

**Solution Manual for Fundamentals of Complex Analysis with
Applications to**

Engineering and Science 3rd Edition by Saff Snider ISBN

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Solution Manual

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CHAPTER 2: Analytic Functions

EXERCISES 2.1: Functions of a Complex Variable

1. a. $w = (3az - 3y + 5 + 1) + i(6zy + 5y + 1)$

b. $w = e^z \cos(z+y)$

c. $w = z^2 + 3$

d. $w = 7e^z - z^2 + 7$

e. $w = e^{3y}(\cos 3y + i \sin 3y)$

f. $w = (e^z + e^{-z})\cos y + i(e^z - e^{-z})\sin y$
 $= 2\cosh z \cos y + i 2\sinh z \sin y$

2. a. C

b. $C \setminus \{0\}$

c. $C \setminus \{i, -i\}$

d. $C \setminus \{1\}$

e. C

f. C

3. a. $\operatorname{Re} w > 5$

b. $\operatorname{Im} w > 0$

c. $|w| > 1$

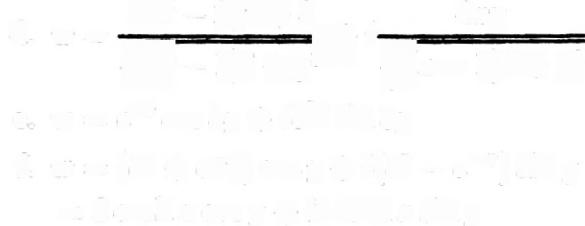
d. The intersection of $|\operatorname{Arg} w| < 2$ and $-\pi < \operatorname{Arg} w < \pi$

4. a. Taking z from 0 to $2a$, the points $z = re^{i\theta}$ traverse the circle

$|z| = r$ exactly once in the counterclockwise direction. For the same values of θ the points $w = \frac{z}{r} = e^{i\theta}$ traverse the circle $|w| = 1$.

$|w| = 1$ exactly once in the clockwise direction, hence the mapping is onto.

- b. For $z = re^{i\theta}$, on the ray $\operatorname{Arg} z = \theta_0$, $w = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-is}$ is on the ray $\operatorname{Arg} w = -s$. Taking values $0 < r < \infty$ shows that this mapping goes onto the ray $\operatorname{Arg} w = -s$.



2-1

1. $z^2 + 1 = 0$
2. $z^2 = 1$
3. $z^2 = 2$
4. $z^2 = 2i$
5. $z^2 = -2$
6. $z^2 = -2i$
7. $z^2 = 2e^{i\pi/3}$
8. $z^2 = 2e^{i\pi/2}$
9. $z^2 = 2e^{i\pi/4}$
10. $z^2 = 2e^{i\pi/6}$
11. $z^2 = 2e^{i\pi/12}$
12. $z^2 = 2e^{i\pi/24}$
13. $z^2 = 2e^{i\pi/48}$
14. $z^2 = 2e^{i\pi/96}$
15. $z^2 = 2e^{i\pi/192}$
16. $z^2 = 2e^{i\pi/384}$
17. $z^2 = 2e^{i\pi/768}$
18. $z^2 = 2e^{i\pi/1536}$
19. $z^2 = 2e^{i\pi/3072}$
20. $z^2 = 2e^{i\pi/6144}$
21. $z^2 = 2e^{i\pi/12288}$
22. $z^2 = 2e^{i\pi/24576}$
23. $z^2 = 2e^{i\pi/49152}$
24. $z^2 = 2e^{i\pi/98304}$
25. $z^2 = 2e^{i\pi/196608}$
26. $z^2 = 2e^{i\pi/393216}$
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4. (($1 - e^{-iz} = 1 - \cos 0 - i\sin 0 = z = 1 + e^{-i\theta}$). $|F(z)| = |z| = \sqrt{1 + e^{-2\theta}}$)
 $= (1 + e^{-2\theta})^{1/2} / \sqrt{2(1 + \cos 0)} = \sqrt{\frac{1 + e^{-2\theta}}{2(1 + \cos 0)}} = \sqrt{\frac{1 + e^{-2\theta}}{2(1 + 2\cos^2 0 - 1)}} = \sqrt{\frac{1 + e^{-2\theta}}{4\cos^2 0}} = \sqrt{\frac{1}{4\cos^2 0}} = \frac{1}{2|\cos \theta|}$
which is a vertical line at $x = \frac{1}{2}$.

5. a. domain: C
range: $C \setminus \{0\}$

b. $f(-z) = e^{-z} = \frac{1}{e^z} = \overline{f(z)}$

- c. circle $|w| = e$
d. ray $\operatorname{Arg} w = n/4$
e. infinite sector $0 < \operatorname{Arg} w < n/4$

6. a. $(\frac{3}{z} - 1)(1 + \frac{1}{z}) = \frac{1}{z}(3 + 1) - n$

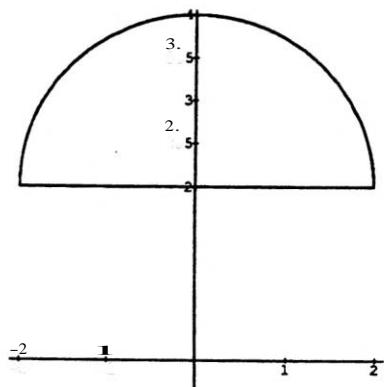
- b. For $z = e^{i\theta}$ on the unit circle $|z| = 1$, $|f(z)| = 5$; $(3 + \frac{1}{e^{i\theta}}) = \cos \theta$.
For all values of θ , this ranges over the real interval $[-1, 1]$.

- c. For $z = re^{i\theta}$ on the circle $|z| = r$, $|f(z)| = \left| \frac{3re^{i\theta} + 1}{r} \right| = \sqrt{(3r\cos \theta + 1)^2 + (3r\sin \theta)^2} = \sqrt{9r^2 + 1 + 6r\cos \theta}$.
Set $x = 3r\cos \theta$ and $y = 3r\sin \theta$. Then $x^2 + y^2 = 9r^2$, which is an ellipse centered at $(-\frac{1}{3}, 0)$ with foci at ± 1 .

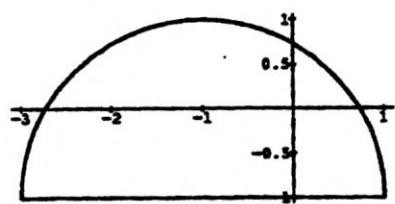
7. a.



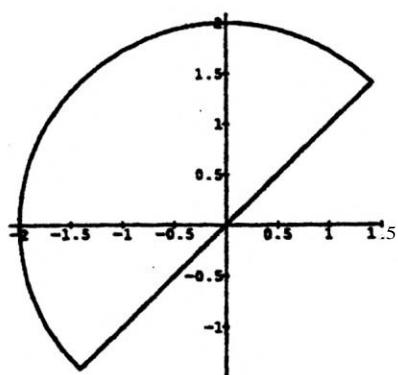
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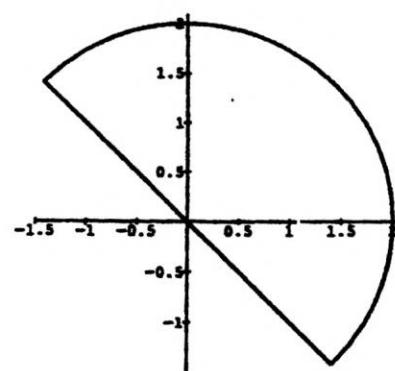
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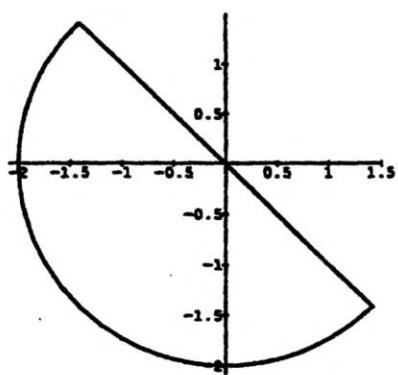
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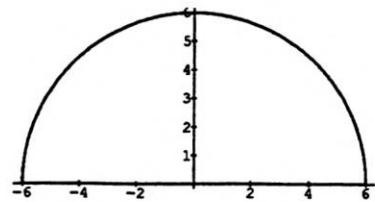
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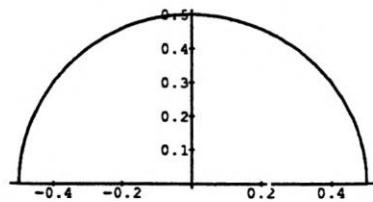
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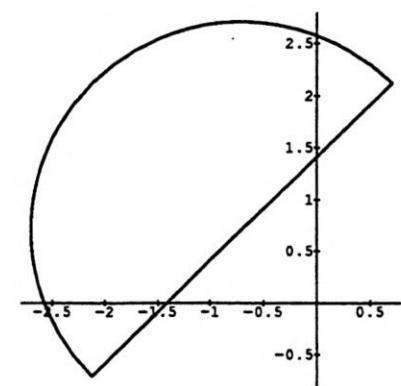
9. a.



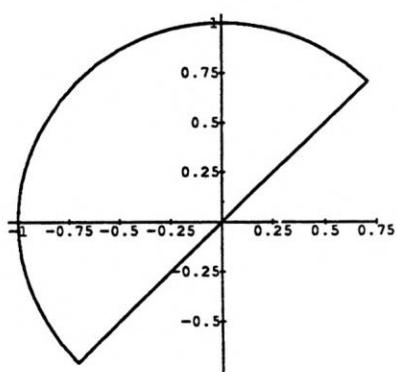
b.



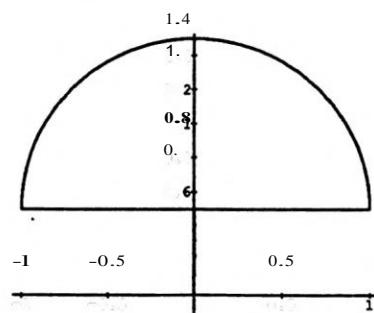
10. a. translate by i , rotate $n/4$



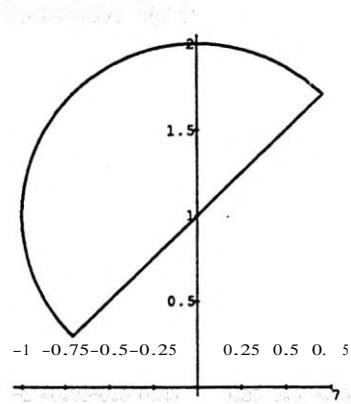
b. reduce by $1/2$, rotate $z/4$



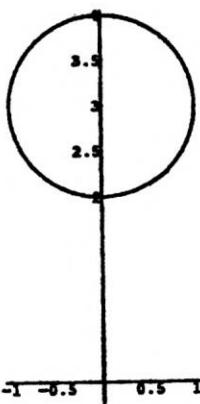
c. translate by i , reduce by $1/2$



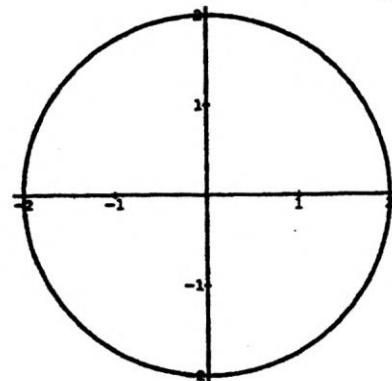
d. reduce by $1/2$, rotate $z/4$,
translate by i



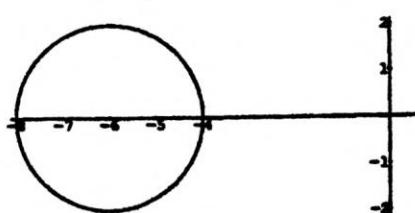
11. a. translate by -3 ,
rotate $-\pi/2$



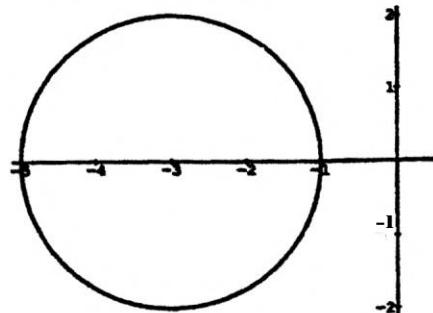
- b. magnify by 2,
rotate $-\pi/2$



- c. translate by -3 ,
magnify by 2



- d. magnify by 2, rotate $-\pi/2$,
translate by -3



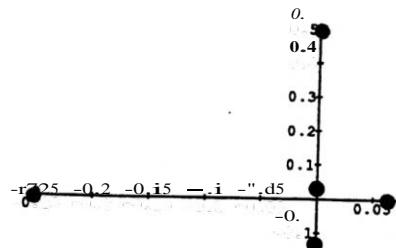
12. Let $\mathbf{a} = p\mathbf{e}$, $F(z) = pz$, $G(z) = \mathbf{e}z$, and $H(z) = z + b$. Then $H(G(F(z))) = \mathbf{a} + b$.

13. (a) $w = u + iv = z = (1 + iy) = 1 - y + i2y$
 $u = 1 - y, v = 2y = y = v/2 = u = 1 - v \rightarrow$ a parabola in the w-plane.
(b) $w = u + iv = z' = (x + iy) = (x + i\cancel{y}) = x - 1/\cancel{y} + 2i$
 $u = x - 1/\cancel{y}, v = 2$ a straight line in the w-plane.
(e) $w = u + iv = z = (1 + e) = (1 + 2e + e^2) = (e + 2 + e)e$
 $= (2 + 2\cos 0)e'' = 2(1 + \cos 0)e''$ a cardioid in the w-plane.
14. (a) $\mathbf{x} = 2\mathbf{z}/(|\mathbf{z}| + 1)$, $\mathbf{x} = 2\mathbf{y}/(|\mathbf{z}| + 1)$, $\mathbf{X} = (|\mathbf{z}| - 1)/(|\mathbf{z}| + 1)$
 $w = e''z = x\cos\theta - y\sin\theta + i(x\sin\theta + y\cos\theta)$, $|w| = |\mathbf{z}|$
 $\mathbf{X} = (x\cos\theta - y\sin\theta)/(|\mathbf{z}|' + 1)$, $\mathbf{X} = (x\sin\theta + y\cos\theta)/(|\mathbf{z}|' + 1)$, $x_3 = x_3$
 $X_1 = (x\cos(-x_2\sin\theta)), X_2 = (x\sin\theta + x_2\cos\theta), X_3 = x_3$ which corresponds to a rotation of an angle θ about the x_3 axis.
(b) $w = -1/z$, $|w| = 1/|z|$, $w = -1/(x+iy) = -x/|z| + iy/|z|$
 $X_1 = -X_1, X_2 = X_2, X_3 = -x_3$ so that (x_1, X_2, θ) is obtained from (X_1, X_2, X_3) by a 180° rotation about the x_2 axis.
15. $w = (1+2)/(1-z) = (1+x+iy)/(1-x-iy) = (1-|z|^2 + i2y)/1-2x + |z|^2$
 $wf = (1+2x+1)/(1-2x+|z|^2)$
 $(1, 42, X_3) = (-x_3, 2, x)$ so that (x_1, X_2, X_3) is obtained by a 90° counterclockwise rotation about the x_2 axis.

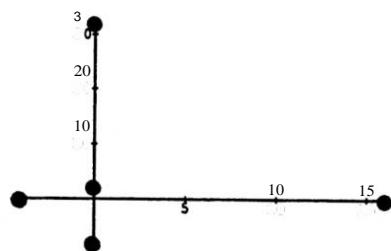
16. $w = (1 - iz)/(1 + iz) = (1 - ix + y)/(1 + ix - y) = (1 - |z| + i2x)/(1 - 2y + |z|)$
 $wf = (1 + 2y + |z|)/(1 - 2y + |z|)$.
 $\zeta_1, X_2, X = (-x_3, -x_1, x_2)$ so that (x_1, X_2, X) is obtained as a 90° counterclockwise rotation about the X axis followed by a 90° counterclockwise rotation about the x_3 axis.
17. Any circle or line in the z -plane corresponds to a line or circle on the stereographic projection onto the Riemann sphere. The function $w = 1/z$ rotates the Riemann sphere 180° about the x_3 axis. Lines and circles on the rotated sphere project to lines and circles in the w -plane. As a result lines and circles in the z -plane map to lines and circles in the w -plane.

EXERCISES 2.2: Limits and Continuity

1. The first five terms are, respectively, $j, \frac{1}{j}, \frac{1}{j^2}, \frac{1}{j^3}, \frac{1}{j^4}$,
sequence converges to 0 in a spiral-like fashion.



2. $2i, -4, -8i, 16, 32i$; divergent because terms grow in modulus without bound.



3. If $\lim_{n \rightarrow \infty} z_n = z_0$, then for any $\epsilon > 0$, there is an integer N such that $|z_n - z_0| < \epsilon$ for all $n > N$. For the same integer N we have $|x_n - x_0| < \epsilon$ and $|y_n - y_0| < \epsilon$ for all $n > N$. Therefore, $\lim_{n \rightarrow \infty} x_n = x_0$ and $\lim_{n \rightarrow \infty} y_n = y_0$.

If $\lim_{n \rightarrow \infty} X_n = x_0$ and $\lim_{n \rightarrow \infty} Y_n = y_0$, then for any $\delta > 0$ and $\epsilon > 0$ there are integers N and N' such

$|x_n - x_0| < \epsilon$; for all $n > N$, and $|y_n - y_0| < \epsilon$ for all $n > N'$. Given any $\epsilon > 0$; let $\epsilon' = \epsilon/2$ and $\epsilon'' = \epsilon/2$. Then $|Z_n - z_0| = |x_n - x_0 + i(y_n - y_0)| \leq |x_n - x_0| + |y_n - y_0| < \epsilon' + \epsilon'' = \epsilon$ for all $n > \max(N, N')$. Thus $\lim_{n \rightarrow \infty} Z_n = z_0$.

4. If $z_n = x_n + iy_n$, $z_0 = x_0 + iy_0$, then $x_n \rightarrow x_0$ and $y_n \rightarrow y_0$ (see Problem 3). $Z_n = X_n + iy_n \rightarrow x_0 + iy_0 = z_0$.

If $z_n = X_n - iy_n$, $z_0 = x_0 - iy_0$, then $X_n \rightarrow x_0$ and $y_n \rightarrow y_0$ (see Problem 3). $Z_n = X_n - iy_n \rightarrow x_0 - iy_0 = z_0$. Thus $z_n \rightarrow z_0$ if and only if $Z_n \rightarrow z_0$.

5 $\lim_{n \rightarrow \infty} z_n = 0 \iff$ There exists an integer N such that

$|z_n| \rightarrow 0 \iff |z_n| < \epsilon$ whenever $n > N \iff |z_n - 0| < \epsilon$ whenever $n > N \iff \lim_{n \rightarrow \infty} |z_n - 0| = 0$, and conversely.

ϵ $\neq 0$ as $n \rightarrow \infty$ by problem 3, since the real-valued sequence $|z_n| \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, if $|z_0| > 1$, then $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$ so Z_n diverges.

7. a. converges to 0

b. does not converge

c. converges to 0

d. converges to $2+i$

e. converges to 0

f. does not converge

8. Given $\epsilon > 0$, choose $\delta = \epsilon/6$. Then whenever $0 < |z - (1+i)| < \delta$,

$$|(z - 1 - i) - (2 + 6i)| = |z - (1 + i)| < \delta = \epsilon/6 = \epsilon$$

9. Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{1 + |z_0|}$. Whenever $0 < |z - (-i)| < \delta$ notice that $|z| > 1 - \delta$ and

$$\left| \frac{1}{z} - i \right| = \left| \left(-\frac{i}{z} \right) (i + z) \right| = \frac{1}{|z|} |z - (-i)| < \left(\frac{1}{1 - \delta} \right) \delta = \epsilon$$

O Given that f and g are continuous at z ,

$$\lim_{z \rightarrow z_0} f(z) \pm g(z) = \lim_{z \rightarrow z_0} f(z) \pm \lim_{z \rightarrow z_0} g(z) = f(z_0) \pm g(z_0)$$

$\Rightarrow f(z) \pm g(z)$ is continuous at z .

$$\lim_{z \rightarrow z_0} f(z)g(z) = \lim_{z \rightarrow z_0} f(z) \lim_{z \rightarrow z_0} g(z) = f(z_0)g(z_0)$$

$\Rightarrow f(z)g(z)$ is continuous at z .

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \rightarrow z_0} f(z)}{\lim_{z \rightarrow z_0} g(z)} = \frac{f(z_0)}{g(z_0)} \quad \text{provided } g(z_0) \neq 0$$

\Rightarrow continuous at z_0

fl. a. $-8i$

b. 5^7

c. $6i$

d. $-1/2$

e. $2z\%$

f. 45

IR. Clearly $\operatorname{Arg} z$ is discontinuous at $z = 0$. Let $a > 0$ be any real number and consider the sequence

$$z_n = -a - i/n \quad n=1,2,\dots, \text{ which converges to } -a.$$

For each n , $-n < \operatorname{Arg} z_n < -n/2$, but $\operatorname{Arg}(-a) = \pi$.

13. $\lim_{z \rightarrow z_0} f(z)$ exists for all $z \neq 0$; f is continuous for all $z \neq 0, -1$; f has a removable discontinuity at $z = 0$.

14. Let z_0 be any complex number. Given $\epsilon > 0$ choose $\delta = \epsilon$. Then whenever $|z - z_0| < \delta$,

$$|g(z) - g(z_0)| = |z - z_0| = |z - z_0| = |z - z_0| < \epsilon.$$

15. Given $\epsilon > 0$ choose δ so that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$. Then, whenever $|z - z_0| < \delta$:

a. $|f(z) - f(z_0)| = |f(z) - f(z_0)| = |f(z) - f(z_0)| < \epsilon$

b. $|\operatorname{Re} f(z) - \operatorname{Re} f(z_0)| = |\operatorname{Re}(f(z) - f(z_0))| < |f(z) - f(z_0)| < \epsilon$

c. $|\operatorname{Im} f(z) - \operatorname{Im} f(z_0)| = |\operatorname{Im}(f(z) - f(z_0))| < |f(z) - f(z_0)| < \epsilon$

d. $|f(z) - f(z_0)| < |f(z) - f(z_0)| < \epsilon$

16 Given $\epsilon > 0$, choose $\delta > 0$ such that $|f(g(z)) - f(g(\zeta))| < \epsilon$ whenever $|g(z) - g(\zeta)| < \delta$. Now choose $\delta > 0$ such that $|g(z) - g(\zeta)| < \delta$ whenever $|z - z_0| < \delta$. Then $|f(g(z)) - f(g(\zeta))| < \epsilon$ whenever $|z - z_0| < \delta$; hence $f(g(z))$ is continuous at z_0 .

11. No: Observe that although $\frac{1}{n} \rightarrow 0$ and $\frac{i}{n} \rightarrow 0$ as $n \rightarrow \infty$,

$$1\left(\frac{1}{n}\right) - 1 + 2i \neq 1\left(\frac{i}{n}\right) - 2 \text{ hence } f \text{ is not analytic.}$$

18. If $\lim_{z \rightarrow z_0} f(z) = w_0$, then given $\epsilon > 0$ there exists $\delta > 0$ such that

$|f(z) - w_0| < \epsilon$ for all $|z - z_0| < \delta$. Notice that

$|f(z) - w_0| = |f(z) - w_1| = |f(z) - w_0| < \epsilon$ for all $|z - z_0| < \delta$. So that $\lim_{z \rightarrow z_0} f(z) = w_0$.

$\lim_{z \rightarrow z_0} (x, y) = \lim_{z \rightarrow z_0} ((f(z) + f(z))/2) = (w_0 + w_0)/2 = 0$.

$\lim_{z \rightarrow z_0} (x, y) = \lim_{z \rightarrow z_0} ((f(z) - f(z))/2i) = (w_0 - w_0)/2i = 0$.

Thus, $\lim_{z \rightarrow z_0} (x, y) = 0$ and $\lim_{z \rightarrow z_0} (x, y) = 0$.

Conversely, if $\lim_{z \rightarrow z_0} (x, y) = 0$ and $\lim_{z \rightarrow z_0} (x, y) = 0$, then (by Theorem I.) $0 + iu_0 = \lim_{z \rightarrow z_0} (x, y) + i\lim_{z \rightarrow z_0} (y, y) = \lim_{z \rightarrow z_0} ((f(z) + f(z))/2) + \lim_{z \rightarrow z_0} ((f(z) - f(z))/2) = \lim_{z \rightarrow z_0} f(z) = w_0$. Also $0 - iv_0 = \lim_{z \rightarrow z_0} (x, y) - i\lim_{z \rightarrow z_0} (x, y) = \lim_{z \rightarrow z_0} ((f(z) + f(z))/2) - \lim_{z \rightarrow z_0} ((f(z) - f(z))/2) = \lim_{z \rightarrow z_0} f(z) = w_0$.

Thus, $\lim_{z \rightarrow z_0} f(z) = w_0$.

$$1. \quad \frac{1-i}{z^2 - 5i} \text{ since } \lim_{z \rightarrow 1} \frac{1-i}{z^2 - 5i} = \frac{1}{2} \text{ and } \lim_{z \rightarrow i} zy = -1.$$

20. For any Z in the complex plane,

$$\begin{aligned} \lim_{z \rightarrow w} e^z &= \lim_{z \rightarrow w} e^{\operatorname{cos} y + i \operatorname{sin} y} = e^{\operatorname{cos} 0 + i \operatorname{sin} 0} \\ &= e \end{aligned}$$

21. a. 1

b. 0

c. $-n/2 + i$

d. 1

22. By contradiction: Suppose $\lim_{z \rightarrow z_0} f(z) = w_0$. Then there is an $\epsilon > 0$ for which there exists a sequence $\{z_n\}$ such that $|z_n - z_0| < \frac{\epsilon}{2}$ but $|f(z_n) - w_0| > \epsilon$. For this sequence, $\lim_{n \rightarrow \infty} z_n = z_0$ but $\lim_{n \rightarrow \infty} f(z_n) \neq w_0$ contrary to hypothesis.

23. If $z \rightarrow \infty$, then for any $M > 0$ there exist an integer N such that $|z| > M$ for all $n > N$. Consider the chordal distance $(|z_n - s|) = 2\sqrt{(|z_n| + 1)^2 - 2|z_n||s|} \leq 2|z_n| < 2/M < \epsilon$ for all $n > N$. Thus $z \rightarrow \infty$ as $n \rightarrow \infty$ is equivalent to $\zeta(z, O) = 0$ as $n \rightarrow \infty$.
24. If $\lim_{z \rightarrow z_0} f(z) = L$, then for any $M > 0$ there exists $\delta > 0$ such that $|f(z)| > M$ for all $|z - z_0| < \delta$. Consider $\zeta(f(z), s) = 2\sqrt{|f(z)f(s)|} \leq 2\sqrt{|f(z)|} < 2/M < \epsilon$ for all $|z - z_0| < \delta$. Thus $\lim_{z \rightarrow z_0} f(z) = L$ is equivalent to $\lim_{z \rightarrow z_0} \zeta(f(z), s) = 0$.
25. (a) ∞ (b) 3 (c) ∞ (d) ∞ (e) ∞ (f) the limit does not exist.

EXERCISES 2.3: Analyticity

1. Let $Az \equiv z - z_0$ so that $Az \rightarrow 0 \Leftrightarrow z \rightarrow z_0$. Then

$$\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = L \iff$$

given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\left| \frac{f(z_0 + Az) - f(z_0)}{Az} - L \right| < \epsilon \text{ whenever } |Az| < \delta$$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - L \right| < \epsilon \text{ whenever } |z - z_0| < \delta$$

$$\frac{f(z) - f(z_0) - L(z - z_0)}{z - z_0} \rightarrow 0 \text{ as } z \rightarrow z_0$$

2. If $f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$, then $f(z) \rightarrow 0$ as $z \rightarrow z_0$ and $f(z_0) + f'(z_0)(z - z_0) + o(z - z_0) = f(z)$.

3. $\zeta(z) = \ln[f(z) + 0] + i[\arg(z - z_0) + \pi]$
 $= f(z) + 0 + i\pi = f(z)$.

4. a. $\lim_{z \rightarrow 0} \frac{\operatorname{Re}(z + Az) - \operatorname{Re}(z)}{Az} = \lim_{z \rightarrow 0} \frac{\operatorname{Re}(Az)}{Az} = \begin{cases} 1, & \text{if } Az = Ax} \\ 0, & \text{if } Az = iAy \end{cases}$

b. $\lim_{z \rightarrow 0} \frac{\operatorname{Im}(z + Az) - \operatorname{Im}(z)}{Az} = \lim_{z \rightarrow 0} \frac{\operatorname{Im}(Az)}{Az} = \begin{cases} 0, & \text{if } Az = -iAy} \\ -i, & \text{if } Az = iAx \end{cases}$

c. Case 1, $z=0$.

$$\lim_{z \rightarrow 0} \frac{|0 + Az| - |0|}{Az} = \lim_{z \rightarrow 0} \frac{\sqrt{A(z)^2 + (Ay)^2}}{Az} = \begin{cases} \pm\infty, & \text{if } Az = Ax} \\ -i, & \text{if } Az = \pm iAy \end{cases}$$

Case 2, $z \neq 0$,

$$\begin{aligned} & \lim_{z \rightarrow 0} \frac{|z + Az| - |z|}{Lz} \\ &= \lim_{z \rightarrow 0} \frac{(z + Az) + (y + Ay) - (z + y)}{(Az + iAy)(/(z + Az) + (y + Ay)) + /T} \\ &= \lim_{z \rightarrow 0} \frac{2Az + (Ax) + 2yAy + (Ay)}{6(Az + iAy)(/(z + Ax) + (y + Ay) + /T)} \\ &= \begin{cases} \sqrt{z^2 + 2^2}, & \text{if } Az = Ax, z \neq 0} \\ \sqrt{z^2 + y^2}, & \text{if } Az = iAy, z \neq 0 \end{cases} \end{aligned}$$

5. Rule 5: $(\pm g)'(z) = \lim_{Ar \rightarrow 0} \frac{(\pm g(z+Ar) - \pm g(z))}{Ar}$

$$\begin{aligned} &= \lim_{Ar \rightarrow 0} \left[\frac{f(z+Ar) - f(z)}{Ar} + \frac{g(z+Ar) - g(z)}{Ar} \right] \\ &= f'(z) \pm g'(z) \end{aligned}$$

Rule 7: $(fg)'(z) = \lim_{Ar \rightarrow 0} \frac{fg(z+Ar) - fg(z)}{Ar}$

$$= \lim_{Az \rightarrow 0} \frac{\{f(z_0 + .6z)g(z_0 + Az) - f(z_0 + .6z)g(z_0)\}}{Lz}$$

$$+ \frac{f(+A)(O)F(lS(O))}{Az}$$

$$= \lim_{Az \rightarrow 0} \frac{\{f(z_0 + .6z)[g(z_0 + .6z) - g(z_0)]\}}{Lz}$$

$$+ g(z_0) \frac{[f(z_0 + .6z) - f(z_0)]}{Lz}\}$$

$$= f'(O)g'(z) + g(z_0)f'(O)$$

6. Let $n > 0$ be an integer.

Then $\frac{d^n}{dz^n} = \frac{d}{dz} \left(\frac{d^{n-1}}{dz^{n-1}} \right) = \frac{d}{dz} \left(\frac{z^n}{z^2} \right) \quad (\text{using Rule 8}) = -nz^{n-2}$.

7. a. $18z + 16z + i$

b. $-12z(z - 3i)$

c. $\frac{-iz + (2+27i)z + 2z + 18}{(iz + 2z + 7)}$

d. $\frac{-(z+2)(5z + (16+i)z - 3 + 8i)}{(z^2 + iz + 1)^5}$

e. $24i(2^o - 1)(2 + i2)(532^o + 28iz^o - 50z - 25i)$

8. Let $z = z_0 + Az$. Then

$$\left| \frac{|f(z) - f(z_0)|}{|z - z_0|} \right| = \left| \frac{\lim_{Az \rightarrow 0} |f(z_0 + .6z) - f(z_0)|}{Lz} \right| = |f'(O)|$$

$$\lim_{z \rightarrow z_0} e^n - t^l \rightarrow -t^l - al = a \mathbf{a}[l^m]$$

$$\arg \left[\lim_{Az \rightarrow 0} \frac{|f(z_0 + .6z) - f(z_0)|}{Lz} \right] = \arg[f'(z_0)]$$

9. a. $2-3i$

b. $\pm i$

c. $\frac{-1 \pm i/15}{2}$

d51

10. $\lim_{\Delta \rightarrow 0} \frac{|Z_0 + Az|^2 - |Z_0|^2}{\Delta}$

$$= \lim_{\Delta \rightarrow 0} \frac{(\bar{z} + A\bar{z})(z + Az) - z\bar{z}}{\Delta}$$

$$= \begin{cases} +z_0 & \text{if } Az = Ar \\ 20 & \text{if } Az = iIy \end{cases}$$

If $z = 0$, then the difference quotient is

$$\lim_{\Delta \rightarrow 0} (0 + 0 + 2) = 0.$$

11. a. nowhere analytic

b. nowhere analytic

c. analytic except at $z = 5$

d. everywhere analytic

e. nowhere analytic

f. analytic except at $z = 0$

g. nowhere analytic

h. **nowhere** analytic

12. The case when $n = 1$ is trivial. Assume that the result holds for all positive integers less than or equal to n and define $Q(z) = P(z)(z - + +)$. Since $Q'(z) = P'(z)(z - + +) + P(z)$, it follows that

$$\frac{Q'(z)}{Q(z)} = \frac{P'(z)}{P(z)} + \frac{1}{z - z_1 + 1} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \dots + \frac{1}{z - z_{n+1}}$$

13. a, b, d, f, and g are always true

$$14. \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \frac{f'(z_0)}{g'(z_0)}$$

Is 2

16. Any point on the line through z and z' has the form

$z = -\frac{1}{2} + i\sqrt{3}(t - t')$, t real (see Section 1.3, Exercise 18). However, $f(z) - f(z') = 0$ but $f'(w) = 3^2 \neq 0$ on the line in question.

$$\begin{aligned} 17. F'(z_0) &= f(z_0)(gh)'(z_0) + f(z_0)gh(z_0) \\ &= f'(z_0)[q(z_0)i'(z_0) + g(z_0)h(z_0)] + f'(z_0)q(z_0)h(z_0) \\ &= f'(z_0)h(z_0) + f(z_0)g'(z_0)h(z_0) + f(z_0)g(z_0)h'(z_0) \end{aligned}$$

EXERCISES 2.4: The Cauchy-Riemann Equations

1. $\frac{\partial u}{\partial x} = \# \alpha = 1$

b. $\alpha = \# \alpha = 0$

c. $\alpha = \# \alpha =$

2. $u = x^2 - y^2$, $v = 2xy$. Then $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial v}{\partial y} = 2x$.

$\frac{\partial u}{\partial y} = -2y$, $\frac{\partial v}{\partial x} = 2y$. Therefore $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$.

but $\frac{\partial v}{\partial y} = 2x = \frac{\partial u}{\partial x}$. Therefore $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$.

only when $x = 0$ or $y = 0$. This means h is differentiable on the axes but h is nowhere analytic since lines are not open sets in the complex plane.

3. $\frac{\partial u}{\partial x} = 6 + 2x$, $\frac{\partial v}{\partial y} = 0$. Since h has parallel derivatives at $(0, 0)$, h is analytic at $(0, 0)$.

exist and are continuous for all x and y , g is analytic. g can be written as $g(z) = 3z^2 + 2z - 1$.

2-1/

$$\begin{aligned} \frac{\partial u}{\partial z} &= \lim_{r \rightarrow 0} \frac{(A_0 - r^0) - a''_0}{1_a} = A_r \cdot o_A \\ \frac{\partial v}{\partial y} &= \lim_{r \rightarrow 0} \frac{(OAy) - u(0,0)}{1_y} = m_a''_0 = O \\ \$a_w\$ &= \$-0 \end{aligned}$$

However, when $\sqrt{z} \rightarrow 0$ through real values ($\sqrt{z} = Aa$)

$$\lim_{r \rightarrow 0} \frac{\sqrt{(0+4Az)} - I(0)}{Lz} = ?$$

while along the real line $y = z$ ($\sqrt{z} = Aa + i1r$)

$$\begin{aligned} \lim_{Ar \rightarrow 0} \frac{4(O \pm Az) - I(O)}{Lz} &= \lim_{Ar \rightarrow 0} \frac{(A)''(A \pm (As!)')}{Ar(1+i)} \\ &= \frac{1}{2}. \end{aligned}$$

Therefore f is not differentiable at $z = 0$.

$$\begin{aligned} f' &= \frac{\partial u}{\partial z} + i \frac{\partial v}{\partial z} \\ &= w'' @ a + m_a r_0 - f; \end{aligned}$$

f is entire because these first partials exist and are continuous for all z and y .

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial z} + i \frac{\partial v}{\partial z} = 2 \cdot 2(\phi + iy)[\cos(2zy) + i\sin(2zy)] \\ &\equiv 2w(z + iy) \\ &= 2z \end{aligned}$$

(This derivative could have been obtained directly, since $f(z) = e_z$.)

6. $z = r\theta = r\cos\theta$ and $y = r\sin\theta$ and

$$f(z) = u(z(r,0), y(r,0)) + iv(z(r,0), y(r,0))$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial y}, \frac{\partial v}{\partial r} = \frac{\partial v}{\partial z}, \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial y}$$

?2-\\$

Similar applications of the chain rule yield

$$\begin{aligned}\frac{\partial}{\partial z} \mathbf{a} &= \frac{\partial u}{\partial x} - " \sin \theta + \frac{\partial v}{\partial y} \cos \theta \\ \text{or } \frac{\partial v}{\partial z} &= \frac{\partial v}{\partial a} \cos(\theta) + \frac{\partial v}{\partial y} \sin \theta \\ \frac{\partial v}{\partial \bar{z}} &= \frac{\partial v}{\partial \bar{z}} (-r \sin \theta) + \frac{\partial v}{\partial y} \cos \theta\end{aligned}$$

Replace the partial derivatives on the right sides of the equations for \mathbf{a} and \mathbf{v} by their Cauchy-Riemann counterparts to obtain:

$$\begin{aligned}\frac{\partial v}{\partial r} - \frac{\partial v}{\partial y} \cos(\theta) \frac{\partial v}{\partial \bar{z}} \sin(\theta) &= \frac{1}{r} \frac{\partial v}{\partial \theta} \\ \text{or } \frac{\partial v}{\partial r} - \frac{\partial v}{\partial y} \cos(\theta) \frac{\partial v}{\partial \theta} \sin(\theta) &= -\frac{1}{r} \frac{\partial v}{\partial \bar{z}}\end{aligned}$$

7. Let $h(z) = f(z) - g(z)$. Then h is analytic in D and $h'(z) = 0$ so h is a constant function.

$$h(z) = c = f(z) - g(z) \Rightarrow f(z) = g(z) + c$$

8. $(z, \theta) \in D \Rightarrow \frac{\partial}{\partial z} = 0$ and $\frac{\partial}{\partial \theta} = 0$. Hence

s

$r \in \mathbb{R}$ (constant), $\theta \in \mathbb{R}$ (constant), $\mathbf{f}(z) = \mathbf{f}(r, \theta)$

$r \in \mathbb{R}$ (constant), $\theta \in \mathbb{R}$ (constant), $\mathbf{f}(z) = \mathbf{f}(r, \theta)$

9. By contradiction. If f is analytic in a domain D then $v(az, y) = 0$ (a constant) $\Rightarrow f$ is constant (by condition 8) $\Rightarrow u$ is constant.

$$0. \operatorname{Im} f(z) = 0 \text{ in } D \Rightarrow \frac{\partial v}{\partial z} = 0 = \frac{\partial v}{\partial \bar{z}} = 0$$

- 1 (However, there is no open set in which $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial \bar{z}}$ is constant).
 $\Rightarrow f(z) = u(z, y) + i v(z, y)$ is constant in D .

11. $\operatorname{Re} f(z) = \frac{1}{2} (f(z) + \bar{f}(z))$ is real valued and analytic if both f and \bar{f} are analytic. Hence $\operatorname{Re} f(z)$ is constant by Exercise 10. It follows that $f(z)$ is constant by Exercise 8.

-1%

12. $|\mathbf{f}(z)|$ constant in $D \Rightarrow |\mathbf{f}(z)\mathbf{f}| = u + v$ is constant in D . $\mathbf{f} = 0$ or $v = 0$ in D , then \mathbf{f} is constant by Exercises 8 and 10. Otherwise,

$$\begin{aligned} \underline{0.5} &= \underline{\partial u}{\partial z} - \underline{\partial v}{\partial z} = 0 \\ \underline{\frac{\partial z}{\partial u}} &= \underline{\frac{\partial u}{\partial z}} - \underline{\frac{\partial v}{\partial z}} = 0 \\ \underline{\frac{\partial z}{\partial y}} &= \underline{\frac{\partial u}{\partial y}} - \underline{\frac{\partial v}{\partial y}} = 0 \\ \underline{\underline{x}} &= \underline{\frac{\partial v}{\partial y}} - \underline{\frac{\partial u}{\partial z}} = 0 \end{aligned}$$

$$\begin{aligned} -\frac{1}{2} \frac{\partial \underline{0.5}}{\partial z} - \underline{\frac{\partial v}{\partial y}} &= (\underline{\partial u} - \underline{\partial v}) \frac{\partial v}{\partial z} \\ -\underline{\frac{\partial v}{\partial z}} &= \underline{\frac{\partial v}{\partial y}} - \underline{\frac{\partial u}{\partial z}} \\ \underline{\underline{0.5}} &= \underline{\underline{\frac{\partial v}{\partial y}}} - \underline{\underline{\frac{\partial u}{\partial z}}} \end{aligned}$$

$\Rightarrow \mathbf{f}$ is constant in D .

13. $|V(z)|$ is analytic and real-valued so the result follows from Exercises 10 and 12.

14. If the line is vertical then $\operatorname{Re} \mathbf{f}(z)$ is constant and this reduces to Problem 8. If the line is not vertical, then $v(z, y) = mu(z, y) + b$, and

$$\begin{aligned} \underline{\underline{0.5}} &= \underline{\frac{\partial u}{\partial z}} - \underline{\frac{\partial v}{\partial y}} + \underline{\frac{\partial u}{\partial y}} - \underline{\frac{\partial v}{\partial z}} \\ \underline{\underline{0.5}} &= \underline{\frac{\partial u}{\partial z}} - \underline{\frac{\partial v}{\partial z}} \end{aligned}$$

It follows that

$$\underline{\underline{0.5}} = \underline{\underline{\frac{\partial u}{\partial z}}} - \underline{\underline{\frac{\partial v}{\partial z}}}$$

Hence $\mathbf{f}(z)$ is constant.

$$15. \quad J(\mathbf{O}, \mathbf{J}_0) = \frac{\partial u}{\partial z} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial z} \cdot \mathbf{a}$$
$$- [\mathbf{z} : \mathbf{a}]' \cdot [\mathbf{a} : \mathbf{z}]$$
$$= \mathbf{f}'(\mathbf{a}) \quad (\text{sing Equation (1)})$$

16. a. $\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}}$ $(g \cdot 2)y!$
 $= -1^2 \cdot (2) \cdot \overline{1} \cdot \overline{\left(\frac{\partial y}{\partial z} \alpha^2 \right)}$
 $\text{or } \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}}$
 $= -\left(\frac{\partial}{\partial z} \overline{A} \right) \left(p + i q \right) \text{te}$
 $= -2 \left(\frac{\partial}{\partial z} v_y \right) - 2 \left(\frac{\partial}{\partial y} + \frac{\partial v}{\partial z} \right)$
b. $\frac{\partial f}{\partial z} = 0 \Rightarrow \alpha = 50^\circ$ and $q = 5^\circ$

$$\alpha = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial x}$$

EXERCISES 2.5: Harmonic Functions

1. a. $u(x, y) = x - y + 2r + \theta \quad \frac{\partial u}{\partial r} = ? = 75y = 4u = 0$
 $v(x, y) = 2y + ? \quad \frac{\partial v}{\partial r} = ? \quad 4A = 0$

b. $u(x, y) = \frac{x}{x^2 + y^2} \cdot \frac{\partial}{\partial x} = \frac{2x(x-3)}{x^2 + y^2} = \frac{\partial u}{\partial x} = Au = 0$
 $v(x, y) = -\frac{y}{x^2 + y^2} \cdot \frac{\partial}{\partial x} = \frac{-2y(x+3)}{x^2 + y^2} = \frac{\partial v}{\partial x} = Av = 0$

c. $u(x, y) = e^{cos \theta} \quad \frac{\partial u}{\partial r} = e^{cos \theta} = 57 = AA = 0$
 $v(x, y) = e^{sin \theta} \quad \frac{\partial v}{\partial r} = e^{sin \theta} = 57 \Rightarrow \Delta = 0$

2. $h(z, y) = \alpha + by - ay$

3. a. $u = \operatorname{Re}(-iz)$, $v = -a + a$, where a is a constant
 b. $= \operatorname{Re}(-ie)$, $v = -e \cos y + a$
 c. $u = \operatorname{Re}(z - i\bar{z})$, $v = -4 - a - (x+y) + a$
 d. It is straightforward to verify that $\Delta u = 0$.

~~Sc~~ $\operatorname{Oz} \cosh y$,

$$\Rightarrow v(z,y) = \int \cos az \cosh y dy = \cos z \sinh y + C_{az}$$

$$\therefore v(z,y) = \sinh y - f; \Rightarrow a = \sinh w + w t = (D) = \infty$$

Thus, $v(z,y) = \cos z \sinh y + a$.

$$\frac{\partial u}{\partial z} = -a = \infty \Rightarrow$$

e. It is straightforward to verify that $Au = 0$.

$$\frac{\partial}{\partial z} = \frac{x}{y} - \frac{y}{y^2} + a$$

$$a. \rightarrow T_{ty} = \partial v = z t_{ty} - W(z) - z = a$$

$$Q9 \quad z' \quad dz$$

Thus, $(s) = ta' + a$.

f. $u = \operatorname{Re}(-ie)$, $v = \sin(2ay) + a$.

4. Suppose v and w are both harmonic conjugates of u , and consider $d(z,y) = w(z,y) - v(z,y)$. Then (using the Cauchy-Riemann equations for v and w),

-#23(3)•

and similarly $\bar{z} = 0$. Hence $C(z,y) = a$, from which it follows that

$$t_0(z,y) = v(z,y) + a.$$

5. If $f(z) = u(z,y) + iv(z,y)$ is analytic then $-if'(z) = v(z,y) - iu(z,y)$ is analytic. Thus $-u$ is a harmonic conjugate of v .

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6. Since $f(z) = u + iv$ is analytic, $\frac{1}{z} f'(z) = \frac{1}{z} (u' - v') + i(v' + u)$ is analytic.

Thus $v' = \operatorname{Im} f'(z)$ is harmonic.

7. $\phi(x, y) = -1$

8. a. Yes, because $\nabla(u + v) = Au + Lv = 0$.

b. No. Take $u = x = -y$ as an example.

c. Yes, because $\nabla(u, v) = \frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial v}{\partial x} \mathbf{j} = \frac{\partial v}{\partial y} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j}$

$$= \frac{\partial}{\partial x}(Au) = \frac{\partial}{\partial x}(0) = 0.$$

9. $\langle kn \rangle = -1$ (is it $\mathbf{I}(\pm \dots)$)

10. Let $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} g_{or} &= \frac{\partial g}{\partial r} = \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial r \partial x} = \frac{\partial^2 u}{\partial r^2} \cos\theta = \frac{\partial^2 u}{\partial r^2} = 0 \\ O\phi &= \frac{\partial g}{\partial \phi} = \frac{\partial}{\partial \phi} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial \phi \partial x} = \frac{\partial^2 u}{\partial \phi^2} \cos\theta = \frac{\partial^2 u}{\partial \phi^2} = 0 \\ O &= \frac{\partial g}{\partial \theta} = \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial \theta \partial x} = \frac{\partial^2 u}{\partial \theta^2} \cos\theta = \frac{\partial^2 u}{\partial \theta^2} = 0 \\ - &= \frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial \phi^2} = \frac{\partial^2 u}{\partial \theta^2} = 0 \\ O\phi &= \frac{\partial^2 u}{\partial r \partial \phi} = \frac{\partial^2 u}{\partial r \partial \theta} = \frac{\partial^2 u}{\partial \phi \partial \theta} = 0 \\ 00 &= \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial z^2} = 0 \\ 00 &= \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial z} = \frac{\partial^2 u}{\partial y \partial z} = 0 \\ 00 &= \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial z^2} = 0 \\ O\phi &= \frac{\partial^2 u}{\partial x \partial \phi} = \frac{\partial^2 u}{\partial y \partial \phi} = \frac{\partial^2 u}{\partial z \partial \phi} = 0 \\ 00 &= \frac{\partial^2 u}{\partial x \partial \theta} = \frac{\partial^2 u}{\partial y \partial \theta} = \frac{\partial^2 u}{\partial z \partial \theta} = 0 \\ 00 &= \frac{\partial^2 u}{\partial \phi \partial \theta} = \frac{\partial^2 u}{\partial \phi^2} = \frac{\partial^2 u}{\partial \theta^2} = 0 \\ \mathbf{a} &= \mathbf{a}_r \mathbf{i} + \mathbf{a}_\theta \mathbf{j} + \mathbf{a}_\phi \mathbf{k} \\ \mathbf{a}_r &= \frac{\partial \mathbf{a}}{\partial r} = \frac{\partial}{\partial r} (r \cos\theta \mathbf{i} + r \sin\theta \mathbf{j}) = \cos\theta \mathbf{i} + \sin\theta \mathbf{j} \\ \mathbf{a}_\theta &= \frac{\partial \mathbf{a}}{\partial \theta} = \frac{\partial}{\partial \theta} (r \cos\theta \mathbf{i} + r \sin\theta \mathbf{j}) = -r \sin\theta \mathbf{i} + r \cos\theta \mathbf{j} \\ \mathbf{a}_\phi &= \frac{\partial \mathbf{a}}{\partial \phi} = \frac{\partial}{\partial \phi} (r \cos\theta \mathbf{i} + r \sin\theta \mathbf{j}) = -r \sin\theta \mathbf{i} + r \cos\theta \mathbf{j} \\ \mathbf{a} &= r \cos\theta \mathbf{i} + r \sin\theta \mathbf{j} + 0 \mathbf{k} \\ \mathbf{a} &= r \cos\theta \mathbf{i} + r \sin\theta \mathbf{j} + 0 \mathbf{k} \end{aligned}$$

$\text{O} \int s).$

6. $\frac{\partial}{\partial x} (\frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial x^2}$ $\frac{\partial}{\partial y} (\frac{\partial v}{\partial y}) = \frac{\partial^2 v}{\partial y^2}$ $\frac{\partial}{\partial z} (\frac{\partial w}{\partial z}) = \frac{\partial^2 w}{\partial z^2}$

7. $\frac{\partial}{\partial x} (\frac{\partial u}{\partial y}) = \frac{\partial^2 u}{\partial y \partial x}$

8. $\frac{\partial}{\partial y} (\frac{\partial u}{\partial z}) = \frac{\partial^2 u}{\partial z \partial y}$

9. $\frac{\partial}{\partial z} (\frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial x \partial z}$

10. $\frac{\partial}{\partial x} (\frac{\partial v}{\partial y}) = \frac{\partial^2 v}{\partial y \partial x}$

11. $\frac{\partial}{\partial y} (\frac{\partial v}{\partial z}) = \frac{\partial^2 v}{\partial z \partial y}$

$\frac{\partial}{\partial z} (\frac{\partial v}{\partial x}) = \frac{\partial^2 v}{\partial x \partial z}$

12. $\frac{\partial}{\partial x} (\frac{\partial w}{\partial y}) = \frac{\partial^2 w}{\partial y \partial x}$

13. $\frac{\partial}{\partial y} (\frac{\partial w}{\partial z}) = \frac{\partial^2 w}{\partial z \partial y}$

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial x} c - \frac{\partial}{\partial x} co \\ &= \frac{\partial}{\partial x} \sin \theta + \frac{\partial}{\partial x} si \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial y} &= \frac{\partial}{\partial y} c - \frac{\partial}{\partial y} co \\ &= \frac{\partial}{\partial y} \sin \theta + \frac{\partial}{\partial y} si \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial z} c - \frac{\partial}{\partial z} co \\ &= \frac{\partial}{\partial z} \sin \theta + \frac{\partial}{\partial z} si \\ &= \frac{\partial}{\partial z} (\sin \theta) - \frac{\partial}{\partial z} (\cos \theta) = \frac{\partial}{\partial z} \sin \theta + \frac{\partial}{\partial z} co \end{aligned}$$

Combining these partial derivatives, one gets

$$\frac{\partial}{\partial r} \left(\frac{1}{r} \right) = \frac{1}{r^2} - \frac{1}{r^2} = 0.$$

11. $\operatorname{Im} f(z) = v - \frac{y}{x^2 + y^2} = 0 \Rightarrow y = 0 \Rightarrow y = y(z + \infty - 1) = 0.$

The points satisfying $|z| = 1$ lie on the circle $|z| = 1$. The points (other than $z = 0$) satisfying $y = 0$ lie on the real axis.

12. $f(z) = z^n = r^n(\cos \theta + i \sin \theta) = r^n(\cos n\theta + i \sin n\theta) \Rightarrow$

$\operatorname{Re} f(z) = r^n \cos n\theta$ and $\operatorname{Im} f(z) = r^n \sin n\theta$ are harmonic since f is analytic.

13. $\phi(z, \theta) = \operatorname{Im} Z = r^n \sin 40^\circ = -4zy + 4ry$

14. Let $\phi(z, \theta) = \operatorname{n}[f(z)] = \ln(+)$

$$\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z} \ln(+).$$

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{(v^2 - u^2) \left[\left(\frac{\partial u}{\partial x} \right)^2 - \left(\frac{\partial v}{\partial x} \right)^2 \right] - 4uv \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}}{(u^2 + v^2)^2} + \frac{u \frac{\partial^2 u}{\partial x^2} + v \frac{\partial^2 v}{\partial x^2}}{u^2 + v^2}$$

A similar calculation yields $\frac{\partial^2 \phi}{\partial y^2}$. By applying Laplace's equation and

the Cauchy-Riemann equations of u and v , the sum simplifies to reveal that $\nabla \phi = 0$.

15. Consider $\operatorname{Re} f(z) = \operatorname{Re}(Az^n + Bz^m + C)$ which is harmonic for $|z| > 0$.

Consider the polar form for z , $z = re^{i\theta}$ and select $n=3$ to agree with the cosine argument. $q(re^{i\theta}) = A\operatorname{Re}(e^{i3\theta}) + B\operatorname{Re}(e^{i3\theta}) + C$

$$q(re^{i\theta}) = A \cos 3\theta + B \cos 3\theta + C = (A+B)\cos 3\theta + C$$

$$(A+B)\cos 30^\circ + C = 0 \Rightarrow A+B=0, C=0$$

$$(A+8B/3)\cos 30^\circ = 5\cos 30^\circ, A = 40/63, B = -40/63$$

$$q(re^{i\theta}) = (40/63)(r - 1)\cos 30^\circ = (40/63) \operatorname{Re}(z - 1)$$

16. $\#+, \theta = ;, z \neq 1 \text{ or } +, v = 1$ are two possibilities.

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17. a. $\phi(z, y) = \operatorname{Re}(z + 5z + 1) = z^2 - 5z + 1$

b. $\phi(z, y) = 2\operatorname{Re}\left(\frac{z}{z+2}\right) = \frac{2z}{z+4y+4}$

18. Let $u = \phi$, $v = -\psi$. Then

$$\begin{aligned} \nabla u &= \mathbf{h}_w - \mathbf{v} \\ \frac{\partial u}{\partial z} &= \dots @ \\ 0 &= 0 \end{aligned}$$

19. $\cos \theta = \operatorname{Re}(\cos 20 + i\sin 20) = \cos(20^\circ) + i\sin(20^\circ) = A \cos 20 + B \sin 20$. In the limit as $r \rightarrow \infty$, $\cos(20^\circ) = 1$, $\sin(20^\circ) = 0$. On the circle $|z|=1$, $r=1$, $A = \cos 20^\circ$, $B = \sin 20^\circ$.

20. In order that $\frac{\partial v}{\partial z} = \operatorname{Im} tv(+, \omega) = \int \frac{\partial u}{\partial z} (n) dn + Z$, Then

$$\begin{aligned} \frac{\partial v}{\partial z} &= \int p \frac{\partial u}{\partial z} (n) dn + W(n) \\ &= \int p \frac{\partial u}{\partial z} (n) dn + v(z) \quad (\text{because } v \text{ is harmonic}) \\ &= \tilde{a}, \tilde{c}, \tilde{d}, \tilde{e}, \tilde{f}, \tilde{g}, \tilde{h}, \tilde{i}, \tilde{j}, \tilde{k}, \tilde{l}, \tilde{m}, \tilde{n}, \tilde{o}, \tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}, \tilde{t}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{x}, \tilde{y}, \tilde{z}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}, \tilde{G}, \tilde{H}, \tilde{I}, \tilde{J}, \tilde{K}, \tilde{L}, \tilde{M}, \tilde{N}, \tilde{O}, \tilde{P}, \tilde{Q}, \tilde{R}, \tilde{S}, \tilde{T}, \tilde{U}, \tilde{V}, \tilde{W}, \tilde{X}, \tilde{Y}, \tilde{Z}. \end{aligned}$$

In order that $\frac{\partial v}{\partial z} = \operatorname{Im} tv(+, \omega)$ it must be true that $w'(z) = 5^{au}(z)$. Thus,

$$w'(z) = \int p \frac{\partial u}{\partial z} (n) dn + a$$

and $v(z) = \int f_y \frac{\partial u}{\partial z} (n) dn + p \frac{\partial u}{\partial z} (n) dn + a$

$v(z) = \int \frac{\partial u}{\partial z} (n) dn - J, \int G.S.O(d, +a)$

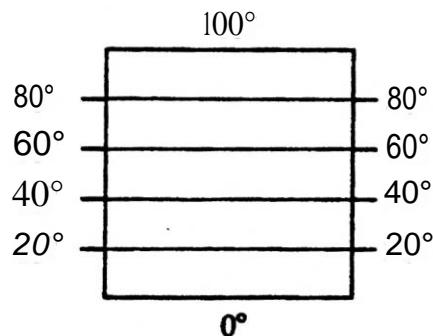
21. It is easily verified that $u = \ln|z|$ satisfies Laplace's equation on $C \setminus \{0\}$ and that $u+iv = \ln|z| + i\operatorname{Arg}(z)$ satisfies the Cauchy-Riemann equations on the domain $D = C \setminus \{\text{nonpositive real axis}\}$, so that

$\operatorname{Arg}(z)$ is a harmonic conjugate of u on D . By Problem 4, any harmonic conjugate of u has to be of the form $\operatorname{Arg}(z) + a$ in D . It is impossible to have a harmonic conjugate of this form that is continuous on $C \setminus \{0\}$.

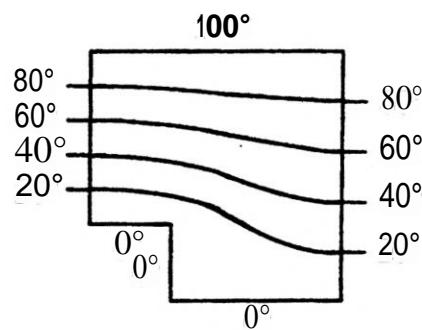
$$= -\phi_{yy}\phi_y + \phi_x\phi_{yx} - \psi_{yy}\psi_y + \psi_x\psi_{yx} = \frac{\partial v}{\partial y}$$

EXERCISES 2.6: Steady-State Temperature as a Harmonic Function.

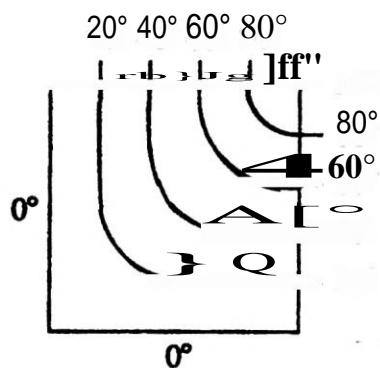
1. a.



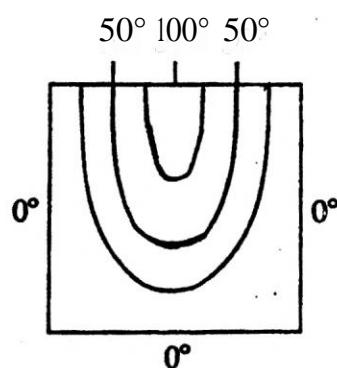
b.



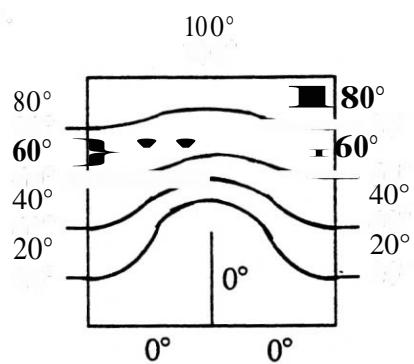
c.



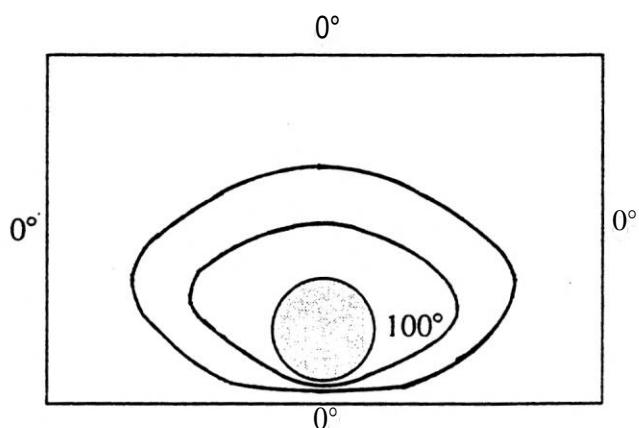
d.



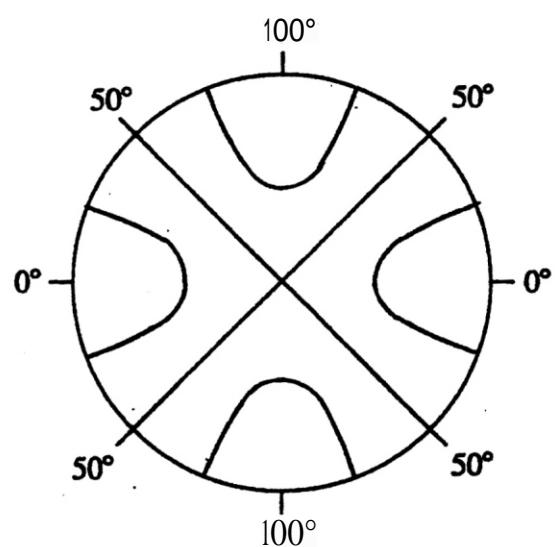
e.



2. This does not violate the maximum principle.



3. This does not violate the maximum principle.



Exercises 2.7

1. $f(z) = z + c$ where c is a real constant.
 $\zeta_1 = \zeta + V(1-40)/2, \quad \% = (1-V(1-4c))/2$
 Only ζ_2 is an attractor for $-3/4 < c < \%$.
2. $f(g) = \mathbf{a}$ nd $f(g) > 1$. Therefore we can pick a real number p between 1 and $|f(g)|$ such that $|f(z) - g| = p|z - g|$ for all z in a sufficiently small disk around g . If any point z_0 in this disk is the seed for an orbit $z_i = f(z_{i-1})$, then we have $|z_i - g| \leq p|z_{i-1} - g| \leq \dots \leq p^i|z_0 - g|$. Because $p > 1$, the point z_i moves away from g until the magnitude of the derivative becomes 1 or less. The orbit is out of the disk.
3. (a) Fixed points are $\zeta_1 = i, \zeta_2 = -i$. Both are repellors.
 (b) Fixed points are $\zeta_1 = 1/2, \zeta_2 = -1/2, \zeta_3 = -i$. Fixed points ζ_1 and ζ_2 are repellors, but fixed point ζ_3 is an attractor.
4. $z_0 = e^{i\theta}$ with θ an irrational real number. $z_n = e^{i2\pi n\theta}$. Because $|z_n| = 1$, the trajectory will follow the unit circle. If iterations p and q coincide, $210.2^\circ - 2n02^\circ = 270(2^\circ - \theta) = 2\pi k$ for some integer k . But because $(2^\circ - \theta)$ is an integer that can be represented by m , the equation $2\pi m = 2\pi k$ is satisfied only if $k = om$ or $\theta = k/m$. Because θ is irrational it cannot be represented by a rational number and no iterations repeat.
5. Fixed points are $\zeta_1 = -1/2 + i\sqrt{3}/2$ (an attractor) and $\zeta_2 = -1/2 - i\sqrt{3}/2$ (a repellor).
6. $f(z) = z^2$. The seed is z_0 . $Z_1 = z_0^2, Z_2 = z_1^2, \dots, Z_n = z_{n-1}^2$. To have an n cycle $z_n = z_0 = z_{n-1}^2$. Or $\sqrt[n]{z_0} = z_{n-1} = 1 = e^{i2\pi k/n}$. Solving gives $z_0 = e^{i2\pi k/(n-1)}$.
7. The cycle is 4. $2(2r/p) = 2n \pmod p \Rightarrow 2 = 1 \pmod p$. $p=3, 5, 15$. 3 will give repeated cycles of length 2. 5 and 15 will give the desired cycles of length 4.
8. Student Matlab: $n=100; c=.253; z0=0; y(1)=z0;$
 $\text{for } k=1:n-1, y(k+1)=y(k)^2+c; \text{end}$
 $\text{plot}(y)$
9. If $|a| < 1$ the whole complex plane is the filled Julia set. If $|a| > 1$ the origin is the filled Julia set.
10. $f(z) = z - F(z)/F'(z)$. $\mathbf{f} \mathbf{g} = -F''U/F' = -F''/F' = 0 = F(K=0)$ with the possible exception of the points where $F'(z) = 0$.
 $f(z) = I - F(z)/F'(z) + F(z)F''(z)/(F'(z)^2) = F(z)F''(z)/F'(z)^2$
 $f(z) = F([F''(z)/F'(z)]) = 0$ where $F'(z) \neq 0$ and every zero of $F(z)$ is an attractor as long as $F'(z) \neq 0$.