Solution Manual for Fundamentals of Complex Analysis with Applications to Engineering and Science 3rd Edition by Saff Snider ISBN 0134689488 9780134689487

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CHAPTER 2: Analytic Functions

EXERCISES 2.1: Functions of a Complex Variable

1. a. $w = (3az --3y + 5 +1) + i(6zy + 5y +1)$

•–et(˅)

cR $\frac{1}{z-i}$ $z+(y-1)$. $\frac{-y+1}{z?+(y--1)}$

d. $\frac{2az--2+3}{\cdot''7f-F+}$. $\frac{4zy}{@-T+7}$

e. $w = e'' \cos 3y + ie \sin 3y$

f. $w = (e + e)\cos y + i(e -- e'')\sin y$
$= 2\cosh z \cos y + i2\sinh a \sin y$

2. a. C
b. C\ {0}
c. C\{i,–i}
d. c\{1}
e. C
f. C

3. a. Rew>5
b. Imw>0
c. |w|2> 1
d. The intersecti on of [o] < 2 and –– < Argw < ⁄2

4. a. Taking 6 from 0 to 2a, the points $z = re''$ traverse the circle

$|z| = r$ exactly once in the counterclockwise direction. For the same values of θ the points $w = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$ traverse the circle $|w| = \frac{1}{r}$ exactly once in the clockwise direction, hence the mapping is onto.

b. For $z = re^{i\theta_0}$ on the ray $\operatorname{Arg} z = \theta_0$, $w = \frac{1}{re^{i\theta_0}} = \frac{1}{r}e^{-i\theta_0}$ is on the ray $\operatorname{Arg} w = -\theta_0$. Taking values $0 < r < \infty$ shows that this mapping goes onto the ray $\operatorname{Arg} w = -\theta_0$.

4 («) 1-1=1 2 020=z=1+e. F) = 1/. = 1I1+e''

=(I+e$^{-i\theta}$)/{2(1+c0s0)} =V -i(%)sin0/(1+cos0)

which is a vertical line at x = /.

5. a. domain: C

range: C \ {0}

b. $f(-z) = e^{z} = \frac{1}{e^{\circ}} = \frac{1}{f(z)}$

c. circle |w| = e

d. ray Arg$w$ = $n/4$

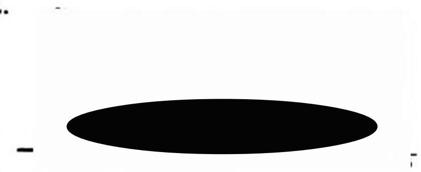
e. infinite sector 0 < Arg$w$ < $n/4$

6. a. (3)-!(+},)-½(+)-n)

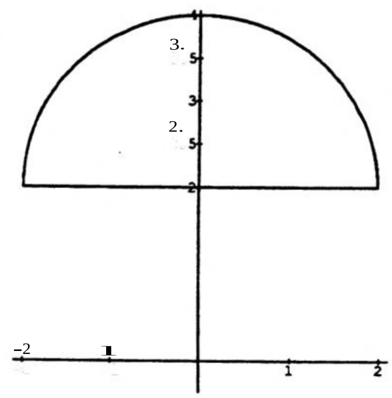
b. For $z = e''$ on the unit circle |z|=1, ()=5;(''+;)= cos0. For all values of 0, this ranges over the real interval [−1, 1].

c. For $z = re''$ on the circle |z| = r, $J()$ = (re''+1/r)= (r+,)cos$o$++,(−;)So. set»g and v equal to the real and imaginary parts of this expression, respectively, one gets a pair of parametric equations that are equivalent to the ellipse [3⁄r+] 7 [EK −?j 1, which has foci at ±1.

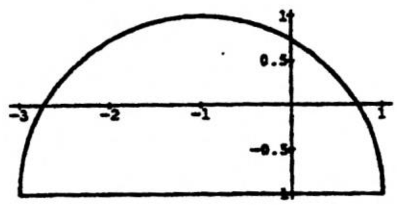
7. a.



b.

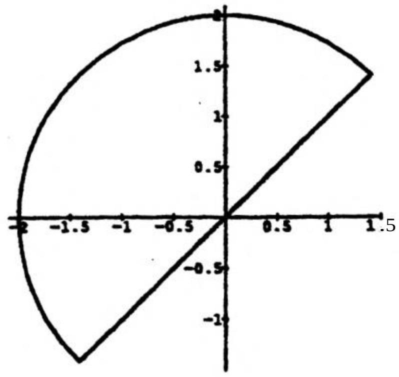


c.



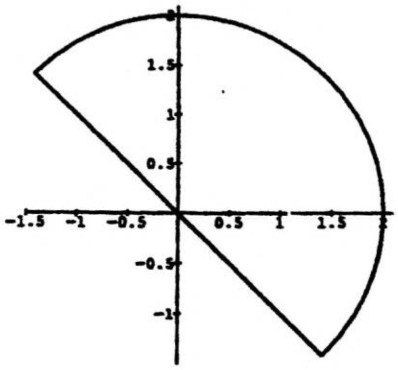
1
0.5
-3
-2
-1
1
-0.5
-1

8. a.



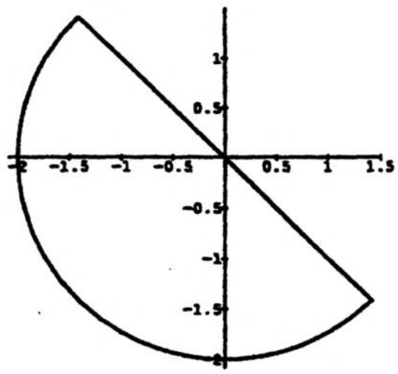
2
1.5
1
0.5
-2
-1.5
-1
-0.5
0.5
1
1.5
-0.5
-1

b.



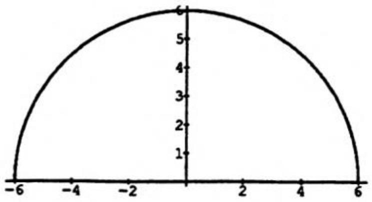
2
1.5
1
0.5
-1.5
-1
-0.5
0.5
1
1.5
2
-0.5
-1

c.

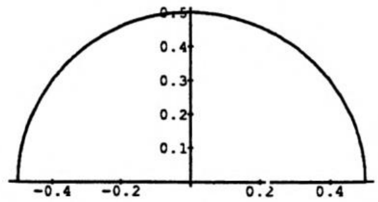


1
0.5
-2
-1.5
-1
-0.5
0.5
1
1.5
-0.5
-1
-1.5
-2

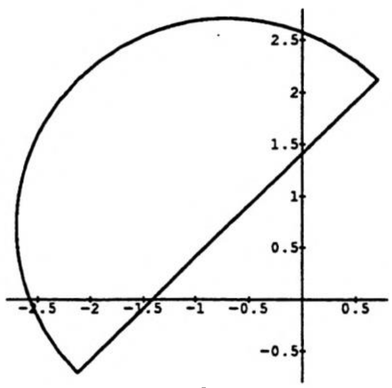
9. a.



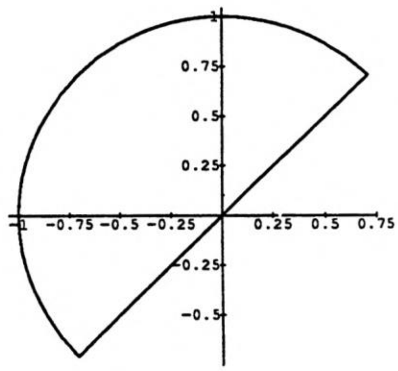
b.



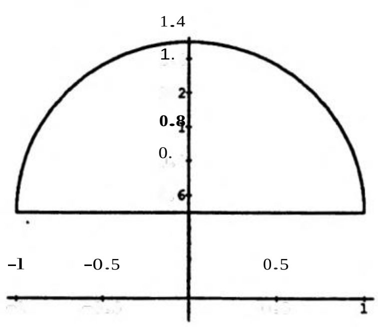
10. a. translate by *i*, rotate n/4



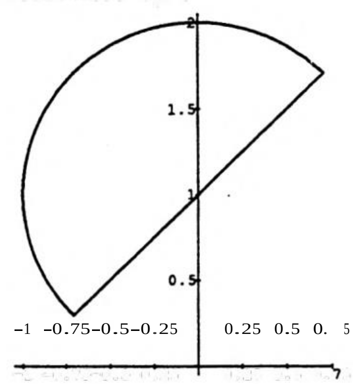
b. reduce by 1/2, rotate z/4



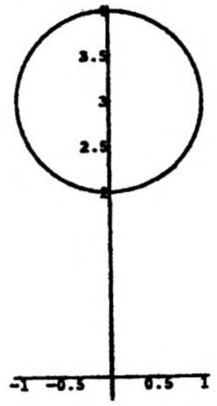
c. translate by *i*, reduce by 1/2



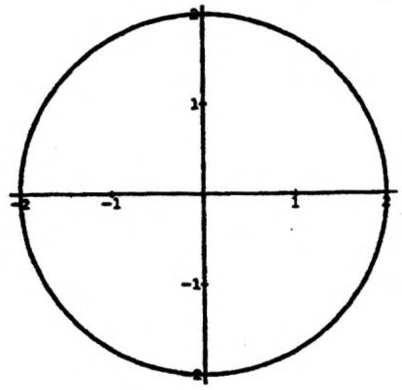
d. reduce by 1/2, rotate z/4, translate by *i*



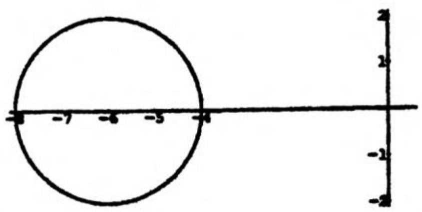
11. **a. translate by --3, rotate --/2**



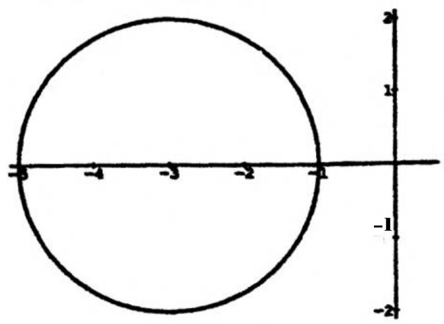
b.magnify by 2, rotate --a/2



c. translate by --3, magnify by 2



d. magnify by 2, rotate --/2, translate by --3



12. **Let *a* = *pe*, F(z) = *pz*, G(z) = ez, and H(z) = *z* + *b*. Then H(G(F(z)))= a + *b*.**

13. (a) w=u+iv=z =(1 +iy)=1−y+i2y
u=1-y,v=2y=y=v/2=u=1-v14 a parabola in the w-plane.
(b) w=u+iv=z' =(x +iy) = (x +i/) =x _ 1/+ 2i
u =x--1/2, v = 2 a straight line in thew-plane.
(e) w=u+iv=z =(+e)=(1 +2e +e)=(e+2+e)e
=(2+2cos0)e" = 2(1 + cos@)e" a cardioid in the w-plane.

14. (a) X =2x/(|zl + 1), x= 2y/(lzl + 1), X = (lzl - 1) /(zl + 1)
w=e"z = xcos−ysin + i(xsin+ycosq), wl = lzl
X = (xcos−ysin@)/(lzl' + 1), X = (xsin+ycosq)/(lzl + 1), Xx3 = Xt3
X1 = (xcos(-x2sin), X2 = (xsin+x2cos(), X3 = x3 which corresponds to a rotation of an angle «p about the x3 axis.
(b) w = -1/z. |wl = 1/lzl. w = -1/(x+iy) = -x/lzl + iy/lzl
X1=-X1,X2=x2, X=-x3 so that (x1, X2,) is obtained from (X, X2, X3) by a 180° rotation about the x2 axis.

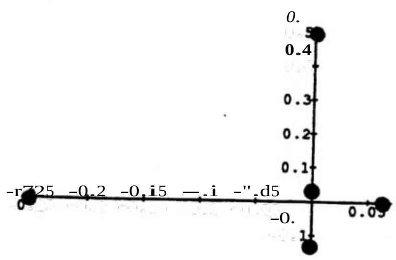
15. w=(l+2)/(1−z)=(I+x +iy)/(1−x−iy)= (1-|zl +i2y)/1−2x + |zl)
wf = (1 +2x +1/)/(1−2x + lz).
(1, 42, X3) = (-x3, 2, x) so that (x1, X, X) is obtained by a 90° counterclockwise rotation about the x2 axis.

16. $w = (1 - iz)/(1 + iz) = (1 - ix + y)/(1 + ix - y) = (1 - |z| + i2x)/(1 - 2y + |z|)$
$w\bar{} = (1 + 2y + |z|)/(1 - 2y + |z|)$.
$(\xi_1, \xi_2, \xi_3) = (-x_3, -x_1, x_2)$ so that (ξ_1, ξ_2, ξ_3) is obtained as a 90° counterclockwise rotation about the X axis followed by a 90° counterclockwise rotation about the x_3 axis.

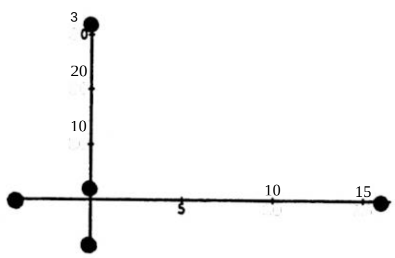
17. Any circle or line in the z-plane corresponds to a line or circle on the stenographic projection onto he Riemann sphere. The function w=1/z rotates the Riemann sphere 180° about the x_1 axis. Lines and circles on the rotated sphere project to lines and circles in the w-plane. As a result lines and circles in the z-plane map to lines and circles in the w-plane.

EXERCISES 2.2: Limits and Continuity

1. The first five terms are, respectively, $\frac{i}{2}$, $-\frac{1}{4}$, $-\frac{i}{8}$, $\frac{1}{16}$ and $\frac{i}{32}$. The sequence converges to 0 in a spiral-like fashion.



2. 2i, −4, −8i, 16, 32i; divergent because terms grow in modulus without bound.



3. Iflim,>>z,=zo, then for any e>0, there is an integer N such that
lz, - zok**e** for all n>N. For the same integer N we have
lx,- xol<=lz, - zok<e and ly, - yok<=lz, - zok<e for all n>N. Therefore,
limn>>. X%=xo and lim»% Y = Yo.

If limns.X%= xo and limn>> J = yo, then for any &; >O and € >O there are integers N and N such
lx,- xol<e; for all n > N, and ly,- yok**e** for all n > N. Given any e > 0; let e; = €/2 and € = e/2. Then
Z,- zok<=e lx,- xol + ly,- yol<€; + € = & for all n > maximum(N,, N»).
Thus limn>> Z, = Zo.

4. Ifz,=x+iy,—zo=xo+ iyo, then x, — xo and y, — yo (see Problem 3).
ZR =X,-iy,—xo- iyo=Zo.

If z,= X,-iy —Zo=xo-iyo, then X — xoand y, — yo(see Problem 3)..
Z,=X+iy,—xo+iyo=zo. Thus z, — zo if and only if Z — Zo.

5 Lig █ l=**0 ==> There exists an integer *N* such that**
llz,] --0] = lz,] ≤ ***e* whenever *n*** > ***N***. => lZ, **--0**] < *e* **whenever**
n>***N***. =» lim,»0, **=0, and conversely.**

€ z,-»0as *n* -» oo by problem 3, since the real-valued sequence
[z3] -» 0 as *n* -+» oo. On the other hand, if]zo] > 1, then *[z]* -» oo as
n -+»oo so *Z* diverges.

7 a. converges to 0

b. does not converge

c. converges to

d. converges to2 +*i*

e. converges to 0

f. does not converge

8. Given *e* > 0, choose *6* = *e/6*. Then whenever
0<*lz*—(1+i)] <*6*,

[6z −4--(2+6i)]=*6]z* --(1 +i)] <6(e/6)=*e*

9, Given *e* > 0, choose *6* = $\frac{t7}{1+¢}$. Whenever 0 < *]z* -- (--i)] < *6* notice
that *[z]*>1--*6* and

$$\left|\frac{1}{z}-i\right| = \left|\left(-\frac{i}{z}\right)(i+z)\right| = \frac{1}{|z|}|z-(-i)| < \left(\frac{1}{1-\delta}\right)\delta = \varepsilon$$

O. **Given that f and *g* are continuous at *z*,**

$\lim_{z \to z_0} f(z) \pm g(z) = \lim_{z \to z_0} f(z) \pm \lim_{z \to z_0} g(z) = f(z_0) \pm g(z_0)$

=> $f(z) \pm g(z)$ **is continuous at *z*.**

$\lim_{z \to z_0} f(z)g(z) = \lim_{z \to z_0} f(z) \lim_{z \to z_0} g(z) = f(z_0)g(z_0)$

=> **f(z)g(z) is continuous at z.**

$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)} = \frac{f(z_0)}{g(z_0)}$ provided $g(z_0) \neq 0$

=> $\frac{f(z)}{g(z)}$ is continuous at z_0

fl, a. --8i

b. $\frac{7}{5}$

c. **6i**

d. −1/2

e. 2z%

f. 45

IR. Clearly Arg *z* is discontinuous at $z = 0$. Let $a > 0$ be any real number and consider the sequence

$$z_n = -a - i/n \quad n = 1, 2, \ldots, \text{ which converges to } -a.$$

For each *n*, $-\pi < \operatorname{Arg} z_n < -\pi/2$, but $\operatorname{Arg}(-a) = \pi$.

13. $\lim_{z \to z_0} f(z)$ exists for all $z \neq$ [illegible]; *f* is continuous for all $z \neq 0, -1$; *f* has a removable discontinuity at $z = 0$.

14. Let z_0 be any complex number. Given $\epsilon > 0$ choose $\delta = \epsilon$. Then whenever $|z - z_0| < \delta$,

$$|g(z) - g(z_0)| = |\bar{z} - \bar{z}_0| = |\overline{z - z_0}| = |z - z_0| < \epsilon.$$

15. Given $\epsilon > 0$ choose δ so that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$. Then, whenever $|z - z_0| < \delta$:

a. $|\overline{f(z)} - \overline{f(z_0)}| = |\overline{f(z) - f(z_0)}| = |f(z) - f(z_0)| < \epsilon$

b. $|\operatorname{Re} f(z) - \operatorname{Re} f(z_0)| = |\operatorname{Re}(f(z) - f(z_0))| < |f(z) - f(z_0)| < \epsilon$

c. $|\operatorname{Im} f(z) - \operatorname{Im} f(z_0)| = |\operatorname{Im}(f(z) - f(z_0))| < |f(z) - f(z_0)| < \epsilon$

d. $||f(z)| - |f(z_0)|| < |f(z) - f(z_0)| < \epsilon$

16. Given $\epsilon > 0$, choose $\delta > 0$ such that $|f(g(z)) - f(g(z_0))| < \epsilon$ whenever $|g(z) - g(z_0)| < \delta$. Now choose $\delta > 0$ such that $|g(z) - g(z_0)| < \delta$ whenever $|z - z_0| < \delta$. Then $|f(g(z)) - f(g(z_0))| < \epsilon$ whenever $|z - z_0| < \delta$; hence $f(g(z))$ is continuous at z_0.

11. No: Observe that although $\frac{1}{n} \to 0$ and $\frac{i}{n} \to 0$ as $n \to \infty$,

1(;)−1+2id1(;)-·2 «ha f(do= not a#wt.

18. If $\lim_{z\to z_0} f(z) = w_0$, then given $\epsilon>0$ there exists $\delta>0$ such that $|f(z)-w_0|<\epsilon$ for all $|z-z_0|<\delta$. Notice that $|\overline{f(z)}-\overline{w_0}| = |\overline{f(z)-w_0}| = |f(z)-w_0|<\epsilon$ for all $|z-z_0|<\delta$. So that $\lim_{z\to z_0}\overline{f(z)} = \overline{w_0}$.
$\lim_{x\to x_0, y\to y_0} A(x,y) = \lim_{z\to z_0} ((f(z)+\overline{f(z)})/2) = (w_0+\overline{w_0})/2 = U_0$.
$\lim_{x\to x_0, y\to y_0} B(x,y) = \lim_{z\to z_0} ((f(z)-\overline{f(z)})/2i) = (w_0-\overline{w_0})/2i = V_0$.
Thus, $\lim_{x\to x_0, y\to y_0} A(x,y) = U_0$ and $\lim_{x\to x_0, y\to y_0} B(x,y) = V_0$.

Conversely, if $\lim_{x\to x_0, y\to y_0} A(x,y) = U_0$ and $\lim_{x\to x_0, y\to y_0} B(x,y) = V_0$, then (by Theorem I.) $U_0 + iV_0 = \lim_{x\to x_0, y\to y_0} A(x,y) + i\lim_{x\to x_0, y\to y_0} B(x,y) = \lim_{z\to z_0} ((f(z)+\overline{f(z)})/2) + \lim_{z\to z_0} ((f(z)-\overline{f(z)})/2) = \lim_{z\to z_0} f(z) = w_0$.
Also $U_0 - iV_0 = \lim_{x\to x_0, y\to y_0} A(x,y) - i\lim_{x\to x_0, y\to y_0} B(x,y) = \lim_{z\to z_0} ((f(z)+\overline{f(z)})/2) - \lim_{z\to z_0} ((f(z)-\overline{f(z)})/2) = \lim_{z\to z_0} \overline{f(z)} = \overline{w_0}$.

Thus, $\lim_{z\to z_0} f(z) = w_0$.

19. $5-i$, since $\lim_{z\to i} \frac{5}{z^2} = -5$ and $\lim_{z\to i} zy = -1$.

20. For any z in the complex plane,

$$\lim e^z = \lim e^x \cos y + i \lim e^x \sin y = e^{x_0}\cos y_0 + i e^{x_0}\sin y_0 = e^{z_0}$$

21. a. 1
b. 0
c. $-\pi/2+i$
d. 1

22. By contradiction: Suppose $\lim_{z\to z_0} f(z) \neq w_0$. Then there is an $\epsilon > 0$ for which there exists a sequence $\{z_n\}$ such that $|z_n - z| < \frac{1}{n}$ but $|f(z_n) - w_0| > \epsilon$. For this sequence, $\lim_{n\to\infty} z_n = z$ but $\lim_{n\to\infty} f(z_n) \neq w_0$, contrary to hypothesis.

23. If z,-» , then for any M>0 there exist an integer N such |z,| >M for al n > N. Consider the chordal distance ((z#.s) = 2//(|z,| + 1) < 2//(|z,|') = 2/|z,| < 2/M < e for all n > N. Thus z,— ∞ as n -» • is equivalent to ((Z,,O) — 0 as n — •

24. If lim,sf(z) =e, then for any M>0 there exis ˜ > 0 such that |f(z)| >M for all |z-z6| < 6. Consider (f(z),s) = 2/Nf(z)f + 1) < 2//(f(z)f) = 2/|f(z)| < 2/M < & for all |z-za| < 6. Thus lim,»,f(z) = - , is equivalent to lim,».y(f(z),s) = 0.

25. (a) oo (b) 3 (c) oo (d) ∞ (f) the limit does not exist.

EXERCISES 2.3: Analyticity

1. Let lAz =z—z0 so that Az -+0 ⇔ z -»z6. Then

$$\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=L\iff$$

given e > 0, there is a 6 > 0 such that

f(z0 +Az) – f(%) / lz —L] <e whenever]/z—0] <6 <%>

f(z)– f(o) / z—z0 —[j < ewhenever]z --z0] < 4>

A()–f() / z-+z0 2 —2Z _L.

2 If '( )AuZ = f(z) = f(z0) / z—z0 —f'(z), then (z) -+O as z -» z0 and f(z0) +f(a)(z – z) + (z)(z – z0)= f(z).

3. 9(=)= lm[f)+1O(– +0) +(=)(—)] =f(z6) +0+0= f(z6).

4. a. $\lim_{\Delta z \to 0} \frac{\text{Re}(z+\Delta z) - \text{Re}(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\text{Re}(\Delta z)}{\Delta z} = \begin{cases} 1, & \text{if } \Delta z = \Delta x \\ 0, & \text{if } \Delta z = i\Delta y \end{cases}$

b. $\lim_{\Delta z \to 0} \frac{\text{Im}(z+\Delta z) - \text{Im}(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\text{Im}(\Delta z)}{\Delta z} = \begin{cases} 0, & \text{if } \Delta z = \Delta x \\ -i, & \text{if } \Delta z = i\Delta y \end{cases}$

c. Case 1, $z=0$.

$\lim_{\Delta z \to 0} \frac{|0+\Delta z| - |0|}{\Delta z} = \lim_{\Delta z \to 0} \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta x + i\Delta y} = \begin{cases} \pm 1, & \text{if } \Delta z = \Delta x \\ \mp i, & \text{if } \Delta z = \pm i\Delta y \end{cases}$

Case 2, $z \neq 0$.

$\lim_{\Delta z \to 0} \frac{|z+\Delta z| - |z|}{\Delta z}$

$= \lim_{\Delta z \to 0} \frac{\sqrt{(x+\Delta x)^2 + (y+\Delta y)^2} - \sqrt{x^2+y^2}}{\Delta x + i\Delta y}$

$= \lim \frac{(x+\Delta x)^2 + (y+\Delta y)^2 - (x^2+y^2)}{(\Delta x + i\Delta y)\left(\sqrt{(x+\Delta x)^2 + (y+\Delta y)^2} + \sqrt{x^2+y^2}\right)}$

$= \lim \frac{2x\Delta x + (\Delta x)^2 + 2y\Delta y + (\Delta y)^2}{(\Delta x + i\Delta y)\left(\sqrt{(x+\Delta x)^2 + (y+\Delta y)^2} + \sqrt{x^2+y^2}\right)}$

$= \begin{cases} \frac{x}{\sqrt{x^2+y^2}}, & \text{if } \Delta z = \Delta x, z \neq 0 \\ \frac{-iy}{\sqrt{x^2+y^2}}, & \text{if } \Delta z = i\Delta y, z \neq 0 \end{cases}$

5. Rule 5: $(f \pm g)'(z_0) = \lim_{\Delta z \to 0} \frac{(f \pm g)(z_0+\Delta z) - (f \pm g)(z_0)}{\Delta z}$

$= \lim_{\Delta z \to 0} \left[\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \pm \frac{g(z_0+\Delta z) - g(z_0)}{\Delta z}\right]$

$= f'(z_0) \pm g'(z_0)$

Rule 7: $(fg)'(z_0) = \lim_{\Delta z \to 0} \frac{fg(z_0+\Delta z) - fg(z_0)}{\Delta z}$

$$= \lim_{Az\to o} \left\{ \frac{f(z_0 + .6.z)g(z_0 + Az) - f(z_0 + .6.z)g(z_0)}{Lz} \right.$$

$$+ \frac{f(+A)()\,f(ls()]}{Az}$$

$$= \lim_{4z\to o} \left\{ f(z_0 + .6.z) \frac{[g(z_0 + .6.z) - g(z_0)]}{L\backslash z} \right.$$

$$+ g(z_0) \frac{[f(z_0 + .6.z) - f(z_0)]\}}{Lz}$$

$$= f(o)g'(z) + g(z_0)f'(o)$$

6. Let $n > 0$ be an integer.

   Then ,+Z'' = ;;g,(;) = — (using Rule 8) $= -nz^{-n-1}$.

7. a. $18z + 16z + i$

   b. $-12z(z - 3i)^{-1}$

   c. $$\frac{-iz + (2+27i)z + 2z + 18}{(iz + 2z + 7)}$$

   d. $$\frac{-(z+2)(5z + (16 + i)z - 3 + 8i)}{(z^2 + iz + 1)^5}$$

   e. $24i(z^{\circ} - 1)(z + i2)(532' + 28iz^{\circ} - 50z - 25i)$

8. Let $z = z_0 + Az$. Then

$$\lim_{z\to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = \left| \lim_{z\to o} \frac{f(z_0 + .6.z) - f(z_0)}{Lz} \right| = |f'(o)|,$$

lim en)- t» -'t- al = aa["l_""]

$$\arg\left[\lim_{Az\to o} \frac{f(z_0 + .6.z) - f(z_0)}{L\backslash z}\right] = \arg[f'(z_0)]$$

9. a. $2-3i$

b. $\pm i$

c. $\dfrac{-1\pm i/15}{2}$

d$\frac{1}{5}$1

10. $\lim_{Ar\to 0} \dfrac{|Z_0 + Az|^2 - |Z_0|^2}{\Delta}$ .2

$= \lim_{r\to 0} \dfrac{(@+A2)(\%+Zz)-z\%z\%}{Lz}$

$= e_{A\to 0}(+\{+s)-\{$ $+z_0$ if $Az = Ar$; $2O\ 20$ if $Az = iZy$

If $z\% = 0$, then the difference quotient is

$$\lim_{A.r\to 0}(0+0+2) = 0.$$

11. a. nowhere analytic

b. nowhere analytic

c. analytic except at $z = 5$

d. everywhere analytic

e. nowhere analytic

f. analytic except at $z = 0$

g. nowhere analytic

h. **nowhere** analytic

12. The case when n == 1 is trivial. Assume that the result holds for all positive integers less than or equal to n and define $Q(z) = P(z)(z-++)$. Since $Q'(z) = P'(z)(z-++)+P(z)$, it follows that

$$\frac{O'(z)}{Q(z)} = \frac{P'(z)}{P(z)} + \frac{1}{z-\,\text{"}\_\,+1} = \frac{1}{2-Z} + \frac{1}{2-Z} + \ldots + \frac{1}{2-Zn+1}$$

13. a, b, d, f, and g are always true

14. li $\frac{f(z)}{g(z)}$ ... $\frac{[f(z)-f(z_0)]/(z-z_0)}{[g(z)-g(z_0)]/(z-z_0)}$ ... $\frac{f'(z_0)}{g'(z_0)}$

15. 2

16. Any point on the line through z and z has the form $z = \frac{1}{\ }+ i83(\ -t), t$ real (see Section 1.3, Exercise 18). However, $f(z) - f(z) = 0$ but $f'(w) = 3 \neq 0$ on the line in question.

17. $F'(z_0) = f(z_0)(gh)'(z_0) + f(z)gh(z_0)$
$= f(O)[q(z_0)i'(z_0) + g'(z)h(z)] + f'(z_0)q(z_0)h(z_0)$
$= f'(o)(z_0)h(z_0) + f(z_0)g'(z)h(z_0) + f(z_0)g(z_0)h'(z_0)$

EXERCISES 2.4: The Cauchy-Riemann Equations

1. $\frac{\partial u}{\partial x} = \ \# \ \frac{\partial v}{\partial y} = 1$

b. $\frac{\partial u}{\partial x} = \ \frac{\partial v}{\partial y} = 0$

c. $\frac{\partial u}{\partial x} = \ \# \ \frac{\partial v}{\partial y} =$

2. $= 3x^2 + 3y^2 - 3 = \frac{\partial v}{\partial y}$ but $\frac{\partial u}{\partial y} = 6xy = \frac{\partial v}{\partial x}$. Therefore $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ only when $= 0$ or $= 0$. This means h is differentiable on the axes but h is nowhere analytic since lines are not open sets in the complex plane.

3. $\frac{\partial u}{\partial x} = 6x + 2 = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -6y = -\frac{\partial v}{\partial x}$. Since these partial derivatives exist and are continuous for all x and y, g is analytic. g can be written as $g(z) = 3z^2 + 2z - 1$.

2–1/

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{(A.0 - ,0}{\Delta a} \ldots$$

$$\frac{?}{\partial y} = \lim_{\Delta y \to 0} \frac{(0,\Delta y) - u(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \ldots$$

s a w $; − = $; − 0

However, when $\Delta z \to 0$ through real values ($\Delta z = \Delta a$)

$$\lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} = \ldots$$

while along the real line $y = x$ ($\Delta z = \Delta a + i\Delta r$)

$$\lim_{\Delta r \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} = \lim_{\Delta r \to 0} \frac{(\Delta)''(\Delta \pm (\Delta \pm \Delta s!)''}{\Delta r(1+i)} = \frac{1}{2}.$$

Therefore f is not differentiable at $z = 0$.

. '';- - ''. tr»-v=»=f;

'';—w''@a+=ar0—f;

f is entire because these first partials exist and are continuous for all x and y.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2 \cdot ?(x+iy)[\cos(2xy) + i\sin(2xy)]$$
$$= 2\% \ldots (x+iy)$$
$$= 2z$$

(This derivative could have been obtained directly, since $f(z) = e$.)

6. $z = re \Rightarrow x = r\cos\theta$ and $y = r\sin\theta$ and

$$f(z) = u(x(r,\theta), y(r,\theta)) + iv(x(r,\theta), y(r,\theta))$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \ldots \frac{\partial u}{\partial x} \ldots \frac{\partial y}{\partial r}$$

?2−\S

Similar applications of the chain rule yield

$$\frac{O}{O} a \frac{\partial u}{\bullet}{}'-\text{"sia)} + \frac{O}{7}{}' cs\,\theta$$

$$\frac{\sim v}{or} = \frac{\sim v}{0a} cos(4\ \frac{0v}{0y} \sin\theta$$

$$\frac{0v}{00} = \frac{0v}{0z}(-r\sin) + \frac{\sim v}{\sim y}\, cos$$

Replace the partial derivatives on the right sides of the equations for $\frac{|N}{\bullet}$ and $:\sim$ by their Cauchy-Riemann counterparts to obtain:

$$\frac{\sim}{0r} - \frac{\sim}{\sim y} cos(J\ \frac{\sim}{0z} \sin t] = \frac{1 \sim v}{r\ 00}$$

$$\frac{\sim v}{or} - \frac{}{\sim y} -cost)4- \frac{O}{O}\sin t\} = -\frac{1 \sim}{r\ 00}$$

7. Let $h(z) = f(z) - g(z)$. Then his analytic in D and $h'(z) = 0$ so his a constant function.

$$h(z) = c = f(z) - g(z) \Rightarrow f(z) = g(z) + c$$

8. $(z,) = c$ in $D =\ \frac{\sim}{\%} = 0$ and $\frac{\sim}{Z,} = {}^\circ = \frac{}{7V}$. Hence

s

$rt - \{ + \{";-oso r$ auat» D.

9. By contradiction. If f is analytic in a domain D then $v(\alpha z, y) = 0$ (a constant) => f is constant (by condition 8) => u is constant.

0. $Im f(z) = 0$ in $D \Rightarrow \frac{0V}{} = \frac{\sim}{} = 0 \Rightarrow \frac{\sim}{} = 0$

1 (However, there is no open set in which $u(z,y) = |z - z|$ is constant). $\frac{}{0z}\ \frac{}{\sim y}\ \frac{}{0z}$

=> $!'(\ )\ z\ / \frac{0U}{7,}\ 7 \cdot \frac{aV}{7a} = 0 \Rightarrow f$ is constant in D.

11. $Re f(z) = \frac{1}{;}[f(z) + f(z)]$ is real valued and analytic if both f and f are analytic. Hence $Re f(z)$ is constant by Exercise 10. It follows that $f(z)$ is constant by Exercise 8.

−1%

12. [f(z)] constant in D =» [f(z)f = u+ V is constant in D. If = 0 or v = 0 in D, then f is constant by Exercises 8 and 10. Otherwise,

0l5 = 0u av
ℤu 42Z =0
0z = ∂u —
au7 0z 0z

~V ~V ~U
Z−−+Zv =−−Zuu +Zv =[
0y ~y ~y −− −−

— −x = ∂v/∂y 0z 0z

— (1/2) y 0\5/0z −tu (1/2) 0\f/0y all=(tut[a]··)−· 0v/0z

~V ~
—a. = = %, o

rt−5;−»

=> f is constant in D.

13. l/(z)l is analytic and real-valued, so the result follows from Exercises 10 and 12.

14. If the line is vertical then Re f(z) is constant and this reduces to Problem 8. If the line is not vertical, then $v(z,y)$ = mu(z,y) + b, and

~
— — zfl 0∂u f0 ∂u + · ∂v —
0z 0z 0y' z

~V 0u 0 ,~v
v ''% ''a ' '%

It follows that

@ @ 0 ~~r %
0 0· 0± 0 ⊄ '0 ''

Hence f(z) is constant.

15. $J(o, J_0)$ = $\frac{bu}{0z}\frac{\sim v}{0y} - \frac{0u}{0y}\frac{v}{0r}\Big|_{\bullet a}$

$$-\left[z:»\right]' \cdot \left[a»\right]$$

$= |f'(a)]$ (sing Equation (1))

16. a. $\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$

-(g e\ (g. 2)y!

− 7. 7 5 ∂y a 2

$-\frac{1}{2}\left(\frac{\partial z}{\partial z} \cdot \frac{\sim}{y}\right) \cdot \frac{1}{2}\left(g - \frac{z}{\partial z}\right)$

$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$

-(z4(p)te

0z 0z 2 + i 0y 21

− 2 0z 0y − 2 (g 0y + $\frac{\partial v}{\partial}$)

b. $\frac{\partial \tilde{f}}{\partial \xi}$ =0so=5d. a and9=5 ± 50 %

~

$^\circ$a2 ′′% $= -\frac{\partial v}{\partial x}$

EXERCISES 2.5: Harmonic Functions

1. a. $u(x,y) = x - y + 2x + \ldots$ $\frac{\partial^2 u}{\partial x^2} = ?= 7 \frac{\partial^2 u}{\partial y^2} = \Delta u = 0$

$v(x,y) = 2y + ?$ $\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2}$ $\Delta v = 0$

b. $u(x,y) = \frac{x}{x^2+y^2}$ $\frac{\partial^2 u}{\partial x^2} = \frac{2x(x^2-3y^2)}{(x^2+y^2)^3} = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \Delta u = 0$

$v(x,y) = -\frac{y}{x^2+y^2}$ $\frac{\partial^2 v}{\partial x^2} = \frac{@?}{(x^2+y^2)^3} = -\frac{\partial^2 v}{\partial y^2} \Rightarrow \Delta v = 0$

c. $u(x,y) = e^x \cos y$ $\frac{\partial^2 u}{\partial x^2} = e^x \cos y = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \Delta u = 0$

$v(x,y) = e^x \sin y$ $\frac{\partial^2 v}{\partial x^2} = e^x \sin y = -\frac{\partial^2 v}{\partial y^2} \Rightarrow \Delta v = 0$

2. $h(x,y) = a + by - ay$

3. a. $u = \text{Re}(-iz)$, $v = -a + a$, where a is a constant

b. $= \text{Re}(-ie)$, $v = -e \cos y + a$

c. $u = \text{Re}(\ldots -iz - z)$, $v = -5(\ldots - a - (r + y)) + a$

d. It is straightforward to verify that $\Delta u = 0$.

$\cos z \cosh y$

$\Rightarrow v(z,y) = \int \cos z \cosh y \, dy = \cos z \sinh y + (az)$

$\ldots = \sinh y = \ldots = \sin w + wt) = () = \ldots$

Thus, $v(z,y) = \cos z \sinh y + a$.

$\frac{\partial u}{\ldots} = \ldots \Rightarrow$

e. It is straightforward to verify that $\Delta u = 0$.

$\ldots + \alpha$

$\ldots - W'(z) = z) = a$

Thus, $\ldots + a$.

f. $u = \text{Re}(-ie)$, $v = e \ldots \cos(2ay) + a$.

4. Suppose v and w are both harmonic conjugates of u, and consider $d(z,y) = w(z,y) - v(z,y)$. Then (using the Cauchy-Riemann equations for v and w),

$$-\#23(3)\bullet$$

and similarly $= 0$. Hence $(z,y) = a$, from which it follows that

$$\text{to}(z,y) = v(z,y) + a.$$

5. If $f(z) = u(z,y) + iv(z,y)$ is analytic then $-if(z) = v(z,y) - iu(z,y)$ is analytic. Thus $-u$ is a harmonic conjugate of v.

6. Since $f(z) = u + iv$ is analytic, $\frac{}{2}$ [f(2) = i(?)+iv is analytic.

Thus v = Im; [f(z)]' is harmonic.

7. ¢(,y)=-+-1

8. a. Yes, because ⁄(u + v) = Au + Lv = 0.

b. No. Take u = , = -y as an example.

c. Yes, because ⁄(u,) =%±±±t± =%±±tz

=0~(Au) = 0/O(0) = 0.

9. «kn)==—1(is is I(±--·))

10. Let $x = r\cos 6$ and $y = r\sin 0$.

g _0gO O@_%%.0• o .O
or O 0r ~y 0r 0a ~y
O¢ O40a O% 0y
O - a, '''' a0. 7,8
O¢ Z , O¢0y .
@.0 a; i a,@

- O¢ o o▪▪ O¢ , O¢ %O
7.z %a.3s@coS+ a,s

0¢ 0¢0z 0¢O 0¢ . 0¢
- —--- + = (--rs nt) I rcost
00 O00 — — i --
0yO0 0r ~y

00 - a. a!''sin@ a,0.a0!" n@)+ ;7'–"cos)
O¢ O¢0¢ . O¢ 0y . 0¢
si

a 0¢ O ! @ O¢0y a" cS+ a(–"sin)
aO 0a cs + a7

@4 , , @@ a..
a. a,a n0cos@ a4 ! osO

, - __,s0) 4 ~ r s).

sin? ?''si c

x^2

0z u_y

sin θ +

Combining these partial derivatives, one gets

4,1@4,1%_4,S_ .
Or ror rO 0z? O

11. Im f(z) = v ——— = o = yaw + — y = y(z + ° — 1) = 0.
x²+y

The points satisfying Z + y --1 = 0 lie on the circle [z] = 1. The points (other than z = 0) satisfying y = 0 lie on the real axis.

12. f(z) = z" = r"(cos0 + i sin0) = r"(cos n6 + i sin n6) =»
Re f(z) = r"cos n and Im f(z) = r" sin n6 are harmonic since f is analytic.

13. ¢(z,) = Im Z = r"sin40 = --4zy +4ry

14. Let ¢(z,) = n[f(z)] = ln(+)

~

¢ _ @S, 4 _ ---- Ou @.
z bu~z Oz ?+?

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{(v^2 - u^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial x}\right)^2\right] - 4uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}}{(u^2+v^2)^2} + \frac{u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 v}{\partial x^2}}{u^2+v^2}$$

A similar calculation yields .~~. By applying Laplace's equation and the Cauchy-Riemann equations of u and v - 82 T 8'q. The sum simplifies to reveal that ∕¢ = 0.

15. Consider r(z) = Re(Az" + B/z") + C which is harmonic for 1 ≤ |z| ≤ 2
Consider the polar form for z. z = re" and select n=3 to agree with the cosine argument. q(re") = Ar'Re(e"") + Br Re(e i3θ) + @.
q(re10) = A r³ cos30 + Br⁻³cos30 + C = (Ar'+B r)cos30 + C

r=1 ⇒ (A+B)c0s30 + C = 0 ⇒ A + B = 0, C = 0.
r=2 ⇒ (A8+B/8)c0s30 = 5cos30. A = 40/63, B = -40/63
q(re") = (40/63)(r' − r)cos30 = (40/63) Re (z − ?)

16. #+,) = ;;z11l − 1or + ,v) = 1]| ae tr possbilities.

17. a. ¢(z,y) = Re(Z + 5z +1) = x^2 — + 5z +1

b. ¢(z,)=2Re ~±2 = ~+ + 4y +4

18. Let u = ¢., v= −¢. Then

~ = − V

7%.=hw

∂u

O = O

19. cosO = cos20 +% = ((z) = ARe r e n +B = Ar cos20 + B. In the limit as r-»6 ((z) =/=» B =/. On the circle lzl=1, r=1. = A =h. q(z) = (%)r'cos20 +% = Re [1/(2z)] +%%.

20 In order the' Ov/75,, = a tv(+,)=], f ~u a(n)dn +Z). Then

v/~a = py O u x 7an)dn + W'(r)

= J p Ou a.5 n)dn + (az) (because is harmonic)

= a, (Ou (0)+ ( ).

In order that ay 7 = a it must be true that w'( )= 5, au (7%0). Thus,

( )= / p Ou 3(O)4c¢ + a

and v(fy Ou p

.)=/ 7(m)dn −J, 7,GS.0)d, +a.

21. It is easily verified that u = ln |z| satisfies Laplace's equation on C\{0} and that $u+iv$ = ln [z]+iArg(z) satisfies the Cauchy-Riemann equations on the domain D = C\ {nonpositive real axis}, so that

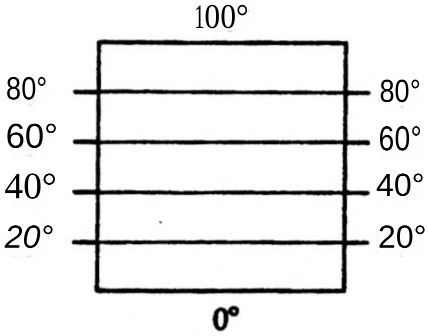
Arg (z) is a harmonic conjugate of u on *D*. By Problem 4, any harmonic conjugate of u has to be of the form Arg(z)+a in *D*. It is impossible to have a harmonic conjugate of this form that is continuous on C \ {0}.

22. $\frac{\partial u}{\partial x} = \phi_{xx}\phi_y + \phi_x\phi_{yx} + \psi_{xx}\psi_y + \psi_x\psi_{yx}$

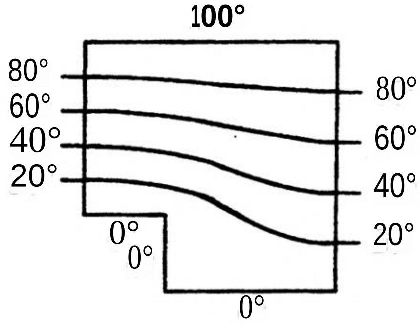
$= -\phi_{yy}\phi_y + \phi_x\phi_{yx} - \psi_{yy}\psi_y + \psi_x\psi_{yx} = \frac{\partial v}{\partial y}$.

EXERCISES 2.6: Steady-State Temperature as a Harmonic Function.

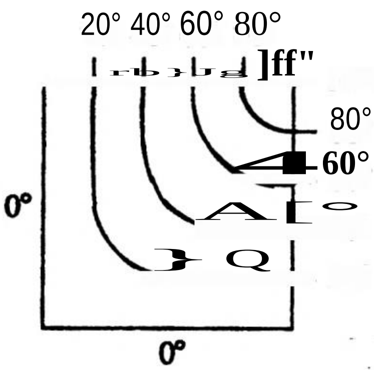
1. a.



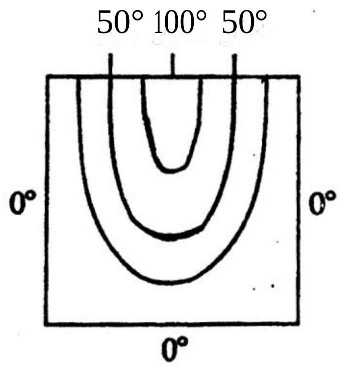
b.



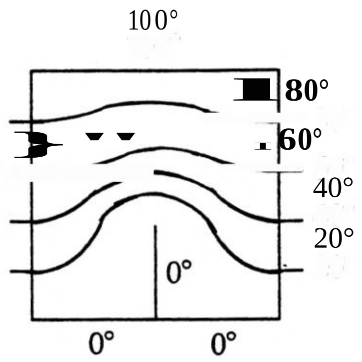
c.



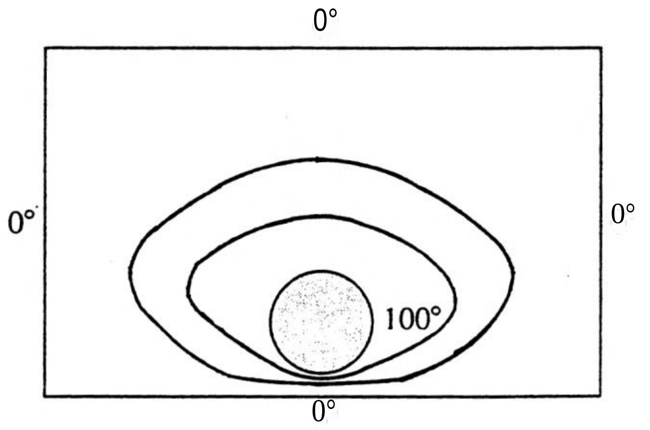
d.



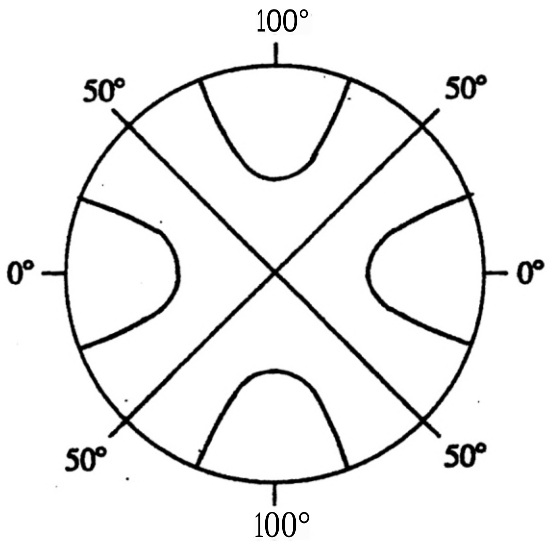
e.



2. This does not violate the maximum principle.



3. This does not **violate the** ma.ximum principle.



Exercises 2.7

1. f(z) =z'+ c where c is a real constant.
ζ1 =(+V(1-40))/2, %=(1-V(1-4c))/2
Only ζ2 is an attractor for -3/4 <c <%.
2. f(g)=and f(g)> 1 Therefore we can pick a real number p between I and lf'(g) l such that lf(z) - gl = plz-gl for all z in a sufficiently small disk around g. If any point zo in this disk is the seed for an orbit z; = f(zo). Z = f(zn), ... z,= f(z,), then we have lz,-gl? plz-gl2 ... 2 p"lz,-gl
Because p>1,the point z, moves away from guntil the magnitude of the derivative becomes 1 or less. The orbit is out of the disk.
3. (a) Fixed points are ζ1 = i, K = -i. Both are repellors.
(b) Fixed points are(= 1/2, (=-1/2,ζ=-I. Fixed points K and + are repellors, but fixed point [is an attractor.
4. zo=e?'' with o an irrational real number. z, = e2ta2n, Because lz,I= I, the trajectory will follow the unit circle. If iterations p and q coincide, 210.2° - 2n02" = 270(2°- 2) = 2rk for some integer k. But because (2°• 2) is an integer that can be represented by m, the equation 2ram=2rk is satisfied only if k=om or a = k/m. Because a is irrational it cannot be represented by a rational number and no iterations repeat.
5. Fixed points are K = -1/2 +i/5/2 (an attractor) and K = -1/2 -iN5/2 (a repellor).
6. f(z) =z. The seed is zo. Z =z¢,z=z6', ... Z,=z(2"). To have an n cycle z,= zo= z6(2"). Or z,/zo = z6(2"-1) = 1 =e. Solving gives zo = e(i2r/(2"-1)).
7. The cycle is4. 2(2r/p) =2n mod p => 2= 1 mod p. p=3,5,15. 3 will give repeated cycles of length 2. 5 and 15 will give the desired cycles of length 4.
8. Student Matlab: n=100;c=.253; z0=0;y(1)=zo;
for k=1:n-1,yk+I)=y(k)2+c;end
plot(y)
9. If lal < 1 the whole complex plane is the filled Julia set. If lal > 1 the origin is the filled Julia set.
10. f(z)=z-F(z)/F'(). fg='-FKU/F(= '= FU/FU)=0 =F(K=0 with the possible exception of the points where F'([) = 0.
f(z) = I-F(/F'(z) + F(z)F"(z)/(F(z)) = F(z)F"()/(F'())
f(= F([F"(K/(F'() = 0 where F'([40 and every zero of F(z) is an attractor as long as F'(~) # 0.