Solution Manual for Linear Algebra with Applications 2nd Edition Holt 1464193347 9781464193347

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Chapter 2

Euclidean Space

2.1 Practice Problems

1. $\mathbf{u} - \mathbf{w} = \begin{bmatrix} -4 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -4-5 \\ 3-0 \\ 4-(-2) \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix}$

$\mathbf{v} + 3\mathbf{w} = \begin{bmatrix} -1 \\ 6 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -1+3(5) \\ 6+3(0) \\ 2+3(-2) \end{bmatrix} = \begin{bmatrix} 14 \\ 6 \\ -4 \end{bmatrix}$

$-2\mathbf{w} + \mathbf{u} + 3\mathbf{v} = -2\begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \\ 4 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} -2(5)+(-4)+3(-1) \\ -2(0)+3+3(6) \\ -2(-2)+4+3(2) \end{bmatrix} = \begin{bmatrix} -17 \\ 21 \\ 14 \end{bmatrix}$

2. (a)
$$\begin{array}{rcrcr} -x_1 & + & 4x_2 & = & 3 \\ 7x_1 & + & 6x_2 & = & 10 \\ 2x_1 & - & 6x_2 & = & 5 \end{array}$$

(b)
$$\begin{array}{rcrcrcr} 3x_1 & & & - & x_3 & = & 4 \\ 4x_1 & - & 2x_2 & + & 2x_3 & = & 7 \\ & - & 5x_2 & + & 9x_3 & = & 11 \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & -6 \end{array}$$

3. (a) $x_1 \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 6 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 0 \end{bmatrix}$

(b) $x_1 \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 2 \\ 12 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 10 \end{bmatrix}$

4. (a)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

(b) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 17 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 13 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

5. (a) $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \iff x_1 \begin{bmatrix} 1 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix} \iff \begin{bmatrix} x_1 + 3x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix} \iff$

the augmented matrix $\begin{bmatrix} 1 & 3 & 5 \\ -5 & 6 & 9 \end{bmatrix}$ has a solution:

$$\begin{bmatrix} 1 & 3 & 5 \\ -5 & 6 & 9 \end{bmatrix} \overset{5R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 21 & 34 \end{bmatrix}$$

From row 2, $21x_2 = 34 \Rightarrow x_2 = \frac{34}{21}$. From row 1, $x_1 + 3(\frac{34}{21}) = 5 \Rightarrow x_1 = \frac{1}{7}$. Thus, b is a linear combination of a_1 and a_2, with $b = \frac{1}{7}a_1 + \frac{34}{21}a_2$.

(b) $x_1a_1 + x_2a_2 + x_2a_2 = b \iff x_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \iff$

$\begin{bmatrix} x_1 - 2x_2 \\ -3x_1 + 3x_2 \\ 8x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -4 \end{bmatrix} \iff$ the augmented matrix $\begin{bmatrix} 1 & -2 & 7 \\ -3 & 3 & 5 \\ 8 & -3 & -4 \end{bmatrix}$ yields a solution.

$$\begin{bmatrix} 1 & -2 & 7 \\ -3 & 3 & 5 \\ 8 & -3 & -4 \end{bmatrix} \overset{\substack{3R_1+R_2\to R_2 \\ -8R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & -2 & 7 \\ 0 & -3 & 26 \\ 0 & 13 & -60 \end{bmatrix} \overset{(\frac{13}{3})R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & -2 & 7 \\ 0 & -3 & 26 \\ 0 & 0 & \frac{158}{3} \end{bmatrix}$$

From the third equation, we have $0 = \frac{158}{3}$, and thus the system does not have a solution. Thus, b is *not* a linear combination of a_1, a_2, and a_3.

6. (a) False. Addition of vectors is associative and commutative.
   (b) True. The scalars may be any real number.
   (c) True. The solutions to a linear system with variables $x_1, \ldots, x_n$ can be expressed as a vector x, which is the sum of a fixed vector with n components and a linear combination of k vectors with n components, where k is the number of free variables.
   (d) False. The Parallelogram Rule gives a geometric interpretation of vector addition.

2.1 Vectors

1. $u - v = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3-(-4) \\ -2-1 \\ 0-5 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -5 \end{bmatrix}$;

   $6w = 6 \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix} = \begin{bmatrix} (6)\,2 \\ (6)\,(-7) \\ (6)\,(-1) \end{bmatrix} = \begin{bmatrix} 12 \\ -42 \\ -6 \end{bmatrix}$

2. $w - u = \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2-3 \\ -7-(-2) \\ -1-0 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -1 \end{bmatrix}$;

   $-5v = (-5) \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} (-5)\,(-4) \\ (-5)\,1 \\ (-5)\,5 \end{bmatrix} = \begin{bmatrix} 20 \\ -5 \\ -25 \end{bmatrix}$

3. $w + 3v = \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2+3\,(-4) \\ -7+3\,(1) \\ -1+3\,(5) \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \\ 14 \end{bmatrix}$;

   $2w - 7v = 2 \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix} - 7 \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 2(2) - 7(-4) \\ 2(-7) - 7(1) \\ 2(-1) - 7(5) \end{bmatrix} = \begin{bmatrix} 32 \\ -21 \\ -37 \end{bmatrix}$$

4. $4\mathbf{w}-\mathbf{u}=4\begin{bmatrix}2\\-7\\-1\end{bmatrix}-\begin{bmatrix}3\\-2\\0\end{bmatrix}=\begin{bmatrix}4(2)-3\\4(-7)-(-2)\\4(-1)-0\end{bmatrix}=\begin{bmatrix}5\\-26\\-4\end{bmatrix}$;

$-2\mathbf{v}+5\mathbf{w}=(-2)\begin{bmatrix}-4\\1\\5\end{bmatrix}+5\begin{bmatrix}2\\-7\\-1\end{bmatrix}=\begin{bmatrix}(-2)(-4)+5(2)\\(-2)(1)+5(-7)\\(-2)(5)+5(-1)\end{bmatrix}=\begin{bmatrix}18\\-37\\-15\end{bmatrix}$

5. $-\mathbf{u}+\mathbf{v}+\mathbf{w}=-\begin{bmatrix}3\\-2\\0\end{bmatrix}+\begin{bmatrix}-4\\1\\5\end{bmatrix}+\begin{bmatrix}2\\-7\\-1\end{bmatrix}=$

$\begin{bmatrix}-3-4+2\\-(-2)+1-7\\-0+5-1\end{bmatrix}=\begin{bmatrix}-5\\-4\\4\end{bmatrix}$;

$2\mathbf{u}-\mathbf{v}+3\mathbf{w}=2\begin{bmatrix}3\\-2\\0\end{bmatrix}-\begin{bmatrix}-4\\1\\5\end{bmatrix}+3\begin{bmatrix}2\\-7\\-1\end{bmatrix}=$

$\begin{bmatrix}2(3)-(-4)+3(2)\\2(-2)-1+3(-7)\\2(0)-5+3(-1)\end{bmatrix}=\begin{bmatrix}16\\-26\\-8\end{bmatrix}$

6. $3\mathbf{u}-2\mathbf{v}+5\mathbf{w}=3\begin{bmatrix}3\\-2\\0\end{bmatrix}-2\begin{bmatrix}-4\\1\\5\end{bmatrix}+5\begin{bmatrix}2\\-7\\-1\end{bmatrix}=$

$\begin{bmatrix}3(3)-2(-4)+5(2)\\3(-2)-2(1)+5(-7)\\3(0)-2(5)+5(-1)\end{bmatrix}=\begin{bmatrix}27\\-43\\-15\end{bmatrix}$;

$-4\mathbf{u}+3\mathbf{v}-2\mathbf{w}=-4\begin{bmatrix}3\\-2\\0\end{bmatrix}+3\begin{bmatrix}-4\\1\\5\end{bmatrix}-2\begin{bmatrix}2\\-7\\-1\end{bmatrix}=$

$\begin{bmatrix}(-4)(3)+3(-4)-2(2)\\(-4)(-2)+3(1)-2(-7)\\(-4)(0)+3(5)-2(-1)\end{bmatrix}=\begin{bmatrix}-28\\25\\17\end{bmatrix}$

7. $\begin{aligned}3x_1-x_2&=8\\2x_1+5x_2&=13\end{aligned}$

8. $\begin{aligned}-x_1+9x_2&=-7\\6x_1-5x_2&=-11\\-4x_1&=3\end{aligned}$

9. $\begin{aligned}-6x_1+5x_2&=4\\5x_1-3x_2+2x_3&=16\end{aligned}$

10. $\begin{aligned}2x_1+5x_3+4x_4&=0\\7x_1+2x_2+x_3+5x_4&=4\\8x_1+4x_2+6x_3+7x_4&=3\\3x_1+2x_2+x_3&=5\end{aligned}$

11. $x_1\begin{bmatrix}2\\-1\end{bmatrix}+x_2\begin{bmatrix}8\\-3\end{bmatrix}+x_3\begin{bmatrix}-4\\5\end{bmatrix}=\begin{bmatrix}-10\\4\end{bmatrix}$

12. $x_1\begin{bmatrix}-2\\1\\7\end{bmatrix}+x_2\begin{bmatrix}5\\-2\\-17\end{bmatrix}+x_3\begin{bmatrix}-10\\3\\34\end{bmatrix}=\begin{bmatrix}4\\-1\\-16\end{bmatrix}$

13. $x_1\begin{bmatrix}1\\2\end{bmatrix}$ − −3 $+x_2$ $\begin{bmatrix}-1\end{bmatrix}$

2
-3 [-3 [[

-1 $-$

$+\, x_3$] $+\, x_4$] 1

6]

1 2

0 $=$ $-$

0 1

5

14. $x_1 \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 9 \\ -5 \\ -2 \end{bmatrix} = \begin{bmatrix} 13 \\ -9 \\ -2 \end{bmatrix}$

15. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

16. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

17. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

18. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} -4 \\ 5 \\ 1 \\ 0 \end{bmatrix}$

19. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -9 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 6 \\ 0 \\ 3 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

20. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -12 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -7 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 14 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

21. $1\mathbf{u} + 0\mathbf{v} = \mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $0\mathbf{u} + 1\mathbf{v} = \mathbf{v} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$, $1\mathbf{u} + 1\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

22. $1\mathbf{u} + 0\mathbf{v} = \mathbf{u} = \begin{bmatrix} 7 \\ 1 \\ -13 \end{bmatrix}$, $0\mathbf{u} + 1\mathbf{v} = \mathbf{v} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$,

$1\mathbf{u} + 1\mathbf{v} = \begin{bmatrix} 7 \\ 1 \\ -13 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ -2 \\ -11 \end{bmatrix}$.

23. $1\mathbf{u} + 0\mathbf{v} + 0\mathbf{w} = \mathbf{u} = \begin{bmatrix} -4 \\ 0 \\ -3 \end{bmatrix}$, $0\mathbf{u} + 1\mathbf{v} + 0\mathbf{w} = \mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}$, $0\mathbf{u} + 0\mathbf{v} + 1\mathbf{w} = \mathbf{w} = \begin{bmatrix} 9 \\ 6 \\ 11 \end{bmatrix}$.

24. $1\mathbf{u} + 0\mathbf{v} + 0\mathbf{w} = \mathbf{u} = \begin{bmatrix} 1 \\ 8 \\ 2 \\ 2 \end{bmatrix}$, $0\mathbf{u} + 1\mathbf{v} + 0\mathbf{w} = \mathbf{v} = \begin{bmatrix} 4 \\ -2 \\ 5 \\ -5 \end{bmatrix}$, $0\mathbf{u} + 0\mathbf{v} + 1\mathbf{w} = \mathbf{w} = \begin{bmatrix} 9 \\ 9 \\ 0 \\ 1 \end{bmatrix}$.

25. $-3\begin{bmatrix} a \\ 3 \end{bmatrix} + 4\begin{bmatrix} -1 \\ b \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix} \Rightarrow \begin{bmatrix} -3a-4 \\ -9+4b \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix} \Rightarrow -3a - 4 = -10$ and $-9 + 4b = 19$.

Solving these equations, we obtain $a = 2$ and $b = 7$.

26. $4\begin{bmatrix} 4 \\ a \end{bmatrix} + 3\begin{bmatrix} -3 \\ 5 \end{bmatrix} - 2\begin{bmatrix} b \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 16-9-2b \\ 4a+15-16 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix} \Rightarrow$

$7 - 2b = -1$ and $4a - 1 = 7$. Solving these equations, we obtain $a = 2$ and $b = 4$.

27. $-\begin{bmatrix} -1 \\ a \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -2 \\ b \end{bmatrix} = \begin{bmatrix} c \\ -7 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1+6 \\ -a-4 \\ -2+2b \end{bmatrix} = \begin{bmatrix} c \\ -7 \\ 8 \end{bmatrix} \Rightarrow$

$7 = c$, $-a - 4 = -7$, and $-2 + 2b = 8$. Solving these equations, we obtain $a = 3$, $b = 5$, and $c = 7$.

28. $-\begin{bmatrix} a \\ -3 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ b \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ c \end{bmatrix} \Rightarrow \begin{bmatrix} -a-1 \\ 3-b \\ -5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ c \end{bmatrix} \Rightarrow$

$-a - 1 = 4$, $3 - b = 2$, and $-5 = c$. Solving these equations, we obtain $a = -5$, $b = 1$, and $c = -5$.

29. $-\begin{bmatrix} 1 \\ 2 \\ a \\ 1 \end{bmatrix} + 2\begin{bmatrix} b \\ 1 \\ -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ c \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ 3 \\ d \end{bmatrix} \Rightarrow \begin{bmatrix} 2b-3 \\ -c \\ -a-9 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ 3 \\ d \end{bmatrix} \Rightarrow$

$2b - 3 = -3$, $-c = -4$, $-a - 9 = 3$, and $5 = d$. Solving these equations, we obtain $a = -12$, $b = 0$, $c = 4$, and $d = 5$.

30. $-\begin{bmatrix} a \\ 4 \\ -2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 1 \\ b \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ c \\ -3 \\ -6 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 3 \\ d \end{bmatrix} \Rightarrow \begin{bmatrix} -a+10-2 \\ -4+2-c \\ 2+2b+3 \\ 1+6+6 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 3 \\ d \end{bmatrix} \Rightarrow$

$-a + 8 = 11$, $-2 - c = -4$, $5 + 2b = 3$, and $13 = d$. Solving these equations, we obtain $a = -3$, $b = -1$, $c = 2$, and $d = 13$.

31. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \Leftrightarrow x_1\begin{bmatrix} -2 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -2x_1 + 7x_2 \\ 5x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \Leftrightarrow$ the augmented matrix $\begin{bmatrix} -2 & 7 & 8 \\ 5 & -3 & 9 \end{bmatrix}$ has a solution:

$$\begin{bmatrix} -2 & 7 & 8 \\ 5 & -3 & 9 \end{bmatrix} \xrightarrow{(5/2)R_1 + R_2 \to R_2} \begin{bmatrix} -2 & 7 & 8 \\ 0 & \frac{29}{2} & 29 \end{bmatrix}$$

From row 2, $\frac{29}{2}x_2 = 29 \Rightarrow x_2 = 2$. From row 1, $-2x_1 + 7(2) = 8 \Rightarrow x_1 = 3$. Hence $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with $\mathbf{b} = 3\mathbf{a}_1 + 2\mathbf{a}_2$.

32. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \Leftrightarrow x_1\begin{bmatrix} 4 \\ -6 \end{bmatrix} + x_2\begin{bmatrix} -6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 4x_1 - 6x_2 \\ -6x_1 + 9x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} \Leftrightarrow$ the augmented matrix $\begin{bmatrix} 4 & -6 & 1 \\ -6 & 9 & -5 \end{bmatrix}$ has a solution:

$$\begin{bmatrix} 4 & -6 & 1 \\ -6 & 9 & -5 \end{bmatrix} \underset{\sim}{\xrightarrow{(3/2)R_1 + R_2 \to R_2}} \begin{bmatrix} 4 & -6 & 1 \\ 0 & 0 & -\frac{7}{2} \end{bmatrix}$$

Because no solution exists, $\mathbf{b}$ is not a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

$\begin{bmatrix} 2 \end{bmatrix}$ $\begin{bmatrix} 0 \end{bmatrix}$ $\begin{bmatrix} 1 \end{bmatrix}$ $\begin{bmatrix} 2x_1 \end{bmatrix}$ $\begin{bmatrix} 1 \end{bmatrix}$

33. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \iff x_1\begin{bmatrix} -3 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \end{bmatrix} \iff \begin{bmatrix} -3x_1 + 3x_2 \\ x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$. The first equation $2x_1 = 1 \Rightarrow x_1 = \frac{1}{2}$. Then the second equation $-3\left(\frac{1}{2}\right) + 3x_2 = -5 \Rightarrow x_2 = -\frac{7}{6}$. We check the third equation, $\frac{1}{2} - 3\left(-\frac{7}{6}\right) = 4 \neq -2$. Hence $\mathbf{b}$ is *not* a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

34. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \iff x_1\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -9 \end{bmatrix} \iff \begin{bmatrix} 2x_1 \\ -3x_1 + 3x_2 \\ x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -9 \end{bmatrix}$. The first equation $2x_1 = 6 \Rightarrow x_1 = 3$. Then the second equation $-3(3) + 3x_2 = 3 \Rightarrow x_2 = 4$. We check the third equation, $3 - 3(4) = -9$. Hence $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with $\mathbf{b} = 3\mathbf{a}_1 + 4\mathbf{a}_2$.

35. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_2\mathbf{a}_2 = \mathbf{b} \iff x_1\begin{bmatrix}1\\2\end{bmatrix} + x_2\begin{bmatrix}-3\\5\end{bmatrix} + x_3\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix} \iff \begin{bmatrix}x_1 - 3x_2 + 2x_3\\2x_1 + 5x_2 + 2x_3\end{bmatrix}$

$= \begin{bmatrix}1\\-2\\3\end{bmatrix} \iff$ the augmented matrix $\begin{bmatrix}1 & -3 & 2 & 1\\2 & 5 & 2 & -2\\1 & -3 & 4 & 3\end{bmatrix}$ yields a solution.

$$\begin{bmatrix}1 & -3 & 2 & 1\\2 & 5 & 2 & -2\\1 & -3 & 4 & 3\end{bmatrix} \underset{\substack{-2R_1+R_2\to R_2\\-R_1+R_3\to R_3}}{\sim} \begin{bmatrix}1 & -3 & 2 & 1\\0 & 11 & -2 & -4\\0 & 0 & 2 & 2\end{bmatrix}$$

From row 3, we have $2x_3 = 2 \Rightarrow x_3 = 1$. From row 2, $11x_2 - 2(1) = -4 \Rightarrow x_2 = -\frac{2}{11}$. From row $\frac{2}{11}$ $\frac{17}{11}$. Hence $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, with $\mathbf{b} = -\frac{17}{11}\mathbf{a}_1 - \frac{2}{11}\mathbf{a}_2 + \mathbf{a}_3$. $x = -$

36. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_2\mathbf{a}_2 = \mathbf{b} \iff x_1\begin{bmatrix}2\\-3\end{bmatrix} + x_2\begin{bmatrix}0\\3\end{bmatrix} + x_3\begin{bmatrix}-2\\-1\end{bmatrix} = \begin{bmatrix}2\\-4\end{bmatrix} \iff$

$\begin{bmatrix}2x_1 - 2x_3\\-3x_1 + 3x_2 - x_3\\x_1 - 3x_2 + 3x_3\end{bmatrix} = \begin{bmatrix}2\\-4\\5\end{bmatrix} \iff$ the augmented matrix $\begin{bmatrix}2 & 0 & -2 & 2\\-3 & 3 & -1 & -4\\1 & -3 & 3 & 5\end{bmatrix}$ yields a solution.

$$\begin{bmatrix}2 & 0 & -2 & 2\\-3 & 3 & -1 & -4\\1 & -3 & 3 & 5\end{bmatrix} \underset{\substack{(3/2)R_1+R_2\to R_2\\(-1/2)R_1+R_3\to R_3}}{\sim} \begin{bmatrix}2 & 0 & -2 & 2\\0 & 3 & -4 & -1\\0 & -3 & 4 & 4\end{bmatrix} \underset{R_2+R_3\to R_3}{\sim} \begin{bmatrix}2 & 0 & -2 & 2\\0 & 3 & -4 & -1\\0 & 0 & 0 & 3\end{bmatrix}$$

From the third equation, we have $0 = 3$, and hence the system does not have a solution. Hence $\mathbf{b}$ is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

37. Using vectors, we calculate

$$(2)\begin{bmatrix}29\\3\\4\end{bmatrix} + (1)\begin{bmatrix}18\\25\\6\end{bmatrix} = \begin{bmatrix}76\\31\\14\end{bmatrix}$$

Hence we have 76 pounds of nitrogen, 31 pounds of phosphoric acid, and 14 pounds of potash.

38. Using vectors, we calculate

$$(4)\begin{bmatrix}29\\3\\4\end{bmatrix} + (7)\begin{bmatrix}18\\25\\6\end{bmatrix} = \begin{bmatrix}242\\187\\58\end{bmatrix}$$

Hence we have 242 pounds of nitrogen, 187 pounds of phosphoric acid, and 58 pounds of potash.

39. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1\begin{bmatrix}29\\3\\4\end{bmatrix} + x_2\begin{bmatrix}18\\25\\6\end{bmatrix} = \begin{bmatrix}112\\81\\26\end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix}29 & 18 & 112\\3 & 25 & 81\\4 & 6 & 26\end{bmatrix} \underset{\substack{(-3/29)R_1+R_2\to R_2\\(-4/29)R_1+R_3\to R_3}}{\sim} \begin{bmatrix}29 & 18 & 112\\0 & \frac{671}{29} & \frac{2013}{29}\\0 & \frac{102}{29} & \frac{306}{29}\end{bmatrix}$$

$$\underset{(-102/671)R_2+R_3\to R_3}{\sim} \begin{bmatrix}29 & 18 & 112\\0 & \frac{671}{29} & \frac{2013}{29}\end{bmatrix}$$

0 0 0

From row 2, we have $\frac{671}{29}x_2 = \frac{2013}{29} \Rightarrow x_2 = 3$. Form row 1, we have $29x_1 + 18(3) = 112 \Rightarrow x_1 = 2$. Thus we need 2 bags of Vigoro and 3 bags of Parker's.

40. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1 \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} = \begin{bmatrix} 285 \\ 284 \\ 78 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 29 & 18 & 285 \\ 3 & 25 & 284 \\ 4 & 6 & 78 \end{bmatrix} \underset{\substack{(-3/29)R_1+R_2\to R_2 \\ (-4/29)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 29 & 18 & 285 \\ 0 & \frac{671}{29} & \frac{7381}{29} \\ 0 & \frac{102}{29} & \frac{1122}{29} \end{bmatrix} \underset{(-102/671)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 29 & 18 & 285 \\ 0 & \frac{671}{29} & \frac{7381}{29} \\ 0 & 0 & 0 \end{bmatrix}$$

From row 2, we have $\frac{671}{29}x_2 = \frac{7381}{29} \Rightarrow x_2 = 11$. Form row 1, we have $29x_1 + 18(11) = 285 \Rightarrow x_1 = 3$. Thus we need 3 bags of Vigoro and 11 bags of Parker's.

41. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1 \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} = \begin{bmatrix} 123 \\ 59 \\ 24 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 29 & 18 & 123 \\ 3 & 25 & 59 \\ 4 & 6 & 24 \end{bmatrix} \underset{\substack{(-3/29)R_1+R_2\to R_2 \\ (-4/29)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 29 & 18 & 123 \\ 0 & \frac{671}{29} & \frac{1342}{29} \\ 0 & \frac{102}{29} & \frac{204}{29} \end{bmatrix} \underset{\substack{(29/671)R_2\to R_2 \\ (-102/29)R_2+R_3\to R_3}}{\sim} \begin{bmatrix} 29 & 18 & 123 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Back substituting gives $x_2 = 2$ and $x_1 = 3$. Hence we need 3 bags of Vigoro and 2 bags of Parker's.

42. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1 \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} = \begin{bmatrix} 159 \\ 109 \\ 36 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 29 & 18 & 159 \\ 3 & 25 & 109 \\ 4 & 6 & 36 \end{bmatrix} \underset{\substack{(-3/29)R_1+R_2\to R_2 \\ (-4/29)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 29 & 18 & 159 \\ 0 & \frac{671}{29} & \frac{2684}{29} \\ 0 & \frac{102}{29} & \frac{408}{29} \end{bmatrix} \underset{\substack{(29/671)R_2\to R_3 \\ (-102/29)R_2+R_3\to R_3}}{\sim} \begin{bmatrix} 29 & 18 & 159 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Back substituting gives $x_2 = 4$ and $x_1 = 3$. Hence we need 3 bags of Vigoro and 4 bags of Parker's.

43. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1\begin{bmatrix}29\\3\\4\end{bmatrix}+x_2\begin{bmatrix}18\\25\\6\end{bmatrix}=\begin{bmatrix}148\\131\\40\end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix}29&18&148\\3&25&131\\4&6&40\end{bmatrix}\begin{matrix}(-3/29)R_1+R_2\to R_2\\(-4/29)R_1+R_3\to R_3\\\sim\end{matrix}\begin{bmatrix}29&18&148\\0&\frac{671}{29}&\frac{3355}{29}\\0&\frac{102}{29}&\frac{568}{29}\end{bmatrix}\begin{matrix}(-102/671)R_2+R_3\to R_3\\\sim\end{matrix}\begin{bmatrix}29&18&148\\0&\frac{671}{29}&\frac{3355}{29}\\0&0&2\end{bmatrix}$$

Since row 3 corresponds to the equation $0 = 2$, the system has no solutions.

44. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1\begin{bmatrix}29\\3\\4\end{bmatrix}+x_2\begin{bmatrix}18\\25\\6\end{bmatrix}=\begin{bmatrix}100\\120\\40\end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix}29&18&100\\3&25&120\\4&6&40\end{bmatrix}\begin{matrix}(-3/29)R_1+R_2\to R_2\\(-4/29)R_1+R_3\to R_3\\\sim\end{matrix}\begin{bmatrix}29&18&100\\0&\frac{671}{29}&\frac{3180}{29}\\0&\frac{102}{29}&\frac{760}{29}\end{bmatrix}\begin{matrix}(-102/671)R_2+R_3\to R_3\\\sim\end{matrix}\begin{bmatrix}29&18&100\\0&\frac{671}{29}&\frac{3180}{29}\\0&0&\frac{6400}{671}\end{bmatrix}$$

Since row 3 is $0 = \frac{6400}{671}$, we conclude that we can not obtain the desired amounts.

45. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1\begin{bmatrix}29\\3\\4\end{bmatrix}+x_2\begin{bmatrix}18\\25\\6\end{bmatrix}=\begin{bmatrix}25\\72\\14\end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix}29&18&25\\3&25&72\\4&6&14\end{bmatrix}\begin{matrix}(-3/29)R_1+R_2\to R_2\\(-4/29)R_1+R_3\to R_3\\\sim\end{matrix}\begin{bmatrix}29&18&25\\0&\frac{671}{29}&\frac{2013}{29}\\0&\frac{102}{29}&\frac{306}{29}\end{bmatrix}\begin{matrix}(-102/671)R_2+R_3\to R_3\\\sim\end{matrix}\begin{bmatrix}29&18&25\\0&\frac{671}{29}&\frac{2013}{29}\\0&0&0\end{bmatrix}$$

From row 2, we have $\frac{671}{29}x_2 = \frac{2013}{29} \Rightarrow x_2 = 3$. From row 1, we have $29x_1 + 18(3) = 25 \Rightarrow x_1 = -1$. Since we can not use a negative amount, we conclude that there is no solution.

46. Let x_1 be the amount of Vigoro, x_2 the amount of Parker's, and then we need

$$x_1 \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} = \begin{bmatrix} 301 \\ 8 \\ 38 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 29 & 18 & 301 \\ 3 & 25 & 8 \\ 4 & 6 & 38 \end{bmatrix} \underset{\sim}{\overset{(-3/29)R_1+R_2\to R_2}{(-4/29)R_1+R_3\to R_3}} \begin{bmatrix} 29 & 18 & 301 \\ 0 & \frac{671}{29} & -\frac{671}{29} \\ 0 & \frac{102}{29} & -\frac{102}{29} \end{bmatrix} \overset{(-102/671)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 29 & 18 & 301 \\ 0 & \frac{671}{29} & -\frac{671}{29} \\ 0 & 0 & 0 \end{bmatrix}$$

From row 2, we have $\frac{671}{29}x_2 = -\frac{671}{29} \Rightarrow x_2 = -1$. Since we can not use a negative amount, we conclude that there is no solution.

47. Let x_1 be the number of cans of Red Bull, and x_2 the number of cans of Jolt Cola, and then we need

$$x_1 \begin{bmatrix} 27 \\ 80 \end{bmatrix} + x_2 \begin{bmatrix} 94 \\ 280 \end{bmatrix} = \begin{bmatrix} 148 \\ 440 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 27 & 94 & 148 \\ 80 & 280 & 440 \end{bmatrix} \overset{(-80/27)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 27 & 94 & 148 \\ 0 & \frac{40}{27} & \frac{40}{27} \end{bmatrix}$$

From row 2, we have $\frac{40}{27}x_2 = \frac{40}{27} \Rightarrow x_2 = 1$. From row 1, $27x_1 + 94(1) = 148 \Rightarrow x_1 = 2$. Thus we need to drink 2 cans of Red Bull and 1 can of Jolt Cola.

48. Let x_1 be the number of cans of Red Bull, and x_2 the number of cans of Jolt Cola, and then we need

$$x_1 \begin{bmatrix} 27 \\ 80 \end{bmatrix} + x_2 \begin{bmatrix} 94 \\ 280 \end{bmatrix} = \begin{bmatrix} 309 \\ 920 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 27 & 94 & 309 \\ 80 & 280 & 920 \end{bmatrix} \overset{(-80/27)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 27 & 94 & 309 \\ 0 & \frac{40}{27} & \frac{40}{9} \end{bmatrix}$$

From row 2, we have $\frac{40}{27}x_2 = \frac{40}{9} \Rightarrow x_2 = 3$. From row 1, $27x_1 + 94(3) = 309 \Rightarrow x_1 = 1$. Thus we need to drink 1 can of Red Bull and 3 cans of Jolt Cola.

49. Let x_1 be the number of cans of Red Bull, and x_2 the number of cans of Jolt Cola, and then we need

$$x_1 \begin{bmatrix} 27 \\ 80 \end{bmatrix} + x_2 \begin{bmatrix} 94 \\ 280 \end{bmatrix} = \begin{bmatrix} 242 \\ 720 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 27 & 94 & 242 \\ 80 & 280 & 720 \end{bmatrix} \overset{(-80/27)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 27 & 94 & 242 \\ 0 & \frac{40}{27} & \frac{80}{27} \end{bmatrix}$$

From row 2, we have $\frac{40}{27}x_2 = \frac{80}{27} \Rightarrow x_2 = 2$. From row 1, $27x_1 + 94(2) = 242 \Rightarrow x_1 = 2$. Thus we need to drink 2 cans of Red Bull and 2 cans of Jolt Cola.

50. Let x_1 be the number of cans of Red Bull, and x_2 the number of cans of Jolt Cola, and then we need

$$x_1 \begin{bmatrix} 27 \\ 80 \end{bmatrix} + x_2 \begin{bmatrix} 94 \\ 280 \end{bmatrix} = \begin{bmatrix} 457 \\ 1360 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 27 & 94 & 457 \\ 80 & 280 & 1360 \end{bmatrix} \overset{(-80/27)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 27 & 94 & 457 \\ 0 & \frac{40}{27} & \frac{160}{27} \end{bmatrix}$$

From row 2, we have $\frac{40}{27}x_2 = \frac{160}{27} \Rightarrow x_2 = 4$. From row 1, $27x_1 + 94(4) = 457 \Rightarrow x_1 = 3$. Thus we need to drink 3 cans of Red Bull and 4 cans of Jolt Cola.

51. Let x_1 be the number of servings of Lucky Charms and x_2 the number of servings of Raisin Bran, and then we need

$$x_1 \begin{bmatrix} 10 \\ 25 \\ 25 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 25 \\ 10 \end{bmatrix} = \begin{bmatrix} 40 \\ 200 \\ 125 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 10 & 2 & 40 \\ 25 & 25 & 200 \\ 25 & 10 & 125 \end{bmatrix} \overset{\substack{(-5/2)R_1+R_2\to R_2 \\ (-5/2)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 10 & 2 & 40 \\ 0 & 20 & 100 \\ 0 & 5 & 25 \end{bmatrix} \overset{(-1/4)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 10 & 2 & 40 \\ 0 & 20 & 100 \\ 0 & 0 & 0 \end{bmatrix}$$

From row 2, we have $20x_2 = 100 \Rightarrow x_2 = 5$. From row 1, $10x_1 + 2(5) = 40 \Rightarrow x_1 = 3$. Thus we need 3 servings of Lucky Charms and 5 servings of Raisin Bran.

52. Let x_1 be the number of servings of Lucky Charms and x_2 the number of servings of Raisin Bran, and then we need

$$x_1 \begin{bmatrix} 10 \\ 25 \\ 25 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 25 \\ 10 \end{bmatrix} = \begin{bmatrix} 34 \\ 125 \\ 95 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 10 & 2 & 34 \\ 25 & 25 & 125 \\ 25 & 10 & 95 \end{bmatrix} \overset{\substack{(-5/2)R_1+R_2\to R_2 \\ (-5/2)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 10 & 2 & 34 \\ 0 & 20 & 40 \\ 0 & 5 & 10 \end{bmatrix} \overset{(-1/4)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 10 & 2 & 34 \\ 0 & 20 & 40 \\ 0 & 0 & 0 \end{bmatrix}$$

From row 2, we have $20x_2 = 40 \Rightarrow x_2 = 2$. From row 1, $10x_1 + 2(2) = 34 \Rightarrow x_1 = 3$. Thus we need 3 servings of Lucky Charms and 2 servings of Raisin Bran.

53. Let x_1 be the number of servings of Lucky Charms and x_2 the number of servings of Raisin Bran, and then we need

$$x_1 \begin{bmatrix} 10 \\ 25 \\ 25 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 25 \\ 10 \end{bmatrix} = \begin{bmatrix} 26 \\ 125 \\ 80 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 10 & 2 & 26 \\ 25 & 25 & 125 \\ 25 & 10 & 80 \end{bmatrix} \overset{\substack{(-5/2)R_1+R_2\to R_2 \\ (-5/2)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 10 & 2 & 26 \\ 0 & 20 & 60 \\ 0 & 5 & 15 \end{bmatrix} \overset{(-1/4)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 10 & 2 & 26 \\ 0 & 20 & 60 \\ 0 & 0 & 0 \end{bmatrix}$$

From row 2, we have $20x_2 = 60 \Rightarrow x_2 = 3$. From row 1, $10x_1 + 2(3) = 26 \Rightarrow x_1 = 2$. Thus we need 2 servings of Lucky Charms and 3 servings of Raisin Bran.

54. Let x_1 be the number of servings of Lucky Charms and x_2 the number of servings of Raisin Bran, and then we need

$$x_1 \begin{bmatrix} 10 \\ 25 \\ 25 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 25 \\ 10 \end{bmatrix} = \begin{bmatrix} 38 \\ 175 \\ 115 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 10 & 2 & 38 \\ 25 & 25 & 175 \\ 25 & 10 & 115 \end{bmatrix} \underset{\substack{(-5/2)R_1+R_2\to R_2 \\ (-5/2)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 10 & 2 & 38 \\ 0 & 20 & 80 \\ 0 & 5 & 20 \end{bmatrix} \underset{(-1/4)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 10 & 2 & 38 \\ 0 & 20 & 80 \\ 0 & 0 & 0 \end{bmatrix}$$

From row 2, we have $20x_2 = 80 \Rightarrow x_2 = 4$. From row 1, $10x_1 + 2(4) = 38 \Rightarrow x_1 = 3$. Thus we need 3 servings of Lucky Charms and 4 servings of Raisin Bran.

55. (a) $\mathbf{a} = \begin{bmatrix} 2000 \\ 8000 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3000 \\ 10000 \end{bmatrix}$

(b) $8\mathbf{b} = (8)\begin{bmatrix} 3000 \\ 10000 \end{bmatrix} = \begin{bmatrix} 24000 \\ 80000 \end{bmatrix}$. The company produces 24000 computer monitors and 80000 flat panel televisions at facility B in 8 weeks.

(c) $6\mathbf{a} + 6\mathbf{b} = 6\begin{bmatrix} 2000 \\ 8000 \end{bmatrix} + 6\begin{bmatrix} 3000 \\ 10000 \end{bmatrix} = \begin{bmatrix} 30000 \\ 108000 \end{bmatrix}$. The company produces 30000 computer monitors and 108000 flat panel televisions at facilities A and B in 6 weeks.

(d) Let x_1 be the number of weeks of production at facility A, and x_2 the number of weeks of production at facility B, and then we need

$$x_1 \begin{bmatrix} 2000 \\ 8000 \end{bmatrix} + x_2 \begin{bmatrix} 3000 \\ 10000 \end{bmatrix} = \begin{bmatrix} 24000 \\ 92000 \end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix} 2000 & 3000 & 24000 \\ 8000 & 10000 & 92000 \end{bmatrix} \underset{(-4)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 2000 & 3000 & 24000 \\ 0 & -2000 & -4000 \end{bmatrix}$$

From row 2, we have $-2000x_2 = -4000 \Rightarrow x_2 = 2$. From row 1, $2000x_1 + 3000(2) = 24000 \Rightarrow x_1 = 9$. Thus we need 9 weeks of production at facility A and 2 weeks of production at facility B.

56. We assume a 5-day work week.

(a) $\mathbf{a} = \begin{bmatrix} 10 \\ 20 \\ 10 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 40 \\ 70 \\ 50 \end{bmatrix}$

(b) $20\mathbf{c} = (20)\begin{bmatrix} 40 \\ 70 \\ 50 \end{bmatrix} = \begin{bmatrix} 800 \\ 1400 \\ 1000 \end{bmatrix}$. The company produces 800 metric tons of PE, 1400 metric tons of PVC, and 1000 metric tons of PS at facility C in 4 weeks.

(c) $10\mathbf{a} + 10\mathbf{b} + 10\mathbf{c} = 10\begin{bmatrix} 10 \\ 20 \\ \end{bmatrix} + 10\begin{bmatrix} 20 \\ 30 \\ \end{bmatrix} + 10\begin{bmatrix} 40 \\ 70 \\ \end{bmatrix} = \begin{bmatrix} 700 \\ 1200 \\ \end{bmatrix}$. The company produces 700

10 40 50 1000

metric tons of PE, 1200 metric tons of PVC, and 1000 metric tons of PS at facilities A,B, and C in 2 weeks.

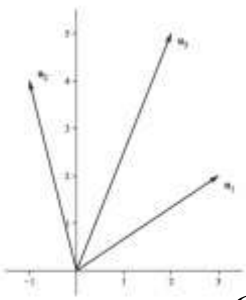
(d) Let x_1 be the number of days of production at facility A, x_2 the number of days of production at facility B, and x_3 the number of days of production at facility C. Then we need

$$x_1\begin{bmatrix}10\\20\\10\end{bmatrix}+x_2\begin{bmatrix}20\\30\\40\end{bmatrix}+x_3\begin{bmatrix}40\\70\\50\end{bmatrix}=\begin{bmatrix}240\\420\\320\end{bmatrix}$$

Solve using the corresponding augmented matrix:

$$\begin{bmatrix}10&20&40&240\\20&30&70&420\\10&40&50&320\end{bmatrix}\underset{-R_1+R_3\to R_3}{\overset{-2R_1+R_2\to R_2}{\sim}}\begin{bmatrix}10&20&40&240\\0&-10&-10&-60\\0&20&10&80\end{bmatrix}\overset{2R_2+R_3\to R_3}{\sim}\begin{bmatrix}10&20&40&240\\0&-10&-10&-60\\0&0&-10&-40\end{bmatrix}$$

From row 3, we have $-10x_3 = -40 \Rightarrow x_3 = 4$. From row 2, $-10x_2 - 10(4) = -60 \Rightarrow x_2 = 2$. From row 1, $10x_1 + 20(2) + 40(4) = 240 \Rightarrow x_1 = 4$. Thus we need 4 days of production at facility A, 2 days of production at facility B, and 4 days of production at facility C.

57. 

$$\bar{v} = \frac{5u_1+3u_2+2u_3}{5+3+2} = \frac{1}{10}\left(5\begin{bmatrix}3\\2\end{bmatrix}+3\begin{bmatrix}-1\\4\end{bmatrix}+2\begin{bmatrix}2\\5\end{bmatrix}\right) = \frac{1}{10}\begin{bmatrix}16\\32\end{bmatrix} = \begin{bmatrix}\frac{8}{5}\\\frac{16}{5}\end{bmatrix}$$

58. $$\bar{v} = \frac{4u_1+1u_2+2u_3+5u_4}{4+1+2+5} = \frac{1}{12}\left(4\begin{bmatrix}-1\\0\\2\end{bmatrix}+1\begin{bmatrix}2\\1\\-3\end{bmatrix}+2\begin{bmatrix}0\\4\\3\end{bmatrix}+5\begin{bmatrix}5\\2\\0\end{bmatrix}\right) = \frac{1}{12}\begin{bmatrix}23\\19\\11\end{bmatrix} = \begin{bmatrix}\frac{23}{12}\\\frac{19}{12}\\\frac{11}{12}\end{bmatrix}$$

59. Let x_1, x_2, and x_3 be the mass of u_1, u_2, and u_3 respectively. Then

$$\bar{v} = \frac{x_1u_1 + x_2u_2 + x_3u_3}{11} = \frac{1}{11}\left(x_1\begin{bmatrix}-1\\3\end{bmatrix}+x_2\begin{bmatrix}3\\-2\end{bmatrix}+x_3\begin{bmatrix}5\\2\end{bmatrix}\right)$$

$$= \begin{bmatrix}-\frac{1}{11}x_1+\frac{3}{11}x_2+\frac{5}{11}x_3\\\frac{3}{11}x_1-\frac{2}{11}x_2+\frac{2}{11}x_3\end{bmatrix} = \begin{bmatrix}\frac{13}{11}\\\frac{16}{11}\end{bmatrix}$$

We obtain the 2 equations, $-x_1+3x_2+5x_3 = 13$ and $3x_1-2x_2+2x_3 = 16$. Together with the equation $x_1+x_2+x_3 = 11$, we have 3 equations and solve the corresponding augmented matrix:

$$\begin{bmatrix}-1&3&5&13\\3&-2&2&16\\1&1&1&11\end{bmatrix}\underset{R_1+R_3\to R_3}{\overset{3R_1+R_2\to R_2}{\sim}}\begin{bmatrix}-1&3&5&13\\0&7&17&55\\0&4&6&24\end{bmatrix}\overset{(-4/7)R_2+R_3\to R_3}{\sim}\begin{bmatrix}-1&3&5&13\\0&7&17&55\\0&0&-\frac{26}{7}&-\frac{52}{7}\end{bmatrix}$$

From row 3, $-\frac{26}{7}x_3 = -\frac{52}{7}$

$\Rightarrow$ $x_3 = 2$. From row 2, $7x_2 + 17(2) = 55$ $\Rightarrow$ $x_2 = 3$. From row 1, $-x_1 + 3(3) + 5(2) = 13$ $\Rightarrow$ $x_1 = 6$.

60. Let $x_1, x_2, x_3,$ and x_4 be the mass of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$ respectively. Then

$$\bar{\mathbf{v}} = \frac{x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + x_4\mathbf{u}_4}{11} = \frac{1}{11}\left(x_1\begin{bmatrix}1\\1\\2\end{bmatrix} + x_2\begin{bmatrix}2\\-1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\3\\2\end{bmatrix} + x_4\begin{bmatrix}-1\\0\\1\end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{1}{11}x_1 + \frac{2}{11}x_2 - \frac{1}{11}x_4 \\ \frac{1}{11}x_1 - \frac{1}{11}x_2 + \frac{3}{11}x_3 \\ \frac{2}{11}x_1 + \frac{2}{11}x_3 + \frac{1}{11}x_4 \end{bmatrix} = \begin{bmatrix} \frac{4}{11} \\ \frac{5}{11} \\ \frac{12}{11} \end{bmatrix}$$

We obtain the 3 equations, $x_1 + 2x_2 - x_4 = 4$, $x_1 - x_2 + 3x_3 = 5$, and $2x_1 + 2x_3 + x_4 = 12$. Together with the equation $x_1 + x_2 + x_3 + x_4 = 11$, we have 4 equations and solve the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 4 \\ 1 & -1 & 3 & 0 & 5 \\ 2 & 0 & 2 & 1 & 12 \\ 1 & 1 & 1 & 1 & 11 \end{bmatrix} \underset{\substack{-R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3 \\ -R_1+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & 2 & 0 & -1 & 4 \\ 0 & -3 & 3 & 1 & 1 \\ 0 & -4 & 2 & 3 & 4 \\ 0 & -1 & 1 & 2 & 7 \end{bmatrix}$$

$$\underset{\substack{(-4/3)R_2+R_3\to R_3 \\ (-1/3)R_2+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & 2 & 0 & -1 & 4 \\ 0 & -3 & 3 & 1 & 1 \\ 0 & 0 & -2 & \frac{5}{3} & \frac{8}{3} \\ 0 & 0 & 0 & \frac{5}{3} & \frac{20}{3} \end{bmatrix}$$

From row 4, $\frac{5}{3}x_4 = \frac{20}{3} \Rightarrow x_4 = 4$. From row 3, $-2x_3 + \frac{5}{3}(4) = \frac{8}{3} \Rightarrow x_3 = 2$. From row 2, $-3x_2 + 3(2) + 4 = 1 \Rightarrow x_2 = 3$. From row 1, $x_1 + 2(3) - 4 = 4 \Rightarrow x_1 = 2$.

61. For example, $\mathbf{u} = (0, 0, -1)$ and $\mathbf{v} = (3, 2, 0)$.

62. For example, $\mathbf{u} = (4, 0, 0, 0)$ and $\mathbf{v} = (0, 2, 0, 1)$.

63. For example, $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (1, 0, 0)$, and $\mathbf{w} = (-2, 0, 0)$.

64. For example, $\mathbf{u} = (1, 0, 0, 0)$, $\mathbf{v} = (1, 0, 0, 0)$, and $\mathbf{w} = (-2, 0, 0, 0)$.

65. For example, $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (2, 0)$.

66. For example, $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (-1, 0)$.

67. For example, $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (2, 0, 0)$, and $\mathbf{w} = (3, 0, 0)$.

68. For example, $\mathbf{u} = (1, 0, 0, 0)$, $\mathbf{v} = (2, 0, 0, 0)$, $\mathbf{w} = (2, 0, 0, 0)$, and $\mathbf{x} = (4, 0, 0, 0)$.

69. Simply, $x_1 = 3$ and $x_2 = -2$.

70. For example, $x_1 - 2x_2 = 1$ and $x_2 + x_3 = 1$.

71. (a) True, since $-2\begin{bmatrix}-3\\5\end{bmatrix} = \begin{bmatrix}(-2)(-3)\\(-2)(5)\end{bmatrix} = \begin{bmatrix}6\\-10\end{bmatrix}$.

(b) False, since $\mathbf{u} - \mathbf{v} = \begin{bmatrix}1\\3\end{bmatrix} - \begin{bmatrix}-4\\2\end{bmatrix} = \begin{bmatrix}1-(-4)\\3-2\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix} \neq \begin{bmatrix}-3\\1\end{bmatrix}$.

72. (a) False. Scalars may be any real number, such as $c = -1$.

(b) True. Vector components and scalars can be any real numbers.

73. (a) True, by Theorem 2.3(b).

(b) False. The sum $c_1 + \mathbf{u}_1$ of a scalar and a vector is undefined.

74. (a) False. A vector can have any initial point.

(b) False. They do not point in opposite directions, as there does not exist $c < 0$ such that $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = c \begin{bmatrix} -2 \\ 4 \\ 8 \end{bmatrix}$.

75. (a) True, by Definition 2.1, where it is stated that vectors can be expressed in column or row form.

(b) True. For any vector $\mathbf{v}$, $\mathbf{0} = 0\mathbf{v}$.

76. (a) True, because $-2(-\mathbf{u}) = (-2)((-1)\mathbf{u}) = ((-2)(-1))\mathbf{u} = 2\mathbf{u}$.

(b) False. For example, $x \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has no solution.

77. (a) False. It works regardless of the quadrant, and can be established algebraically for vectors positioned anywhere.

(b) False. Because vector addition is commutative, one can order the vectors in either way for the Tip-to-Tail Rule.

78. (a) False. For instance, if $\mathbf{u} = (2, 1)$ and $\mathbf{v} = (-1, 3)$, then $\mathbf{u} - \mathbf{v} = (3, -2)$ while $-\mathbf{u} + \mathbf{v} = (-3, 2)$. (The difference $\mathbf{u} - \mathbf{v}$ is found by adding $\mathbf{u}$ to $-\mathbf{v}$.)

(b) True, as long as the vectors have the same number of components.

79. (a) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$. Then $(a + b)\mathbf{u} = (a + b)\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} (a+b)u_1 \\ (a+b)u_2 \\ \vdots \\ (a+b)u_n \end{bmatrix}$

$$= \begin{bmatrix} au_1 + bu_1 \\ au_2 + bu_2 \\ \vdots \\ au_n + bu_n \end{bmatrix} = \begin{bmatrix} au_1 \\ au_2 \\ \vdots \\ au_n \end{bmatrix} + \begin{bmatrix} bu_1 \\ bu_2 \\ \vdots \\ bu_n \end{bmatrix} = a\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + b\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = a\mathbf{u} + b\mathbf{u}.$$

(b) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$. Then

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

$$= \begin{bmatrix} (u_1 + v_1) + w_1 \\ (u_2 + v_2) + w_2 \\ \vdots \\ (u_n + v_n) + w_n \end{bmatrix} = \begin{bmatrix} u_1 + (v_1 + w_1) \\ u_2 + (v_2 + w_2) \\ \vdots \\ u_n + (v_n + w_n) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}\right) = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

(c) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$. Then $a(b\mathbf{u}) = a\left(b\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\right) = a\begin{bmatrix} bu_1 \\ bu_2 \\ \vdots \\ bu_n \end{bmatrix}$

$$= \begin{bmatrix} a(bu_1) \\ a(bu_2) \\ \vdots \\ a(bu_n) \end{bmatrix} = \begin{bmatrix} (ab)u_1 \\ (ab)u_2 \\ \vdots \\ (ab)u_n \end{bmatrix} = (ab)\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = (ab)\mathbf{u}.$$

(d) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$. Then $\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \left(-\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\right)$

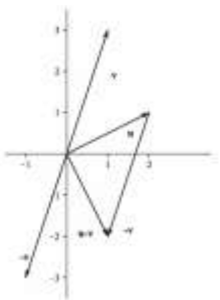
$$= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{bmatrix} = \begin{bmatrix} u_1 - u_1 \\ u_2 - u_2 \\ \vdots \\ u_n - u_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}.$$

(e) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$. Then $\mathbf{u} + \mathbf{0} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 + 0 \\ u_2 + 0 \\ \vdots \\ u_n + 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \mathbf{u}$. Likewise,

$$\mathbf{0} + \mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} 0 + u_1 \\ 0 + u_2 \\ \vdots \\ 0 + u_n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \mathbf{u}.$$

(f) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$. Then $1\mathbf{u} = (1)\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} (1)u_1 \\ (1)u_2 \\ \vdots \\ (1)u_n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \mathbf{u}$.

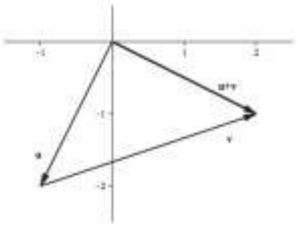
80. Using, for example, $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

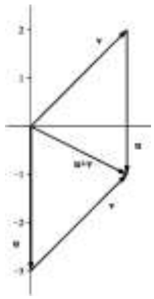


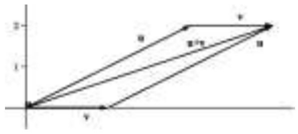
The vector $\mathbf{u} - \mathbf{v} =$

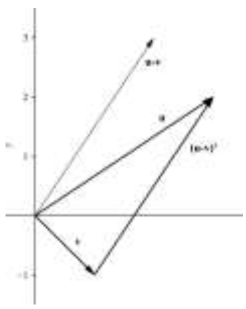
$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is the translation of the vector w' which has initial point the tip of u and terminal point the tip of v, as in Figure 6.

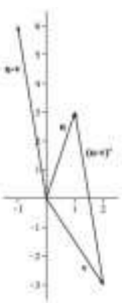
81.

82. 

83. 

84. 

85. 

86. 

87. We obtain the three equations $2x_1 + 2x_2 + 5x_3 = 0$, $7x_1 + 4x_2 + x_3 = 3$, and $3x_1 + 2x_2 + 6x_3 = 5$. Using a computer algebra system to solve this system, we get $x_1 = 4$, $x_2 = -6.5$, and $x_3 = 1$.

88. We obtain the four equations $x_1+4x_2-4x_3+5x_4 = 1$, $-3x_1+3x_2+2x_3+2x_4 = 7$, $2x_1+2x_2-3x_3-4x_4 = 2$, and $x_2 + x_3 = -6$. Using a computer algebra system to solve this system, we get $x_1 = -7.5399$, $x_2 = -1.1656$, $x_3 = -4.8344$, and $x_4 = -1.2270$. (Solving this system exactly, we obtain $x_1 = -\frac{1229}{163}$, $x_2 = -\frac{190}{163}$, $x_3 = -\frac{788}{163}$, and $x_4 = -\frac{200}{163}$.)

2.2 Practice Problems

1. (a) $0\mathbf{u}_1+0\mathbf{u}_2 = 0\begin{bmatrix}2\\-3\end{bmatrix}+0\begin{bmatrix}4\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$, $1\mathbf{u}_1+0\mathbf{u}_2 = 1\begin{bmatrix}2\\-3\end{bmatrix}+0\begin{bmatrix}4\\1\end{bmatrix} = \begin{bmatrix}2\\-3\end{bmatrix}$, $0\mathbf{u}_1+1\mathbf{u}_2 = 0\begin{bmatrix}2\\-3\end{bmatrix}+1\begin{bmatrix}4\\1\end{bmatrix} = \begin{bmatrix}4\\1\end{bmatrix}$

   (b) $0\mathbf{u}_1+0\mathbf{u}_2 = 0\begin{bmatrix}6\\1\\4\end{bmatrix}+0\begin{bmatrix}-2\\3\\-3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $1\mathbf{u}_1+0\mathbf{u}_2 = 1\begin{bmatrix}6\\1\\4\end{bmatrix}+0\begin{bmatrix}-2\\3\\-3\end{bmatrix} = \begin{bmatrix}6\\1\\4\end{bmatrix}$, $0\mathbf{u}_1+1\mathbf{u}_2 = 0\begin{bmatrix}6\\1\\4\end{bmatrix}+1\begin{bmatrix}-2\\3\\-3\end{bmatrix} = \begin{bmatrix}-2\\3\\-3\end{bmatrix}$

2. Set $x_1\mathbf{u}_1+x_2\mathbf{u}_2 = \mathbf{b} \Rightarrow x_1\begin{bmatrix}1\\2\\-2\end{bmatrix}+x_2\begin{bmatrix}0\\4\\3\end{bmatrix} = \begin{bmatrix}-1\\2\\5\end{bmatrix} \Rightarrow \begin{bmatrix}x_1\\2x_1+4x_2\\-2x_1+3x_2\end{bmatrix} = \begin{bmatrix}-1\\2\\5\end{bmatrix}$. From the first equation, $x_1 = -1$. Then the second equation is $2(-1) + 4x_2 = 2 \Rightarrow x_2 = 1$. The third equation is now $-2(-1)+3(1) = 5 \Rightarrow 5 = 5$. So $\mathbf{b}$ is in the span of $\{\mathbf{u}_1,\mathbf{u}_2\}$, with $(-1)\mathbf{u}_1 + (1)\mathbf{u}_2 = \mathbf{b}$.

3. (a) $A = \begin{bmatrix}7 & -2 & -2\\-1 & 7 & 4\\3 & -1 & -2\end{bmatrix}$, $\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}6\\11\\1\end{bmatrix}$

   (b) $A = \begin{bmatrix}4 & -3 & -1 & 5\\3 & 12 & 6 & 0\end{bmatrix}$, $\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}0\\10\end{bmatrix}$

4. (a) Row-reduce to echelon form:

$$\begin{bmatrix}2 & 3\\-1 & -2\end{bmatrix} \overset{(1/2)R_1+R_2\to R_2}{\sim} \begin{bmatrix}2 & 3\\0 & -\frac{1}{2}\end{bmatrix}$$

   There is not a row of zeros, so every choice of $\mathbf{b}$ is in the span of the columns of the given matrix and, therefore, the columns of the matrix span $\mathbf{R}^2$.

   (b) Row-reduce to echelon form:

$$\begin{bmatrix}4 & 1\\1 & -3\end{bmatrix} \overset{(-1/4)R_1+R_2\to R_2}{\sim} \begin{bmatrix}4 & 1\\0 & -\frac{13}{4}\end{bmatrix}$$

Since there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of the columns of the given matrix, and therefore the columns of the matrix span $\mathbf{R}^2$.

5. (a) Row-reduce to echelon form:

$$\begin{bmatrix} 1 & 3 & -1 \\ -1 & -2 & 3 \\ 0 & 2 & 5 \end{bmatrix} \overset{R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \overset{-2R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

There is not a row of zeros, so every choice of $\mathbf{b}$ is in the span of the columns of the given matrix and, therefore, the columns of the matrix span $\mathbf{R}^3$.

(b) Row-reduce to echelon form:

$$\begin{bmatrix} 2 & 0 & 6 \\ 1 & -2 & 1 \\ -1 & 4 & 1 \end{bmatrix} \overset{\substack{(-1/2)R_1+R_2\to R_2 \\ (1/2)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 2 & 0 & 6 \\ 0 & -2 & -2 \\ 0 & 4 & 4 \end{bmatrix} \overset{2R_2+R_3\to R_3}{\sim} \begin{bmatrix} 2 & 0 & 6 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Because there is a row of zeros, there exists a vector $\mathbf{b}$ that is not in the span of the columns of the matrix and, therefore, the columns of the matrix do not span $\mathbf{R}^3$.

6. (a) False. If the vectors span $\mathbf{R}^3$, then vectors have three components, and cannot span $\mathbf{R}^2$.

(b) True. Every vector $\mathbf{b}$ in $\mathbf{R}^2$ can be written as

$$\begin{aligned} \mathbf{b} &= x_1\mathbf{u}_1 + x_2\mathbf{u}_2 \\ &= \frac{x_1}{2}(2\mathbf{u}_1) + \frac{x_2}{3}(3\mathbf{u}_2) \end{aligned}$$

which shows that $\{2\mathbf{u}_1, 3\mathbf{u}_2\}$ spans $\mathbf{R}^2$.

(c) True. Every vector $\mathbf{b}$ in $\mathbf{R}^3$ can be written as $\mathbf{b} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3$. So $A\mathbf{x} = \mathbf{b}$ has the solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(d) True. Every vector $\mathbf{b}$ in $\mathbf{R}^2$ can be written as $\mathbf{b} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + 0\mathbf{u}_3$, so $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans $\mathbf{R}^2$.

2.2 Span

1. $0\mathbf{u}_1 + 0\mathbf{u}_2 = 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 0\begin{bmatrix} 9 \\ 15 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $1\mathbf{u}_1 + 0\mathbf{u}_2 = 1\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 0\begin{bmatrix} 9 \\ 15 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $0\mathbf{u}_1 + 1\mathbf{u}_2 = 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 1\begin{bmatrix} 9 \\ 15 \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \end{bmatrix}$

2. $0\mathbf{u}_1 + 0\mathbf{u}_2 = 0\begin{bmatrix} -2 \\ 7 \end{bmatrix} + 0\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $1\mathbf{u}_1 + 0\mathbf{u}_2 = 1\begin{bmatrix} -2 \\ 7 \end{bmatrix} + 0\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$, $0\mathbf{u}_1 + 1\mathbf{u}_2 = 0\begin{bmatrix} -2 \\ 7 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

3. $0\mathbf{u}_1 + 0\mathbf{u}_2 = 0\begin{bmatrix}2\\5\\-3\end{bmatrix} + 0\begin{bmatrix}1\\0\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $1\mathbf{u}_1 + 0\mathbf{u}_2 = 1\begin{bmatrix}2\\5\\-3\end{bmatrix} + 0\begin{bmatrix}1\\0\\4\end{bmatrix} = \begin{bmatrix}2\\5\\-3\end{bmatrix}$, $0\mathbf{u}_1 + 1\mathbf{u}_2 = 0\begin{bmatrix}2\\5\\-3\end{bmatrix} + 1\begin{bmatrix}1\\0\\4\end{bmatrix} = \begin{bmatrix}1\\0\\4\end{bmatrix}$

4. $0\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 = 0\begin{bmatrix}0\\5\\-2\end{bmatrix} + 0\begin{bmatrix}1\\2\\6\end{bmatrix} + 0\begin{bmatrix}-6\\7\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $1\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 = 1\begin{bmatrix}0\\5\\-2\end{bmatrix} + 0\begin{bmatrix}1\\2\\6\end{bmatrix} +$

$\begin{bmatrix}-6\\ \\2\end{bmatrix} \begin{bmatrix}0\\ \\-2\end{bmatrix}$ $\begin{bmatrix}0\\ \\-2\end{bmatrix} \begin{bmatrix}1\\ \\6\end{bmatrix} \begin{bmatrix}-6\\ \\2\end{bmatrix} \begin{bmatrix}1\\ \\6\end{bmatrix}$

5. $0\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 = 0\begin{bmatrix}2\\0\\0\end{bmatrix} + 0\begin{bmatrix}4\\1\\6\end{bmatrix} + 0\begin{bmatrix}-4\\0\\7\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $1\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 = 1\begin{bmatrix}2\\0\\0\end{bmatrix} + 0\begin{bmatrix}4\\1\\6\end{bmatrix} + 0\begin{bmatrix}-4\\0\\7\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix}$, $0\mathbf{u}_1 + 1\mathbf{u}_2 + 0\mathbf{u}_3 = 0\begin{bmatrix}2\\0\\0\end{bmatrix} + 1\begin{bmatrix}4\\1\\6\end{bmatrix} + 0\begin{bmatrix}-4\\0\\7\end{bmatrix} = \begin{bmatrix}4\\1\\6\end{bmatrix}$

6. $0\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 = 0\begin{bmatrix}0\\1\\3\\0\end{bmatrix} + 0\begin{bmatrix}-1\\8\\-5\\2\end{bmatrix} + 0\begin{bmatrix}12\\-1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$, $1\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 = 1\begin{bmatrix}0\\1\\3\\0\end{bmatrix} + 0\begin{bmatrix}-1\\8\\-5\\2\end{bmatrix} + 0\begin{bmatrix}12\\-1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\3\\0\end{bmatrix}$, $0\mathbf{u}_1 + 1\mathbf{u}_2 + 0\mathbf{u}_3 = 0\begin{bmatrix}0\\1\\3\\0\end{bmatrix} + 1\begin{bmatrix}-1\\8\\-5\\2\end{bmatrix} + 0\begin{bmatrix}12\\-1\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\8\\-5\\2\end{bmatrix}$

7. Set $x_1\mathbf{a}_1 = \mathbf{b} \Rightarrow x_1\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}9\\-15\end{bmatrix} \Rightarrow \begin{bmatrix}3x_1\\5x_1\end{bmatrix} = \begin{bmatrix}9\\-15\end{bmatrix}$.

From the first component, $x_1 = 3$, but from the second component $x_1 = -3$. Thus $\mathbf{b}$ is not in the span of $\mathbf{a}_1$.

8. Set $x_1\mathbf{a}_1 = \mathbf{b} \Rightarrow x_1\begin{bmatrix}10\\-15\end{bmatrix} = \begin{bmatrix}-30\\45\end{bmatrix} \Rightarrow \begin{bmatrix}10x_1\\-15x_1\end{bmatrix} = \begin{bmatrix}-30\\45\end{bmatrix}$.

From the first component, $x_1 = -3$, and from the second component $x_1 = 3$. Thus $\mathbf{b} = -3\mathbf{a}_1$, and $\mathbf{b}$ is in the span of $\mathbf{a}_1$.

9. Set $x_1\mathbf{a}_1 = \mathbf{b} \Rightarrow x_1\begin{bmatrix}4\\-2\\10\end{bmatrix} = \begin{bmatrix}2\\-1\\-5\end{bmatrix} \Rightarrow \begin{bmatrix}4x_1\\-2x_1\\10x_1\end{bmatrix} = \begin{bmatrix}2\\-1\\-5\end{bmatrix}$.

From the first and second components, $x_1 = \frac{1}{2}$, but from the third component $x_1 = -\frac{1}{2}$. Thus $\mathbf{b}$ is not in the span of $\mathbf{a}_1$.

10. Set $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \Rightarrow x_1\begin{bmatrix}-1\\3\end{bmatrix} \; -\; 1 + x_2 \quad \begin{bmatrix}-2\\-3\end{bmatrix}$

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$$\begin{bmatrix} -x_1 - 2x_2 \\ 3x_1 - 3x_2 \\ -x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} -6 \\ 9 \\ 2 \end{bmatrix}$$

. We obtain 3 equations and row-reduce the associated augmented matrix to determine if there are solutions.

$$\begin{bmatrix} -1 & -2 & -6 \\ 3 & -3 & 9 \\ -1 & 6 & 2 \end{bmatrix} \begin{matrix} 3R_1+R_2\to R_2 \\ -R_1+R_3\to R_3 \\ \sim \end{matrix} \begin{bmatrix} -1 & -2 & -6 \\ 0 & -9 & -9 \\ 0 & 8 & 8 \end{bmatrix} \begin{matrix} (8/9)R_2+R_3\to R_3 \\ \sim \end{matrix} \begin{bmatrix} -1 & -2 & -6 \\ 0 & -9 & -9 \\ 0 & 0 & 0 \end{bmatrix}$$

From the second row, $-9x_2 = -9 \Rightarrow x_2 = 1$. From row 1, $-x_1 - 2(1) = -6 \Rightarrow x_1 = 4$. We conclude $\mathbf{b}$ is in the span of $\mathbf{a}_1$ and $\mathbf{a}_2$, with $\mathbf{b} = 4\mathbf{a}_1 + \mathbf{a}_2$.

11. Set $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \Rightarrow x_1 \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 8 \\ -7 \end{bmatrix} = \begin{bmatrix} -10 \\ -8 \\ 7 \end{bmatrix} \Rightarrow$

$\begin{bmatrix} -x_1 + 2x_2 \\ 4x_1 + 8x_2 \\ -3x_1 - 7x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ -8 \\ 7 \end{bmatrix}$. We obtain 3 equations and row-reduce the associated augmented matrix to determine if there are solutions.

$$\begin{bmatrix} -1 & 2 & -10 \\ 4 & 8 & -8 \\ -3 & -7 & 7 \end{bmatrix} \begin{matrix} 4R_1+R_2\to R_2 \\ -3R_1+R_3\to R_3 \\ \sim \end{matrix} \begin{bmatrix} -1 & 2 & -10 \\ 0 & 16 & -48 \\ 0 & -13 & 37 \end{bmatrix} \begin{matrix} (13/16)R_2+R_3\to R_3 \\ \sim \end{matrix} \begin{bmatrix} -1 & 2 & -10 \\ 0 & 16 & -48 \\ 0 & 0 & -2 \end{bmatrix}$$

From the third row, $0 = -2$, and hence there are no solutions. We conclude that there do not exist x_1 and x_2 such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$, and therefore $\mathbf{b}$ is not in the span of $\mathbf{a}_1$ and $\mathbf{a}_2$.

12. Set $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \Rightarrow x_1 \begin{bmatrix} 3 \\ 1 \\ -2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 1 \\ 5 \end{bmatrix} \Rightarrow$

$\begin{bmatrix} 3x_1 - 4x_2 \\ x_1 + 2x_2 \\ -2x_1 + 3x_2 \\ -x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 1 \\ 5 \end{bmatrix}$. We obtain 4 equations and row-reduce the associated augmented matrix to determine if there are solutions.

$$\begin{bmatrix} 3 & -4 & 0 \\ 1 & 2 & 10 \\ -2 & 3 & 1 \\ -1 & 3 & 5 \end{bmatrix} \begin{matrix} (-1/3)R_1+R_2\to R_2 \\ (2/3)R_1+R_3\Rightarrow R_3 \\ (1/3)R_1+R_4\to R_4 \\ \sim \end{matrix} \begin{bmatrix} 3 & -4 & 0 \\ 0 & \frac{10}{3} & 10 \\ 0 & \frac{1}{3} & 1 \\ 0 & \frac{5}{3} & 5 \end{bmatrix} \begin{matrix} (-1/10)R_2+R_4\to R_3 \\ (-1/2)R_3+R_4\to R_4 \\ \sim \end{matrix} \begin{bmatrix} 3 & -4 & 0 \\ 0 & \frac{10}{3} & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From the second row, $\frac{10}{3}x_2 = 10 \Rightarrow x_2 = 3$. From row 1, $3x_1 - 4(3) = 0 \Rightarrow x_1 = 4$. We conclude $\mathbf{b}$ is in the span of $\mathbf{a}_1$ and $\mathbf{a}_2$, with $\mathbf{b} = 4\mathbf{a}_1 + 3\mathbf{a}_2$.

13. $A = \begin{bmatrix} 2 & 8 & -4 \end{bmatrix}$

−1
−3
5 , x = [x_1] [−10
x_2 , b]
x_3 =
4

14. $A = \begin{bmatrix} -2 & 5 & -10 \\ 1 & -2 & 3 \\ 7 & -17 & 34 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ -16 \end{bmatrix}$

15. $A = \begin{bmatrix} 1 & -1 & -3 & -1 \\ -2 & 2 & 6 & 2 \\ -3 & -3 & 10 & 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}$

16. $A = \begin{bmatrix} -5 & 9 \\ 3 & -5 \\ 1 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 13 \\ -9 \\ -2 \end{bmatrix}$

17. $x_1 \begin{bmatrix} 5 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$

18. $x_1 \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 4 \\ -13 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

19. $x_1 \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -5 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 7 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 2 \end{bmatrix}$

20. $x_1 \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -9 \\ 4 \\ -7 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \\ 2 \end{bmatrix}$

21. Row-reduce to echelon form:

$$\begin{bmatrix} 15 & -6 \\ -5 & 2 \end{bmatrix} \overset{(1/3)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 15 & -6 \\ 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there exists a vector $\mathbf{b}$ which is not in the span of the columns of A, and therefore the columns of A do not span $\mathbf{R}^2$.

22. Row-reduce to echelon form:

$$\begin{bmatrix} 4 & -12 \\ 2 & 6 \end{bmatrix} \overset{(-1/2)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 4 & -12 \\ 0 & 12 \end{bmatrix}$$

Since there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of the columns of A, and therefore the columns of A span $\mathbf{R}^2$.

23. Row-reduce to echelon form:

$$\begin{bmatrix} 2 & 1 & 0 \\ 6 & -3 & -1 \end{bmatrix} \overset{-3R_1+R_2\to R_2}{\sim} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -6 & -1 \end{bmatrix}$$

Since there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of the columns of A, and therefore the columns of A span $\mathbf{R}^2$.

24. Row-reduce to echelon form:

$$\begin{bmatrix} 1 & 0 & 5 \end{bmatrix} \overset{2R_1+R_2\to R_2}{} \begin{bmatrix} 1 & 0 & 5 \end{bmatrix}$$

$$-2 \quad 2 \quad 7 \qquad \sim \qquad 0 \quad 2 \quad 17$$

Since there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of A, and therefore the columns of A span $\mathbb{R}^2$.

25. Row-reduce to echelon form:

$$\begin{bmatrix} 3 & 1 & 0 \\ 5 & -2 & -1 \\ 4 & -4 & -3 \end{bmatrix} \underset{\sim}{\overset{(-5/3)R_1+R_2 \to R_2}{(-4/3)R_1+R_3 \to R_3}} \begin{bmatrix} 3 & 1 & 0 \\ 0 & -\frac{11}{3} & -1 \\ 0 & -\frac{16}{3} & -3 \end{bmatrix} \underset{\sim}{(-16/11)R_2+R_3 \to R_3} \begin{bmatrix} 3 & 1 & 0 \\ 0 & -\frac{11}{3} & -1 \\ 0 & 0 & -\frac{17}{11} \end{bmatrix}$$

Since there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of the columns of A, and therefore the columns of A span $\mathbf{R}^3$.

26. Row-reduce to echelon form:

$$\begin{bmatrix} 1 & 2 & 8 \\ -2 & 3 & 7 \\ 3 & -1 & 1 \end{bmatrix} \underset{\sim}{\overset{2R_1+R_2 \to R_2}{-3R_1+R_3 \to R_3}} \begin{bmatrix} 1 & 2 & 8 \\ 0 & 7 & 23 \\ 0 & -7 & -23 \end{bmatrix} \underset{\sim}{R_2+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 8 \\ 0 & 7 & 23 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there exists a vector $\mathbf{b}$ which is not in the span of A, and therefore the columns of A do not span $\mathbf{R}^3$.

27. Row-reduce to echelon form:

$$\begin{bmatrix} 2 & 1 & -3 & 5 \\ 1 & 4 & 2 & 6 \\ 0 & 3 & 3 & 3 \end{bmatrix} \underset{\sim}{(-1/2)R_1+R_2 \to R_2} \begin{bmatrix} 2 & 1 & -3 & 5 \\ 0 & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} \\ 0 & 3 & 3 & 3 \end{bmatrix} \underset{\sim}{(-6/7)R_2+R_3 \to R_3} \begin{bmatrix} 2 & 1 & -3 & 5 \\ 0 & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there exists a vector $\mathbf{b}$ which is not in the span of the columns of A, and therefore the columns of A do not span $\mathbf{R}^3$.

28. Row-reduce to echelon form:

$$\begin{bmatrix} -4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 8 \\ 5 & -1 & 1 & -4 \end{bmatrix} \underset{\sim}{R_2 \leftrightarrow R_3} \begin{bmatrix} -4 & -7 & 1 & 2 \\ 5 & -1 & 1 & -4 \\ 0 & 0 & 3 & 8 \end{bmatrix} \underset{\sim}{(5/4)R_1+R_2 \to R_2} \begin{bmatrix} -4 & -7 & 1 & 2 \\ 0 & -\frac{39}{4} & \frac{9}{4} & -\frac{3}{2} \\ 0 & 0 & 3 & 8 \end{bmatrix}$$

Since there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of A, and therefore the columns of A span $\mathbf{R}^3$.

29. Row-reduce A to echelon form:

$$\begin{bmatrix} 3 & -4 \\ 4 & 2 \end{bmatrix} \underset{\sim}{(-4/3)R_1+R_2 \to R_2} \begin{bmatrix} 3 & -4 \\ 0 & \frac{22}{3} \end{bmatrix}$$

Since there is not a row of zeros, for every choice of $\mathbf{b}$ there is a solution of $A\mathbf{x} = \mathbf{b}$.

30. Row-reduce A to echelon form:

$$\begin{bmatrix} -9 & 21 \\ 6 & -14 \end{bmatrix} \underset{\sim}{(2/3)R_1+R_2\to R_2} \begin{bmatrix} -9 & 21 \\ 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there is a choice of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

31. Since the number of columns, $m = 2$, is less than $n = 3$, the columns of A do not span $\mathbf{R}^3$, and by Theorem 2.9, there is a choice of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

32. Row-reduce A to echelon form.

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & -1 \\ 1 & 0 & 5 \end{bmatrix} \underset{\sim}{\begin{matrix} 2R_1+R_2\to R_2 \\ -R_1+R_3\to R_3 \end{matrix}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \underset{\sim}{-R_2+R_3\to R_3} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there is a choice of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

33. Row-reduce A to echelon form:

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -1 & -1 \\ 5 & -4 & -3 \end{bmatrix} \underset{\sim}{\begin{matrix} (1/3)R_1+R_2\to R_2 \\ (5/3)R_1+R_3\to R_3 \end{matrix}} \begin{bmatrix} -3 & 2 & 1 \\ 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & -\frac{2}{3} & -\frac{4}{3} \end{bmatrix} \underset{\sim}{(-2)R_2+R_3\to R_3} \begin{bmatrix} -3 & 2 & 1 \\ 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there is a choice of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

34. Since the number of columns, $m = 3$, is less than $n = 4$, the columns of A do not span $\mathbf{R}^4$, and by Theorem 2.11, there is a choice of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution.

35. $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not in span $\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}\right\}$, since span $\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}\right\}$ = span $\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$ and $\mathbf{b} = c\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ for any scalar c.

36. $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not in span $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \end{bmatrix}\right\}$, since span $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \end{bmatrix}\right\}$ = span $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$ and $\mathbf{b} = c\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ for any scalar c.

37. $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in span $\left\{\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}$, since $c_1\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solutions.

38. $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in span $\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}\right\}$, since $c_1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solutions.

39. $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in span $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\right\}$, because span $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\right\}$ = span $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

c $_2$ for any scalar c.

40. $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 15 \\ -10 \end{bmatrix} \right\}$, because span $\left\{ \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 15 \\ -10 \end{bmatrix} \right\}$ $=$ span $\left\{ \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$ and $\mathbf{b} = c \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ for any scalar c.

41. $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ 10 \end{bmatrix}, \begin{bmatrix} 7 \\ -14 \end{bmatrix} \right\}$, because span $\left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ 10 \end{bmatrix}, \begin{bmatrix} 7 \\ -14 \end{bmatrix} \right\}$ $=$ span $\left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\}$ and $\mathbf{b} = c \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ for any scalar c.

42. $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} 4 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -6 \\ -15 \end{bmatrix} \right\}$, because span $\left\{ \begin{bmatrix} 4 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -6 \\ -15 \end{bmatrix} \right\}$ $=$ span $\left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$ and $\mathbf{b} = c \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ for any scalar c.

43. $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} \right\}$, because $c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solutions.

44. $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \right\}$, because $c_1 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solutions.

45. $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix} \right\}$, because $c_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solutions.

46. $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, because $c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solutions.

47. $h = 3$, since when $h = 3$ the vectors $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ are parallel and do not span $\mathbf{R}^2$.

48. $h = \frac{12}{5}$, since when $h = \frac{12}{5}$ the vectors $\begin{bmatrix} -3 \\ \frac{12}{5} \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -4 \end{bmatrix}$ are parallel and do not span $\mathbf{R}^2$.

49. $h = 4$. This value for h was determined by row-reducing

$$\begin{bmatrix} 2 & h & 1 \\ 4 & 8 & 2 \\ 5 & 10 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & h & 1 \\ 0 & 8-2h & 0 \\ 0 & 0 & \frac{7}{2} \end{bmatrix}$$

Then $c_1 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} h \\ 8 \\ 10 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ has a solution provided $h = 4$.

5 10 6 z

50. $h = -27$. This value for h was determined by row-reducing

$$\begin{bmatrix} -1 & 4 & 1 \\ h & -2 & -3 \\ 7 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & 1 \\ 0 & 33 & 9 \\ 0 & 0 & -\frac{1}{11}h - \frac{27}{11} \end{bmatrix}$$

Then $c_1 \begin{bmatrix} -1 \\ h \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ has a solution provided $h = -27$.

51. $\mathbf{u}_1 = (1,0,0)$, $\mathbf{u}_2 = (0,1,0)$, $\mathbf{u}_3 = (0,0,1)$, $\mathbf{u}_4 = (1,1,1)$

52. $\mathbf{u}_1 = (1,0,0,0)$, $\mathbf{u}_2 = (0,1,0,0)$, $\mathbf{u}_3 = (0,0,1,0)$, $\mathbf{u}_4 = (0,0,0,1)$

53. $\mathbf{u}_1 = (1,0,0)$, $\mathbf{u}_2 = (2,0,0)$, $\mathbf{u}_3 = (3,0,0)$, $\mathbf{u}_4 = (4,0,0)$

54. $\mathbf{u}_1 = (1,0,0,0)$, $\mathbf{u}_2 = (2,0,0,0)$, $\mathbf{u}_3 = (3,0,0,0)$, $\mathbf{u}_4 = (4,0,0,0)$

55. $\mathbf{u}_1 = (1,0,0)$, $\mathbf{u}_2 = (0,1,0)$

56. $\mathbf{u}_1 = (0,1,0,0)$, $\mathbf{u}_2 = (0,0,1,0)$, $\mathbf{u}_3 = (0,0,0,1)$

57. $\mathbf{u}_1 = (1,-1,0)$, $\mathbf{u}_2 = (1,0,-1)$

58. $\mathbf{u}_1 = (1,-1,0,0)$, $\mathbf{u}_2 = (1,0,-1,0)$, $\mathbf{u}_3 = (1,0,0,-1)$

59. (a) True, by Theorem 2.9.
    (b) False, the zero vector can be included with any set of vectors which already span $\mathbf{R}^n$.

60. (a) False, since every column of A may be a zero column.
    (b) False, by Example 5.

61. (a) False. Consider $A = [1]$.
    (b) True, by Theorem 2.11.

62. (a) True, the span of a set of vectors can only increase (with respect to set containment) when adding a vector to the set.
    (b) False. Consider $\mathbf{u}_1 = (0,0,0)$, $\mathbf{u}_2 = (1,0,0)$, $\mathbf{u}_3 = (0,1,0)$, and $\mathbf{u}_4 = (0,0,1)$.

63. (a) False. Consider $\mathbf{u}_1 = (0,0,0)$, $\mathbf{u}_2 = (1,0,0)$, $\mathbf{u}_3 = (0,1,0)$, and $\mathbf{u}_4 = (0,0,1)$.
    (b) True. The span of $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ will be a subset of the span of $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$.

64. (a) True. span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\} \subseteq$ span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$ is always true. If a vector $\mathbf{w} \in$ span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$, then since $\mathbf{u}_4$ is a linear combination of $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$, we can express $\mathbf{w}$ as a linear combination of just the vectors $\mathbf{u}_1,\mathbf{u}_2$, and $\mathbf{u}_3$. Hence $\mathbf{w}$ is in span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$, and we have span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\} \subseteq$ span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$.
    (b) False. If $\mathbf{u}_4$ is a linear combination of $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ then span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$ = span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$. (See problem 61, and the solutions to problems 43 and 45 for examples.)

65. (a) False. Consider $\mathbf{u}_1 = (1,0,0,0)$, $\mathbf{u}_2 = (0,1,0,0)$, $\mathbf{u}_3 = (0,0,1,0)$, and $\mathbf{u}_4 = (0,0,0,1)$.
    (b) True. Since $\mathbf{u}_4 \in$ span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$, but $\mathbf{u}_4 \notin$ span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$.

66. (a) True, because $c_1\mathbf{0}+c_2\mathbf{u}_1+c_3\mathbf{u}_2+c_4\mathbf{u}_3 = c_2\mathbf{u}_1+c_3\mathbf{u}_2+c_4\mathbf{u}_3$, span $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ = span $\{\mathbf{0},\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$.
    (b) False, because span $\{\mathbf{u}_1,\mathbf{u}_2\}$ = span $\{\mathbf{u}_1\} \neq \mathbf{R}^2$, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin$ span $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

67. (a) Cannot possibly span $\mathbf{R}^3$, since $m = 1 < n = 3$.
    (b) Cannot possibly span $\mathbf{R}^3$, since $m = 2 < n = 3$.
    (c) Can possibly span $\mathbf{R}^3$. For example, $\mathbf{u}_1 = (1,0,0)$, $\mathbf{u}_2 = (0,1,0)$, $\mathbf{u}_3 = (0,0,1)$.
    (d) Can possibly span $\mathbf{R}^3$. For example, $\mathbf{u}_1 = (1,0,0)$, $\mathbf{u}_2 = (0,1,0)$, $\mathbf{u}_3 = (0,0,1)$, $\mathbf{u}_4 = (0,0,0)$.

68. (a) Cannot possibly span $\mathbf{R}^3$, since $m = 1 < n = 3$.
    (b) Cannot possibly span $\mathbf{R}^3$, since $m = 1 < n = 3$.

(c) Can possibly span $\mathbb{R}^3$. For example, $\mathbf{u}_1 = (1,0,0)$, $\mathbf{u}_2 = (0,1,0)$, $\mathbf{u}_3 = (0,0,1)$.

(d) Can possibly span $\mathbb{R}^3$. For example, $\mathbf{u}_1 = (1,0,0)$, $\mathbf{u}_2 = (0,1,0)$, $\mathbf{u}_3 = (0,0,1)$, $\mathbf{u}_4 = (0,0,0)$.

69. Let $\mathbf{w} \in \text{span}\{\mathbf{u}\}$, then $\mathbf{w} = x_1\mathbf{u} = \left(\frac{x_1}{c}\right)(c\mathbf{u})$, so $\mathbf{w} \in \text{span}\{c\mathbf{u}\}$ and thus $\text{span}\{\mathbf{u}\} \subseteq \text{span}\{c\mathbf{u}\}$. Now let $\mathbf{w} \in \text{span}\{c\mathbf{u}\}$, then $\mathbf{w} = x_1(c\mathbf{u}) = (x_1c)(\mathbf{u})$, so $\mathbf{w} \in \text{span}\{\mathbf{u}\}$ and thus $\text{span}\{c\mathbf{u}\} \subseteq \text{span}\{\mathbf{u}\}$. Together, we conclude $\text{span}\{\mathbf{u}\} = \text{span}\{c\mathbf{u}\}$.

70. Let $\mathbf{w} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\mathbf{w} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 = \left(\frac{x_1}{c_1}\right)(c_1\mathbf{u}_1) + \left(\frac{x_2}{c_2}\right)(c_2\mathbf{u}_2)$, so $\mathbf{w} \in \text{span}\{c_1\mathbf{u}_1, c_2\mathbf{u}_2\}$ and thus $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\} \subseteq \text{span}\{c_1\mathbf{u}_1, c_2\mathbf{u}_2\}$. Now let $\mathbf{w} \in \text{span}\{c_1\mathbf{u}_1, c_2\mathbf{u}_2\}$, then $\mathbf{w} = x_1(c_1\mathbf{u}_1) + x_2(c_2\mathbf{u}_2) = (x_1c_1)(\mathbf{u}_1) + (x_2c_2)(\mathbf{u}_2)$, so $\mathbf{w} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and thus $\text{span}\{c_1\mathbf{u}_1, c_2\mathbf{u}_2\} \subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Together, we conclude $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \text{span}\{c_1\mathbf{u}_1, c_2\mathbf{u}_2\}$.

71. We may let $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ and $S_2 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{u}_{m+1}, \ldots \mathbf{u}_n\}$ where $m \le n$. Let $\mathbf{w} \in \text{span}(S_1)$, then

$$\begin{aligned} \mathbf{w} &= x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m \\ &= x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_m\mathbf{u}_m + 0\mathbf{u}_{m+1} + \cdots + 0\mathbf{u}_n \end{aligned}$$

and thus $\mathbf{w} \in \text{span}(S_2)$. We conclude that $\text{span}(S_1) \subseteq \text{span}(S_2)$.

72. Let $\mathbf{b} \in \mathbb{R}^2$, then $\mathbf{b} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2$ for some scalars x_1 and x_2 because $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \mathbb{R}^2$. We can rewrite $\mathbf{b} = \frac{x_1+x_2}{2}(\mathbf{u}_1 + \mathbf{u}_2) + \frac{x_1-x_2}{2}(\mathbf{u}_1 - \mathbf{u}_2)$, thus $\mathbf{b} \in \text{span}\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2\}$. Since $\mathbf{b}$ was arbitrary, $\text{span}\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2\} = \mathbb{R}^2$.

73. Let $\mathbf{b} \in \mathbb{R}^3$, then $\mathbf{b} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3$ for some scalars x_1, x_2, and x_3 because $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \mathbb{R}^3$. We can rewrite $\mathbf{b} = \frac{x_1+x_2-x_3}{2}(\mathbf{u}_1 + \mathbf{u}_2) + \frac{x_1-x_2+x_3}{2}(\mathbf{u}_1 + \mathbf{u}_3) + \frac{-x_1+x_2+x_3}{2}(\mathbf{u}_2 + \mathbf{u}_3)$, thus $\mathbf{b} \in \text{span}\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3, \mathbf{u}_2 + \mathbf{u}_3\}$. Since $\mathbf{b}$ was arbitrary, $\text{span}\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3, \mathbf{u}_2 + \mathbf{u}_3\} = \mathbb{R}^3$.

74. If $\mathbf{b}$ is in $\text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$, then by Theorem 2.11 the linear system corresponding to the augmented matrix

$$[\mathbf{u}_1 \ \cdots \ \mathbf{u}_m \ \ \mathbf{b}]$$

has at least one solution. Since $m > n$, this system has more variables than equations. Hence the echelon form of the system will have free variables, and since the system is consistent this implies that it has infinitely many solutions.

75. Let $A = [\mathbf{u}_1 \cdots \mathbf{u}_m]$ and suppose $A \sim B$, where B is in echelon form. Since $m < n$, the last row of B must consist of zeros. Form B_1 by appending to B the vector $\mathbf{e} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$, so that $B_1 = [B \ \ \mathbf{e}]$. If B_1 is viewed as an augmented matrix, then the bottom row corresponds to the equation $0 = 1$, so the corresponding linear system is inconsistent. Now reverse the row operations used to transform A to B, and apply these to B_1. Then the resulting matrix will have the form $[A \ \ \mathbf{e}']$. This implies that $\mathbf{e}'$ is not in the span of the columns of A, as required.

76. [$(a) \Rightarrow (b)$] Since $\mathbf{b} \in \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ there exists scalars $x_1, x_2, \ldots, x_m$ such that $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_m\mathbf{a}_m$, which is statement (b).
[$(b) \Rightarrow (c)$] The linear system corresponding to $[\ \mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \cdots \ \ \mathbf{a}_m \ \ \mathbf{b}\ ]$ can be expressed by the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_m\mathbf{a}_m = \mathbf{b}$. By (b), $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_m\mathbf{a}_m = \mathbf{b}$ has a solution, hence we conclude that linear system corresponding to $[\ \mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \cdots \ \ \mathbf{a}_m \ \ \mathbf{b}\ ]$ has a solution.

[$(c) \Rightarrow (d)$] $A\mathbf{x} = \mathbf{b}$ has a solution provided the augmented matrix $[\ A \ \ \mathbf{b}\ ]$ has a solution. In terms of the columns of A, this is true if the augmented matrix $[\ \mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \cdots \ \ \mathbf{a}_m \ \ \mathbf{b}\ ]$ has a solution. This is what (c) implies, hence $A\mathbf{x} = \mathbf{b}$ has a solution.

[$(d) \Rightarrow (a)$] If $A\mathbf{x} = \mathbf{b}$ has a solution, then $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_m\mathbf{a}_m = \mathbf{b}$ where $A = [\ \mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \cdots \ \ \mathbf{a}_m\ ]$ and $\mathbf{x} = (x_1, x_2, \ldots, x_m)$. Thus $\mathbf{b} \in \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$.

77. True. Using a computer algebra system, the row-reduced echelon form of the matrix with the given vectors as columns does not have any zero rows. Hence the vectors span $\mathbf{R}^3$.
78. False. Using a computer algebra system, the row-reduced echelon form of the matrix with the given vectors as columns does have a zero row. Hence the vectors do not span $\mathbf{R}^3$.
79. False. Using a computer algebra system, the row-reduced echelon form of the matrix with the given vectors as columns does have a zero row. Hence the vectors do not span $\mathbf{R}^4$.
80. True. Using a computer algebra system, the row-reduced echelon form of the matrix with the given vectors as columns does not have any zero rows. Hence the vectors span $\mathbf{R}^4$.

2.3 Practice Problems

Section 2.3

1. (a) Consider $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} 2 & 4 & 0 \\ -3 & 1 & 0 \end{bmatrix} \overset{(3/2)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 2 & 4 & 0 \\ 0 & 7 & 0 \end{bmatrix}$$

The only solution is the trivial solution, so the vectors are linearly independent.

(b) Consider $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} 6 & -2 & 0 \\ 1 & 3 & 0 \\ 4 & -3 & 0 \end{bmatrix} \overset{\substack{(-1/6)R_1+R_2\to R_2 \\ (-2/3)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 6 & -2 & 0 \\ 0 & \frac{10}{3} & 0 \\ 0 & -\frac{5}{3} & 0 \end{bmatrix} \overset{(1/2)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 6 & -2 & 0 \\ 0 & \frac{10}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The only solution is the trivial solution, so the vectors are linearly independent.

2. (a) We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 5 & 0 \\ 3 & -4 & 0 \end{bmatrix} \overset{-3R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1 & 5 & 0 \\ 0 & -19 & 0 \end{bmatrix}$$

The only solution is the trivial solution, so the columns of the matrix are linearly independent.

(b) We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 4 & 0 \\ -3 & 7 & 2 & 0 \end{bmatrix} \overset{\substack{-2R_1+R_2\to R_2 \\ 3R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 7 & 11 & 0 \end{bmatrix} \overset{(7/2)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

There is only the trivial solution; the columns of the matrix are linearly independent.

3. (a) We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 8 & 4 & 0 \end{bmatrix} \overset{-2R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because there exist nontrivial solutions, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

(b) We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ -2 & 2 & 1 & 0 & 0 \end{bmatrix} \overset{\substack{R_1+R_2\to R_2 \\ 2R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 2 & -1 & 2 & 0 \end{bmatrix} \overset{2R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & -3 & 6 & 0 \end{bmatrix}$$

Because there exist nontrivial solutions, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

4. (a) False, because $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is linearly independent in $\mathbf{R}^3$ but does not span $\mathbf{R}^3$.

(b) True, by the Unifying Theorem.

(c) True. Because $\mathbf{u}_1 - 4\mathbf{u}_2 = 4\mathbf{u}_2 - 4\mathbf{u}_2 = \mathbf{0}$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly dependent.

(d) False. Suppose $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then the columns of A are linearly dependent, and $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has no solutions.

2.3 Linear Independence

1. Consider $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} 3 & -1 & 0 \\ -2 & -4 & 0 \end{bmatrix} \overset{(2/3)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 3 & -1 & 0 \\ 0 & -\frac{14}{3} & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the vectors are linearly independent.

2. Consider $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} 6 & -4 & 0 \\ -15 & -10 & 0 \end{bmatrix} \overset{(5/2)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 6 & -4 & 0 \\ 0 & -20 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the vectors are linearly independent.

3. Consider $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} 7 & 5 & 0 \\ 1 & -3 & 0 \\ -13 & 2 & 0 \end{bmatrix} \overset{\substack{(-1/7)R_1+R_2\to R_2 \\ (13/7)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 7 & 5 & 0 \\ 0 & -\frac{26}{7} & 0 \\ 0 & \frac{79}{7} & 0 \end{bmatrix} \overset{(79/26)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 7 & 5 & 0 \\ 0 & -\frac{26}{7} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the vectors are linearly independent.

4. Consider $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} -4 & -2 & -8 & 0 \\ 0 & -1 & 2 & 0 \\ -3 & 5 & -19 & 0 \end{bmatrix} \overset{(-3/4)R_1+R_3\to R_3}{\sim} \begin{bmatrix} -4 & -2 & -8 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & \frac{13}{2} & -13 & 0 \end{bmatrix}$$

$$\overset{(13/2)R_2+R_3\to R_3}{\sim} \begin{bmatrix} -4 & -2 & -8 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the vectors are not linearly independent.

5. Consider $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} 3 & 0 & 2 & 0 \\ -1 & 4 & 4 & 0 \\ 2 & 1 & 7 & 0 \end{bmatrix} \underset{\sim}{\overset{(1/3)R_1+R_2\to R_2 \atop (-2/3)R_1+R_3\to R_3}{}} \begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & 4 & \frac{14}{3} & 0 \\ 0 & 1 & \frac{17}{3} & 0 \end{bmatrix} \underset{\sim}{\overset{(-1/4)R_2+R_3\to R_3}{}} \begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & 4 & \frac{14}{3} & 0 \\ 0 & 0 & \frac{9}{2} & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the vectors are linearly independent.

6. Consider $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$, and solve using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 4 & -1 & 0 \\ 8 & -2 & 2 & 0 \\ 3 & 5 & 0 & 0 \\ 3 & -5 & 1 & 0 \end{bmatrix} \underset{\sim}{\overset{-8R_1+R_2\to R_2 \atop {-3R_1+R_3\to R_3 \atop -3R_1+R_4\to R_4}}{}} \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & -34 & 10 & 0 \\ 0 & -7 & 3 & 0 \\ 0 & -17 & 4 & 0 \end{bmatrix}$$

$$\underset{\sim}{\overset{(-7/34)R_2+R_3\to R_3 \atop (-1/2)R_2+R_4\to R_4}{}} \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & -34 & 10 & 0 \\ 0 & 0 & \frac{16}{17} & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \underset{\sim}{\overset{(17/16)R_3+R_4\to R_4}{}} \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & -34 & 10 & 0 \\ 0 & 0 & \frac{16}{17} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the vectors are linearly independent.

7. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 15 & -6 & 0 \\ -5 & 2 & 0 \end{bmatrix} \underset{\sim}{\overset{(2/3)R_1+R_2\to R_2}{}} \begin{bmatrix} 15 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the columns of A are not linearly independent.

8. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 4 & -12 & 0 \\ 2 & 6 & 0 \end{bmatrix} \underset{\sim}{\overset{(-1/2)R_1+R_2\to R_2}{}} \begin{bmatrix} 4 & -12 & 0 \\ 0 & 12 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of A are linearly independent.

9. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 5 & -7 & 0 \end{bmatrix} \underset{\sim}{\overset{2R_1+R_2\to R_2 \atop -5R_1+R_3\to R_3}{}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -7 & 0 \end{bmatrix} \underset{\sim}{\overset{(7/2)R_2+R_3\to R_3}{}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There is only the trivial solution, the columns of A are linearly independent.

10. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ -4 & 5 & -5 & 0 \\ -1 & 2 & 1 & 0 \end{bmatrix} \underset{\sim}{\overset{4R_1+R_2\to R_2}{R_1+R_3\to R_3}} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \overset{-R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are trivial solutions, the columns of A are linearly dependent.

11. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 5 & -2 & -1 & 0 \\ 4 & -4 & -3 & 0 \end{bmatrix} \underset{\sim}{\overset{(-5/3)R_1+R_2\to R_2}{(-4/3)R_1+R_3\to R_3}} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & -\frac{11}{3} & -1 & 0 \\ 0 & -\frac{16}{3} & -3 & 0 \end{bmatrix} \overset{(-16/11)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & -\frac{11}{3} & -1 & 0 \\ 0 & 0 & -\frac{17}{11} & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of A are linearly independent.

12. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} -4 & -7 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 5 & -1 & 1 & 0 \\ 8 & 2 & -4 & 0 \end{bmatrix} \overset{R_2\leftrightarrow R_4}{\sim} \begin{bmatrix} -4 & -7 & 1 & 0 \\ 8 & 2 & -4 & 0 \\ 5 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\underset{\sim}{\overset{(5/4)R_1+R_3\to R_3}{2R_1+R_2\Rightarrow R_2}} \begin{bmatrix} -4 & -7 & 1 & 0 \\ 0 & -12 & -2 & 0 \\ 0 & -\frac{39}{4} & \frac{9}{4} & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\overset{(-13/16)R_2+R_3\to R_3}{\sim} \begin{bmatrix} -4 & -7 & 1 & 0 \\ 0 & -12 & -2 & 0 \\ 0 & 0 & \frac{31}{8} & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\overset{(-24/31)R_3+R_4\to R_4}{\sim} \begin{bmatrix} -4 & -7 & 1 & 0 \\ 0 & -12 & -2 & 0 \\ 0 & 0 & \frac{31}{8} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of A are linearly independent.

13. We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} -3 & 5 & 0 \\ 4 & 1 & 0 \end{bmatrix} \overset{(4/3)R_1+R_2\to R_2}{\sim} \begin{bmatrix} -3 & 5 & 0 \\ 0 & \frac{23}{3} & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the homogeneous equation $A\mathbf{x} = 0$ has only the trivial solution.

14. We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} 12 & 10 & 0 \end{bmatrix} \overset{(-1/2)R_1+R_2\to R_2}{} \begin{bmatrix} 12 & 10 & 0 \end{bmatrix}$$

6 5 0 ~ 0 0 0

Since there exist nontrivial solutions, the homogeneous equation $A\mathbf{x} = 0$ has nontrivial solutions.

15. We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} 8 & 1 & 0 \\ 0 & -1 & 0 \\ -3 & 2 & 0 \end{bmatrix} \underset{\sim}{(3/8)R_1+R_3 \to R_3} \begin{bmatrix} 8 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & \frac{19}{8} & 0 \end{bmatrix} \underset{\sim}{(19/8)R_2+R_3 \to R_3} \begin{bmatrix} 8 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the homogeneous equation $A\mathbf{x} = 0$ has only the trivial solution.

16. We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} -3 & 2 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 5 & -4 & -3 & 0 \end{bmatrix} \underset{\sim}{\substack{(1/3)R_1+R_2 \to R_2 \\ (5/3)R_1+R_3 \to R_3}} \begin{bmatrix} -3 & 2 & 1 & 0 \\ 0 & -\frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} & -\frac{4}{3} & 0 \end{bmatrix} \underset{\sim}{-2R_2+R_3 \to R_3} \begin{bmatrix} -3 & 2 & 1 & 0 \\ 0 & -\frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the homogeneous equation $A\mathbf{x} = 0$ has nontrivial solutions.

17. We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} -1 & 3 & 1 & 0 \\ 4 & -3 & -1 & 0 \\ 3 & 0 & 5 & 0 \end{bmatrix} \underset{\sim}{\substack{4R_1+R_2 \to R_2 \\ 3R_1+R_3 \to R_3}} \begin{bmatrix} -1 & 3 & 1 & 0 \\ 0 & 9 & 3 & 0 \\ 0 & 9 & 8 & 0 \end{bmatrix} \underset{\sim}{-R_2+R_3 \to R_3} \begin{bmatrix} -1 & 3 & 1 & 0 \\ 0 & 9 & 3 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

The homogeneous equation $A\mathbf{x} = 0$ has only the trivial solution.

18. We solve the homogeneous equation using the corresponding augmented matrix:

$$\begin{bmatrix} 2 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ -5 & 3 & -9 & 0 \\ 3 & 0 & 9 & 0 \end{bmatrix} \underset{\sim}{\substack{(5/2)R_1+R_3 \to R_3 \\ (-3/2)R_1+R_4 \to R_4}} \begin{bmatrix} 2 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -\frac{9}{2} & -9 & 0 \\ 0 & \frac{9}{2} & 9 & 0 \end{bmatrix} \underset{\sim}{\substack{(9/2)R_2+R_3 \to R_3 \\ (-9/2)R_2+R_3 \to R_3}} \begin{bmatrix} 2 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the homogeneous equation $A\mathbf{x} = 0$ has nontrivial solutions.

19. Linearly dependent. Notice that $\mathbf{u} = 2\mathbf{v}$, so $\mathbf{u} - 2\mathbf{v} = 0$.

20. Linearly independent. The vectors are not scalar multiples of each other.

21. Linearly dependent. Apply Theorem 2.14.

22. Linearly independent. The vectors are not scalar multiples of each other.

23. Linearly dependent. Any collection of vectors containing the zero vector must be linearly dependent.

24. Linearly dependent. Since $\mathbf{u} = \mathbf{v}$, $\mathbf{u} - \mathbf{v} = \mathbf{0}$.

25. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 6 & 1 & 0 \\ 2 & 7 & 0 \\ -5 & 0 & 0 \end{bmatrix} \underset{\substack{(-1/3)R_1+R_2\to R_2 \\ (5/6)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 6 & 1 & 0 \\ 0 & \frac{20}{3} & 0 \\ 0 & \frac{5}{6} & 0 \end{bmatrix} \underset{(-1/8)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 6 & 1 & 0 \\ 0 & \frac{20}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of the matrix are linearly independent. By Theorem 2.15, none of the vectors is in the span of the other vectors.

26. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 7 & 1 & 3 & 0 \\ -1 & 6 & 0 & 0 \end{bmatrix} \underset{\substack{(-7/2)R_1+R_2\to R_2 \\ (1/2)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -\frac{5}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{13}{2} & \frac{1}{2} & 0 \end{bmatrix} \underset{(13/5)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -\frac{5}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{4}{5} & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of the matrix are linearly independent. By Theorem 2.15, none of the vectors is in the span of the other vectors.

27. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 4 & 3 & -5 & 0 \\ -1 & 5 & 7 & 0 \\ 3 & -2 & -7 & 0 \end{bmatrix} \underset{\substack{(1/4)R_1+R_2\to R_2 \\ (-3/4)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 4 & 3 & -5 & 0 \\ 0 & \frac{23}{4} & \frac{23}{4} & 0 \\ 0 & -\frac{17}{4} & -\frac{13}{4} & 0 \end{bmatrix} \underset{(17/23)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 4 & 3 & -5 & 0 \\ 0 & \frac{23}{4} & \frac{23}{4} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of the matrix are linearly independent. By Theorem 2.15, none of the vectors is in the span of the other vectors.

28. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 7 & 3 & 1 & 0 \\ 8 & 5 & -2 & 0 \\ 4 & 2 & 0 & 0 \end{bmatrix} \underset{\substack{(-7)R_1+R_2\to R_2 \\ (-8)R_1+R_3\to R_3 \\ (-4)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 10 & -20 & 0 \\ 0 & 13 & -26 & 0 \\ 0 & 6 & -12 & 0 \end{bmatrix} \underset{\substack{(-13/10)R_2+R_3\to R_3 \\ (-3/5)R_2+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 10 & -20 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the columns of the matrix are linearly dependent. By Theorem 2.15, one of the vectors is in the span of the other vectors.

29. We row–reduce to echelon form:

$$\begin{bmatrix} 2 & -1 \end{bmatrix} \underset{-(1/2)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 2 & -1 \end{bmatrix}$$

1 0 ∼ 0 $\frac{1}{2}$

Because the echelon form has a pivot in every row, by Theorem 2.9 $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^2$.

30. We row–reduce to echelon form:

$$\begin{bmatrix} 4 & 1 \\ -8 & 2 \end{bmatrix} \overset{2R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

Because the echelon form has a pivot in every row, by Theorem 2.9 $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^2$.

31. We row–reduce to echelon form:

$$\begin{bmatrix} 6 & -9 \\ -4 & 6 \end{bmatrix} \overset{(2/3)R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 6 & -9 \\ 0 & 0 \end{bmatrix}$$

Because the echelon form does not have a pivot in every row, by Theorem 2.9 $A\mathbf{x} = \mathbf{b}$ does not have a solution for all $\mathbf{b}$ in $\mathbf{R}^2$.

32. We row–reduce to echelon form:

$$\begin{bmatrix} 1 & -2 \\ 2 & 7 \end{bmatrix} \overset{-2R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 11 \end{bmatrix}$$

Because the echelon form has a pivot in every row, by Theorem 2.9 $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^2$.

33. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 4 & 5 & 0 \end{bmatrix} \overset{\substack{(-1/2)R_1+R_2 \to R_2 \\ (3/2)R_1+R_3 \to R_3}}{\sim} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & \frac{5}{2} & 5 & 0 \end{bmatrix} \overset{-5R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the columns of the matrix are linearly dependent. By The Unifying Theorem, $A\mathbf{x} = \mathbf{b}$ does not have a unique solution for all $\mathbf{b}$ in $\mathbf{R}^3$.

34. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 3 & 4 & 7 & 0 \\ 7 & -1 & 6 & 0 \\ -2 & 0 & 2 & 0 \end{bmatrix} \overset{\substack{(-7/3)R_1+R_2 \to R_2 \\ (2/3)R_1+R_3 \to R_3}}{\sim} \begin{bmatrix} 3 & 4 & 7 & 0 \\ 0 & -\frac{31}{3} & -\frac{31}{3} & 0 \\ 0 & \frac{8}{3} & \frac{20}{3} & 0 \end{bmatrix} \overset{(8/31)R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 3 & 4 & 7 & 0 \\ 0 & -\frac{31}{3} & -\frac{31}{3} & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of the matrix are linearly independent. By The Unifying Theorem, $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^3$.

35. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 3 & -2 & 1 & 0 \\ -4 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \end{bmatrix} \overset{\substack{(4/3)R_1+R_2 \to R_2 \\ (5/3)R_1+R_3 \to R_3}}{\sim} \begin{bmatrix} 3 & -2 & 1 & 0 \\ 0 & -\frac{5}{3} & \frac{4}{3} & 0 \\ 0 & -\frac{10}{3} & \frac{8}{3} & 0 \end{bmatrix}$$

$$\underset{\sim}{-2R_2+R_3\to R_3}\begin{bmatrix} 3 & -2 & 1 & 0 \\ 0 & -\,_3 & _3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there exist nontrivial solutions, the columns of the matrix are linearly dependent. By The Unifying Theorem, $A\mathbf{x} = \mathbf{b}$ does not have a unique solution for all $\mathbf{b}$ in $\mathbf{R}^3$.

36. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 4 & 7 & 0 \end{bmatrix} \overset{-2R_1+R_3\to R_3}{\sim} \begin{bmatrix} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 10 & 11 & 0 \end{bmatrix} \overset{-10R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Since the only solution is the trivial solution, the columns of the matrix are linearly independent. By The Unifying Theorem, $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^3$.

37. $\mathbf{u} = (1, 0, 0, 0)$, $\mathbf{v} = (0, 1, 0, 0)$, $\mathbf{w} = (1, 1, 0, 0)$

38. $\mathbf{u} = (1, 0, 0, 0, 0)$, $\mathbf{v} = (0, 1, 0, 0, 0)$, $\mathbf{w} = (0, 0, 1, 0, 0)$

39. $\mathbf{u} = (1, 0)$, $\mathbf{v} = (2, 0)$, $\mathbf{w} = (3, 0)$

40. $\mathbf{u} = (1, 0)$, $\mathbf{v} = (0, 1)$, $\mathbf{w} = (1, 1)$

41. $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (0, 1, 0)$, $\mathbf{w} = (1, 1, 0)$

42. $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (0, 1, 0)$, $\mathbf{w} = (0, 0, 1)$, $\mathbf{x} = (0, 0, 0)$. The collection is linearly dependent, and $\mathbf{x}$ is a *trivial* linear combination of the other vectors, so Theorem 2.15 is not violated.

43. (a) False. For example, $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (2, 0)$ are linearly dependent but do not span $\mathbf{R}^2$.
    (b) False. For example, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ spans $\mathbf{R}^2$, but is not linearly independent.

44. (a) True, by Theorem 2.14.
    (b) False. For example, $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$ does not span $\mathbf{R}^2$.

45. (a) False. For example, $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and has a pivot in every row, but the columns of A are not linearly independent.
    (b) True. If every column has a pivot, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and therefore the columns of A are linearly independent.

46. (a) False. If $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, but the columns of A are linearly dependent.
    (b) False. For example, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has linearly dependent columns, and the columns of A do not span $\mathbf{R}^2$.

47. (a) False. For example, $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 0 & 0 \end{bmatrix}$ has more rows than columns but the columns are linearly dependent.
    (b) False. For example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ has more columns than rows, but the columns are linearly dependent. (Theorem 2.14 can also be applied here to show that no matrix with more columns than rows can have linearly independent columns.)

48. (a) False. $A\mathbf{x} = \mathbf{0}$ corresponds to $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$, and by linear independence, each $x_i = 0$.

(b) False. For example, if $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $A\mathbf{x} = \mathbf{b}$ has no solution.

49. (a) False. Consider for example $\mathbf{u}_4 = \mathbf{0}$.

(b) True. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent, then $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{0}$ with at least one of the $x_i = 0$. Since $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{0} \Rightarrow x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + 0\mathbf{u}_4 = \mathbf{0}$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent.

50. (a) True. Consider $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{0}$. If one of the $x_i = 0$, then $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + 0\mathbf{u}_4 = \mathbf{0}$ would imply that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent, a contradiction. Hence each $x_i = 0$, and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent.

(b) False. Consider $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (0, 1, 0)$, $\mathbf{u}_3 = (0, 0, 1)$, $\mathbf{u}_4 = (0, 0, 0)$.

51. (a) False. If $\mathbf{u}_4 = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3$, then $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 - \mathbf{u}_4 = \mathbf{0}$, and since the coefficient of $\mathbf{u}_4$ is -1, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent.

(b) True. If $\mathbf{u}_4 = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3$, then $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 - \mathbf{u}_4 = \mathbf{0}$, and since the coefficient of $\mathbf{u}_4$ is -1, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent.

52. (a) False. Consider $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (1, 0, 0)$, $\mathbf{u}_3 = (1, 0, 0)$, $\mathbf{u}_4 = (0, 1, 0)$.

(b) False. Consider $\mathbf{u}_1 = (1, 0, 0, 0)$, $\mathbf{u}_2 = (0, 1, 0, 0)$, $\mathbf{u}_3 = (0, 0, 1, 0)$, $\mathbf{u}_4 = (0, 0, 0, 1)$.

53. (a), (b), and (c). For example, consider $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (1, 0, 0)$, and $\mathbf{u}_3 = (1, 0, 0)$. (d) cannot be linearly independent, by Theorem 2.14.

54. Only (c), since to span $\mathbf{R}^3$ we need at least 3 vectors, and to be linearly independent in $\mathbf{R}^3$ we can have at most 3 vectors.

55. Consider $x_1(c_1\mathbf{u}_1) + x_2(c_2\mathbf{u}_2) + x_3(c_3\mathbf{u}_3) = \mathbf{0}$. Then $(x_1c_1)\mathbf{u}_1 + (x_2c_2)\mathbf{u}_2 + (x_3c_3)\mathbf{u}_3 = \mathbf{0}$, and since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent, $x_1c_1 = 0$, $x_2c_2 = 0$, and $x_3c_3 = 0$. Since each $c_i = 0$, we must have each $x_i = 0$. Hence, $\{c_1\mathbf{u}_1, c_2\mathbf{u}_2, c_3\mathbf{u}_3\}$ is linearly independent.

56. Consider $x_1(\mathbf{u} + \mathbf{v}) + x_2(\mathbf{u} - \mathbf{v}) = \mathbf{0}$. This implies $(x_1 + x_2)\mathbf{u} + (x_1 - x_2)\mathbf{v} = \mathbf{0}$. Since $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, $x_1 + x_2 = 0$ and $x_1 - x_2 = 0$. Solving this system, we obtain $x_1 = 0$ and $x_2 = 0$. Thus $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is linearly independent.

57. Consider $x_1(\mathbf{u}_1 + \mathbf{u}_2) + x_2(\mathbf{u}_1 + \mathbf{u}_3) + x_3(\mathbf{u}_2 + \mathbf{u}_3) = \mathbf{0}$. This implies $(x_1 + x_2)\mathbf{u}_1 + (x_1 + x_3)\mathbf{u}_2 + (x_2 + x_3)\mathbf{u}_3 = \mathbf{0}$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent, $x_1 + x_2 = 0$, $x_1 + x_3 = 0$, and $x_2 + x_3 = 0$. Solving this system, we obtain $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$. Thus $\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3, \mathbf{u}_2 + \mathbf{u}_3\}$ is linearly independent.

58. We can, by re-indexing, consider the non-empty subset as $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ where $1 \le n \le m$. Let $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n = \mathbf{0}$, then $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n + 0\mathbf{u}_{n+1} + \cdots + 0\mathbf{u}_m = \mathbf{0}$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n, \mathbf{u}_{n+1}, \ldots, \mathbf{u}_m\}$ is linearly independent, every $x_i = 0$, $1 \le i \le n$. Therefore, $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is linearly independent.

59. Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is linearly dependent set, and we add vectors to form a new set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n, \ldots \mathbf{u}_m\}$. There exist x_i with a least one $x_i = 0$ such that $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n = \mathbf{0}$. Thus $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n + 0\mathbf{u}_{n+1} + \cdots + 0\mathbf{u}_m = \mathbf{0}$, and so $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n, \ldots \mathbf{u}_m\}$ is linearly dependent.

60. Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, there exists scalars x_1, x_2, x_3 such that $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$, and at least one $x_i = 0$. If $x_3 = 0$, then $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$ with either x_1 or x_2 nonzero, contradicting $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent. Hence $x_3 = 0$, and we may write then $\mathbf{w} = (-x_1/x_3)\mathbf{u} + (-x_2/x_3)\mathbf{v}$, and therefore $\mathbf{w}$ is in the span of $\{\mathbf{u}, \mathbf{v}\}$.

61. $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if and only if there exist scalars x_1 and x_2, not both zero, such that $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$. If $x_1 = 0$, then $\mathbf{u} = (-x_2/x_1)\mathbf{v} = c\mathbf{v}$. If $x_2 = 0$, then $\mathbf{v} = (-x_1/x_2)\mathbf{u} = c\mathbf{u}$.

62. Let $\mathbf{u}_i$ be the vector in the i^{th} nonzero row of A. Suppose the pivot in row i occurs in column k_i. Let r be the number of pivots, and consider $x_1\mathbf{u}_1+\cdots x_r\mathbf{u}_r=0$. Since A is in echelon form, the k_1 component of $\mathbf{u}_i$ for $i\geq 2$ must be 0. Hence when we equate the k_1 component of $x_1\mathbf{u}_1+\cdots x_r\mathbf{u}_r=0$ we obtain $x_1=0$. Applying the same argument to the k_2 component now with the equation $x_2\mathbf{u}_2+\cdots x_r\mathbf{u}_r=0$ we conclude that $x_2=0$. Continuing in this way we see that $x_i=0$ for all i, and hence the nonzero rows of A are linearly independent.

63. Suppose $A=[\,\mathbf{a}_1\ \ \mathbf{a}_2\ \ \ldots\ \ \mathbf{a}_m\,]$, $\mathbf{x}=(x_1,x_2,\ldots,x_m)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_m)$. Then we have $\mathbf{x}-\mathbf{y}=(x_1-y_1,x_2-y_2,\ldots,x_m-y_m)$, and thus

$$\begin{aligned}A(\mathbf{x}-\mathbf{y})&=(x_1-y_1)\,\mathbf{a}_1+(x_2-y_2)\,\mathbf{a}_2+\cdots+(x_m-y_m)\,\mathbf{a}_m\\&=(x_1\mathbf{a}_1+x_2\mathbf{a}_2+\cdots+x_m\mathbf{a}_m)-(y_1\mathbf{a}_1+y_2\mathbf{a}_2+\cdots+y_m\mathbf{a}_m)\\&=A\mathbf{x}-A\mathbf{y}\end{aligned}$$

64. Since $\mathbf{u}_1=0$ and $\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_m\}$ is linearly dependent, there exists a smallest index r such that $\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_r\}$ is linearly independent but $\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_r,\mathbf{u}_{r+1}\}$ is linearly dependent. Consider $x_1\mathbf{u}_1+\cdots+x_r\mathbf{u}_r+x_{r+1}\mathbf{u}_{r+1}=0$. Since $\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_r,\mathbf{u}_{r+1}\}$ is linearly dependent, at least one of the $x_i=0$. If $x_{r+1}=0$, then $x_1\mathbf{u}_1+\cdots+x_r\mathbf{u}_r=0$, which implies that $x_i=0$ for all $i\leq r$ since $\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_r\}$ is linearly independent. But this contradicts that some $x_i=0$, and so we must have $x_{r+1}=0$. Thus we may write $\mathbf{u}_{r+1}=(-x_1/x_{r+1})\,\mathbf{u}_1+\cdots+(-x_r/x_{r+1})\,\mathbf{u}_r$. We select those subscripts i with $x_i=0$ (there must be at least one, otherwise $\mathbf{u}_{r+1}=0$, a contradiction), and rewrite $\mathbf{u}_{r+1}=(-x_{k_1}/x_{r+1})\,\mathbf{u}_{k_1}+\cdots+\left(-x_{k_p}/x_{r+1}\right)\mathbf{u}_{k_p}$. We now have a vector $\mathbf{u}_{r+1}$ written as a linear combination of a subset of the remaining vectors, with nonzero coefficients. Since $\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_r\}$ is linearly independent, this subset of vectors $\{\mathbf{u}_{k_1},\mathbf{u}_{k_2},\ldots,\mathbf{u}_{k_p}\}$ is also linearly independent (see exercise 56). Finally, these coefficients are unique, since if $(-x_{k_1}/x_{r+1})\,\mathbf{u}_{k_1}+\cdots+\left(-x_{k_p}/x_{r+1}\right)\mathbf{u}_{k_p}=y_1\mathbf{u}_{k_1}+\cdots+y_p\mathbf{u}_{k_p}$, then $(y_1-x_{k_1}/x_{r+1})\,\mathbf{u}_{k_1}+\cdots+\left(y_p-x_{k_p}/x_{r+1}\right)\mathbf{u}_{k_p}=0$, and by linear independence of $\{\mathbf{u}_{k_1},\mathbf{u}_{k_2},\ldots,\mathbf{u}_{k_p}\}$, each $y_i-x_{k_i}/x_{r+1}=0$, and thus $y_i=x_{k_i}/x_{r+1}$.

65. Using a computer algebra system, the vectors are linearly independent.

66. Using a computer algebra system, the vectors are linearly dependent.

67. Using a computer algebra system, the vectors are linearly independent.

68. Using a computer algebra system, the vectors are linearly dependent.

69. We row–reduce to using computer software to obtain

$$\begin{bmatrix}2&1&-1&3\\-5&3&1&2\\-1&2&-2&1\\1&-2&0&-3\\0&&&\\3&1&-4&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&1\\0&1&0&2\\0&0&1&1\\0&0&0&\\0&0&0&0\end{bmatrix}$$

So, because $A\mathbf{x}=0$ has infinitely many solutions, we conclude that the vectors are linearly dependent.

70. We row–reduce to using computer software to obtain

$$\begin{bmatrix}4&2&-3&0\\2&3&2&2\\-1&1&1&-1\\5&-1&1&3\\2&0&1&2\end{bmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

So, because $A\mathbf{x}=0$ has only the trivial solution, we conclude that the vectors are linearly independent.

71. Using a computer algebra system, $A\mathbf{x}=\mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^3$.

72. Using a computer algebra system, $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^3$.

73. Using a computer algebra system, $A\mathbf{x} = \mathbf{b}$ does not have a unique solution for all $\mathbf{b}$ in $\mathbf{R}^4$.

74. Using a computer algebra system, $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^4$.

Chapter 2 Supplementary Exercises

1. $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$;

$3\mathbf{w} = 3\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ -15 \\ 21 \end{bmatrix}$

2. $\mathbf{v}-\mathbf{w} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}$;

$-4\mathbf{u} = -4\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \\ -8 \end{bmatrix}$

3. $2\mathbf{w}+3\mathbf{v} = 2\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 17 \end{bmatrix}$;

$2\mathbf{u}-5\mathbf{w} = 2\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} - 5\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 19 \\ -31 \end{bmatrix}$

4. $3\mathbf{v}+2\mathbf{u} = 3\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \\ 7 \end{bmatrix}$;

$-2\mathbf{u}+4\mathbf{w} = -2\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ -14 \\ 24 \end{bmatrix}$

5. $2\mathbf{u}+\mathbf{v}+3\mathbf{w} = 2\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ -17 \\ 26 \end{bmatrix}$;

$\mathbf{u}-3\mathbf{v}+2\mathbf{w} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} - 3\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 9 \\ -25 \\ 13 \end{bmatrix}$

6. $\mathbf{u}-2\mathbf{v}+4\mathbf{w} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 9 \\ -31 \\ 28 \end{bmatrix}$;

$-3\mathbf{u}+\mathbf{v}-2\mathbf{w} = -3\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} -7 \\ 23 \\ -19 \end{bmatrix}$

7. $$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & 1 \\ -3x_1 & + & 4x_2 & = & -5 \\ 2x_1 & + & x_2 & = & 7 \end{array}$$

8. $x_1 + x_2 = 4$

$$\begin{array}{rcrcr} -5x_1 & - & 3x_2 & = & -8 \\ 7x_1 & + & 2x_2 & = & -2 \end{array}$$

9. $0\begin{bmatrix}1\\-3\\2\end{bmatrix}+0\begin{bmatrix}-2\\4\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$; $1\begin{bmatrix}1\\-3\\2\end{bmatrix}+0\begin{bmatrix}-2\\4\\1\end{bmatrix}=\begin{bmatrix}1\\-3\\2\end{bmatrix}$;

$0\begin{bmatrix}1\\-3\\2\end{bmatrix}+1\begin{bmatrix}-2\\4\\1\end{bmatrix}=\begin{bmatrix}-2\\4\\1\end{bmatrix}$

10. $0\begin{bmatrix}1\\-5\\7\end{bmatrix}+0\begin{bmatrix}-2\\4\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$; $1\begin{bmatrix}1\\-5\\7\end{bmatrix}+0\begin{bmatrix}-2\\4\\1\end{bmatrix}=\begin{bmatrix}1\\-5\\7\end{bmatrix}$;

$0\begin{bmatrix}1\\-5\\7\end{bmatrix}+1\begin{bmatrix}-2\\4\\1\end{bmatrix}=\begin{bmatrix}-2\\4\\1\end{bmatrix}$

11. $x_1\mathrm{u}+x_2\mathrm{v}=\mathrm{w} \iff x_1\begin{bmatrix}1\\-3\\2\end{bmatrix}+x_2\begin{bmatrix}-2\\4\\1\end{bmatrix}=\begin{bmatrix}1\\-5\\7\end{bmatrix} \iff \begin{bmatrix}x_1-2x_2\\-3x_1+4x_2\\2x_1+x_2\end{bmatrix}$

$=\begin{bmatrix}1\\-5\\7\end{bmatrix} \iff$ the augmented matrix $\begin{bmatrix}1&-2&1\\-3&4&-5\\2&1&7\end{bmatrix}$ has a solution:

$$\begin{bmatrix}1&-2&1\\-3&4&-5\\2&1&7\end{bmatrix}\underset{-2R_1+R_3\to R_3}{\overset{3R_1+R_2\to R_2}{\sim}}\begin{bmatrix}1&-2&1\\0&-2&-2\\0&5&5\end{bmatrix}\overset{(5/2)R_2+R_3\to R_3}{\sim}\begin{bmatrix}1&-2&1\\0&-2&-2\\0&0&0\end{bmatrix}$$

Because a solution exists, w is a linear combination of u and v.

12. $x_1\mathrm{w}+x_2\mathrm{u}=\mathrm{v} \iff x_1\begin{bmatrix}1\\-5\\7\end{bmatrix}+x_2\begin{bmatrix}1\\-3\\2\end{bmatrix}=\begin{bmatrix}-2\\4\\1\end{bmatrix} \iff$

$\begin{bmatrix}x_1+x_2\\-5x_1-3x_2\\7x_1+2x_2\end{bmatrix}=\begin{bmatrix}-2\\4\\1\end{bmatrix} \iff$ the augmented matrix $\begin{bmatrix}1&1&-2\\-5&-3&4\\7&2&1\end{bmatrix}$ has a solution:

$$\begin{bmatrix}1&1&-2\\-5&-3&4\\7&2&1\end{bmatrix}\underset{-7R_1+R_3\to R_3}{\overset{5R_1+R_2\to R_2}{\sim}}\begin{bmatrix}1&1&-2\\0&2&-6\\0&-5&15\end{bmatrix}\overset{(5/2)R_2+R_3\to R_3}{\sim}\begin{bmatrix}1&1&-2\\0&2&-6\\0&0&0\end{bmatrix}$$

Because a solution exists, v is a linear combination of w and u.

13. Because w is in the span of u and v, by Exercise 11, $\{\mathrm{u},\mathrm{v},\mathrm{w}\}$ is linearly dependent.

14. Because $\{\mathrm{u},\mathrm{v},\mathrm{w}\}$ is linearly dependent, by Exercise 13, span $\{\mathrm{u},\mathrm{v},\mathrm{w}\}=\mathbf{R}^3$.

15. $x_1\begin{bmatrix}4\\1\end{bmatrix}+x_2\begin{bmatrix}13\\-7\end{bmatrix}+x_3\begin{bmatrix}-1\\4\end{bmatrix}=\begin{bmatrix}-7\\12\end{bmatrix}$

16. $x_1\begin{bmatrix}3\\-1\\-3\end{bmatrix}+x_2\begin{bmatrix}-2\\5\\0\end{bmatrix}+x_3\begin{bmatrix}-1\\0\\10\end{bmatrix}+x_4\begin{bmatrix}2\\1\\-3\end{bmatrix}=\begin{bmatrix}0\\-7\\2\end{bmatrix}$

17. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

18. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix}$

19. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -5 \\ 0 \\ 8 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

20. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

21. $2\begin{bmatrix} -3 \\ a \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} b \\ 4 \end{bmatrix} = \begin{bmatrix} -2b-7 \\ 2a-8 \end{bmatrix}$, so we have the equations $-2b - 7 = -2$, and $2a - 8 = 5$.

We solve these and obtain $a = \frac{13}{2}$ and $b = -\frac{5}{2}$.

22. $-\begin{bmatrix} a \\ 1 \\ -2 \end{bmatrix} + 3\begin{bmatrix} 3 \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 9-a \\ 3b-1 \\ 2 \end{bmatrix}$, so we have the equations $9 - a = 1$, $3b - 1 = -4$, and $2 = c$. We solve these and obtain $a = 8$, $b = -1$, and $c = 2$.

23. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \iff x_1\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -11 \\ 10 \end{bmatrix} \iff \begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 3x_2 \\ 4x_1 - x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -11 \\ 10 \end{bmatrix} \iff$ the augmented matrix $\begin{bmatrix} 1 & 2 & -1 \\ -2 & 3 & -11 \\ 4 & -1 & 10 \end{bmatrix}$ yields a solution.

$$\begin{bmatrix} 1 & 2 & -1 \\ -2 & 3 & -11 \\ 4 & -1 & 10 \end{bmatrix} \underset{-4R_1+R_3 \to R_3}{\overset{2R_1+R_2 \to R_2}{\sim}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 7 & -13 \\ 0 & -9 & 14 \end{bmatrix} \overset{(9/7)R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 7 & -13 \\ 0 & 0 & -\frac{19}{7} \end{bmatrix}$$

From the third row, we have $0 = -\frac{19}{7}$, and hence the system does not have a solution. Hence $\mathbf{b}$ is not a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

24. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b} \iff x_1\begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 0 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 5 \\ \end{bmatrix} \iff \begin{bmatrix} x_1 - 2x_3 \\ -3x_1 + 2x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ \vdots \end{bmatrix} \iff \begin{bmatrix} 1 & 0 & -2 & -2 \\ -3 & 2 & 0 & -4 \\ \vdots \end{bmatrix}$

$-3x_2 + 3x_3$ = 5 yields
$2x_1 + x_2 - x_3$
$\Leftrightarrow$ the augmented matrix

0 −1 3 5
3 1 −1 3
2

a solution.

$$\begin{bmatrix} 1 & 0 & -2 & -2 \\ -3 & 2 & 0 & -4 \\ 0 & -1 & 3 & 5 \\ 2 & 1 & -1 & 3 \end{bmatrix} \overset{\substack{3R_1+R_2\to R_2 \\ -2R_1+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 2 & -6 & -10 \\ 0 & -1 & 3 & 5 \\ 0 & 1 & 3 & 7 \end{bmatrix}$$

$$\overset{\substack{(1/2)R_2+R_3\to R_3 \\ -(1/2)R_2+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 2 & -6 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 12 \end{bmatrix}$$

From row 4, $6x_3 = 12 \Rightarrow x_3 = 2$. From row 2, $2x_2 - 6(2) = -10 \Rightarrow x_2 = 1$. From row 1, $x_1 - 2(2) = -2 \Rightarrow x_1 = 2$. We conclude $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ with $\mathbf{b} = 2\mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3$.

25. $A = \begin{bmatrix} 2 & 3 & -8 & 1 \\ 6 & -1 & 4 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$

26. $A = \begin{bmatrix} 3 & -1 & -7 \\ -4 & 5 & 0 \\ -8 & 2 & 6 \\ 1 & 3 & 9 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ -4 \\ 3 \\ 7 \end{bmatrix}$

27. Set $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \Rightarrow x_1 \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 7 \end{bmatrix} \Rightarrow$

$\begin{bmatrix} 3x_1 + x_2 \\ -x_1 + 4x_2 \\ -2x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 7 \end{bmatrix}$. We obtain 3 equations and row-reduce the associated augmented matrix to determine if there are solutions.

$$\begin{bmatrix} 3 & 1 & -1 \\ -1 & 4 & 5 \\ -2 & 5 & 7 \end{bmatrix} \overset{\substack{(1/3)R_1+R_2\to R_2 \\ (2/3)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 3 & 1 & -1 \\ 0 & \frac{13}{3} & \frac{14}{3} \\ 0 & \frac{17}{3} & \frac{19}{3} \end{bmatrix}$$

$$\overset{-(17/13)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 3 & 1 & -1 \\ 0 & \frac{13}{3} & \frac{14}{3} \\ 0 & 0 & \frac{3}{13} \end{bmatrix}$$

From the third row, $0 = \frac{3}{13}$, and hence there are no solutions. We conclude that there do not exist x_1 and x_2 such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$, and therefore $\mathbf{b}$ is not in the span of $\mathbf{a}_1$ and $\mathbf{a}_2$.

28. Set $x_1\mathbf{a}_1+x_2\mathbf{a}_2+x_3\mathbf{a}_3 = \mathbf{b} \Rightarrow x_1 \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -7 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 - x_2 + 2x_3 \\ 3x_1 + 2x_2 + 2x_3 \\ x_1 + 3x_2 \\ 4x_2 - x_3 \end{bmatrix} =$

$\begin{bmatrix} -3 \\ 4 \\ -7 \\ 1 \end{bmatrix}$. We obtain 4 equations and row-reduce the associated augmented matrix to determine if there are solutions.

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & 2 & 2 & 4 \\ 1 & 3 & 0 & -7 \\ 0 & 4 & -1 & 1 \end{bmatrix} \overset{\substack{-3R_1+R_2\to R_2 \\ -R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -4 & 13 \\ 0 & 4 & -2 & -4 \\ 0 & 4 & -1 & 1 \end{bmatrix}$$

$$\overset{\substack{(-4/5)R_2+R_4\to R_3 \\ (-4/5)R_3+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -4 & 13 \\ 0 & 0 & \frac{6}{5} & -\frac{72}{5} \\ 0 & 0 & \frac{11}{5} & \frac{47}{5} \end{bmatrix}$$

$(-11/6)R_3+R_4\to R_4$

$$\begin{bmatrix} & & 5 & -5 \\ 1 & -1 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 5 & -4 & 13 \\ 0 & 0 & \frac{6}{5} & -\frac{72}{5} \\ 0 & 0 & 0 & 17 \end{bmatrix}$$

From the third row, $0 = 17$,and hence there are no solutions. We conclude that there do not exist x_1, x_2, and x_3 such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$, and therefore $\mathbf{b}$ is not in the span of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

29. $\{\mathbf{a}_1\}$ does not span $\mathbf{R}^2$, by Theorem 2.9, because $m = 1 < 2 = n$.

30. Row-reduce to echelon form:

$$\begin{bmatrix} 6 & -2 \\ -9 & 3 \end{bmatrix} \overset{(3/2)R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 6 & -2 \\ 0 & 0 \end{bmatrix}$$

Because there is a row of zeros, there exists a vector $\mathbf{b}$ which is not in the span of the columns of the matrix, and therefore $\{\mathbf{a}_1, \mathbf{a}_2\}$ does not span $\mathbf{R}^2$.

31. Row-reduce to echelon form:

$$\begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix} \overset{-2R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 1 & -3 \\ 0 & 11 \end{bmatrix}$$

Because there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of the columns of the given matrix, and therefore $\{\mathbf{a}_1, \mathbf{a}_2\}$ spans $\mathbf{R}^2$.

32. Row-reduce to echelon form:

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 4 \end{bmatrix} \overset{-3R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Because there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of the columns of the given matrix, and therefore $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ spans $\mathbf{R}^2$.

33. $\{\mathbf{a}_1\}$ does not span $\mathbf{R}^3$, by Theorem 2.9, because $m = 1 < 3 = n$.

34. $\{\mathbf{a}_1, \mathbf{a}_2\}$ does not span $\mathbf{R}^3$, by Theorem 2.9, because $m = 2 < 3 = n$.

35. Row-reduce to echelon form:

$$\begin{bmatrix} 1 & -3 & 4 \\ 2 & -5 & 6 \\ 5 & 4 & 11 \end{bmatrix} \underset{-5R_1+R_3 \to R_3}{\overset{-2R_1+R_2 \to R_2}{\sim}} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -2 \\ 0 & 19 & -9 \end{bmatrix} \overset{-19R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 29 \end{bmatrix}$$

Because there is not a row of zeros, every choice of $\mathbf{b}$ is in the span of the columns of the given matrix, and therefore $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ spans $\mathbf{R}^3$.

36. Row-reduce to echelon form:

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ -3 & 2 & -5 & 2 \\ 1 & -2 & -1 & -6 \end{bmatrix} \underset{-R_1+R_3 \to R_3}{\overset{3R_1+R_2 \to R_2}{\sim}} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & -1 & -2 & -4 \\ 0 & -1 & -2 & -4 \end{bmatrix} \overset{-R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there exists a vector $\mathbf{b}$ which is not in the span of the columns of the matrix, and therefore $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ does not span $\mathbf{R}^3$.

37. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -2 & 0 \\ -5 & 9 & 0 \end{bmatrix} \overset{5R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Because the only solution is the trivial solution, the set of column vectors, $\{a_1, a_2\}$, is linearly independent.

38. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 9 & -6 & 0 \\ -6 & 4 & 0 \end{bmatrix} \xrightarrow{(2/3)R_1+R_2\to R_2} \begin{bmatrix} 9 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Because there exist nontrivial solutions, the set of column vectors, $\{a_1, a_2\}$, is not linearly independent.

39. By Theorem 2.14, because $m = 3 > 2 = n$, the set $\{a_1, a_2, a_3\}$ is not linearly independent.

40. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -2 & 0 \\ 6 & 3 & 0 \\ -2 & 0 & 0 \end{bmatrix} \xrightarrow[2R_1+R_3\to R_3]{-6R_1+R_2\to R_2} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 15 & 0 \\ 0 & -4 & 0 \end{bmatrix} \xrightarrow{(4/15)R_2+R_3\to R_3} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Because the only solution is the trivial solution, the set of column vectors, $\{a_1, a_2\}$, is linearly independent.

41. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -2 & 0 \\ 4 & -8 & 0 \\ -5 & 10 & 0 \end{bmatrix} \xrightarrow[5R_1+R_3\to R_3]{-4R_1+R_2\to R_2} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Because there exist nontrivial solutions, the set of column vectors, $\{a_1, a_2\}$, is not linearly independent.

42. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -2 & 2 & 0 \\ -1 & 3 & -5 & 0 \\ 3 & 4 & 9 & 0 \end{bmatrix} \xrightarrow[-3R_1+R_3\to R_3]{R_1+R_2\to R_2} \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 10 & 3 & 0 \end{bmatrix} \xrightarrow{-10R_2+R_3\to R_3} \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 33 & 0 \end{bmatrix}$$

Because the only solution is the trivial solution, the set of column vectors, $\{a_1, a_2, a_3\}$, is linearly independent.

43. We solve the homogeneous system of equations using the corresponding augmented matrix:

$$\begin{bmatrix} 3 & -2 & 0 & 0 \\ 0 & 3 & 9 & 0 \\ 2 & -4 & -8 & 0 \end{bmatrix} \xrightarrow{(-2/3)R_1+R_3\to R_3} \begin{bmatrix} 3 & -2 & 0 & 0 \\ 0 & 3 & 9 & 0 \\ 0 & -\frac{8}{3} & -8 & 0 \end{bmatrix} \xrightarrow{(8/9)R_2+R_3\to R_3} \begin{bmatrix} 3 & -2 & 0 & 0 \\ 0 & 3 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because there exist nontrivial solutions, the set of column vectors, $\{a_1, a_2, a_3\}$, is not linearly independent.

44. By Theorem 2.14, because $m = 4 > 3 = n$, the set $\{a_1, a_2, a_3, a_4\}$ is not linearly independent.