic (by the result of problem 2.4)

3. $\sin[n]$ is not periodic. There is no integer N such that $\sin[n+N] = \sin[n]$ for all n.

Problem 2.9

1)

$$
P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 \left| e^{j(2\pi f_0 t + \theta)} \right|^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 dt = \lim_{T\to\infty} \frac{1}{T} A^2 T = A^2
$$

Thus $x(t) = Ae^{j(2\pi f_0 t + \theta)}$ is a power-type signal and its power content is A^2.

2)

$$
P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 \cos^2(2\pi f_0 t + \theta)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{A^2}{2} dt + \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{A^2}{2} \cos(4\pi f_0 t + 2\theta)\, dt
$$

As $T \to \infty$, the there will be no contribution by the second integral. Thus the signal is a power-type signal and its power content is $\frac{A^2}{2}$.

3)

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u_{-1}^2(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_0^{\frac{T}{2}} dt = \lim_{T\to\infty} \frac{1}{T}\frac{T}{2} = \frac{1}{2}$$

Thus the unit step signal is a power-type signal and its power content is $1/2$

4)

$$E_x = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt = \lim_{T\to\infty} \int_0^{\frac{T}{2}} K^2 t^{-\frac{1}{2}} dt = \lim_{T\to\infty} 2K^2 t^{\frac{1}{2}}\Big|_0^{T/2}$$

$$= \lim_{T\to\infty} \sqrt{2}K^2 T^{\frac{1}{2}} = \infty$$

Thus the signal is not an energy-type signal.

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_0^{\frac{T}{2}} K^2 t^{-\frac{1}{2}} dt$$

$$= \lim_{T\to\infty} \frac{1}{T} 2K^2 t^{\frac{1}{2}}\Big|_0^{T/2} = \lim_{T\to\infty} \frac{1}{T} 2K^2 (T/2)^{\frac{1}{2}} = \lim_{T\to\infty} \sqrt{2}K^2 T^{-\frac{1}{2}} = 0$$

Since P_x is not bounded away from zero it follows by definition that the signal is not of the power- type (recall that power-type signals should satisfy $0 < P_x < \infty$).

Problem 2.10

$$\Lambda(t) = \begin{cases} t+1, & -1 \le t \le 0 \\ -t+1, & 0 \le t \le 1 \\ 0, & \text{o.w.} \end{cases} \qquad u_{-1}(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}$$

Thus, the signal $x(t) = \Lambda(t)u_{-1}(t)$ is given by

$$x(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ -t+1 & 0 \le t \le 1 \\ 0 & t \ge 1 \end{cases} \Rightarrow x(-t) = \begin{cases} 0 & t \le -1 \\ t+1 & -1 \le t < 0 \\ 1/2 & t = 0 \\ 0 & t > 0 \end{cases}$$

The even and the odd part of $x(t)$ are given by

$$\frac{x(t)+x(-t)}{2} \quad \frac{1}{}$$

$$x_e(t) \;=\; \frac{}{2} \;=\; \frac{}{2}\Lambda(t)$$

$$x_o(t) \;=\; \frac{x(t) - x(-t)}{2} = \begin{cases} 0 & t \le -1 \\ \frac{-t-1}{2} & -1 \le t < 0 \\ 0 & t = 0 \\ \frac{-t+1}{2} & 0 < t \le 1 \\ 0 & 1 \le t \end{cases}$$

Problem 2.11

1) Suppose that

$$x(t) = x_e^1(t) + x_o^1(t) = x_e^2(t) + x_o^2(t)$$

with $x_e^1(t)$, $x_e^2(t)$ even signals and $x_o^1(t)$, $x_o^1(t)$ odd signals. Then, $x(-t) = x_e^1(t) - x_o^1(t)$ so that

$$\begin{aligned} x_e^1(t) &= \frac{x(t) + x(-t)}{2} \\ &= \frac{x_e^2(t) + x_o^2(t) + x_e^2(-t) + x_o^2(-t)}{2} \\ &= \frac{2x_e^2(t) + x_o^2(t) - x_o^2(t)}{2} = x_e^2(t) \end{aligned}$$

Thus $x_e^1(t) = x_e^2(t)$ and $x_o^1(t) = x(t) - x_e^1(t) = x(t) - x_e^2(t) = x_o^2(t)$

2) Let $x_e^1(t)$, $x_e^2(t)$ be two even signals and $x_o^1(t)$, $x_o^2(t)$ be two odd signals. Then,

$$y(t) = x_e^1(t)x_e^2(t) \Rightarrow y(-t) = x_e^1(-t)x_e^2(-t) = x_e^1(t)x_e^2(t) = y(t)$$

$$z(t) = x_o^1(t)x_o^2(t) \Rightarrow z(-t) = x_o^1(-t)x_o^2(-t) = (-x_o^1(t))(-x_o^2(t)) = z(t)$$

Thus the product of two even or odd signals is an even signal. For $v(t) = x_e^1(t)x_o^1(t)$ we have

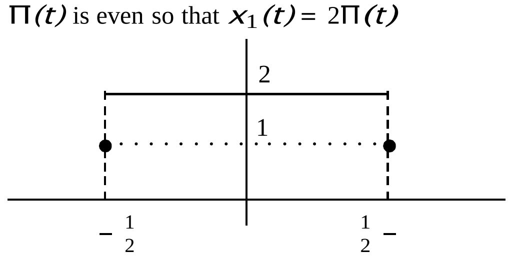
$$v(-t) = x_e^1(-t)x_o^1(-t) = x_e^1(t)(-x_o^1(t)) = -x_e^1(t)x_o^1(t) = -v(t)$$

Thus the product of an even and an odd signal is an odd signal.

3) One trivial example is $t + 1$ and $\frac{t^2}{t+1}$.

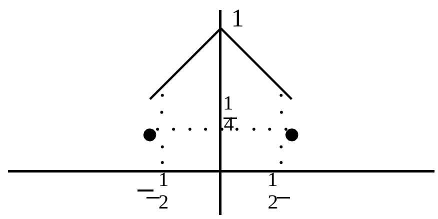
Problem 2.12

1) $x_1(t) = \Pi(t) + \Pi(-t)$. The signal $\Pi(t)$ is even so that $x_1(t) = 2\Pi(t)$

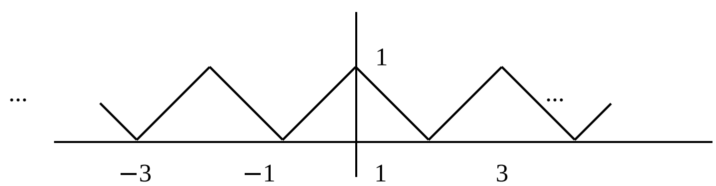


2)

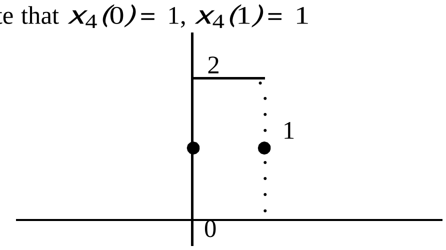
$$x_2(t) = \Lambda(t) \cdot \Pi(t) = \begin{cases} 0, & t < -1/2 \\ 1/4, & t = -1/2 \\ t+1, & -1/2 < t \leq 0 \\ -t+1, & 0 \leq t < 1/2 \\ 1/4, & t = 1/2 \\ 0, & 1/2 < t \end{cases}$$



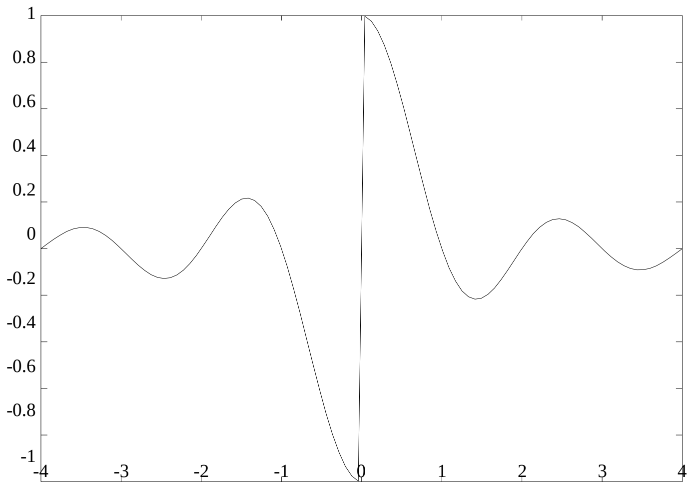
3) $x_3(t) = \sum_{n=-\infty}^{\infty} \Lambda(t - 2n)$



4) $x_4(t) = \text{sgn}(t) + \text{sgn}(1 - t)$. Note that $x_4(0) = 1$, $x_4(1) = 1$



5) $x_5(t) = \text{sinc}(t)\text{sgn}(t)$. Note that $x_5(0) = 0$.



Problem 2.13

1) The value of the expression $\text{sinc}(t)\delta(t)$ can be found by examining its effect on a function $\phi(t)$ through the integral

$$\int_{-\infty}^{\infty} \phi(t)\text{sinc}(t)\delta(t) = \phi(0)\text{sinc}(0) = \text{sinc}(0) \int_{-\infty}^{\infty} \phi(t)\delta(t)$$

Thus $\text{sinc}(t)\delta(t)$ has the same effect as the function $\text{sinc}(0)\delta(t)$ and we conclude that

$$x_1(t) = \text{sinc}(t)\delta(t) = \text{sinc}(0)\delta(t) = \delta(t)$$

2) $\text{sinc}(t)\delta(t-3) = \text{sinc}(3)\delta(t-3) = 0.$

3)

$$\begin{aligned} x_3(t) &= \Lambda(t) \star \sum_{n=-\infty}^{\infty} \delta(t-2n) \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \Lambda(t-\tau)\delta(\tau-2n)d\tau \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \Lambda(\tau-t)\delta(\tau-2n)d\tau \\ &= \sum_{n=-\infty}^{\infty} \Lambda(t-2n) \end{aligned}$$

4)

$$
\begin{aligned}
x_4(t) &= \Lambda(t) \star \delta'(t) = \int_{-\infty}^{\infty} \Lambda(t-\tau)\delta'(\tau)d\tau \\
&= (-1)\frac{d}{d\tau}\Lambda(t-\tau)\Big|_{\tau=0} = \Lambda'(t) = \begin{cases} 0 & t < -1 \\ \frac{1}{2} & t = -1 \\ 1 & -1 < t < 0 \\ 0 & t = 0 \\ -1 & 0 < t < 1 \\ -\frac{1}{2} & t = 1 \\ 0 & 1 < t \end{cases}
\end{aligned}
$$

5) $x_5(t) = \cos\left(2t + \frac{\pi}{3}\right)\delta(3t) = \frac{1}{3}\cos\left(2t + \frac{\pi}{3}\right)\delta(t) = \frac{1}{3}\cos\left(\frac{\pi}{3}\right)\delta(t)$. Hence $x_5(t) = \frac{1}{6}\delta(t)$.

6)

$$
x_6(t) = \delta(5t) \star \delta(4t) = \frac{1}{5}\delta(t) \star \frac{1}{4}\delta(t) = \frac{1}{20}\delta(t)
$$

7)

$$
\int_{-\infty}^{\infty} \text{sinc}(t)\delta(t)dt = \text{sinc}(0) = 1
$$

8)

$$
\int_{-\infty}^{\infty} \text{sinc}(t+1)\delta(t)dt = \text{sinc}(1) = 0
$$

Problem 2.14

The impulse signal can be defined in terms of the limit

$$
\delta(t) = \lim_{\tau \to 0} \frac{1}{2\tau} e^{-\frac{|t|}{\tau}}
$$

But $e^{-\frac{|t|}{\tau}}$ is an even function for every τ so that $\delta(t)$ is even. Since $\delta(t)$ is even, we obtain

$$
\delta(t) = \delta(-t) \Longrightarrow \delta'(t) = -\delta'(-t)
$$

Thus, the function $\delta'(t)$ is odd. For the function $\delta^{(n)}(t)$ we have

$$
\int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(-t)dt = (-1)^n \int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(t)dt
$$

d

where we have used the differentiation chain rule

$$\frac{d}{dt}\delta^{(k-1)}(-t) = \frac{d}{d(-t)}\delta^{(k-1)}(-t)\frac{}{dt}(-t) = (-1)\delta^{(k)}(-t)$$

Thus, if $n = 2l$ (even)

$$\int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(-t)dt = \int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(t)dt$$

and the function $\delta^{(n)}(t)$ is even. If $n = 2l + 1$ (odd), then $(-1)^n = -1$ and

$$\int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(-t)dt = -\int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(t)dt$$

from which we conclude that $\delta^{(n)}(t)$ is odd.

Problem 2.15

$$x(t) \star \delta^{(n)}(t) = \int_{-\infty}^{\infty} x(\tau)\delta^{(n)}(t-\tau)\, d\tau$$

The signal $\delta^{(n)}(t)$ is even if n is even and odd if n is odd. Consider first the case that $n = 2l$. Then,

$$x(t) \star \delta^{(2l)}(t) = \int_{-\infty}^{\infty} x(\tau)\delta^{(2l)}(\tau - t)\, d\tau = (-1)^{2l} \left. \frac{d^{2l}}{d\tau^{2l}} x(\tau) \right|_{\tau=t} = \frac{d^n}{dt^n} x(t)$$

If n is odd then,

$$\begin{aligned} x(t) \star \delta^{(2l+1)}(t) &= \int_{-\infty}^{\infty} x(\tau)(-1)\delta^{(2l+1)}(\tau - t)\, d\tau = (-1)(-1)^{2l+1} \left. \frac{d^{2l+1}}{d\tau^{2l+1}} x(\tau) \right|_{\tau=t} \\ &= \frac{d^n}{dt^n} x(t) \end{aligned}$$

In both cases

$$x(t) \star \delta^{(n)}(t) = \frac{d^n}{dt^n} x(t)$$

The convolution of $x(t)$ with $u_{-1}(t)$ is

$$x(t) \star u_{-1}(t) = \int_{-\infty}^{\infty} x(\tau)u_{-1}(t-\tau)d\tau$$

But $u_{-1}(t-\tau) = 0$ for $\tau > t$ so that

$$x(t) \star u_{-1}(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Problem 2.16

1) Nonlinear, since the response to $x(t) = 0$ is not $y(t) = 0$ (this is a necessary condition for linearity of a system, see also problem 2.21).

2) Nonlinear, if we multiply the input by constant -1, the output does not change. In a linear system the output should be scaled by -1.

3) Linear, the output to any input zero, therefore for the input $\alpha x_1(t) + \beta x_2(t)$ the output is zero which can be considered as $\alpha y_1(t) + \beta y_2(t) = \alpha \times 0 + \beta \times 0 = 0$. This is a linear combination of the corresponding outputs to $x_1(t)$ and $x_2(t)$.
4) Nonlinear, the output to $x(t) = 0$ is not zero.
5) Nonlinear. The system is not homogeneous for if $\alpha < 0$ and $x(t) > 0$ then $y(t) = T[\alpha x(t)] = 0$ whereas $z(t) = \alpha T[x(t)] = \alpha$.

6) Linear. For if $x(t) = \alpha x_1(t) + \beta x_2(t)$ then

$$\begin{aligned} T[\alpha x_1(t) + \beta x_2(t)] &= (\alpha x_1(t) + \beta x_2(t))e^{-t} \\ &= \alpha x_1(t)e^{-t} + \beta x_2(t)e^{-t} = \alpha T[x_1(t)] + \beta T[x_2(t)] \end{aligned}$$

7) Linear. For if $x(t) = \alpha x_1(t) + \beta x_2(t)$ then

$$\begin{aligned} T[\alpha x_1(t) + \beta x_2(t)] &= (\alpha x_1(t) + \beta x_2(t))u(t) \\ &= \alpha x_1(t)u(t) + \beta x_2(t)u(t) = \alpha T[x_1(t)] + \beta T[x_2(t)] \end{aligned}$$

8) Linear. We can write the output of this feedback system as

$$y(t) = x(t) + y(t-1) = \sum_{n=0}^{\infty} x(t-n)$$

Then for $x(t) = \alpha x_1(t) + \beta x_2(t)$

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} (\alpha x_1(t-n) + \beta x_2(t-n)) \\ &= \alpha \sum_{n=0}^{\infty} x_1(t-n) + \beta \sum_{n=0}^{\infty} x_2(t-n)) \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

9) Linear. Assuming that only a finite number of jumps occur in the interval $(-\infty, t]$ and that the magnitude of these jumps is finite so that the algebraic sum is well defined, we obtain

$$y(t) = T[\alpha x(t)] = \sum_{n=1}^{N} \alpha J_x(t_n) = \alpha \sum_{n=1}^{N} J_x(t_n) = \alpha T[x(t)]$$

where N is the number of jumps in $(-\infty, t]$ and $J_x(t_n)$ is the value of the jump at time instant t_n, that is

$$J_x(t_n) = \lim_{\varrho \to 0} (x(t_n + \varrho) - x(t_n - \varrho))$$

For $x(t) = x_1(t) + x_2(t)$ we can assume that $x_1(t)$, $x_2(t)$ and $x(t)$ have the same number of jumps and at the same positions. This is true since we can always add new jumps of magnitude zero to the already existing ones. Then for each $t_n, J_x(t_n) = J_{x_1}(t_n) + J_{x_2}(t_n)$ and

$$y(t) = \sum_{n=1}^{N} J_x(t_n) = \sum_{n=1}^{N} J_x(t_n) + \sum_{n=1}^{N} J_x(t_n)$$

so that the system is additive.

$n=1$ $n=$ ^{1} 1 $n=1$ ^{2}

Problem 2.17

Only if ($\Rightarrow$)

If the system T is linear then

$$T[\alpha x_1(t) + \beta x_2(t)] = \alpha T[x_1(t)] + \beta T[x_2(t)]$$

for all α, β and $x(t)$'s. If we set $\beta = 0$, then

$$T[\alpha x_1(t)] = \alpha T[x_1(t)]$$

so that the system is homogeneous. If we let $\alpha = \beta = 1$, we obtain

$$T[x_1(t) + x_2(t)] = T[x_1(t)] + T[x_2(t)]$$

and thus the system is additive. If ($\Leftarrow$)
Suppose that both conditions 1) and 2) hold. Thus the system is homogeneous and additive. Then

$$\begin{aligned} & T[\alpha x_1(t) + \beta x_2(t)] \\ = \; & T[\alpha x_1(t)] + T[\beta x_2(t)] \text{ (additive system)} \\ = \; & \alpha T[x_1(t)] + \beta T[x_2(t)] \text{ (homogeneous system)} \end{aligned}$$

Thus the system is linear.

Problem 2.18

1. Neither homogeneous nor additive.
2. Neither homogeneous nor additive.
3. Homogeneous and additive.
4. Neither homogeneous nor additive.
5. Neither homogeneous nor additive.
6. Homogeneous but not additive.
7. Neither homogeneous nor additive.
8. Homogeneous and additive.
9. Homogeneous and additive.

10. Homogeneous and additive.

11. Homogeneous and additive.

12. Homogeneous and additive.

13. Homogeneous and additive.

14. Homogeneous and additive.

Problem 2.19

We first prove that

$$\mathrm{T}\,[nx(t)] = n\mathrm{T}\,[x(t)]$$

for $n \in \mathbb{N}$. The proof is by induction on n. For $n = 2$ the previous equation holds since the system is additive. Let us assume that it is true for n and prove that it holds for $n + 1$.

$$\begin{aligned}
&\mathrm{T}\,[(n+1)x(t)] \\
&= \mathrm{T}\,[nx(t) + x(t)] \\
&= \mathrm{T}\,[nx(t)] + \mathrm{T}\,[x(t)] \quad \text{(additive property of the system)} \\
&= n\mathrm{T}\,[x(t)] + \mathrm{T}\,[x(t)] \quad \text{(hypothesis, equation holds for } n\text{)} \\
&= (n+1)\mathrm{T}\,[x(t)]
\end{aligned}$$

Thus $\mathrm{T}\,[nx(t)] = n\mathrm{T}\,[x(t)]$ for every n. Now, let

$$x(t) = my(t)$$

This implies that

$$\mathrm{T}\left[\frac{x(t)}{m}\right] = \mathrm{T}\,[y(t)]$$

and since $\mathrm{T}\,[x(t)] = \mathrm{T}\,[my(t)] = m\mathrm{T}\,[y(t)]$ we obtain

$$\mathrm{T}\left[\frac{x(t)}{m}\right] = \frac{1}{m}\mathrm{T}\,[x(t)]$$

Thus, for an arbitrary rational $\alpha = \frac{k}{\lambda}$ we have

$$\mathrm{T}\left[\frac{k}{\lambda}x(t)\right] = \mathrm{T}\left[k\,\frac{x(t)}{\lambda}\right] = k\mathrm{T}\left[\frac{x(t)}{\lambda}\right] = \frac{k}{\lambda}\mathrm{T}\,[x(t)]$$

Problem 2.20

Clearly, for any α

$$y(t) = T[\alpha x(t)] = \begin{cases} \frac{\alpha^2 x^2(t)}{\alpha x'(t)} & x'(t) \neq 0 \\ 0 & x'(t) = 0 \end{cases} = \alpha \begin{cases} \frac{x^2(t)}{x'(t)} & x'(t) \neq 0 \\ 0 & x'(t) = 0 \end{cases} = \alpha T[x(t)]$$

Thus the system is homogeneous and if it is additive then it is linear. However, if $x(t) = x_1(t) + x_2(t)$ then $x'(t) = x'_1(t) + x'_2(t)$ and

$$\frac{(x_1(t)+x_2(t))^2}{x'_1(t)+x'_2(t)} \neq \frac{x_1^2(t)}{x'_1(t)} + \frac{x_2^2(t)}{x'_2(t)}$$

for some $x_1(t), x_2(t)$. To see this let $x_2(t) = c$ (a constant signal). Then

$$T[x_1(t)+x_2(t)] = \frac{(x_1(t)+c)^2}{x'_1(t)} = \frac{x_1^2(t)+2cx_1(t)+c^2}{x'_1(t)}$$

and

$$T[x_1(t)] + T[x_2(t)] = \frac{x_1^2(t)}{x'_1(t)}$$

Thus $T[x_1(t)+x_2(t)] \neq T[x_1(t)] + T[x_2(t)]$ unless $c = 0$. Hence the system is nonlinear since the additive property has to hold for every $x_1(t)$ and $x_2(t)$.

As another example of a system that is homogeneous but non linear is the system described by

$$T[x(t)] = \begin{cases} x(t) + x(t-1) & x(t)x(t-1) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Clearly $T[\alpha x(t)] = \alpha T[x(t)]$ but $T[x_1(t) + x_2(t)] \neq T[x_1(t)] + T[x_2(t)]$

Problem 2.21

Any zero input signal can be written as $0 \cdot x(t)$ with $x(t)$ an arbitrary signal. Then, the response of the linear system is $y(t) = \mathrm{L}[0 \cdot x(t)]$ and since the system is homogeneous (linear system) we obtain

$$y(t) = \mathrm{L}[0 \cdot x(t)] = 0 \cdot \mathrm{L}[x(t)] = 0$$

Thus the response of the linear system is identically zero.

Problem 2.22

For the system to be linear we must have

$$\mathrm{T}\,[\alpha x_1(t) + \beta x_2(t)] = \alpha \mathrm{T}\,[x_1(t)] + \beta \mathrm{T}\,[x_2(t)]$$

for every α, β and $x(t)$'s.

$$\begin{aligned} \mathrm{T}\,[\alpha x_1(t) + \beta x_2(t)] &= (\alpha x_1(t) + \beta x_2(t))\cos(2\pi f_0 t) \\ &= \alpha x_1(t)\cos(2\pi f_0 t) + \beta x_2(t)\cos(2\pi f_0 t) \\ &= \alpha \mathrm{T}\,[x_1(t)] + \beta \mathrm{T}\,[x_2(t)] \end{aligned}$$

Thus the system is linear. In order for the system to be time-invariant the response to $x(t-t_0)$ should be $y(t-t_0)$ where $y(t)$ is the response of the system to $x(t)$. Clearly $y(t-t_0) = x(t-t_0)\cos(2\pi f_0(t-t_0))$ and the response of the system to $x(t-t_0)$ is $y'(t) = x(t-t_0)\cos(2\pi f_0 t)$. Since $\cos(2\pi f_0 (t-t_0))$ is not equal to $\cos(2\pi f_0 t)$ for all t, t_0 we conclude that $y'(t) \neq y(t-t_0)$ and thus the system is time-variant.

Problem 2.23

1) False. For if $T_1[x(t)] = x^3(t)$ and $T_2[x(t)] = x^{1/3}(t)$ then the cascade of the two systems is the identity system $T[x(t)] = x(t)$ which is known to be linear. However, both $T_1[\cdot]$ and $T_2[\cdot]$ are nonlinear.

2) False. For if

$$T_1[x(t)] = \begin{cases} tx(t) & t \neq 0 \\ 0 & t = 0 \end{cases} \qquad T_2[x(t)] = \begin{cases} \frac{1}{t}x(t) & t \neq 0 \\ 0 & t = 0 \end{cases}$$

Then $T_2[T_1[x(t)]] = x(t)$ and the system which is the cascade of $T_1[\cdot]$ followed by $T_2[\cdot]$ is time- invariant , whereas both $T_1[\cdot]$ and $T_2[\cdot]$ are time variant.

3) False. Consider the system

$$y(t) = T[x(t)] = \begin{cases} x(t) & t \geq 0 \\ 1 & t < 0 \end{cases}$$

Then the output of the system $y(t)$ depends only on the input $x(\tau)$ for $\tau \leq t$ This means that the system is causal. However the response to a causal signal, $x(t) = 0$ for $t \leq 0$, is nonzero for negative values of t and thus it is not causal.

Problem 2.24

1) Time invariant: The response to $x(t - t_0)$ is $2x(t - t_0) + 3$ which is $y(t - t_0)$.

2) Time varying the response to $x(t - t_0)$ is $(t + 2)x(t - t_0)$ but $y(t - t_0) = (t - t_0 + 2)x(t - t_0)$, obviously the two are not equal.

3) Time-varying system. The response $y(t - t_0)$ is equal to $x(-(t - t_0)) = x(-t + t_0)$. However the response of the system to $x(t - t_0)$ is $z(t) = x(-t - t_0)$ which is not equal to $y(t - t_0)$

4) Time-varying system. Clearly

$$y(t) = x(t)u_{-1}(t) \Longrightarrow y(t - t_0) = x(t - t_0)u_{-1}(t - t_0)$$

However, the response of the system to $x(t - t_0)$ is $z(t) = x(t - t_0)u_{-1}(t)$ which is not equal to $y(t - t_0)$

5) Time-invariant system. Clearly

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \Longrightarrow y(t - t_0) = \int_{-\infty}^{t-t_0} x(\tau)d\tau$$

The response of the system to $x(t - t_0)$ is

$$\int^{t} \qquad \int^{t-t_0}$$

$$z(t) = \int_{-\infty} x(\tau - t_0)d\tau = \int_{-\infty} x(v)dv = y(t - t_0)$$

where we have used the change of variable $v = \tau - t_0$.

6) Time-invariant system. Writing $y(t)$ as $\sum_{n=-\infty}^{\infty} x(t-n)$ we get

$$y(t-t_0) = \sum_{n=-\infty}^{\infty} x(t-t_0-n) = T[x(t-t_0)]$$

Problem 2.25

The differentiator is a LTI system (see examples 2.19 and 2.1.21 in book). It is true that the output of a system which is the cascade of two LTI systems does not depend on the order of the systems. This can be easily seen by the commutative property of the convolution

$$h_1(t) \star h_2(t) = h_2(t) \star h_1(t)$$

Let $h_1(t)$ be the impulse response of a differentiator, and let $y(t)$ be the output of the system $h_2(t)$ with input $x(t)$. Then,

$$\begin{aligned} z(t) &= h_2(t) \star x'(t) = h_2(t) \star (h_1(t) \star x(t)) \\ &= h_2(t) \star h_1(t) \star x(t) = h_1(t) \star h_2(t) \star x(t) \\ &= h_1(t) \star y(t) = y'(t) \end{aligned}$$

Problem 2.26

The integrator is is a LTI system (why?). It is true that the output of a system which is the cascade of two LTI systems does not depend on the order of the systems. This can be easily seen by the commutative property of the convolution

$$h_1(t) \star h_2(t) = h_2(t) \star h_1(t)$$

Let $h_1(t)$ be the impulse response of an integrator, and let $y(t)$ be the output of the system $h_2(t)$ with input $x(t)$. Then,

$$\begin{aligned} z(t) &= h_2(t) \star \int_{-\infty}^{t} x(\tau)\,d\tau = h_2(t) \star (h_1(t) \star x(t)) \\ &= h_2(t) \star h_1(t) \star x(t) = h_1(t) \star h_2(t) \star x(t) \\ &= h_1(t) \star y(t) = \int_{-\infty}^{t} y(\tau)\,d\tau \end{aligned}$$

Problem 2.27

The output of a LTI system is the convolution of the input with the impulse response of the system.
Thus,

$$\delta(t) = \int_{-\infty}^{\infty} h(\tau) e^{-\alpha(t-\tau)} u_{-1}(t-\tau) d\tau = \int_{-\infty}^{t} h(\tau) e^{-\alpha(t-\tau)} d\tau$$

Differentiating both sides with respect to t we obtain

$$\delta'(t) = (-\alpha)e^{-\alpha t}\int_{-\infty}^{t} h(\tau)e^{\alpha\tau}d\tau + e^{-\alpha t}\frac{d}{dt}\left[\int_{-\infty}^{t} h(\tau)e^{\alpha\tau}d\tau\right]$$

$$= (-\alpha)\delta(t) + e^{-\alpha t}h(t)e^{\alpha t} = (-\alpha)\delta(t) + h(t)$$

Thus

$$h(t) = \alpha\delta(t) + \delta'(t)$$

The response of the system to the input $x(t)$ is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)\left[\alpha\delta(t-\tau) + \delta'(t-\tau)\right]d\tau \\ &= \alpha\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)\delta'(t-\tau)d\tau \\ &= \alpha x(t) + \frac{d}{dt}x(t) \end{aligned}$$

Problem 2.28

For the system to be causal the output at the time instant t_0 should depend only on $x(t)$ for $t \le t_0$.

$$y(t_0) = \frac{1}{2T}\int_{t_0-T}^{t_0+T} x(\tau)d\tau = \frac{1}{2T}\int_{t_0-T}^{t_0} x(\tau)d\tau + \frac{1}{2T}\int_{t_0}^{t_0+T} x(\tau)d\tau$$

We observe that the second integral on the right side of the equation depends on values of $x(\tau)$ for τ greater than t_0. Thus the system is non causal.

Problem 2.29

Consider the system

$$y(t) = T[x(t)] = \begin{cases} x(t) & x(t) \neq 0 \\ 1 & x(t) = 0 \end{cases}$$

This system is causal since the output at the time instant t depends only on values of $x(\tau)$ for $\tau \le t$ (actually it depends only on the value of $x(\tau)$ for $\tau = t$, a stronger condition.) However, the response of the system to the impulse signal $\delta(t)$ is one for $t < 0$ so that the impulse response of the system is nonzero for $t < 0$.

Problem 2.30

1. Noncausal: Since for $t < 0$ we do not have $sinc(t) = 0$.

2. This is a rectangular signal of width 6 centered at $t_0 = 3$, for negative t's it is zero, therefore the system is causal.

3. The system is causal since for negative t's $h(t) = 0$.

Problem 2.31

The output $y(t)$ of a LTI system with impulse response $h(t)$ and input signal $u_{-1}(t)$ is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u_{-1}(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)u_{-1}(t-\tau)d\tau + \int_{t}^{\infty} h(\tau)u_{-1}(t-\tau)d\tau$$

But $u_{-1}(t-\tau) = 1$ for $\tau < t$ so that

$$\int_{-\infty}^{t} h(\tau)u_{-1}(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

Similarly, since $u_{-1}(t-\tau) = 0$ for $\tau < t$ we obtain

$$\int_{t}^{\infty} h(\tau)u_{-1}(t-\tau)d\tau = 0$$

Combining the previous integrals we have

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u_{-1}(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

Problem 2.32

Let $h(t)$ denote the the impulse response of a differentiator. Then for every input signal

$$x(t) \star h(t) = \frac{d}{dt}x(t)$$

If $x(t) = \delta(t)$ then the output of the differentiator is its impulse response. Thus,

$$\delta(t) \star h(t) = h(t) = \delta'(t)$$

The output of the system to an arbitrary input $x(t)$ can be found by convolving $x(t)$ with $\delta'(t)$. In this case

$$y(t) = x(t) \star \delta'(t) = \int_{-\infty}^{\infty} x(\tau)\delta'(t-\tau)d\tau = \frac{d}{dt}x(t)$$

Assume that the impulse response of a system which delays its input by t_0 is $h(t)$. Then the response to the input $\delta(t)$ is

$$\delta(t) \star h(t) = \delta(t-t_0)$$

However, for every $x(t)$

$$\delta(t) \star x(t) = x(t)$$

so that $h(t) = \delta(t-t_0)$. The output of the system to an arbitrary input $x(t)$ is

$$y(t) = x(t) \star \delta(t - t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t - t_0 - \tau)d\tau = x(t - t_0)$$

Problem 2.33

The response of the system to the signal $\alpha x_1(t) + \beta x_2(t)$ is

$$y_1(t) = \int_{t-T}^{t} (\alpha x_1(\tau) + \beta x_2(\tau))d\tau = \alpha \int_{t-T}^{t} x_1(\tau)d\tau + \beta \int_{t-T}^{t} x_2(\tau)d\tau$$

Thus the system is linear. The response to $x(t - t_0)$ is

$$y_1(t) = \int_{t-T}^{t} x(\tau - t_0)d\tau = \int_{t-t_0-T}^{t-t_0} x(v)dv = y(t - t_0)$$

where we have used the change of variables $v = \tau - t_0$. Thus the system is time invariant. The impulse response is obtained by applying an impulse at the input.

$$h(t) = \int_{t-T}^{t} \delta(\tau)d\tau = \int_{-\infty}^{t} \delta(\tau)d\tau - \int_{-\infty}^{t-T} \delta(\tau)d\tau = u_{-1}(t) - u_{-1}(t - T)$$

Problem 2.34

1)

$$e^{-t}u_{-1}(t) \star e^{-t}u_{-1}(t) = \int_{-\infty}^{\infty} e^{-\tau}u_{-1}(\tau)e^{-(t-\tau)}u_{-1}(t-\tau)d\tau = \int_0^t e^{-t}d\tau$$

$$= \begin{cases} te^{-t} & t > 0 \\ 0 & t < 0 \end{cases}$$

2)

$$x(t) = \Pi(t) \star \Lambda(t) = \int_{-\infty}^{\infty} \Pi(\theta)\Lambda(t-\theta)d\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Lambda(t-\theta)d\theta = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \Lambda(v)dv$$

$$t \leq -\frac{3}{2} \Rightarrow x(t) = 0$$

$$-\frac{3}{2} < t \leq -\frac{1}{2} \Rightarrow x(t) = \int_{-1}^{t+\frac{1}{2}} (v+1)dv = \left(\frac{1}{2}v^2 + v\right)\Big|_{-1}^{t+\frac{1}{2}} = \frac{1}{2}t^2 + \frac{3}{2}t + \frac{9}{8}$$

$$-\frac{1}{2} < t \leq \frac{1}{2} \Rightarrow x(t) = \int_{t-\frac{1}{2}}^{0} (v+1)dv + \int_{0}^{t+\frac{1}{2}} (-v+1)dv$$

$$= \left(\frac{1}{2}v^2 + v\right)\Big|_{t-\frac{1}{2}}^{0} \quad \frac{1}{2}$$

$+ (-\frac{}{2} v$ $\Big|_0^{t+} = -t^2 + \quad + v)$ $\frac{3}{4}$

$$\frac{1}{2} < t \le \frac{3}{2} \implies x(t) = \int_{t-\frac{1}{2}}^{1} (-v + 1) dv = (-\frac{1}{2} v^2 + v) \Big|_{t-\frac{1}{2}}^{1} = \frac{1}{2} t^2 - \frac{3}{2} t + \frac{9}{8}$$

$$\frac{3}{2} < t \implies x(t) = 0$$

Thus,

$$x(t) = \begin{cases} 0 & t \leq -\frac{3}{2} \\ \frac{1}{2}t^2 + \frac{3}{2}t + \frac{9}{8} & -\frac{3}{2} < t \leq -\frac{1}{2} \\ -t^2 + \frac{3}{4} & -\frac{1}{2} < t \leq \frac{1}{2} \\ \frac{1}{2}t^2 - \frac{3}{2}t + \frac{9}{8} & \frac{1}{2} < t \leq \frac{3}{2} \\ 0 & \frac{3}{2} < t \end{cases}$$

Problem 2.35

The output of a LTI system with impulse response $h(t)$ is

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Using the first formula for the convolution and observing that $h(\tau) = 0$, $\tau < 0$ we obtain

$$y(t) = \int_{-\infty}^{0} x(t-\tau)h(\tau)d\tau + \int_{0}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{0}^{\infty} x(t-\tau)h(\tau)d\tau$$

Using the second formula for the convolution and writing

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau + \int_{t}^{\infty} x(\tau)h(t-\tau)d\tau$$

we obtain

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

The last is true since $h(t-\tau) = 0$ for $t < \tau$ so that $\int_t^{\infty} x(\tau)h(t-\tau)d\tau = 0$

Problem 2.36

In order for the signals $\psi_n(t)$ to constitute an orthonormal set of signals in $[\alpha, \alpha + T_0]$ the following condition should be satisfied

$$\langle \psi_n(t), \psi_m(t) \rangle = \int_{\alpha}^{\alpha+T_0} \psi_n(t)\psi_m^*(t)dt = \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

But

$$\langle \psi_n(t), \psi_m(t) \rangle = \int_{\alpha}^{\alpha+T_0} \frac{1}{\sqrt{T_0}} e^{j2\pi \frac{n}{T_0} t} \frac{1}{\sqrt{T_0}} e^{-j2\pi \frac{m}{T_0} t} dt$$

$$= \frac{1}{T_0} \int_{\alpha}^{\alpha + T_0} e^{j2\pi \frac{(n-m)}{T_0} t} dt$$

If $n = m$ then $e^{j2\pi \frac{(n-m)}{T_0} t} = 1$ so that

$$\langle \psi_n(t), \psi_n(t) \rangle = \frac{1}{T_0} \int_{\alpha}^{\alpha + T_0} dt = \frac{1}{T_0} t \Big|_{\alpha}^{\alpha + T_0} = 1$$

When $n \neq m$ then,

$$\langle \psi_n(t), \psi_m(t) \rangle = \frac{1}{j2\pi(n-m)} e^{j2\pi(n-m)x/T_0} \Big|_{\alpha}^{\alpha+T_0} = 0$$

Thus, $\langle \psi_n(t), \psi_n(t) \rangle = \delta_{mn}$ which proves that $\psi_n(t)$ constitute an orthonormal set of signals.

Problem 2.37

1) Since $(a-b)^2 \geq 0$ we have that

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

with equality if $a = b$. Let

$$A = \left[\sum_{i=1}^{n} \alpha_i^2\right]^{\frac{1}{2}}, \qquad B = \left[\sum_{i=1}^{n} \beta_i^2\right]^{\frac{1}{2}}$$

Then substituting α_i/A for a and β_i/B for b in the previous inequality we obtain

$$\frac{\alpha_i}{A}\frac{\beta_i}{B} \leq \frac{1}{2}\frac{\alpha_i^2}{A^2} + \frac{1}{2}\frac{\beta_i^2}{B^2}$$

with equality if $\frac{\alpha_i}{\beta_i} = \frac{A}{B} = k$ or $\alpha_i = k\beta_i$ for all i. Summing both sides from $i = 1$ to n we obtain

$$\sum_{i=1}^{n} \frac{\alpha_i\beta_i}{AB} \leq \frac{1}{2}\sum_{i=1}^{n} \frac{\alpha_i^2}{A^2} + \frac{1}{2}\sum_{i=1}^{n} \frac{\beta_i^2}{B^2}$$

$$= \frac{1}{2A^2}\sum_{i=1}^{n} \alpha_i^2 + \frac{1}{2B^2}\sum_{i=1}^{n} \beta_i^2 = \frac{1}{2A^2}A^2 + \frac{1}{2B^2}B^2 = 1$$

Thus,

$$\frac{1}{AB}\sum_{i=1}^{n} \alpha_i\beta_i \leq 1 \Rightarrow \sum_{i=1}^{n} \alpha_i\beta_i \leq \left[\sum_{i=1}^{n} \alpha_i^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} \beta_i^2\right]^{\frac{1}{2}}$$

Equality holds if $\alpha_i = k\beta_i$, for $i = 1, \ldots, n$.

2) The second equation is trivial since $|x_i y_i^*| = |x_i||y_i^*|$. To see this write x_i and y_i in polar coordinates as $x_i = \rho_{x_i} e^{j\theta_{x_i}}$ and $y_i = \rho_{y_i} e^{j\theta_{y_i}}$. Then, $|x_i y_i^*| = |\rho_{x_i}\rho_{y_i} e^{j(\theta_{x_i}-\theta_{y_i})}| = \rho_{x_i}\rho_{y_i} = |x_i||y_i| = |x_i||y_i^*|$. We turn now to prove the first inequality. Let z_i be any complex with real and

imaginary components $z_{i,R}$ and $z_{i,I}$ respectively. Then,

$$\left|\sum_{i=1}^{n} z_i\right|^2 = \left|\sum_{i=1}^{n} z_{i,R} + j\sum_{i=1}^{n} z_{i,I}\right|^2 = \left(\sum_{i=1}^{n} z_{i,R}\right)^2 + \left(\sum_{i=1}^{n} z_{i,I}\right)^2$$

$$= \sum_{i=1}^{n}\sum_{m=1}^{n} (z_{i,R} z_{m,R} + z_{i,I} z_{m,I})$$

Since $(z_{i,R}z_{m,I} - z_{m,R}z_{i,I})^2 \geq 0$ we obtain

$$(z_{i,R}z_{m,R} + z_{i,I}z_{m,I})^2 \leq (z_{i,R}^2 + z_{i,I}^2)(z_{m,R}^2 + z_{m,I}^2)$$

Using this inequality in the previous equation we get

$$\left|\sum_{i=1}^{n} z_i\right|^2 = \sum_{i=1}^{n}\sum_{m=1}^{n}(z_{i,R}z_{m,R} + z_{i,I}z_{m,I})$$

$$\leq \sum_{i=1}^{n}\sum_{m=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}(z_{m,R}^2 + z_{m,I}^2)^{\frac{1}{2}}$$

$$= \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)\left(\sum_{m=1}^{n}(z_{m,R}^2 + z_{m,I}^2)^{\frac{1}{2}}\right) = \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)^2$$

Thus

$$\left|\sum_{i=1}^{n} z_i\right|^2 \leq \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)^2 \quad \text{or} \quad \left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n}|z_i|$$

The inequality now follows if we substitute $z_i = x_i y_i^*$. Equality is obtained if $\frac{z_{i,R}}{z_{i,I}} = \frac{z_{m,R}}{z_{m,I}} = k_1$ or $\angle z_i = \angle z_m = \theta$.

3) From 2) we obtain

$$\left|\sum_{i=1}^{n} x_i y_i^*\right|^2 \leq \sum_{i=1}^{n}|x_i||y_i|$$

But $|x_i|$, $|y_i|$ are real positive numbers so from 1)

$$\sum_{i=1}^{n}|x_i||y_i| \leq \left(\sum_{i=1}^{n}|x_i|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}|y_i|^2\right)^{\frac{1}{2}}$$

Combining the two inequalities we get

$$\left|\sum_{i=1}^{n} x_i y_i^*\right|^2 \leq \left(\sum_{i=1}^{n}|x_i|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}|y_i|^2\right)^{\frac{1}{2}}$$

From part 1) equality holds if $\alpha_i = k\beta_i$ or $|x_i| = k|y_i|$ and from part 2) $x_i y_i^* = |x_i y_i^*|e^{j\theta}$. Therefore, the two conditions are

- $|x_i| = k|y_i|$
- $\angle x_i - \angle y_i = \theta$

which imply that for all i, $x_i = Ky_i$ for some complex constant K.

4) The same procedure can be used to prove the Cauchy-Schwartz inequality for integrals. An easier approach is obtained if one considers the inequality

$$|x(t) + \alpha y(t)| \geq 0, \qquad \text{for all } \alpha$$

Then

$$
\begin{aligned}
0 &\leq \int_{-\infty}^{\infty} |x(t)+\alpha y(t)|^2 dt = \int_{-\infty}^{\infty} (x(t)+\alpha y(t))(x^*(t)+\alpha^* y^*(t))dt \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt + \alpha \int_{-\infty}^{\infty} x^*(t)y(t)dt + \alpha^* \int_{-\infty}^{\infty} x(t)y^*(t)dt + |a|^2 \int_{-\infty}^{\infty} |y(t)|^2 dt
\end{aligned}
$$

The inequality is true for $\int_{-\infty}^{\infty} x^*(t)y(t)dt = 0$. Suppose that $\int_{-\infty}^{\infty} x^*(t)y(t)dt \neq 0$ and set

$$
\alpha = -\frac{\int_{-\infty}^{\infty} |x(t)|^2 dt}{\int_{-\infty}^{\infty} x^*(t)y(t)dt}
$$

Then,

$$
0 \leq -\int_{-\infty}^{\infty} |x(t)|^2 dt + \frac{[\int_{-\infty}^{\infty} |x(t)|^2 dt]^2 \int_{-\infty}^{\infty} |y(t)|^2 dt}{|\int_{-\infty}^{\infty} x(t)y^*(t)dt|^2}
$$

and

$$
\left|\int_{-\infty}^{\infty} x(t)y^*(t)dt\right| \leq \left[\int_{-\infty}^{\infty} |x(t)|^2 dt\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} |y(t)|^2 dt\right]^{\frac{1}{2}}
$$

Equality holds if $x(t) = -\alpha y(t)$ a.e. for some complex α.

Problem 2.38

1)

$$
\begin{aligned}
\epsilon^2 &= \int_{-\infty}^{\infty} \left| x(t) - \sum_{i=1}^{N} \alpha_i \phi_i(t) \right|^2 dt \\
&= \int_{-\infty}^{\infty} \left(x(t) - \sum_{i=1}^{N} \alpha_i \phi_i(t) \right)\left(x^*(t) - \sum_{j=1}^{N} \alpha_j^* \phi_j^*(t) \right) dt \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{\infty} \phi_i(t)x^*(t)dt - \sum_{j=1}^{N} \alpha_j^* \int_{-\infty}^{\infty} \phi_j^*(t)x(t)dt \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j^* \int_{-\infty}^{\infty} \phi_i(t)\phi_j^* dt
\end{aligned}
$$

$$= \int_{-\infty}^{\infty} |x(t)|^2 \, dt + \sum_{i=1} |\alpha_i|^2 - \sum_{i=1} \alpha_i \int_{-\infty}^{\infty} \phi_i(t) x^*(t) dt - \sum_{j=1} \alpha_j^* \int_{-\infty}^{\infty} \phi_j^*(t) x(t) dt$$

Completing the square in terms of α_i we obtain

$$\epsilon^2 = \int_{-\infty}^{\infty} |x(t)|^2 \, dt - \sum_{i=1}^{N} \left| \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2 + \sum_{i=1}^{N} \left| \alpha_i - \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2$$

The first two terms are independent of α's and the last term is always positive. Therefore the minimum is achieved for

$$\alpha_i = \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt$$

which causes the last term to vanish.

2) With this choice of α_i's

$$\begin{aligned}
\epsilon^2 &= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \left| \int_{-\infty}^{\infty} \phi_i^*(t)x(t)dt \right|^2 \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} |\alpha_i|^2
\end{aligned}$$

Problem 2.39

1) Using Euler's relation we have

$$\begin{aligned}
x_1(t) &= \cos(2\pi t) + \cos(4\pi t) \\
&= \frac{1}{2}\left(e^{j2\pi t} + e^{-j2\pi t} + e^{j4\pi t} + e^{-j4\pi t}\right)
\end{aligned}$$

Therefore for $n = \pm 1, \pm 2$, $x_{1,n} = \frac{1}{2}$ and for all other values of n, $x_{1,n} = 0$.

2) Using Euler's relation we have

$$\begin{aligned}
x_2(t) &= \cos(2\pi t) - \cos(4\pi t + \pi/3) \\
&= \frac{1}{2}\left(e^{j2\pi t} + e^{-j2\pi t} - e^{j(4\pi t + \pi/3)} - e^{-j(4\pi t + \pi/3)}\right) \\
&= \frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t} + \frac{1}{2}e^{-j2\pi/3}e^{j4\pi t} + \frac{1}{2}e^{j2\pi/3}e^{-j4\pi t}
\end{aligned}$$

from this we conclude that $x_{2,\pm 1} = \frac{1}{2}$ and $x_{2,2} = x^*_{2,-2} = \frac{1}{2}e^{-j2\pi/3}$, and for all other values of n, $x_{2,n} = 0$.

3) We have $x_3(t) = 2\cos(2\pi t) - \sin(4\pi t) = 2\cos(2\pi t) + \cos(4\pi t + \pi/2)$. Using Euler's relation as in parts 1 and 2 we see that $x_{3,\pm 1} = 1$ and $x_{3,2} = x^*_{3,-2} = j$, and for all other values of n, $x_{3,n} = 0$.

4) The signal $x_4(t)$ is periodic with period $T_0 = 2$. Thus

$$\begin{aligned}
x_{4,n} &= \frac{1}{2}\int_{-1}^{1} \Lambda(t)e^{-j2\pi\frac{n}{2}t}dt = \frac{1}{2}\int_{-1}^{1} \Lambda(t)e^{-j\pi nt}dt \\
&= \frac{1}{2}\int_{-1}^{0}(t+1)e^{-j\pi nt}dt + \frac{1}{2}\int_{0}^{1}(-t+1)e^{-j\pi nt}dt \\
&= \frac{1}{2}\left(\frac{j}{\pi n}te^{-j\pi nt}\Big|_{-1}^{0} + \frac{1}{\pi^2 n^2}e^{-j\pi nt}\Big|_{-1}^{0}\right) + \frac{j}{2\pi n}e^{-j\pi nt}\Big|_{-1}^{0} \cdots
\end{aligned}$$

$$-\frac{1}{2}\frac{j}{\pi n} t e^{-j\pi n t} + \frac{1}{\pi^2 n^2} e^{-j\pi n t} + \frac{j}{2\pi n} e^{-j\pi n t}$$

$$\frac{1}{\pi^2 n^2} - \frac{1}{2\pi^2 n^2}\left(e^{j\pi n} + e^{-j\pi n}\right) = \frac{1}{\pi^2 n^2}\left(1 - \cos(\pi n)\right)$$

When $n = 0$ then

$$x_{4,0} = \frac{1}{2}\int_{-1}^{1} \Lambda(t)dt = \frac{1}{2}$$

Thus

$$x_4(t) = \frac{1}{2} + 2\sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2}(1 - \cos(\pi n))\cos(\pi n t)$$

5) The signal $x_5(t)$ is periodic with period $T_0 = 1$. For $n = 0$

$$x_{5,0} = \int_0^1 (-t+1)dt = \left(-\frac{1}{2}t^2 + t\right)\Big|_0^1 = \frac{1}{2}$$

For $n \neq 0$

$$\begin{aligned} x_{5,n} &= \int_0^1 (-t+1)e^{-j2\pi n t}dt \\ &= -\left(\frac{j}{2\pi n}te^{-j2\pi n t} + \frac{1}{4\pi^2 n^2}e^{-j2\pi n t}\right)\Big|_0^1 + \frac{j}{2\pi n}e^{-j2\pi n t}\Big|_0^1 \\ &= -\frac{j}{2\pi n} \end{aligned}$$

Thus,

$$x_5(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{\pi n}\sin 2\pi n t$$

6) The signal $x_6(t)$ is real even and periodic with period $T_0 = \frac{1}{2f_0}$. Hence, $x_{6,n} = a_{8,n}/2$ or

$$\begin{aligned} x_{6,n} &= 2f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0 t)\cos(2\pi n 2 f_0 t)dt \\ &= f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1+2n)t)dt + f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1-2n)t)dt \\ &= \frac{1}{2\pi(1+2n)}\sin(2\pi f_0(1+2n)t)\Big|_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} + \frac{1}{2\pi(1-2n)}\sin(2\pi f_0(1-2n)t)\Big|_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \\ &= \frac{(-1)^n}{\pi}\left[\frac{1}{(1+2n)} + \frac{1}{(1-2n)}\right] \end{aligned}$$

Problem 2.40

It follows directly from the uniqueness of the decomposition of a real signal in an even and odd part. Nevertheless for a real periodic signal

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi \frac{n}{T_0} t) + b_n \sin(2\pi \frac{n}{T_0} t)$$

The even part of $x(t)$ is

$$\begin{aligned}
x_e(t) &= \frac{x(t)+x(-t)}{2} \\
&= \frac{1}{2}\left[a_0 + \sum_{n=1}^{\infty} a_n\left(\cos(2\pi\frac{n}{T_0}t)+\cos(-2\pi\frac{n}{T_0}t)\right)\right. \\
&\qquad \left. + b_n\left(\sin(2\pi\frac{n}{T_0}t)+\sin(-2\pi\frac{n}{T_0}t)\right)\right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi\frac{n}{T_0}t)
\end{aligned}$$

The last is true since $\cos(\theta)$ is even so that $\cos(\theta)+\cos(-\theta) = 2\cos\theta$ whereas the oddness of $\sin(\theta)$ provides $\sin(\theta)+\sin(-\theta) = \sin(\theta)-\sin(\theta) = 0$.

The odd part of $x(t)$ is

$$\begin{aligned}
x_o(t) &= \frac{x(t)-x(-t)}{2} \\
&- \sum_{n=1}^{\infty} b_n \sin(2\pi\frac{n}{T_0}t)
\end{aligned}$$

Problem 2.41

1) The signal $y(t) = x(t-t_0)$ is periodic with period $T = T_0$.

$$\begin{aligned}
y_n &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t-t_0)e^{-j2\pi\frac{n}{T_0}t}dt \\
&= \frac{1}{T_0}\int_{\alpha-t_0}^{\alpha-t_0+T_0} x(v)e^{-j2\pi\frac{n}{T_0}(v+t_0)}dv \\
&= e^{-j2\pi\frac{n}{T_0}t_0}\frac{1}{T_0}\int_{\alpha-t_0}^{\alpha-t_0+T_0} x(v)e^{-j2\pi\frac{n}{T_0}v}dv \\
&= x_n e^{-j2\pi\frac{n}{T_0}t_0}
\end{aligned}$$

where we used the change of variables $v = t - t_0$

2) For $y(t)$ to be periodic there must exist T such that $y(t+mT) = y(t)$. But $y(t+T) = x(t+T)e^{j2\pi f_0 t}e^{j2\pi f_0 T}$ so that $y(t)$ is periodic if $T = T_0$ (the period of $x(t)$) and $f_0 T = k$ for some k in $\mathbb{Z}$. In this case

$$y_n = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}e^{j2\pi f_0 t}dt$$

$$= \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) e^{-j2\pi \frac{(n-k)}{T_0} t} dt = x_{n-k}$$

3) The signal $y(t)$ is periodic with period $T = T_0/\alpha$.

$$
\begin{aligned}
y_n &= \frac{1}{T}\int_{\beta}^{\beta+T} y(t)e^{-j2\pi\frac{n}{T}t}dt = \frac{\alpha}{T_0}\int_{\beta}^{\beta+\frac{T_0}{\alpha}} x(\alpha t)e^{-j2\pi\frac{n\alpha}{T_0}t}dt \\
&= \frac{1}{T_0}\int_{\beta\alpha}^{\beta\alpha+T_0} x(v)e^{-j2\pi\frac{n}{T_0}v}dv = x_n
\end{aligned}
$$

where we used the change of variables $v = \alpha t$.

Problem 2.42

$$
\begin{aligned}
\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)y^*(t)dt &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0}\sum_{n=-\infty}^{\infty} x_n e^{\frac{j2\pi n}{T_0}t}\sum_{m=-\infty}^{\infty} y_m^* e^{-\frac{j2\pi m}{T_0}t}dt \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n y_m^* \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} e^{\frac{j2\pi(n-m)}{T_0}t}dt \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n y_m^*\delta_{mn} = \sum_{n=-\infty}^{\infty} x_n y_n^*
\end{aligned}
$$

Problem 2.43

a) The signal is periodic with period T. Thus

$$
\begin{aligned}
x_n &= \frac{1}{T}\int_0^T e^{-t}e^{-j2\pi\frac{n}{T}t}dt = \frac{1}{T}\int_0^T e^{-(j2\pi\frac{n}{T}+1)t}dt \\
&= -\frac{1}{T\left(j2\pi\frac{n}{T}+1\right)}e^{-(j2\pi\frac{n}{T}+1)t}\Big|_0^T = -\frac{1}{j2\pi n+T}\left[e^{-(j2\pi n+T)}-1\right] \\
&= \frac{1}{j2\pi n+T}[1-e^{-T}] = \frac{T-j2\pi n}{T^2+4\pi^2n^2}[1-e^{-T}]
\end{aligned}
$$

If we write $x_n = \frac{a_n - jb_n}{2}$ we obtain the trigonometric Fourier series expansion coefficients as

$$
a_n = \frac{2T}{T^2+4\pi^2n^2}[1-e^{-T}], \qquad b_n = \frac{4\pi n}{T^2+4\pi^2n^2}[1-e^{-T}]
$$

b) The signal is periodic with period $2T$. Since the signal is odd we obtain $x_0 = 0$. For $n \neq 0$

$$
x_n = \frac{1}{2T}\int_{-T}^{T} x(t)e^{-j2\pi\frac{n}{2T}t}dt = \frac{1}{2T}\int_{-T}^{T}\frac{t}{T}e^{-j2\pi\frac{n}{2T}t}dt
$$

$$= \frac{1}{2T^2}\int_{-T}^{T} t e^{-j\pi \frac{n}{T} t}\, dt$$

$$= \frac{1}{2T^2}\left(\frac{jT}{\pi n} t e^{-\frac{j\pi n t}{T}} + \frac{T^2}{\pi^2 n^2} e^{-\frac{j\pi n t}{T}} \right)\Bigg|_{-T}^{T}$$

$$= \frac{1}{2T^2}\left[\frac{jT^2}{\pi n} e^{-j\pi n} + \frac{T^2}{\pi^2 n^2} e^{-j\pi n} + \frac{jT^2}{\pi n} e^{j\pi n} - \frac{T^2}{\pi^2 n^2} e^{j\pi n} \right]$$

$$= \frac{j}{\pi n}(-1)^n$$

The trigonometric Fourier series expansion coefficients are:

$$a_n = 0, \qquad b_n = (-1)^{n+1}\frac{2}{\pi n}$$

c) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{3}{2}$$

If $n \neq 0$ then

$$\begin{aligned}
x_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{j}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{-\frac{T}{2}}^{\frac{T}{2}} + \frac{j}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{-\frac{T}{4}}^{\frac{T}{4}} \\
&= \frac{j}{2\pi n}\left[e^{-j\pi n} - e^{j\pi n} + e^{-j\pi\frac{n}{2}} - e^{j\pi\frac{n}{2}}\right] \\
&= \frac{1}{\pi n}\sin(\pi\frac{n}{2}) = \frac{1}{2}\operatorname{sinc}(\frac{n}{2})
\end{aligned}$$

Note that $x_n = 0$ for n even and $x_{2l+1} = \frac{1}{\pi(2l+1)}(-1)^l$. The trigonometric Fourier series expansion coefficients are:

$$a_0 = 3, \quad a_{2l} = 0, \quad a_{2l+1} = \frac{2}{\pi(2l+1)}(-1)^l, \quad b_n = 0, \ \forall n$$

d) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{2}{3}$$

If $n \neq 0$ then

$$\begin{aligned}
x_n &= \frac{1}{T}\int_0^T x(t)e^{-j2\pi\frac{n}{T}t}dt = \frac{1}{T}\int_0^{\frac{T}{3}}\frac{3}{T}te^{-j2\pi\frac{n}{T}t}dt \\
&+ \frac{1}{T}\int_{\frac{T}{3}}^{\frac{2T}{3}} e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{\frac{2T}{3}}^{T}(-\frac{3}{T}t + 3)e^{-j2\pi\frac{n}{T}t}dt
\end{aligned}$$

$3 \quad jT \quad j2\pi\frac{n}{T}t$

The trigonometric Fourier series expansion coefficients are:

$$= \frac{3}{T^2}\left[\frac{jT}{2\pi n} t e^{-\frac{j2\pi n t}{T}} + \frac{T^2}{4\pi^2 n^2} e^{-\frac{j2\pi n t}{T}}\right]_0^{\frac{T}{3}}$$

$$-\frac{3}{T^2}\left[\frac{jT}{2\pi n} t e^{-\frac{j2\pi n t}{T}} + \frac{T^2}{4\pi^2 n^2} e^{-\frac{j2\pi n t}{T}}\right]_{\frac{T}{3}}^{\frac{2T}{3}}$$

$$+\left[\frac{j}{2\pi n} e^{-\frac{j2\pi n t}{T}}\right]_{\frac{T}{3}}^{\frac{2T}{3}} + \left[\frac{3jT}{T\,2\pi n} e^{-\frac{j2\pi n t}{T}}\right]_{\frac{2T}{3}}^{T}$$

$$= \frac{3}{2\pi^2 n^2}\left[\cos\left(\frac{2\pi n}{3}\right) - 1\right]$$

The trigonometric Fourier series expansion coefficients are:

$$a_0 = \frac{4}{3}, \quad a_n = \frac{3}{\pi^2 n^2}\left[\cos\left(\frac{2\pi n}{3}\right) - 1\right], \quad b_n = 0, \ \forall n$$

e) The signal is periodic with period T. Since the signal is odd $x_0 = a_0 = 0$. For $n \neq 0$

$$\begin{aligned}
x_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{1}{T}\int_{-\frac{T}{2}}^{-\frac{T}{4}} -e^{-j2\pi\frac{n}{T}t}dt \\
&\quad + \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \frac{4}{T} t e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{\frac{T}{4}}^{\frac{T}{2}} e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{4}{T^2}\left(\frac{jT}{2\pi n} t e^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2} e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{-\frac{T}{4}}^{\frac{T}{4}} \\
&\quad - \frac{1}{T}\frac{jT}{2\pi n} e^{-j2\pi\frac{n}{T}t}\Bigg|_{-\frac{T}{2}}^{-\frac{T}{4}} + \frac{1}{T}\frac{jT}{2\pi n} e^{-j2\pi\frac{n}{T}t}\Bigg|_{\frac{T}{4}}^{\frac{T}{2}} \\
&= \frac{j}{\pi n}\left[(-1)^n - \frac{2\sin(\frac{\pi n}{2})}{\pi n}\right] = \frac{j}{\pi n}\left[(-1)^n - \text{sinc}(\frac{n}{2})\right]
\end{aligned}$$

For n even, $\text{sinc}(\frac{n}{2}) = 0$ and $x_n = \frac{j}{\pi n}$. The trigonometric Fourier series expansion coefficients are:

$$a_n = 0, \ \forall n, \quad b_n = \begin{cases} -\frac{1}{\pi l} & n = 2l \\ \frac{2}{\pi(2l+1)}\left[1 + \frac{2(-1)^l}{\pi(2l+1)}\right] & n = 2l+1 \end{cases}$$

f) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_{-\frac{T}{3}}^{\frac{T}{3}} x(t)dt = 1$$

For $n \neq 0$

$$\begin{aligned}
x_n &= \frac{1}{T}\int_{-\frac{T}{3}}^{0}\left(\frac{3}{T}t + 2\right)e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{0}^{\frac{T}{3}}\left(-\frac{3}{T}t + 2\right)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{3}{T^2}\left(\frac{jT}{2\pi n} t e^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2} e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{-\frac{T}{3}}^{0} \\
&\quad - \frac{3}{T^2}\left(\frac{jT}{2\pi n} t e^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2} e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{0}^{\frac{T}{3}}
\end{aligned}$$

The trigonometric Fourier series expansion coefficients are: $-\frac{T}{T^2}\,\frac{}{2\pi n} t e^{-\frac{}{T}} + \frac{}{4\pi^2 n^2} e^{-\frac{}{T}}$

$$+\frac{2jT}{T2\pi n}e^{-\frac{j2\pi nt}{T}}\Big|_{-\frac{T}{3}}^{0}+\frac{2jT}{T2\pi n}e^{-\frac{j2\pi nt}{T}}\Big|_{0}^{\frac{T}{3}}$$

$$=\frac{3}{\pi^2 n^2}\left(\frac{1}{2}-\cos\left(\frac{2\pi n}{3}\right)\right)+\frac{1}{\pi n}\sin\left(\frac{2\pi n}{3}\right)$$

The trigonometric Fourier series expansion coefficients are:

$$a_0 = 2, \quad a_n = 2\left[\frac{3}{\pi^2 n^2}\left(\frac{1}{2}-\cos\left(\frac{2\pi n}{3}\right)\right)+\frac{1}{\pi n}\sin\left(\frac{2\pi n}{3}\right)\right], \quad b_n = 0, \forall n$$

Problem 2.44

1) $H(f) = 10\Pi(\frac{f}{4})$. The system is bandlimited with bandwidth $W = 2$. Thus at the output of the system only the frequencies in the band $[-2, 2]$ will be present. The gain of the filter is 10 for all f in $(-2, 2)$ and 5 at the edges $f = \pm 2$.

a) Since the period of the signal is $T = 1$ we obtain

$$\begin{aligned} y(t) &= 10[\frac{a_0}{2} + a_1\cos(2\pi t) + b_1\sin(2\pi t)] \\ &\quad + 5[a_2\cos(2\pi 2t) + b_2\sin(2\pi 2t)] \end{aligned}$$

With

$$a_n = \frac{2}{1+4\pi^2 n^2}[1 - e^{-1}], \qquad b_n = \frac{4\pi n}{1+4\pi^2 n^2}[1 - e^{-1}]$$

we obtain

$$\begin{aligned} y(t) &= (1 - e^{-1})\left(20 + \frac{20}{1+4\pi^2}\cos(2\pi t) + \frac{40\pi}{1+4\pi^2}\sin(2\pi t)\right. \\ &\quad \left. + \frac{10}{1+16\pi^2}\cos(2\pi 2t) + \frac{40\pi}{1+16\pi^2}\sin(2\pi 2t)\right) \end{aligned}$$

b) Since the period of the signal is $2T = 2$ and $a_n = 0$, for all n, we have

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(2\pi \frac{n}{2} t)$$

The frequencies $\frac{n}{2}$ should satisfy $|\frac{n}{2}| \leq 2$ or $n \leq 4$. With $b_n = (-1)^{n+1}\frac{2}{\pi n}$ we obtain

$$\begin{aligned} y(t) &= \frac{20}{\pi}\sin(\frac{2\pi t}{2}) - \frac{20}{2\pi}\sin(2\pi t) \\ &\quad + \frac{20}{3\pi}\sin(\frac{2\pi 3t}{2}) - \frac{10}{4\pi}\sin(2\pi 2t) \end{aligned}$$

c) The period of the signal is $T = 1$ and

$$a_0 = 3, \quad a_{2l} = 0, \quad a_{2l+1} = \frac{2}{\pi(2l+1)}(-1)^l, \quad b_n = 0, \ \forall n$$

Hence,

$$x(t) = \frac{3}{2} + \sum_{l=0}^{\infty} a_{2l+1}\cos(2\pi(2l+1)t)$$

At the output of the channel only the frequencies for which $2l + 1 \le 2$ will be present so that

$$y(t) = 10\frac{3}{2} + 10\frac{2}{\pi}\cos(2\pi t)$$

d) Since $b_n = 0$ for all n, and the period of the signal is $T = 1$, we have

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi n t)$$

With $a_0 = \frac{4}{3}$ and $a_n = \frac{3}{\pi^2 n^2}[\cos(\frac{2\pi n}{3}) - 1]$ we obtain

$$\begin{aligned} y(t) &= \frac{20}{3} + \frac{30}{\pi^2}(\cos(\frac{2\pi}{3}) - 1)\cos(2\pi t) \\ &\quad + \frac{15}{4\pi^2}(\cos(\frac{4\pi}{3}) - 1)\cos(2\pi 2t) \\ &= \frac{20}{3} - \frac{45}{\pi^2}\cos(2\pi t) - \frac{45}{8\pi^2}\cos(2\pi 2t) \end{aligned}$$

e) With $a_n = 0$ for all n, $T = 1$ and

$$b_n = \begin{cases} -\frac{1}{\pi l} & n = 2l \\ \frac{2}{\pi(2l+1)}[1 + \frac{2(-1)^l}{\pi(2l+1)}] & n = 2l+1 \end{cases}$$

we obtain

$$\begin{aligned} y(t) &= 10 b_1 \sin(2\pi t) + 5 b_2 \sin(2\pi 2t) \\ &= 10\frac{2}{\pi}(1 + \frac{2}{\pi})\sin(2\pi t) - 5\frac{1}{\pi}\sin(2\pi 2t) \end{aligned}$$

f) Similarly with the other cases we obtain

$$\begin{aligned} y(t) &= 10 + 10 \cdot 2\left[\frac{3}{\pi^2}(\frac{1}{2} - \cos(\frac{2\pi}{3})) + \frac{1}{\pi}\sin(\frac{2\pi}{3})\right]\cos(2\pi t) \\ &\quad + 5 \cdot 2\left[\frac{3}{4\pi^2}(\frac{1}{2} - \cos(\frac{4\pi}{3})) + \frac{1}{2\pi}\sin(\frac{4\pi}{3})\right]\cos(2\pi 2t) \\ &= 10 + 20\left[\frac{3}{\pi^2} + \frac{\sqrt{3}}{2\pi}\right]\cos(2\pi t) + 10\left[\frac{3}{4\pi^2} - \frac{\sqrt{3}}{4\pi}\right]\cos(2\pi 2t) \end{aligned}$$

2) In general

$$y(t) = \sum_{n=-\infty}^{\infty}$$

$x_n H($

n $-$ $\frac{n}{}$

$)e^{j2\pi_T}$

$\frac{t}{T}$

The DC component of the input signal and all frequencies higher than 4 will be cut off.

a) For this signal $T = 1$ and $x_n = \frac{1-j2\pi n}{1+4\pi^2 n^2}(1-e^{-1})$. Thus,

$$
\begin{aligned}
y(t) &= \frac{1-j2\pi}{1+4\pi^2}(1-e^{-1})(-j)e^{j2\pi t} + \frac{1-j2\pi 2}{1+4\pi^2 4}(1-e^{-1})(-j)e^{j2\pi 2t} \\
&\quad + \frac{1-j2\pi 3}{1+4\pi^2 9}(1-e^{-1})(-j)e^{j2\pi 3t} + \frac{1-j2\pi 4}{1+4\pi^2 16}(1-e^{-1})(-j)e^{j2\pi 4t} \\
&\quad + \frac{1+j2\pi}{1+4\pi^2}(1-e^{-1})je^{-j2\pi t} + \frac{1+j2\pi 2}{1+4\pi^2 4}(1-e^{-1})je^{-j2\pi 2t} \\
&\quad + \frac{1+j2\pi 3}{1+4\pi^2 9}(1-e^{-1})je^{-j2\pi 3t} + \frac{1+j2\pi 4}{1+4\pi^2 16}(1-e^{-1})je^{-j2\pi 4t} \\
&= (1-e^{-1})\sum_{n=1}^{4}\frac{2}{1+4\pi^2 n^2}\left(\sin(2\pi n t) - 2\pi n\cos(2\pi n t)\right)
\end{aligned}
$$

b) With $T = 2$ and $x_n = \frac{j}{\pi n}(-1)^n$ we obtain

$$
\begin{aligned}
y(t) &= \sum_{n=1}^{8}\frac{j}{\pi n}(-1)^n(-j)e^{j\pi n t} + \sum_{n=-8}^{-1}\frac{j}{\pi n}(-1)^n j e^{j\pi n t} \\
&= \sum_{n=1}^{8}\frac{(-1)^n}{\pi n}e^{j\pi n t} + \sum_{n=-8}^{-1} -\frac{1}{\pi n}(-1)^n j e^{j\pi n t}
\end{aligned}
$$

c) In this case

$$x_{2l} = 0, \qquad x_{2l+1} = \frac{1}{\pi(2l+1)}(-1)^l$$

Hence

$$
\begin{aligned}
y(t) &= \frac{1}{\pi}(-j)e^{j2\pi t} + \frac{1}{3\pi}(-1)(-j)e^{j2\pi 3t} \\
&\quad + \frac{1}{-\pi}(-1)je^{-j2\pi t} + \frac{1}{-3\pi}je^{-j2\pi 3t} \\
&= \frac{1}{2\pi}\sin(2\pi t) - \frac{1}{6\pi}\sin(2\pi 3t)
\end{aligned}
$$

d) $x_0 = \frac{2}{3}$ and $x_n = \frac{3}{2\pi n^2}\left(\cos\left(\frac{2\pi n}{3}\right) - 1\right)$. Thus

$$y(t) = \sum^{4}$$

$$\sum_{n=1}^{4} \frac{3}{2\pi n^2}\left(\cos\left(\frac{2\pi n}{3}\right)-1\right)(-j)e^{j2\pi nt} + \sum_{n=-4}^{-1} \frac{3}{2\pi n^2}\left(\cos\left(\frac{2\pi n}{3}\right)-1\right) j e^{j2\pi nt}$$

e) With $x_n = \frac{j}{\pi n}((-1)^n - \text{sinc}(\frac{n}{2}))$ we obtain

$$y(t) = \sum_{n=1}^{4} \frac{1}{\pi n}\left((-1)^n - \text{sinc}\left(\frac{n}{2}\right)\right) + \sum_{n=-4}^{-1} \frac{-1}{\pi n}\left((-1)^n - \text{sinc}\left(\frac{n}{2}\right)\right)$$

f) Working similarly with the other cases we obtain

$$y(t) = \sum_{n=1}^{4} \frac{3}{\pi^2 n^2}\left[\frac{1}{2} - \cos\left(\frac{2\pi n}{3}\right) + \frac{1}{\pi n}\sin\left(\frac{2\pi n}{3}\right)\right](-j)e^{j2\pi nt}$$
$$+ \sum_{n=-4}^{-1} \frac{3}{\pi^2 n^2}\left[\frac{1}{2} - \cos\left(\frac{2\pi n}{3}\right) + \frac{1}{\pi n}\sin\left(\frac{2\pi n}{3}\right)\right] je^{j2\pi nt}$$

Problem 2.45

Using Parseval's relation (Equation 2.2.38), we see that the power in the periodic signal is given by $\sum_{n=-\infty}^{\infty} |x_n|^2$. Since the signal has finite power

$$\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} |x(t)|^2 dt = K < \infty$$

Thus, $\sum_{n=-\infty}^{\infty} |x_n|^2 = K < \infty$. The last implies that $|x_n| \to 0$ as $n \to \infty$. To see this write

$$\sum_{n=-\infty}^{\infty} |x_n|^2 = \sum_{n=-\infty}^{-M} |x_n|^2 + \sum_{n=-M}^{M} |x_n|^2 + \sum_{n=M}^{\infty} |x_n|^2$$

Each of the previous terms is positive and bounded by K. Assume that $|x_n|^2$ does not converge to zero as n goes to infinity and choose $\epsilon = 1$. Then there exists a subsequence of x_n, x_{n_k}, such that

$$|x_{n_k}| > \epsilon = 1, \qquad \text{for } n_k > N \geq M$$

Then

$$\sum_{n=M}^{\infty} |x_n|^2 \geq \sum_{n=N}^{\infty} |x_n|^2 \geq \sum_{n_k} |x_{n_k}|^2 = \infty$$

This contradicts our assumption that $\sum_{n=M}^{\infty} |x_n|^2$ is finite. Thus $|x_n|$, and consequently x_n, should converge to zero as $n \to \infty$.

Problem 2.46

1) Using the Fourier transform pair

$$e^{-\alpha|t|} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2 + (2\pi f)^2} = \frac{2\alpha}{4\pi^2} \frac{1}{\frac{\alpha^2}{4\pi^2} + f^2}$$

and the duality property of the Fourier transform: $X(f) = \mathcal{F}[x(t)] \Rightarrow x(-f) = \mathcal{F}[X(t)]$ we obtain

$$\frac{2\alpha}{4\pi^2}\mathbf{F}\left\{\frac{1}{\frac{\alpha^2}{4\pi^2}+t^2}\right\} = e^{-\alpha|f|}$$

With $\alpha = 2\pi$ we get the desired result

$$\mathbf{F}\left\{\frac{1}{1+t^2}\right\} = \pi e^{-2\pi|f|}$$

2)

$$\begin{aligned} \mathcal{F}[x(t)] &= \mathcal{F}[\Pi(t-3)+\Pi(t+3)] \\ &= \text{sinc}(f)e^{-j2\pi f3} + \text{sinc}(f)e^{j2\pi f3} \\ &= 2\text{sinc}(f)\cos(2\pi 3 f) \end{aligned}$$

3) $\mathcal{F}[\Pi(t/4)] = 4\,\text{sinc}(4f)$, hence $\mathcal{F}[4\Pi(t/4)] = 16\,\text{sinc}(4f)$. Using modulation property of FT we have $\mathcal{F}[4\Pi(t/4)\cos(2\pi f_0 t)] = 8\,\text{sinc}(4(f-f_0)) + 8\,\text{sinc}(4(f+f_0))$.

4)

$$\mathcal{F}[t\,\text{sinc}(t)] = \frac{1}{\pi}\mathcal{F}[\sin(\pi t)] = \frac{j}{2\pi}\left[\delta\left(f+\frac{1}{2}\right) - \delta\left(f-\frac{1}{2}\right)\right]$$

The same result is obtain if we recognize that multiplication by t results in differentiation in the frequency domain. Thus

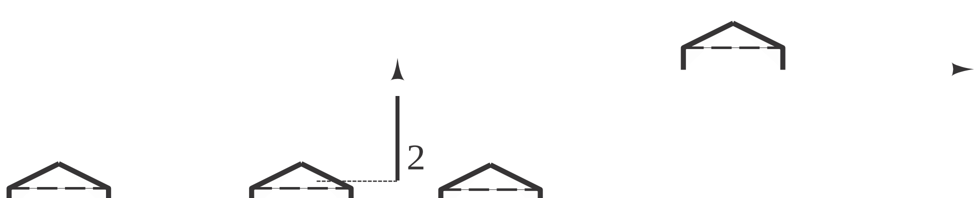
$$\mathcal{F}[t\,\text{sinc}] = \frac{j}{2\pi}\frac{d}{df}\Pi(f) = \frac{j}{2\pi}\left[\delta\left(f+\frac{1}{2}\right) - \delta\left(f-\frac{1}{2}\right)\right]$$

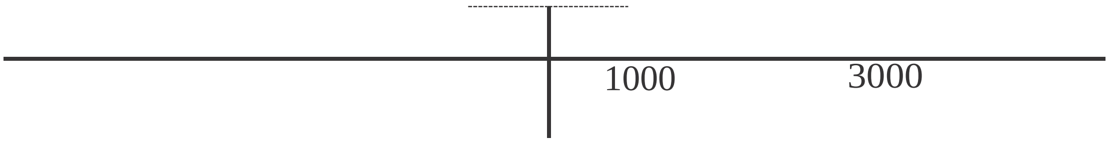
5)

$$\begin{aligned} \mathcal{F}[t\cos(2\pi f_0 t)] &= \frac{j}{2\pi}\frac{d}{df}\left[\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right] \\ &= \frac{j}{4\pi}\left[\delta'(f-f_0) + \delta'(f+f_0)\right] \end{aligned}$$

Problem 2.47

$x_1(t) = -x(t) + x(t)\cos(2000\pi t) + x(t)(1+\cos(6000\pi t))$ or $x_1(t) = x(t)\cos(2000\pi t) + x(t)\cos(6000\pi t)$. Using modulation property, we have $X_1(f) = \frac{1}{2}X(f-1000) + \frac{1}{2}X(f+1000) + \frac{1}{2}X(f-3000) + \frac{1}{2}X(f+3000)$. The plot is given below:




1000
3000

Problem 2.48

$$\mathcal{F}[\frac{1}{2}(\delta(t+\frac{1}{2})+\delta(t-\frac{1}{2}))] = \int_{-\infty}^{\infty}\frac{1}{2}(\delta(t+\frac{1}{2})+\delta(t-\frac{1}{2}))e^{-j2\pi ft}dt$$

$$= \frac{1}{2}(e^{-j\pi f}+e^{-j\pi f}) = \cos(\pi f)$$

Using the duality property of the Fourier transform:

$$X(f) = \mathcal{F}[x(t)] \Longrightarrow x(f) = \mathcal{F}[X(-t)]$$

we obtain

$$\mathcal{F}[\cos(-\pi t)] = \mathcal{F}[\cos(\pi t)] = \frac{1}{2}(\delta(f+\frac{1}{2})+\delta(f-\frac{1}{2}))$$

Note that $\sin(\pi t) = \cos(\pi t + \frac{\pi}{2})$. Thus

$$\mathcal{F}[\sin(\pi t)] = \mathcal{F}[\cos(\pi(t+\frac{1}{2}))] = \frac{1}{2}(\delta(f+\frac{1}{2})+\delta(f-\frac{1}{2}))e^{j\pi f}$$

$$= \frac{1}{2}e^{j\pi\frac{1}{2}}\delta(f+\frac{1}{2}) + \frac{1}{2}e^{-j\pi\frac{1}{2}}\delta(f-\frac{1}{2})$$

$$= \frac{j}{2}\delta(f+\frac{1}{2}) - \frac{j}{2}\delta(f-\frac{1}{2})$$

Problem 2.49

a) We can write $x(t)$ as $x(t) = 2\Pi(\frac{t}{4}) - 2\Lambda(\frac{t}{2})$. Then

$$\mathcal{F}[x(t)] = \mathcal{F}[2\Pi(\frac{t}{4})] - \mathcal{F}[2\Lambda(\frac{t}{2})] = 8\text{sinc}(4f) - 4\text{sinc}^2(2f)$$

b)

$$x(t) = 2\Pi(\frac{t}{4}) - \Lambda(t) \Longrightarrow \mathcal{F}[x(t)] = 8\text{sinc}(4f) - \text{sinc}^2(f)$$

c)

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \int_{-1}^{0}(t+1)e^{-j2\pi ft}dt + \int_{0}^{1}(t-1)e^{-j2\pi ft}dt$$

$$= j \quad 1 \; ! \quad e^{j2\pi ft}\Big|^{0} \quad \frac{j}{} \quad e^{j2\pi ft}\Big|^{0}$$

$$\overline{2\pi f}\, t + \overline{4\pi^2 f^2} \quad e^- \quad {}_{-1} + 2\pi f\, e^- \quad {}_{-1}$$

$$+ \quad \frac{j}{2\pi f}\, t + \frac{1}{4\pi^2 f^2} \;!\; e^{-j2\pi f t}\Big|_0^1 - \frac{j}{2\pi f} e^{-j2\pi f t}\Big|_0^1$$

$$= \frac{j}{\pi f}(1 - \sin(\pi f))$$

d) We can write $x(t)$ as $x(t) = \Lambda(t+1) - \Lambda(t-1)$. Thus

$$X(f) = \text{sinc}^2(f)e^{j2\pi f} - \text{sinc}^2(f)e^{-j2\pi f} = 2j\,\text{sinc}^2(f)\sin(2\pi f)$$

e) We can write $x(t)$ as $x(t) = \Lambda(t+1) + \Lambda(t) + \Lambda(t-1)$. Hence,

$$X(f) = \text{sinc}^2(f)(1 + e^{j2\pi f} + e^{-j2\pi f}) = \text{sinc}^2(f)(1 + 2\cos(2\pi f))$$

f) We can write $x(t)$ as

$$x(t) = \left[\Pi\left(2f_0\left(t - \frac{1}{4f_0}\right)\right) - \Pi\left(2f_0\left(t - \frac{1}{4f_0}\right)\right)\right]\sin(2\pi f_0 t)$$

Then

$$\begin{aligned}
X(f) &= \left[\frac{1}{2f_0}\text{sinc}\left(\frac{f}{2f_0}\right)e^{-j2\pi\frac{1}{4f_0}f} - \frac{1}{2f_0}\text{sinc}\left(\frac{f}{2f_0}\right)e^{j2\pi\frac{1}{4f_0}f}\right] \\
&\quad \star \frac{j}{2}(\delta(f+f_0) - \delta(f+f_0)) \\
&= \frac{1}{2f_0}\text{sinc}\left(\frac{f+f_0}{2f_0}\right)\sin\left(\pi\frac{f+f_0}{2f_0}\right) - \frac{1}{2f_0}\text{sinc}\left(\frac{f-f_0}{2f_0}\right)\sin\left(\pi\frac{f-f_0}{2f_0}\right)
\end{aligned}$$

Problem 2.50

(Convolution theorem:)

$$\mathcal{F}[x(t) \star y(t)] = \mathcal{F}[x(t)]\mathcal{F}[y(t)] = X(f)Y(f)$$

Thus

$$\begin{aligned}
\text{sinc}(t) \star \text{sinc}(t) &= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t) \star \text{sinc}(t)]] \\
&= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t)] \cdot \mathcal{F}[\text{sinc}(t)]] \\
&= \mathcal{F}^{-1}[\Pi(f)\Pi(f)] = \mathcal{F}^{-1}[\Pi(f)] \\
&= \text{sinc}(t)
\end{aligned}$$

Problem 2.51

$$\begin{aligned}
\mathcal{F}[x(t)y(t)] &= \int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}\,dt \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X(\theta)e^{j2\pi\theta t}\,d\theta\right)y(t)e^{-j2\pi ft}\,dt \\
&= \int_{-\infty}^{\infty} X(\theta)\left(\int_{-\infty}^{\infty} y(t)e^{-j2\pi(f-\theta)t}\,dt\right)d\theta
\end{aligned}$$

$$= \int_{-\infty} X(\theta)Y(f-\theta)d\theta = X(f) \star Y(f)$$

Problem 2.52

1) Clearly

$$\begin{aligned} x_1(t+kT_0) &= \sum_{n=-\infty}^{\infty} x(t+kT_0-nT_0) = \sum_{n=-\infty}^{\infty} x(t-(n-k)T_0) \\ &= \sum_{m=-\infty}^{\infty} x(t-mT_0) = x_1(t) \end{aligned}$$

where we used the change of variable $m = n-k$.

2)

$$x_1(t) = x(t) \star \sum_{n=-\infty}^{\infty} \delta(t-nT_0)$$

This is because

$$\int_{-\infty}^{\infty} x(\tau) \sum_{n=-\infty}^{\infty} \delta(t-\tau-nT_0)d\tau = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-nT_0)d\tau = \sum_{n=-\infty}^{\infty} x(t-nT_0)$$

3)

$$\begin{aligned} \mathcal{F}[x_1(t)] &= \mathcal{F}[x(t) \star \sum_{n=-\infty}^{\infty} \delta(t-nT_0)] = \mathcal{F}[x(t)]\mathcal{F}[\sum_{n=-\infty}^{\infty} \delta(t-nT_0)] \\ &= X(f)\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f-\frac{n}{T_0}) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_0})\delta(f-\frac{n}{T_0}) \end{aligned}$$

Problem 2.53

1) By Parseval's theorem

$$\int_{-\infty}^{\infty} \mathrm{sinc}^5(t)dt = \int_{-\infty}^{\infty} \mathrm{sinc}^3(t)\mathrm{sinc}^2(t)dt = \int_{-\infty}^{\infty} \Lambda(f)T(f)df$$

where

$$T(f) = \mathcal{F}[\mathrm{sinc}^3(t)] = \mathcal{F}[\mathrm{sinc}^2(t)\mathrm{sinc}(t)] = \Pi(f) \star \Lambda(f)$$

But

$$\Pi(f) \star \Lambda(f) = \int_{\infty}$$

$$\Pi(\theta)\Lambda(f-\theta)d\theta = \int_{-\infty}^{\frac{1}{2}} \Lambda(f-\theta)d\theta = \int_{-\frac{1}{2}}^{f+\frac{1}{2}} \Lambda(v)dv$$

For $f \leq -\frac{3}{2} \Rightarrow T(f) = 0$

For $-\frac{3}{2} < f \leq -\frac{1}{2} \Rightarrow T(f) = \int_{-1}^{f+\frac{1}{2}} (v+1)dv = (\frac{1}{2}v^2 + v)\Big|_{-1}^{f+\frac{1}{2}} = \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8}$

For $-\frac{1}{2} < f \leq \frac{1}{2} \Rightarrow T(f) = \int_{f-\frac{1}{2}}^{0} (v+1)dv + \int_{0}^{f+\frac{1}{2}} (-v+1)dv$

$$= (\frac{1}{2}v^2 + v)\Big|_{f-\frac{1}{2}}^{0} + (-\frac{1}{2}v^2 + v)\Big|_{0}^{f+\frac{1}{2}} = -f^2 + \frac{3}{4}$$

For $\frac{1}{2} < f \leq \frac{3}{2} \Rightarrow T(f) = \int_{f-\frac{1}{2}}^{1} (-v+1)dv = (-\frac{1}{2}v^2 + v)\Big|_{f-\frac{1}{2}}^{1} = \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8}$

For $\frac{3}{2} < f \Rightarrow T(f) = 0$

Thus,

$$T(f) = \begin{cases} 0 & f \leq -\frac{3}{2} \\ \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8} & -\frac{3}{2} < f \leq -\frac{1}{2} \\ -f^2 + \frac{3}{4} & -\frac{1}{2} < f \leq \frac{1}{2} \\ \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8} & \frac{1}{2} < f \leq \frac{3}{2} \\ 0 & \frac{3}{2} < f \end{cases}$$

Hence,

$$\int_{-\infty}^{\infty} \Lambda(f)T(f)df = \int_{-1}^{-\frac{1}{2}} (\frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8})(f+1)df + \int_{-\frac{1}{2}}^{0} (-f^2 + \frac{3}{4})(f+1)df$$

$$+ \int_{0}^{\frac{1}{2}} (-f^2 + \frac{3}{4})(-f+1)df + \int_{\frac{1}{2}}^{1} (\frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8})(-f+1)df$$

$$= \frac{41}{64}$$

2)

$$\int_{0}^{\infty} e^{-\alpha t}\operatorname{sinc}(t)dt = \int_{-\infty}^{\infty} e^{-\alpha t}u_{-1}(t)\operatorname{sinc}(t)dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{\alpha + j2\pi f}\Pi(f)df = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\alpha + j2\pi f}df$$

$$= \frac{1}{j2\pi}\ln(\alpha + j2\pi f)\Big|_{-1/2}^{1/2} = \frac{1}{j2\pi}\ln(\frac{\alpha + j\pi}{\alpha - j\pi}) = \frac{1}{\pi}\tan^{-1}\frac{\pi}{\alpha}$$

—

3)

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}\cos(\beta t)dt &= \int_{-\infty}^\infty e^{-\alpha t}u_{-1}(t)\cos(\beta t)dt \\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{1}{\alpha + j2\pi f}(\delta(f - \frac{\beta}{2\pi}) + \delta(f + \frac{\beta}{2\pi}))dt \\
&= \frac{1}{2}[\frac{1}{\alpha + j\beta} + \frac{1}{\alpha - j\beta}] = \frac{\alpha}{\alpha^2 + \beta^2}
\end{aligned}$$

Problem 2.54

Using the convolution theorem we obtain

$$\begin{aligned} Y(f) &= X(f)H(f) = \left(\frac{1}{\alpha + j2\pi f}\right)\left(\frac{1}{\beta + j2\pi f}\right) \\ &= \frac{1}{(\beta-\alpha)}\frac{1}{\alpha + j2\pi f} - \frac{1}{(\beta-\alpha)}\frac{1}{\beta + j2\pi f} \end{aligned}$$

Thus

$$y(t) = \mathcal{F}^{-1}[Y(f)] = \frac{1}{(\beta-\alpha)}[e^{-\alpha t} - e^{-\beta t}]u_{-1}(t)$$

If $\alpha = \beta$ then $X(f) = H(f) = \frac{1}{\alpha + j2\pi f}$. In this case

$$y(t) = \mathcal{F}^{-1}[Y(f)] = \mathcal{F}^{-1}\left[\left(\frac{1}{\alpha + j2\pi f}\right)^2\right] = te^{-\alpha t}u_{-1}(t)$$

The signal is of the energy-type with energy content

$$\begin{aligned} E_y &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} |y(t)|^2 dt = \lim_{T\to\infty} \int_0^{\frac{T}{2}} \frac{1}{(\beta-\alpha)^2}(e^{-\alpha t} - e^{-\beta t})^2 dt \\ &= \lim_{T\to\infty} \frac{1}{(\beta-\alpha)^2}\left[-\frac{1}{2\alpha}e^{-2\alpha t}\Big|_0^{T/2} - \frac{1}{2\beta}e^{-2\beta t}\Big|_0^{T/2} + \frac{2}{(\alpha+\beta)}e^{-(\alpha+\beta)t}\Big|_0^{T/2} \right] \\ &= \frac{1}{(\beta-\alpha)^2}\left[\frac{1}{2\alpha} + \frac{1}{2\beta} - \frac{2}{\alpha+\beta}\right] = \frac{1}{2\alpha\beta(\alpha+\beta)} \end{aligned}$$

Problem 2.55

$$x_\alpha(t) = \begin{cases} x(t) & \alpha \le t < \alpha + T_0 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$X_\alpha(f) = \int_{-\infty}^{\infty} x_\alpha(t)e^{-j2\pi ft}dt = \int_\alpha^{\alpha+T_0} x(t)e^{-j2\pi ft}dt$$

Evaluating $X_\alpha(f)$ for $f = \frac{n}{T_0}$ we obtain

$$X_\alpha\left(\frac{n}{T_0}\right) = \int_\alpha^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}dt$$

$$X_\alpha\left(\frac{n}{T_0}\right) = \int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi nt/T_0}\,dt = T_0 x_n$$

where x_n are the coefficients in the Fourier series expansion of $x(t)$. Thus $X_\alpha(\frac{n}{T_0})$ is independent of the choice of α.

Problem 2.56

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x(t-nT_s) &= x(t) \star \sum_{n=-\infty}^{\infty} \delta(t-nT_s) = \frac{1}{T_s} x(t) \star \sum_{n=-\infty}^{\infty} e^{j2\pi \frac{n}{T_s} t} \\
&= \frac{1}{T_s}\mathcal{F}^{-1}\left[X(f)\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_s}\right)\right] \\
&= \frac{1}{T_s}\mathcal{F}^{-1}\left[\sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right)\delta\left(f-\frac{n}{T_s}\right)\right] \\
&= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right) e^{j2\pi \frac{n}{T_s} t}
\end{aligned}$$

If we set $t = 0$ in the previous relation we obtain Poisson's sum formula

$$\sum_{n=-\infty}^{\infty} x(-nT_s) = \sum_{m=-\infty}^{\infty} x(mT_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right)$$

Problem 2.57

1) We know that

$$e^{-\alpha|t|} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$$

Applying Poisson's sum formula with $T_s = 1$ we obtain

$$\sum_{n=-\infty}^{\infty} e^{-\alpha|n|} = \sum_{n=-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + 4\pi^2 n^2}$$

2) Use the Fourier transform pair $\Pi(t) \to \text{sinc}(f)$ in the Poisson's sum formula with $T_s = K$. Then

$$\sum_{n=-\infty}^{\infty} \Pi(nK) = \frac{1}{K}\sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n}{K}\right)$$

But $\Pi(nK) = 1$ for $n = 0$ and $\Pi(nK) = 0$ for $|n| \geq 1$ and $K \in \{1, 2, \ldots\}$. Thus the left side of the previous relation reduces to 1 and

$$K = \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n}{K}\right)$$

3) Use the Fourier transform pair $\Lambda(t) \rightarrow \text{sinc}^2(f)$ in the Poisson's sum formula with $T_s = K$. Then

$$\sum_{n=-\infty}^{\infty} \Lambda(nK) = \frac{1}{K} \sum_{n=-\infty}^{\infty} \text{sinc}^2\left(\frac{n}{K}\right)$$

Reasoning as before we see that $\sum_{n=-\infty}^{\infty} \Lambda(nK) = 1$ since for $K \in \{1, 2, \ldots\}$

$$\Lambda(nK) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, $K = \sum_{n=-\infty}^{\infty} \text{sinc}^2\left(\frac{n}{K}\right)$

Problem 2.58

Let $H(f)$ be the Fourier transform of $h(t)$. Then

$$H(f)\mathcal{F}[e^{-\alpha t}u_{-1}(t)] = \mathcal{F}[\delta(t)] \Longrightarrow H(f)\frac{1}{\alpha + j2\pi f} = 1 \Longrightarrow H(f) = \alpha + j2\pi f$$

The response of the system to $e^{-\alpha t}\cos(\beta t)u_{-1}(t)$ is

$$y(t) = \mathcal{F}^{-1}\left[H(f)\mathcal{F}[e^{-\alpha t}\cos(\beta t)u_{-1}(t)]\right]$$

But

$$\begin{aligned}
\mathcal{F}[e^{-\alpha t}\cos(\beta t)u_{-1}(t)] &= \mathcal{F}[\frac{1}{2}e^{-\alpha t}u_{-1}(t)e^{j\beta t} + \frac{1}{2}e^{-\alpha t}u_{-1}(t)e^{-j\beta t}] \\
&= \frac{1}{2}\left[\frac{1}{\alpha + j2\pi(f - \frac{\beta}{2\pi})} + \frac{1}{\alpha + j2\pi(f + \frac{\beta}{2\pi})}\right]
\end{aligned}$$

so that

$$Y(f) = \mathcal{F}[y(t)] = \frac{\alpha + j2\pi f}{2}\left[\frac{1}{\alpha + j2\pi(f - \frac{\beta}{2\pi})} + \frac{1}{\alpha + j2\pi(f + \frac{\beta}{2\pi})}\right]$$

Using the linearity property of the Fourier transform, the Convolution theorem and the fact that $\delta'(t) \xleftrightarrow{\mathcal{F}} j2\pi f$ we obtain

$$\begin{aligned}
y(t) &= \alpha e^{-\alpha t}\cos(\beta t)u_{-1}(t) + (e^{-\alpha t}\cos(\beta t)u_{-1}(t)) \star \delta'(t) \\
&= e^{-\alpha t}\cos(\beta t)\delta(t) - \beta e^{-\alpha t}\sin(\beta t)u_{-1}(t) \\
&= \delta(t) - \beta e^{-\alpha t}\sin(\beta t)u_{-1}(t)
\end{aligned}$$

Problem 2.59

1) Using the result of Problem 2.50 we have $\text{sinc}(t) \star \text{sinc}(t) = \text{sinc}(t)$.

2)

$$\begin{aligned}
y(t) &= x(t) \star h(t) = x(t) \star (\delta(t) + \delta'(t)) \\
&= x(t) + \frac{d}{dt}x(t)
\end{aligned}$$

With $x(t) = e^{-\alpha|t|}$ we obtain $y(t) = e^{-\alpha|t|} - \alpha e^{-\alpha|t|}\text{sgn}(t)$.

3)

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= \int_0^t e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau = e^{-\beta t}\int_0^t e^{-(\alpha-\beta)\tau}d\tau
\end{aligned}
$$

If $\quad \alpha = \beta \Rightarrow y(t) = te^{-\alpha t}u_{-1}(t)$

$$\alpha \neq \beta \Rightarrow y(t) = e^{-\beta t}\frac{1}{\beta - \alpha}e^{-(\alpha-\beta)t}\Big|_0^t u_{-1}(t) = \frac{1}{\beta - \alpha}\left[e^{-\alpha t} - e^{-\beta t}\right]u_{-1}(t)$$

Problem 2.60

Let the response of the LTI system be $h(t)$ with Fourier transform $H(f)$. Then, from the convolution theorem we obtain

$$Y(f) = H(f)X(f) \Longrightarrow \Lambda(f) = \Pi(f)H(f)$$

However, this relation cannot hold since $\Pi(f) = 0$ for $\frac{1}{2} < |f|$ whereas $\Lambda(f) \neq 0$ for $1 < |f| \leq 1/2$.

Problem 2.61

1) No. The input $\Pi(t)$ has a spectrum with zeros at frequencies $f = k$, ($k \neq 0$, $k \in \mathbb{Z}$) and the information about the spectrum of the system at those frequencies will not be present at the output. The spectrum of the signal $\cos(2\pi t)$ consists of two impulses at $f = \pm 1$ but we do not know the response of the system at these frequencies.

2)

$$\begin{aligned}
h_1(t) \star \Pi(t) &= \Pi(t) \star \Pi(t) = \Lambda(t) \\
h_2(t) \star \Pi(t) &= (\Pi(t) + \cos(2\pi t)) \star \Pi(t) \\
&= \Lambda(t) + \frac{1}{2}\mathcal{F}^{-1}\left[\delta(f-1)\text{sinc}^2(f) + \delta(f+1)\text{sinc}^2(f)\right] \\
&= \Lambda(t) + \frac{1}{2}\mathcal{F}^{-1}\left[\delta(f-1)\text{sinc}^2(1) + \delta(f+1)\text{sinc}^2(-1)\right] \\
&= \Lambda(t)
\end{aligned}$$

Thus both signals are candidates for the impulse response of the system.

3) $\mathcal{F}[u_{-1}(t)] = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$. Thus the system has a nonzero spectrum for every f and all the frequencies of the system will be excited by this input. $\mathcal{F}[e^{-at}u_{-1}(t)] = \frac{1}{a+j2\pi f}$. Again the spectrum is nonzero for all f and the response to this signal uniquely determines the system. In general the

spectrum of the input must not vanish at any frequency. In this case the influence of the system will be present at the output for every frequency.

Problem 2.62

$$
\begin{aligned}
\mathcal{F}[A\widehat{\sin(2\pi f_0 t + \theta)}] &= -j\text{sgn}(f)A\left[-\frac{1}{2j}\delta(f+f_0)e^{j2\pi f\frac{\theta}{2f_0}} + \frac{1}{2j}\delta(f-f_0)e^{-j2\pi f\frac{\theta}{2f_0}}\right] \\
&= \frac{A}{2}\left[\text{sgn}(-f_0)\delta(f+f_0)e^{j2\pi f\frac{\theta}{2f_0}} - \text{sgn}(-f_0)\delta(f-f_0)e^{-j2\pi f\frac{\theta}{2f_0}}\right] \\
&= -\frac{A}{2}\left[\delta(f+f_0)e^{j2\pi f\frac{\theta}{2f_0}} + \delta(f-f_0)e^{-j2\pi f\frac{\theta}{2f_0}}\right] \\
&= -A\mathcal{F}[\cos(2\pi f_0 t + \theta)]
\end{aligned}
$$

Thus, $A\widehat{\sin(2\pi f_0 t + \theta)} = -A\cos(2\pi f_0 t + \theta)$

Problem 2.63

Taking the Fourier transform of $\widehat{e^{j2\pi f_0 t}}$ we obtain

$$\mathcal{F}[\widehat{e^{j2\pi f_0 t}}] = -j\text{sgn}(f)\delta(f-f_0) = -j\text{sgn}(f_0)\delta(f-f_0)$$

Thus,

$$\widehat{e^{j2\pi f_0 t}} = \mathcal{F}^{-1}[-j\text{sgn}(f_0)\delta(f-f_0)] = -j\text{sgn}(f_0)e^{j2\pi f_0 t}$$

Problem 2.64

$$
\begin{aligned}
\mathcal{F}\left[\widehat{\frac{d}{dt}x(t)}\right] &= \mathcal{F}[x(t) \star \widehat{\delta'}(t)] = -j\text{sgn}(f)\mathcal{F}[x(t) \star \delta'(t)] \\
&= -j\text{sgn}(f)j2\pi f X(f) = 2\pi f\text{sgn}(f)X(f) \\
&= 2\pi|f|X(f)
\end{aligned}
$$

Problem 2.65

We need to prove that $\widehat{x'}(t) = (\hat{x}(t))'$.

$$
\begin{aligned}
\mathcal{F}[\widehat{x'}(t)] &= \mathcal{F}[x(t) \star \widehat{\delta'}(t)] = -j\text{sgn}(f)\mathcal{F}[x(t) \star \delta'(t)] = -j\text{sgn}(f)X(f)j2\pi f \\
&= \mathcal{F}[\hat{x}(t)]j2\pi f = \mathcal{F}[(\hat{x}(t))']
\end{aligned}
$$

Taking the inverse Fourier transform of both sides of the previous relation we obtain, $x'(t) = (x(t))'$

Problem 2.66

1) The spectrum of the output signal $y(t)$ is the product of $X(f)$ and $H(f)$. Thus,

$$Y(f) = H(f)X(f) = X(f)A(f_0)e^{j(\theta(f_0)+(f-f_0)\theta'(f)|_{f=f_0})}$$

$y(t)$ is a narrowband signal centered at frequencies $f = \pm f_0$. To obtain the lowpass equivalent signal we have to shift the spectrum (positive band) of $y(t)$ to the right by f_0. Hence,

$$Y_l(f) = u(f+f_0)X(f+f_0)A(f_0)e^{j(\theta(f_0)+f\theta'(f)|_{f=f_0})} = X_l(f)A(f_0)e^{j(\theta(f_0)+f\theta'(f)|_{f=f_0})}$$

2) Taking the inverse Fourier transform of the previous relation, we obtain

$$\begin{aligned} y_l(t) &= \mathcal{F}^{-1}\left[X_l(f)A(f_0)e^{j\theta(f_0)}e^{jf\theta'(f)|_{f=f_0}}\right] \\ &= A(f_0)x_l(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0}) \end{aligned}$$

With $y(t) = \text{Re}[y_l(t)e^{j2\pi f_0 t}]$ and $x_l(t) = V_x(t)e^{j\Theta_x(t)}$ we get

$$\begin{aligned} y(t) &= \text{Re}[y_l(t)e^{j2\pi f_0 t}] \\ &= \text{Re}\left[A(f_0)x_l(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0})e^{j\theta(f_0)}e^{j2\pi f_0 t}\right] \\ &= \text{Re}\left[A(f_0)V_x(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0})e^{j2\pi f_0 t}e^{j\Theta_x(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0})}\right] \\ &= A(f_0)V_x(t-t_g)\cos(2\pi f_0 t + \theta(f_0) + \Theta_x(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0})) \\ &= A(f_0)V_x(t-t_g)\cos(2\pi f_0(t+\frac{\theta(f_0)}{2\pi f_0}) + \Theta_x(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0})) \\ &= A(f_0)V_x(t-t_g)\cos(2\pi f_0(t-t_p) + \Theta_x(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0})) \end{aligned}$$

where

$$t_g = -\frac{1}{2\pi}\theta'(f)|_{f=f_0}, \qquad t_p = -\frac{1}{2\pi}\frac{\theta(f_0)}{f_0} = -\frac{1}{2\pi}\frac{\theta(f)}{f}\bigg|_{f=f_0}$$

3) t_g can be considered as a time lag of the envelope of the signal, whereas t_p is the time corresponding to a phase delay of $\frac{1}{2\pi}\frac{\theta(f_0)}{f_0}$.

Problem 2.67

1) We can write $H_\theta(f)$ as follows

$$H_\theta(f) = \begin{cases} \cos\theta - j\sin\theta & f > 0 \\ 0 & f = 0 \\ \cos\theta + j\sin\theta & f < 0 \end{cases} = \cos\theta - j\,\mathrm{sgn}(f)\sin\theta$$

Thus,

$$h_\theta(t) = \mathcal{F}^{-1}[H_\theta(f)] = \cos\theta\,\delta(t) + \frac{1}{\pi t}\sin\theta$$

2)

$$\begin{aligned} x_\theta(t) &= x(t) \star h_\theta(t) = x(t) \star \left(\cos\theta\,\delta(t) + \frac{1}{\pi t}\sin\theta\right) \\ &= \cos\theta\, x(t) \star \delta(t) + \sin\theta\,\frac{1}{\pi t} \star x(t) \\ &= \cos\theta\, x(t) + \sin\theta\,\hat{x}(t) \end{aligned}$$

3)

$$\begin{aligned} \int_{-\infty}^{\infty} |x_\theta(t)|^2 dt &= \int_{-\infty}^{\infty} |\cos\theta\, x(t) + \sin\theta\,\hat{x}(t)|^2 dt \\ &= \cos^2\theta \int_{-\infty}^{\infty} |x(t)|^2 dt + \sin^2\theta \int_{-\infty}^{\infty} |\hat{x}(t)|^2 dt \\ &\quad + \cos\theta\sin\theta \int_{-\infty}^{\infty} x(t)\hat{x}^*(t)dt + \cos\theta\sin\theta \int_{-\infty}^{\infty} x^*(t)\hat{x}(t)dt \end{aligned}$$

But $\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{x}(t)|^2 dt = \mathcal{E}_x$ and $\int_{-\infty}^{\infty} x(t)\hat{x}^*(t)dt = 0$ since $x(t)$ and $\hat{x}(t)$ are orthogonal.
Thus,

$$\mathcal{E}_{x_\theta} = \mathcal{E}_x(\cos^2\theta + \sin^2\theta) = \mathcal{E}_x$$

Computer Problems

Computer Problem 2.1

1) To derive the Fourier series coefficients in the expansion of $x(t)$, we have

$$x_n = \frac{1}{4}\int_{-1}^{1} e^{-j2\pi nt/4}\, dt \tag{2.1}$$

$$= \frac{\quad}{-2j\pi n}\left(e^{-j2\pi n/4} - e^{j2\pi n/4}\right)$$

$$= \frac{1}{2}\,\text{sinc}\left(\frac{n}{2}\right) \tag{2.2}$$

where $\text{sinc}(x)$ is defined as

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \tag{2.3}$$

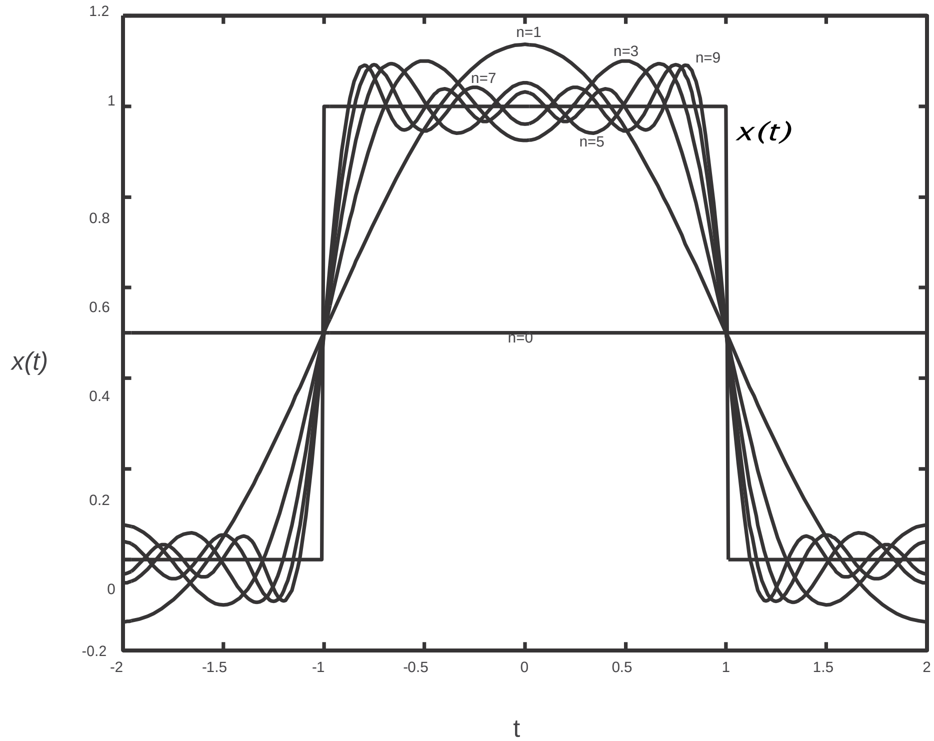


Figure 2.1: Various Fourier series approximations for the rectangular pulse

2) Obviously, all the x_n's are real (since $x(t)$ is real and even), so

$$\begin{cases} a_n = \operatorname{sinc}\left(\frac{n}{2}\right) \\ b_n = 0 \end{cases} \tag{2.4}$$

$$\begin{cases} c_n = \left|\operatorname{sinc}\left(\frac{n}{2}\right)\right| \\ \theta_n = 0, \pi \end{cases}$$

Note that for even n's, $x_n = 0$ (with the exception of $n = 0$, where $a_0 = c_0 = 1$ and $x_0 = \frac{1}{2}$). Using these coefficients, we have

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{2} \operatorname{sinc}\left(\frac{n}{2}\right) e^{j2\pi nt/4} \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \operatorname{sinc}\left(\frac{n}{2}\right) \cos\left(2\pi t \frac{n}{4}\right) \end{aligned} \tag{2.5}$$

A plot of the Fourier series approximations to this signal over one period for $n = 0, 1, 3, 5, 7, 9$ is shown in Figure 2.1.

3) Note that x_n is always real. Therefore, depending on its sign, the phase is either zero or π. The magnitude of the x_n's is $\frac{1}{2}\,\text{sinc}\left(\frac{n}{2}\right)$. The discrete and phase spectrum are shown in Figure 2.2.

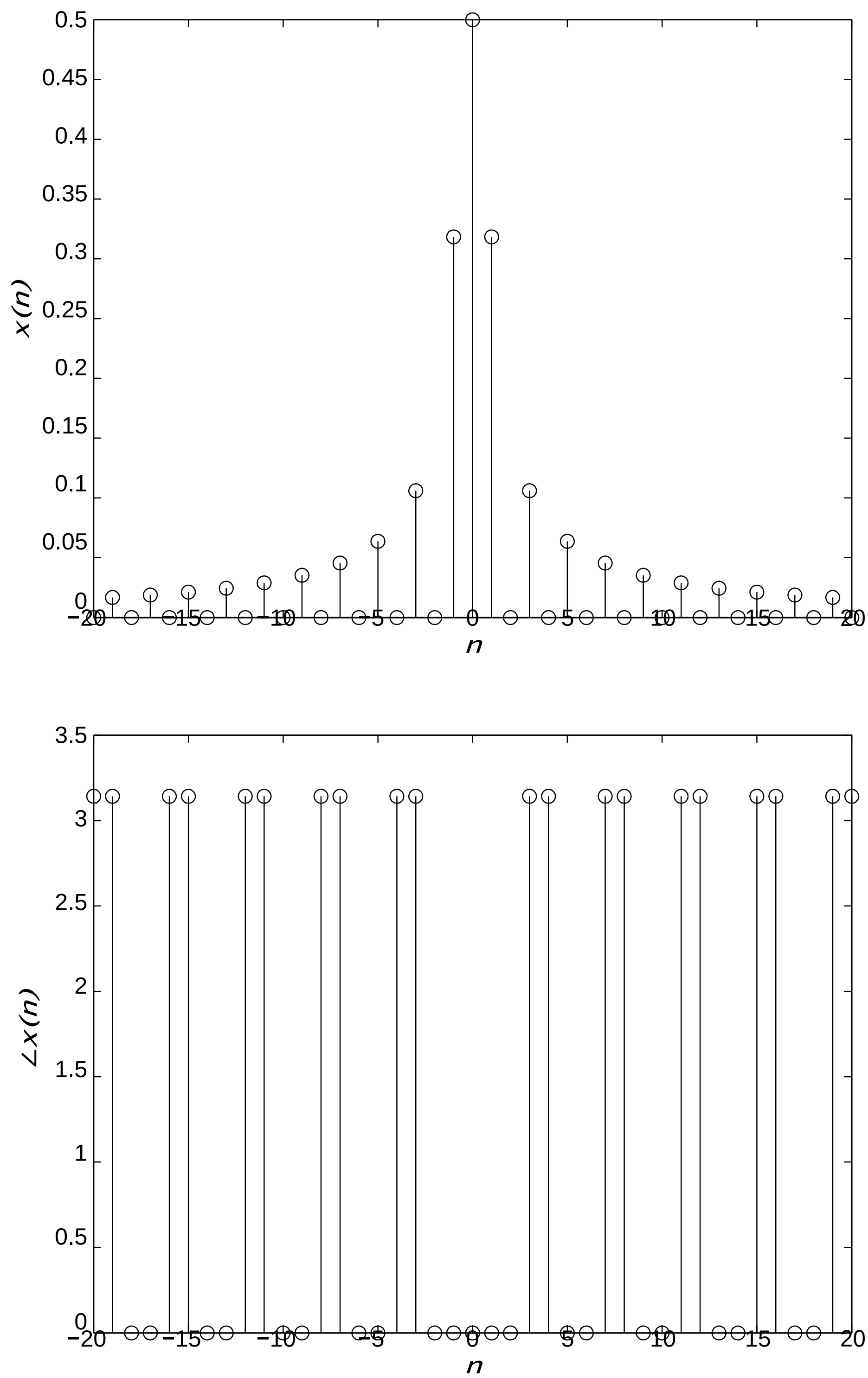


Figure 2.2: The discrete and phase spectrum of the signal

Computer Problem 2.2

1) We have

$$
\begin{aligned}
x_n &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j2\pi nt/T_0}dt && (2.6)\\
&= \frac{1}{2}\int_{-1}^{1} \Lambda(t)e^{-j\pi nt}dt && (2.7)\\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \Lambda(t)e^{-j\pi nt}dt && (2.8)\\
&= \frac{1}{2}\mathcal{F}[\Lambda(t)]_{f=n/2} && (2.9)\\
&= \frac{1}{2}\,\mathrm{sinc}^2\left(\frac{n}{2}\right) && (2.10)\\
& && (2.11)
\end{aligned}
$$

where we have used the facts that $\Lambda(t)$ vanishes outside the $[-1, 1]$ interval and that the Fourier transform of $\Lambda(t)$ is $\mathrm{sinc}^2(f)$. This result can also be obtained by using the expression for $\Lambda(t)$ and integrating by parts. Obviously, we have $x_n = 0$ for all even values of n except for $n = 0$.

2) A plot of the discrete spectrum of $x(t)$ is presented in Figure 2.3

3) A plot of the discrete spectrum $\{y_n\}$ is presented in Figure 2.4

The MATLAB script for this problem is given next.

```
% MATLAB script for Computer Problem 2.2.
echo on
n=[-20:1:20];
% Fourier series coefficients of x(t) vector
x=.5*(sinc(n/2)).^2;
% sampling interval
ts=1/40;
% time vector
t=[-.5:ts:1.5];
% impulse response
fs=1/ts;
h=[zeros(1,20),t(21:61),zeros(1,20)];
% transfer function
H=fft(h)/fs;
% frequency resolution
df=fs/80; f=[0:df:fs]-fs/2;
% rearrange H
H1=fftshift(H);
y=x.*H1(21:61);
% Plotting commands follow.
```

55

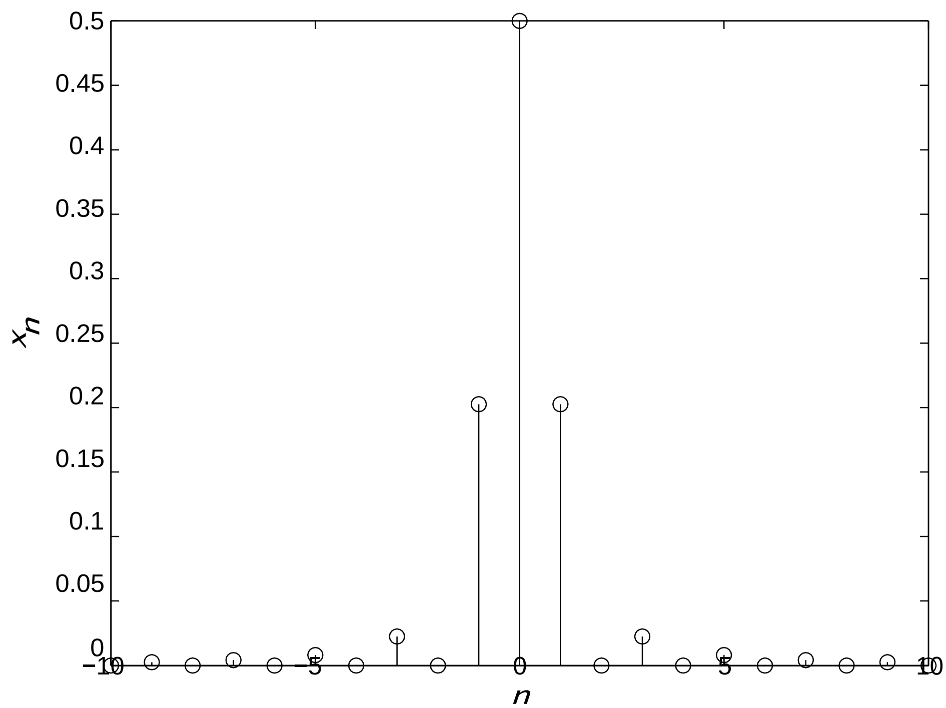


Figure 2.3: The discrete spectrum of the signal

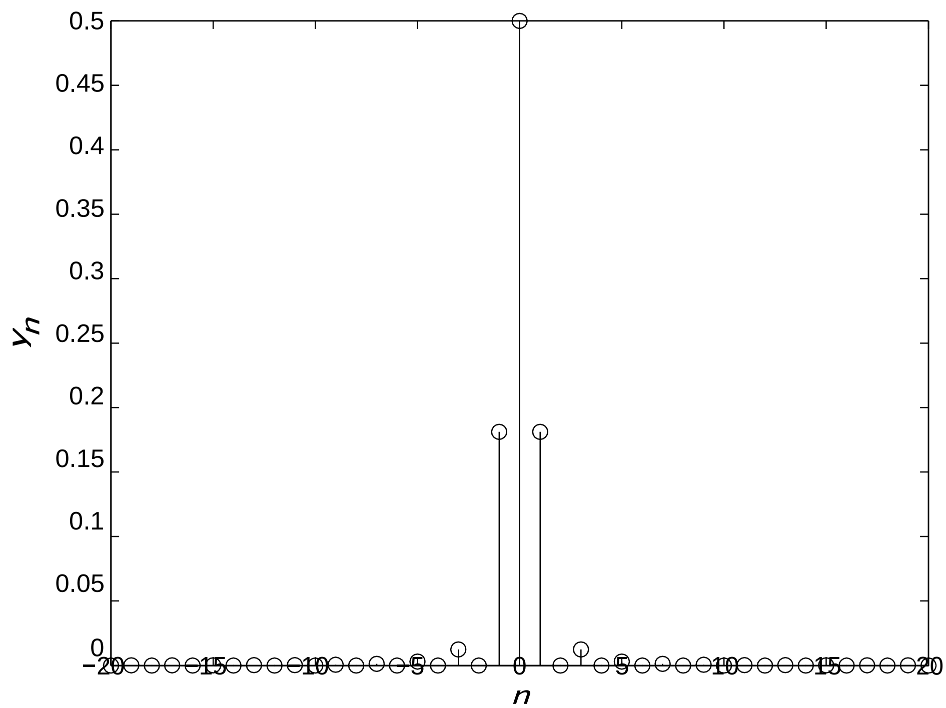


Figure 2.4: The discrete spectrum of the signal

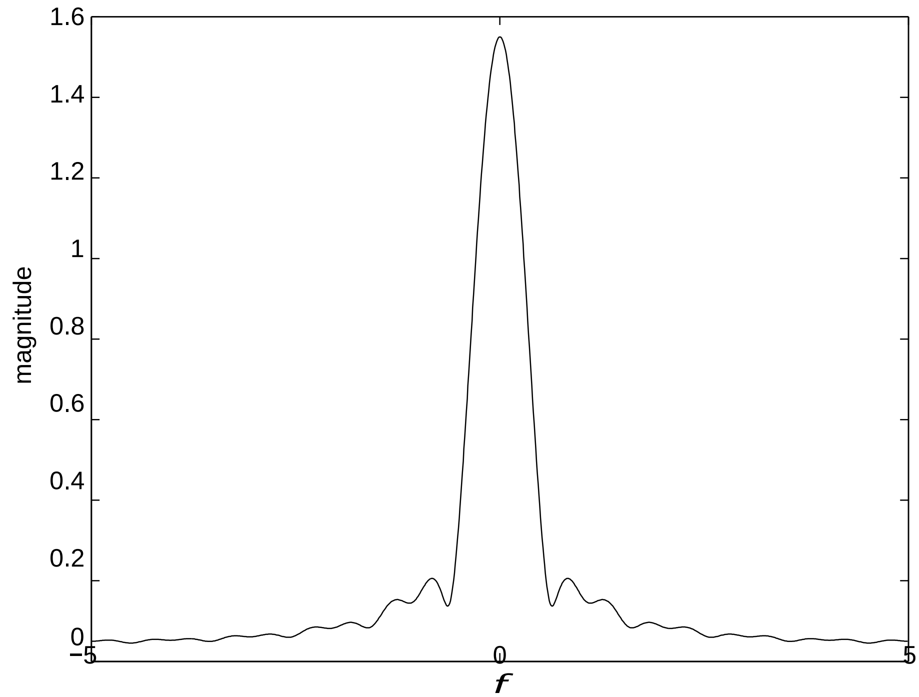


Figure 2.5: The common magnitude spectrum of the signals $x_1(t)$ and $x_2(t)$

Computer Problem 2.3

The common magnitude spectrum is presented in Figure 2.5. The two phase spectrum of the two signals plotted on the same axes are given in Figure 2.6.

The MATLAB script for this problem follows.

```
% MATLAB script for Computer Problem 2.3.
df=0.01;
fs=10;
ts=1/fs;
t=[−5:ts:5];
x1=zeros(size(t));
x1(41:51)=t(41:51)+1;
x1(52:61)=ones(size(x1(52:61)));
x2=zeros(size(t));
x2(51:71)=x1(41:61);
[X1,x11,df1]=fftseq(x1,ts,df);
[X2,x21,df2]=fftseq(x2,ts,df);
X11=X1/fs;
X21=X2/fs;
f=[0:df1:df1*(length(x11)−1)]−fs/2;
plot(f,fftshift(abs(X11)))
figure
plot(f(500:525),fftshift(angle(X11(500:525))),f(500:525),fftshift(angle(X21(500:525))),'−−')
```

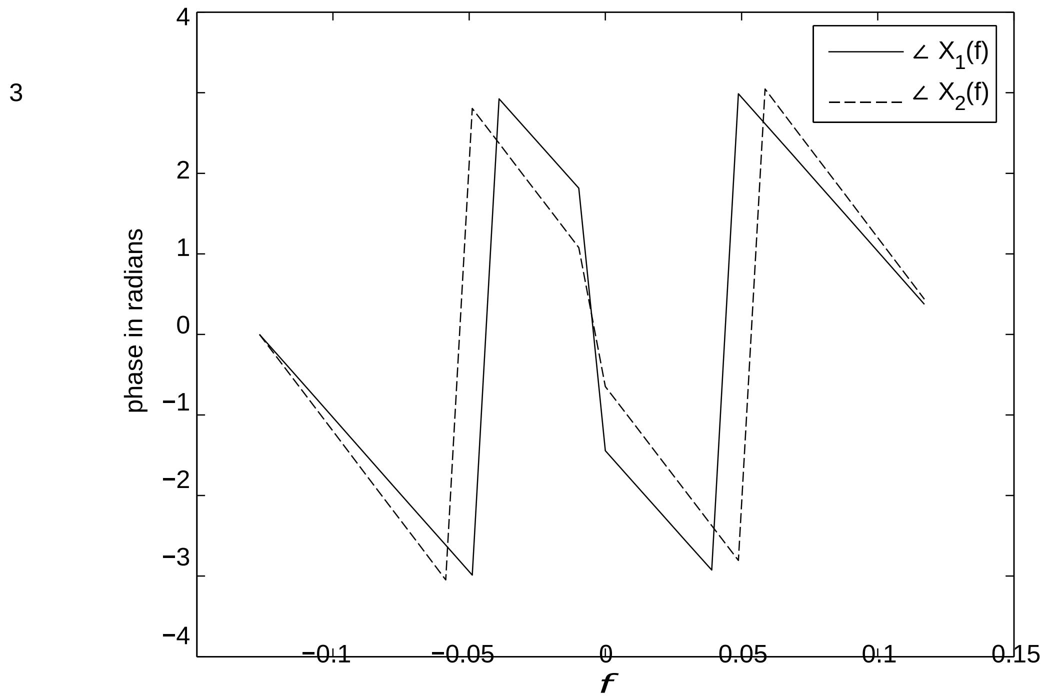


Figure 2.6: The phase spectrum of the signals $\Delta x_1(t)$ and $\Delta x_2(t)$

Computer Problem 2.4

The Fourier transform of the signal $x(t)$ is

$$\frac{1}{1+j2\pi f}$$

Figures 2.7 and 2.8 present the magnitude and phase spectrum of the input signal $x(t)$.
2) The fourier transform of the output signal $y(t)$ is

$$y(f) = \begin{cases} \frac{1}{1+j2\pi f} & |f| \leq 1.5 \\ 0 & \text{otherwise} \end{cases}$$

The magnitude and phase spectrum of $y(t)$ is given in Figures 2.9 and 2.10, respectively.
3) The inverse Fourier transform of the output signal is parented in Figure 2.11

The MATLAB script for this problem is given next

```
% MATLAB script for Computer Problem 2.4.
df= 0.01;
f = -4:df:4;
x_f = 1./(1+2*pi*i*f);
plot(f, abs(x_f));
figure;
plot(f, angle(x_f));
indH = find(abs(f) <= 1.5);
H_f = zeros(1, length(x_f));
H_f(indH) = cos(pi*f(indH)./3);
y_f = x_f.*H_f;
figure;
plot(f,abs(y_f));
axis([-1.5 1.5 0 16]);
figure;
plot(f, angle(y_f));

y_f(401) = 10^30;
y_t = ifft(y_f, 'symmetric');
figure;
plot(y_t)
```

Computer Problem 2.5

Choosing the sampling interval to be $t_s = 0.001$ s, we have a sampling frequency of $f_s = 1/t_s = 1000$ Hz. Choosing a desired frequency resolution of $df = 0.5$ Hz, we have the following.

1) Plots of the signal and its magnitude spectrum are given in Figures 2.12 and 2.13, respectively. Plots are generated by Matlab.

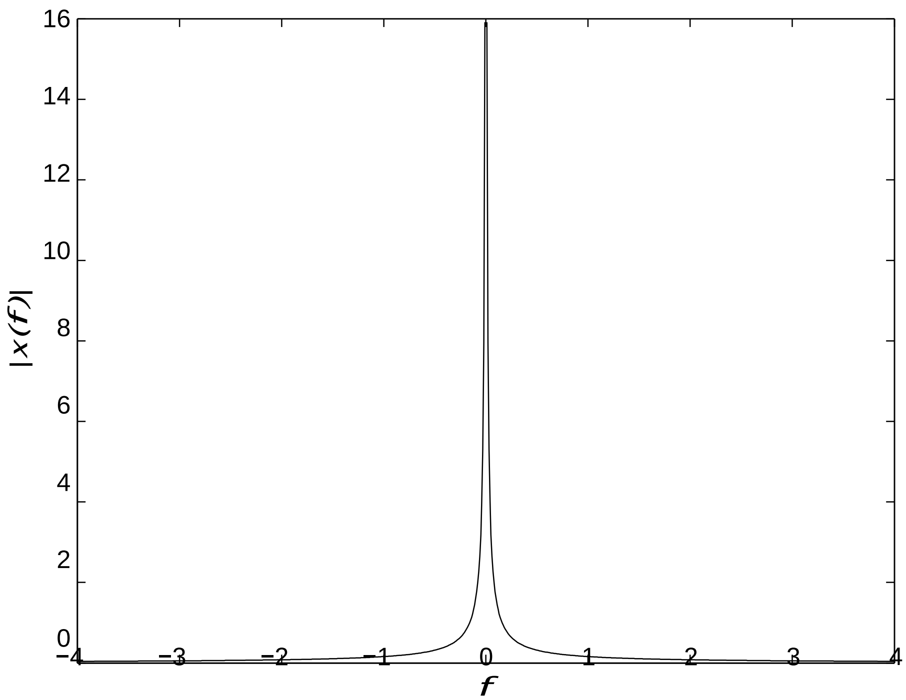


Figure 2.7: Magnitude spectrum of $x(t)$

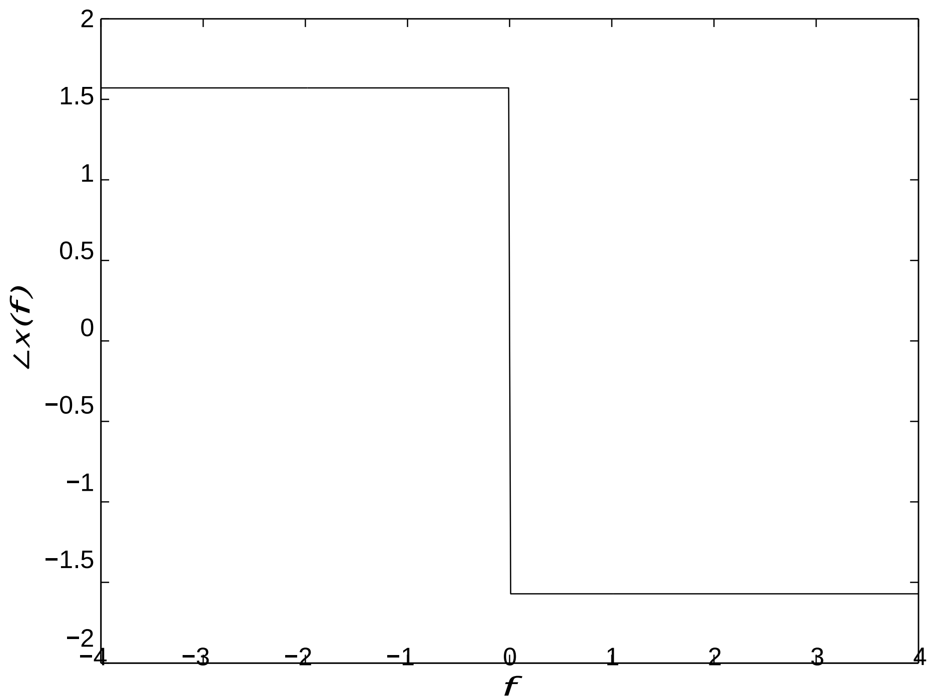


Figure 2.8: Phase spectrum of $x(t)$

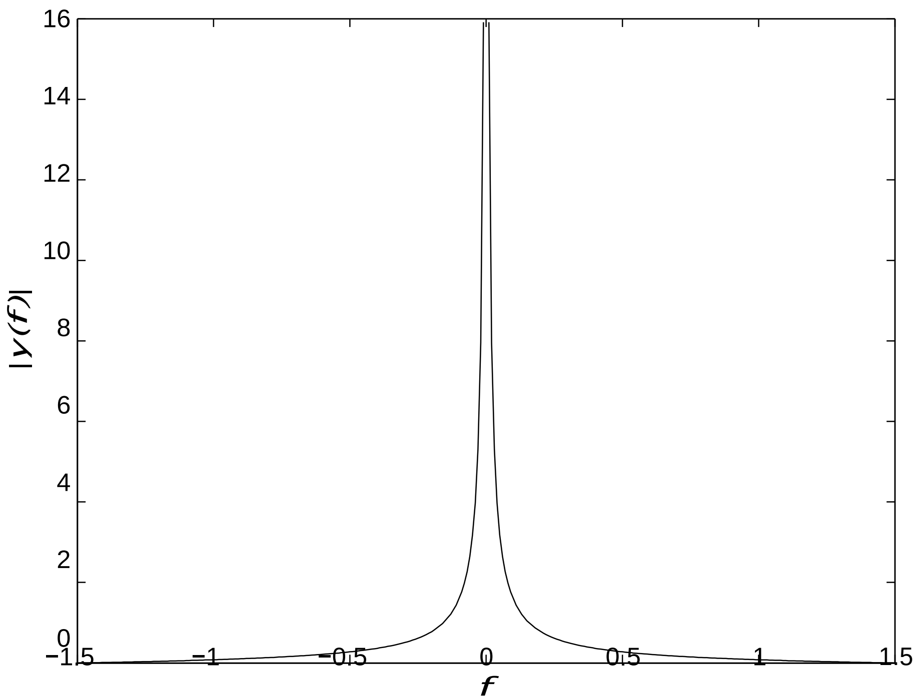


Figure 2.9: Magnitude spectrum of $y(t)$

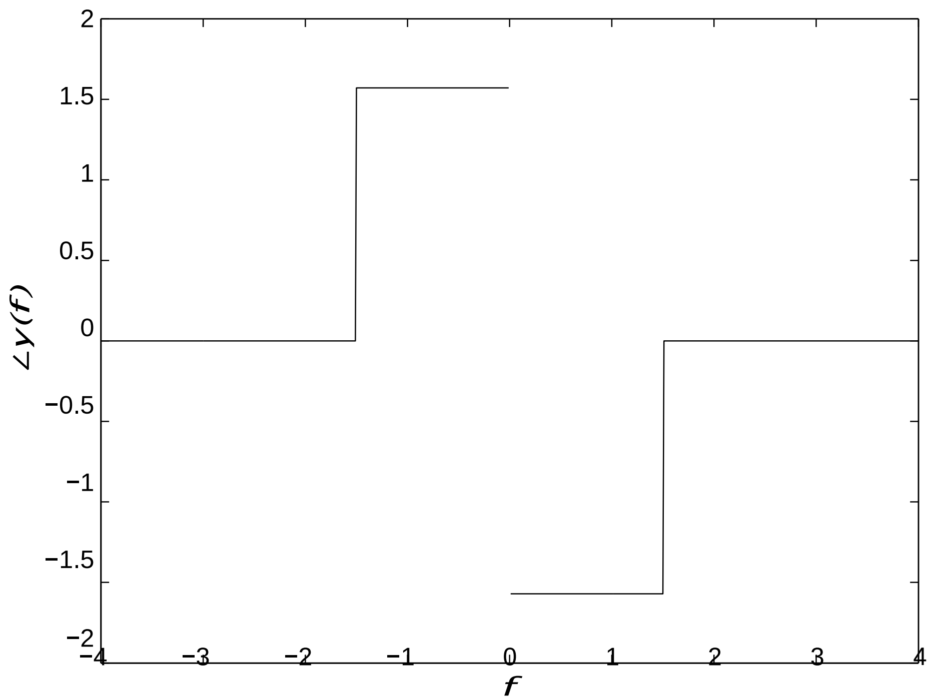


Figure 2.10: Phase spectrum of $y(t)$

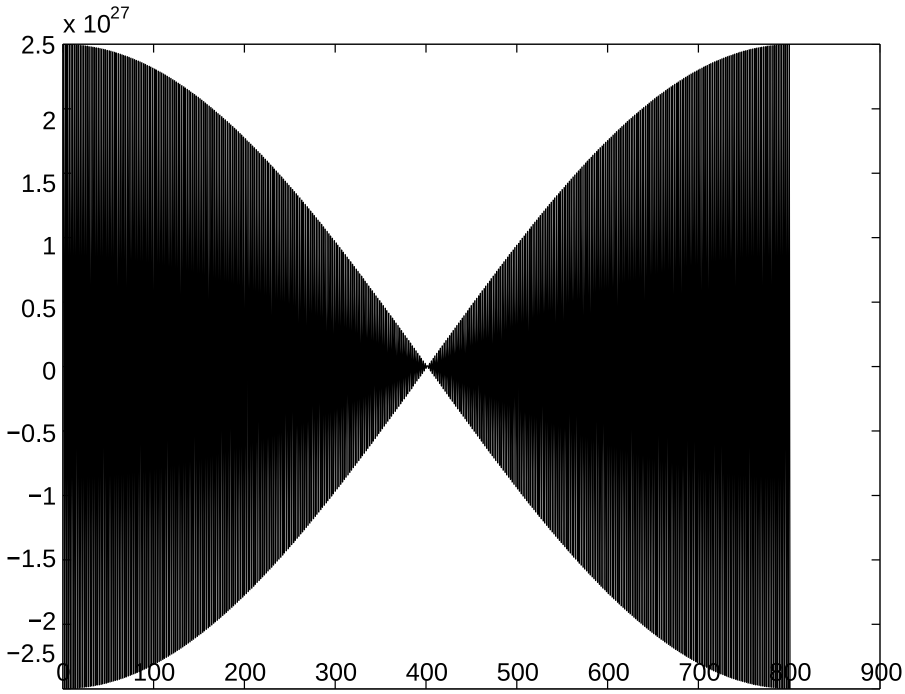


Figure 2.11: Inverse Fourier transform

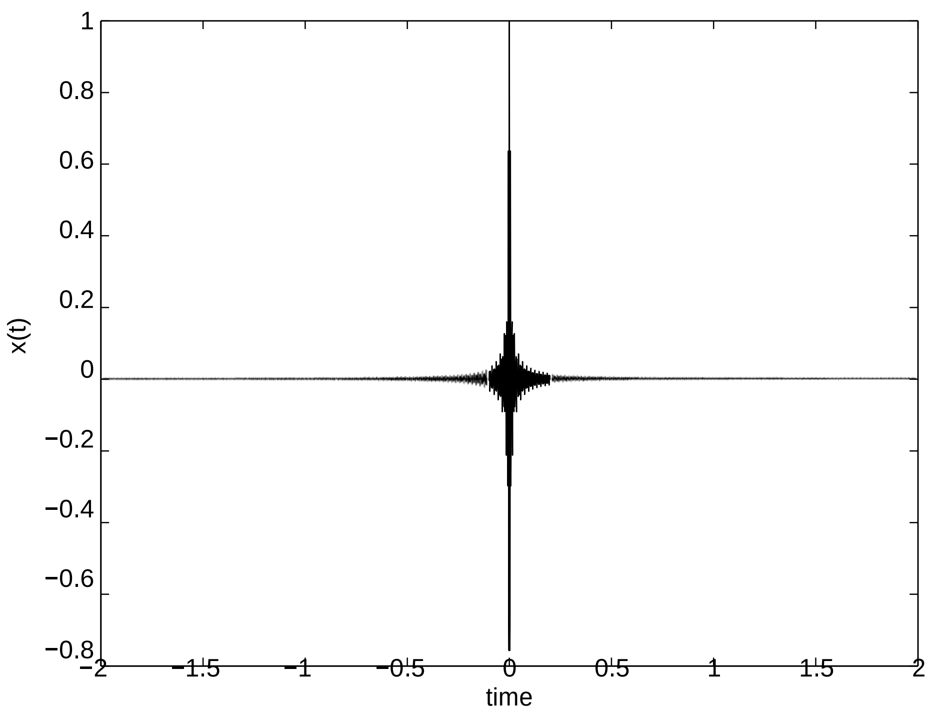


Figure 2.12: The signal $x(t)$

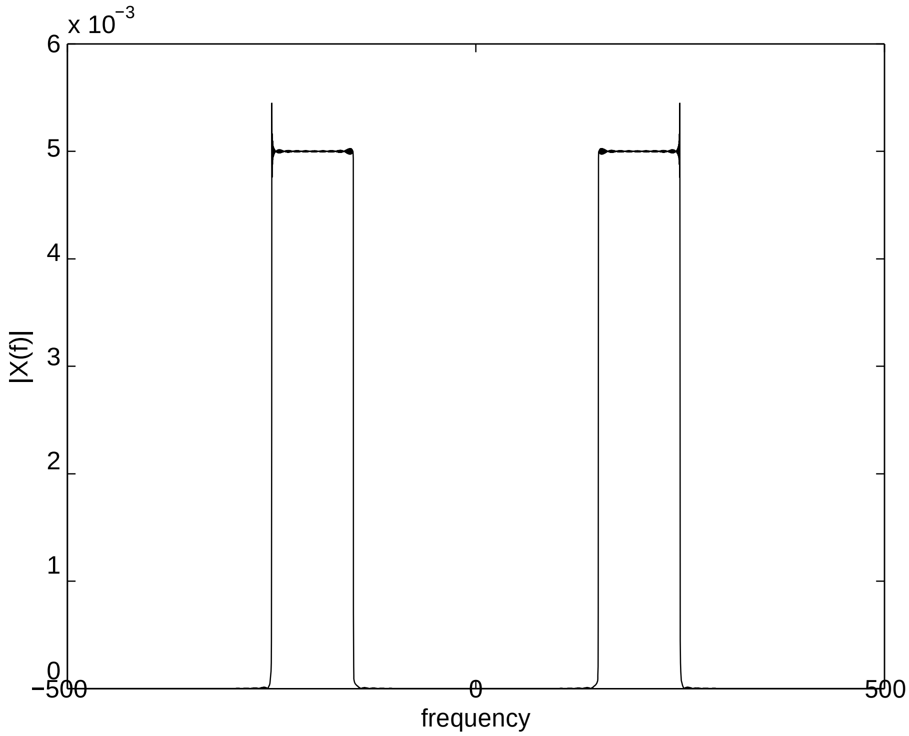


Figure 2.13: The magnitude spectrum of $x(t)$

2) Choosing $f_0 = 200$ Hz, we find the lowpass equivalent to $x(t)$ by using the loweq.m function. Then using fftseq.m, we obtain its spectrum; we plot its magnitude spectrum in Figure 2.14. The MATLAb functions loweq.m and fftseq.m are given next.

```
function [M,m,df]=fftseq(m,ts,df)
%               [M,m,df]=fftseq(m,ts,df)
%               [M,m,df]=fftseq(m,ts)
%FFTSEQ          generates M, the FFT of the sequence m.
%               The sequence is zero-padded to meet the required frequency resolution df.
%               ts is the sampling interval. The output df is the final frequency resolution.
%               Output m is the zero-padded version of input m. M is the FFT.
fs=1/ts;
if nargin == 2
  n1=0;
else n1=fs/df;
end n2=length(m);
n=2^(max(nextpow2(n1),nextpow2(n2)));
M=fft(m,n);
m=[m,zeros(1,n-n2)];
df=fs/n;
```

```
function xl=loweq(x,ts,f0)
%               xl=loweq(x,ts,f0)
%LOWEQ           returns the lowpass equivalent of the signal x
%               f0 is the center frequency.
%               ts is the sampling interval.
% t=[0:ts:ts*(length(x)-1)];
z=hilbert(x);
xl=z.*exp(-j*2*pi*f0*t);
```

It is seen that the magnitude spectrum is an even function in this case because we can write

$$x(t) = Re[\text{sinc}(100t)e^{j\times 400\pi t}] \tag{2.12}$$

Comparing this to

$$x(t) = Re[x_l(t)e^{j2\pi\times f_0 t}] \tag{2.13}$$

we conclude that

$$x_l(t) = sinc(100t) \tag{2.14}$$

which means that the lowpass equivalent signal is a real signal in this case. This, in turn, means that $x_c(t) = x_l(t)$ and $x_s(t) = 0$. Also, we conclude that

$$\begin{cases} V(t) = |x_c(t)| \\ \Theta = \begin{cases} 0, & x_c(t) \geq 0 \\ \pi, & x_c(t) < 0 \end{cases} \end{cases} \tag{2.15}$$

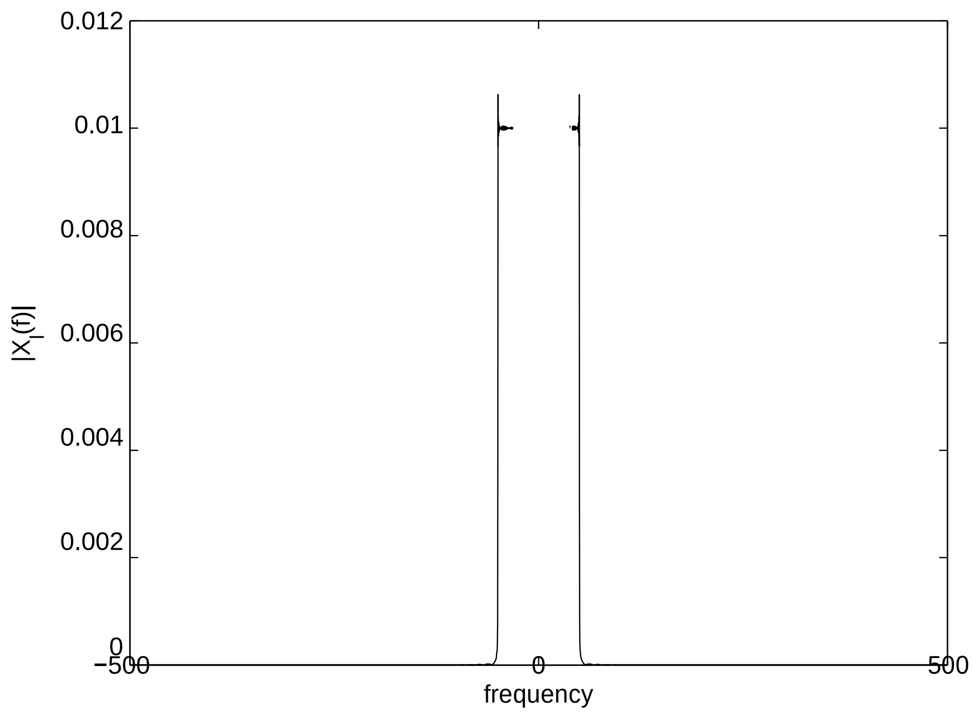


Figure 2.14: The magnitude spectrum of $x_l(t)$

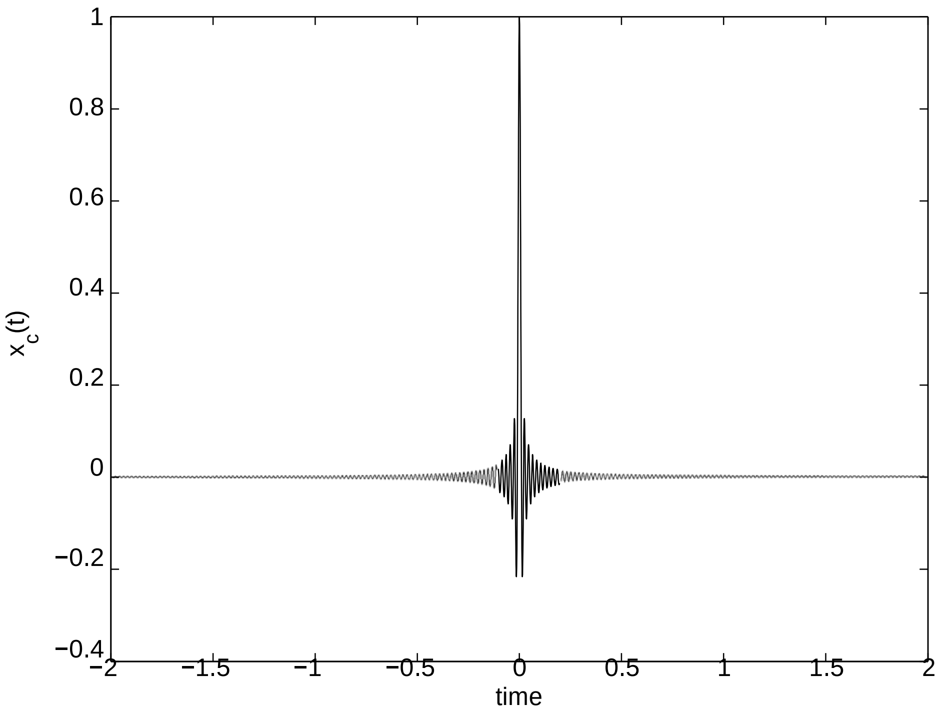


Figure 2.15: The signal $x_C(t)$

Plots of $x_c(t)$ and $V(t)$ are given in Figures 2.15 and 2.16, respectively. Note that choosing f_0 to be the frequency with respect to which $X(f)$ is symmetric result in these figures.

Computer Problem 2.6

The Remez algorithm requires that we specify the length of the FIR filter M, the passband edge frequency f_p, the stopband edge frequency f_s, and the ratio δ_2/δ_1. Here, δ_1 and δ_2 denote passband and stopband ripples, respectively. The filter length M can be approximated using

$$\hat{M} = \frac{-20\log_{10}\sqrt{\delta_1\delta_2} - 13}{14.6\Delta f} + 1$$

where Δf is the transition bandwidth $\Delta f = f_s - f_p$

1) Figure 2.17 shows the impulse response coefficients of the FIR filter.

2) Figures 2.18 and 2.19 show the magnitude and phase of the frequency response of the filter, respectively.

The MATLAB script for this problem is given next

```
% MATLAB script for Computer Problem 2.6.
fp = 0.4;
```

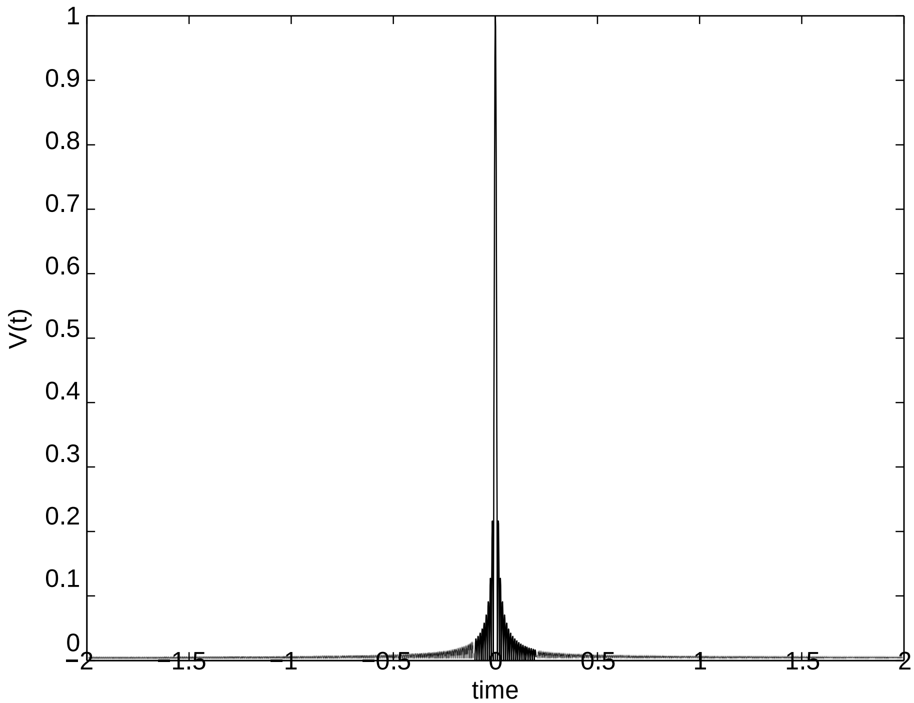


Figure 2.16: The signal $V(t)$

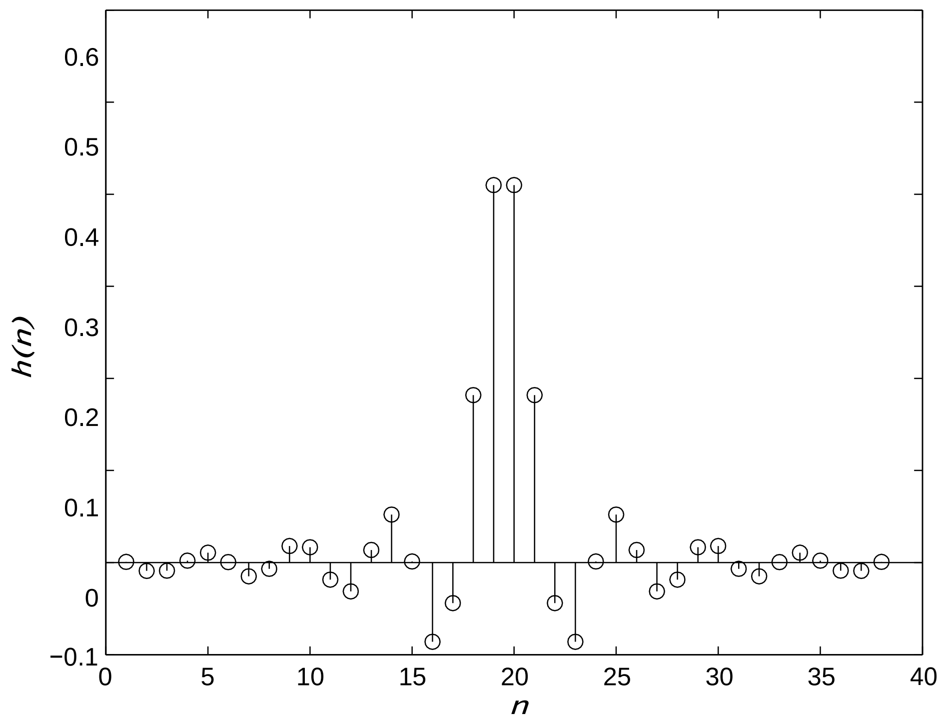


Figure 2.17: Impulse response coefficients of the FIR filter

```
fs = 0.5;
df = fs−fp;
Rp = 0.5;
As = 40;
delta1=(10^(Rp/20)−1)/(10^(Rp/20)+1);
delta2=(1+delta1)*(10^(−As/20));
%Calculate approximate filter length
Mhat=ceil((−20*log10(sqrt(delta1*delta2))−13)/(14.6*df)+1);
f=[0 fp fs 1];
m=[1 1 0 0];
w=[delta2/delta1 1];
h=remez(Mhat+20,f,m,w);
[H,W]=freqz(h,[1],3000);
db = 20*log10(abs(H));
% plot results
stem(h);
figure;
plot(W/pi, db)
figure;
plot(W/pi, angle(H));
```

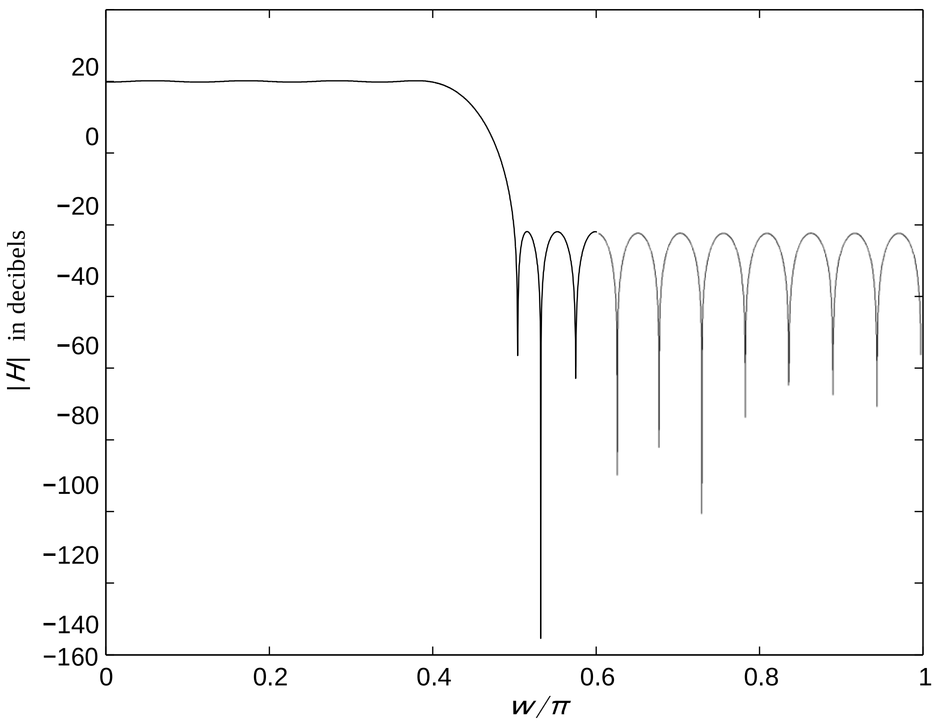


Figure 2.18: Magnitude of the frequency response of the FIR filter

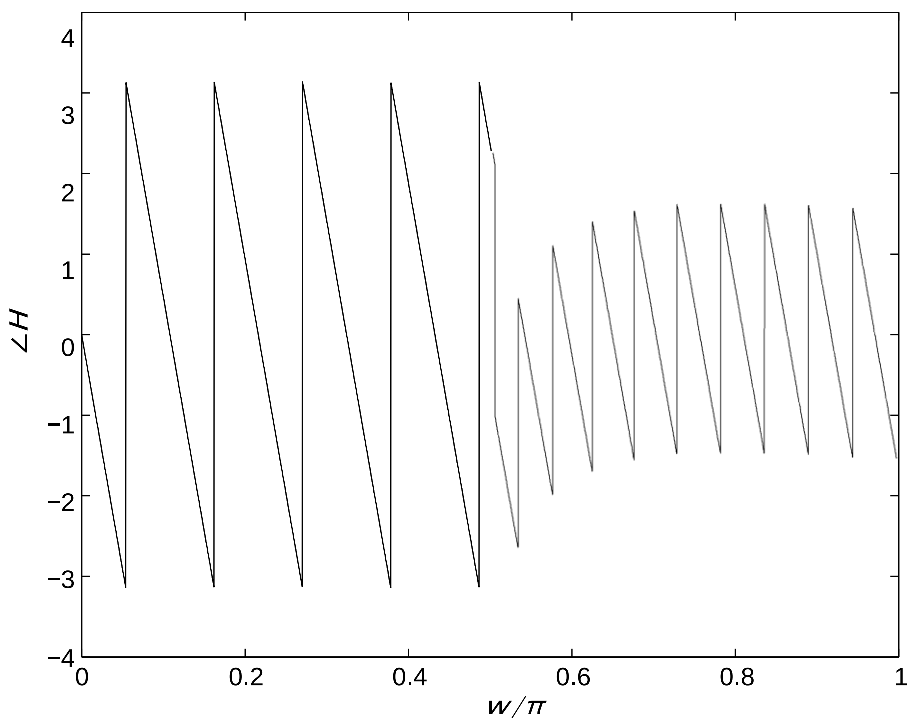


Figure 2.19: Phase of the frequency response of the FIR filter

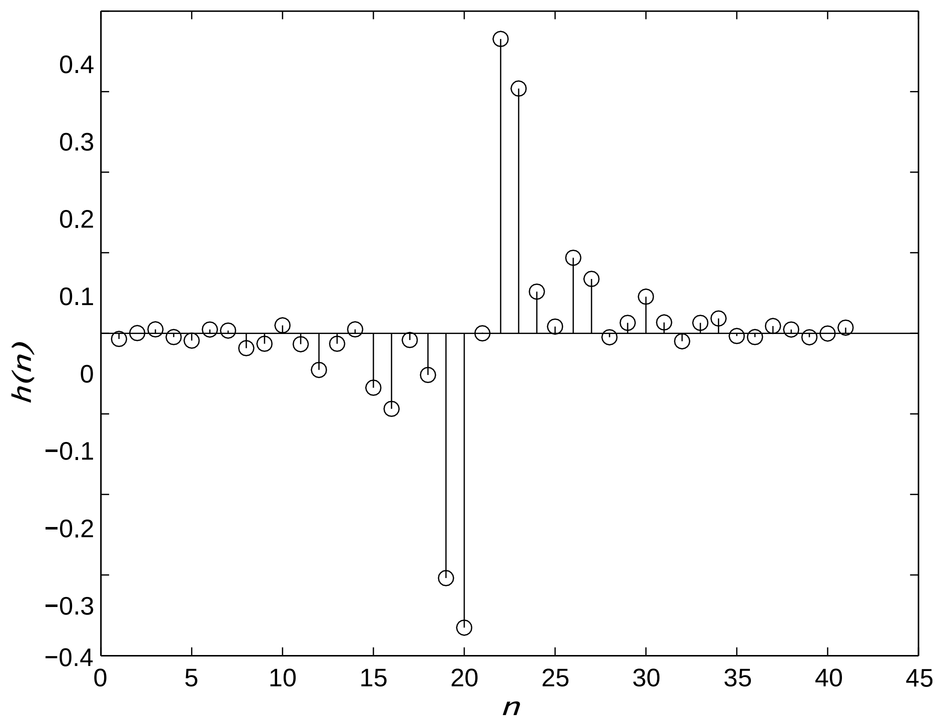


Figure 2.20: The impulse response coefficients of the filter

Computer Problem 2.7

1) The impulse response coefficients of the filter is presented in Figure 2.20.

2) The magnitude of the frequency response of the filter is given in Figure 2.21. The MATLAB script for this problem is given next

---

```
% MATLAB script for Computer Problem 2.7.
f=[0 0.01 0.1 0.5 0.6 1];
m=[0 0 1 1 0 0]; delta1
= 0.01;
delta2 = 0.01;
df = 0.1 - 0.01;
Mhat=ceil((-20*log10(sqrt(delta1*delta2))-13)/(14.6*df)+1);
w=[1 delta2/delta1 1];
h=remez(Mhat+20,f,m,w,'hilbert');

[H,W]=freqz(h,[1],3000);
db = 20*log10(abs(H));
% plot results
stem(h);
```

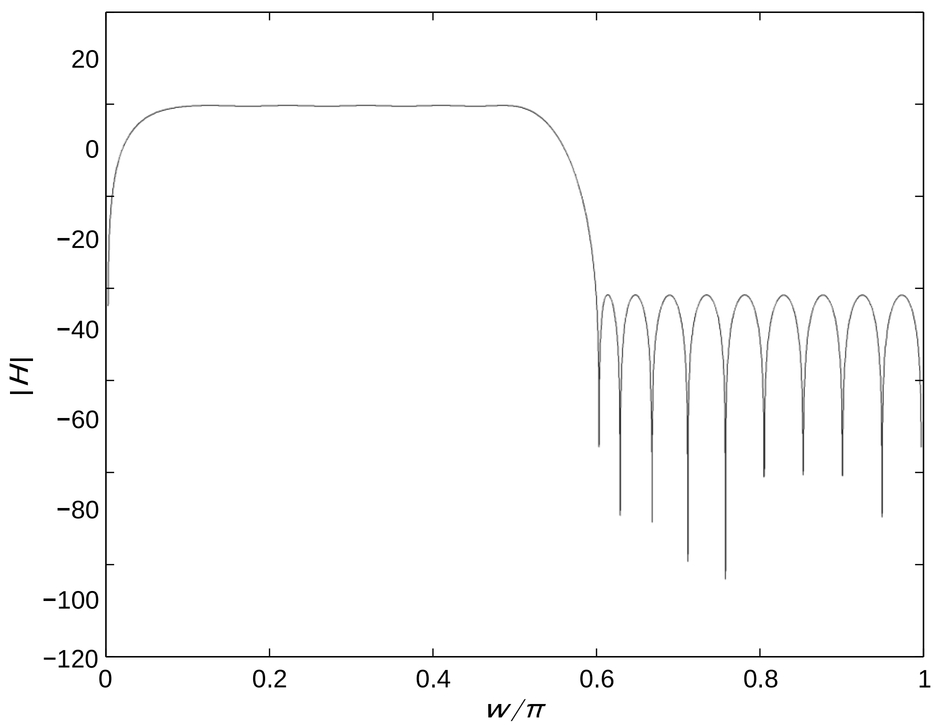


Figure 2.21: The magnitude of the frequency response of the filter

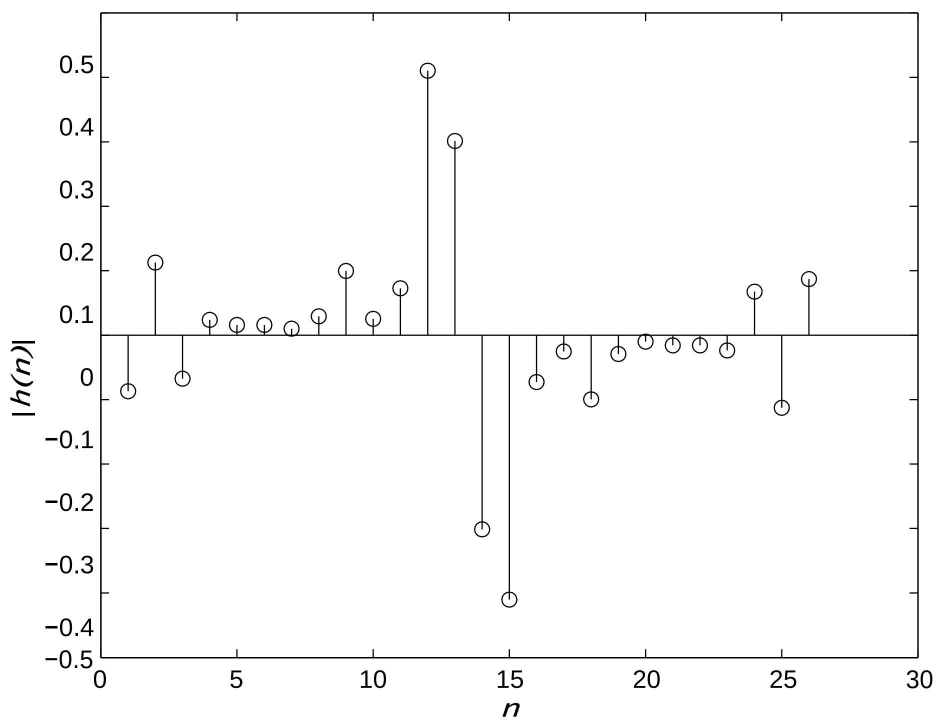


Figure 2.22: Impulse response of the filter

```
figure;
plot(W/pi, db)
figure;
plot(W/pi, angle(H));
```

Computer Problem 2.8

1) The impulse response of the filter is given in Figure 2.22.

2) The magnitude of the frequency response of the filter is presented in Figure 2.23.

3) The filter output $y(n)$ and $x(n)$ are presented in Figure 2.24. It should be noted that $y(n)$ is the derivative of $x(n)$.

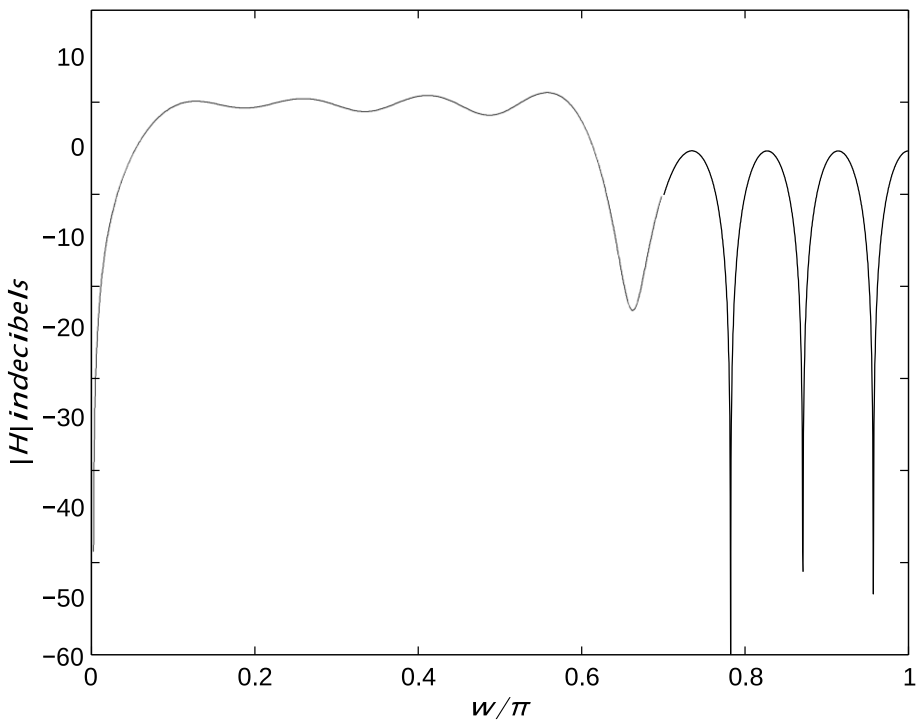


Figure 2.23: Magnitude of the frequency response of the filter

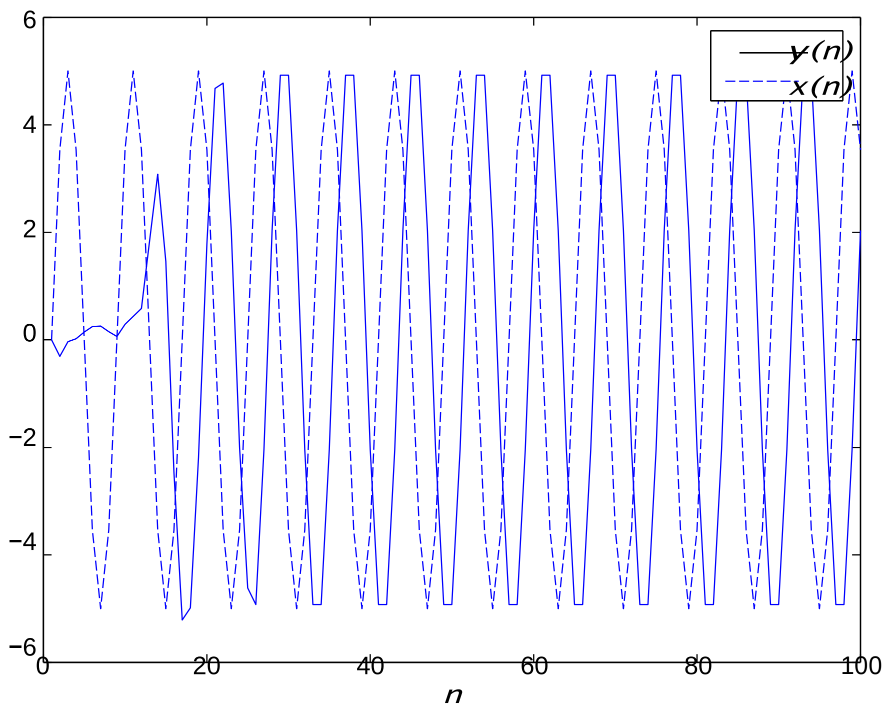


Figure 2.24: Signals $x(n)$ and $y(n)$