Solution Manual for Mathematics for Elementary School Teachers 6th Edition Bassarear Moss 1305071360 9781305071360

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Chapter 2 Fundamental Concepts

SECTION 2.1 Sets

1. **a.** $0 \notin \emptyset$ or $0 \notin \{\ \}$ **b.** $3 \notin B$

2. **a.** $D \not\subseteq E$ **b.** $A \subseteq U$

3. **a.** {e, l, m, n, t, a, r, y} and {x | x is a letter in the word "elementary"}
or {x | x is one of these letters: e, l, m, n, t, a, r, y}.

b. {Spain, Portugal, France, Ireland, United Kingdom (England/Scotland), Western Russia, Germany, Italy, Austria, Switzerland, Belgium, Netherlands, Estonia, Latvia, Denmark, Sweden, Norway, Finland, Poland, Bulgaria, Yugoslavia, The Czech Republic, Slovakia, Romania, Greece, Macedonia, Albania, Croatia, Hungary, Bosnia and Herzegovina, Ukraine, Belarus, Lithuania}.
Also {x | x is a country in Europe}.

c. {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97}.
Also {x | x is a prime less than 100}.

d. The set of fractions between 0 and 1 is infinite.
{x | x is a fraction between zero and one}.

e. {name1, name2, name3, etc.}.
{x | x is a student in this class}.

4. **a.** $\subset$ **b.** $\in$ **c.** $\subset$ **d.** $\subset$

e. True **f.** False; red is an element, not a set.

g. False; gray is not in set *S*. **h.** True

5. **a.** $\in$; 3 is an element of the set. **b.** $\subset$; $\{3\}$ is a subset of the set.

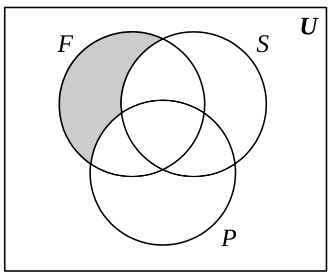
c. $\in$; $\{1\}$ is an element of this set of sets. **d.** $\subset$; $\{a\}$ is a subset of the set.

e. $\not\subset$ or $\notin$; $\{ab\}$ $\not\subset$ is neither a subset nor an element.

f. $\subset$; the null set is a subset of every set.

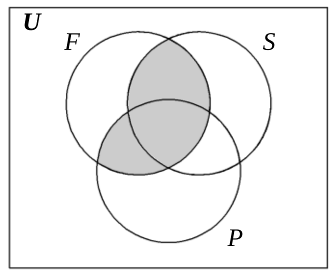
6. **a.** 64 **b.** A set with *n* elements has 2^n subsets.

7. **a.**



$F \cap \left(\overline{S \cup P}\right)$ or $F \cap \overline{S} \cap \overline{P}$

b.



American females who smoke and/or have a health problem.

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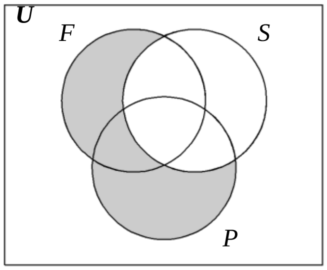
c.



Nonsmokers who either are
female or have health problems.

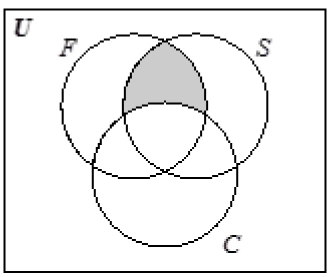
d. $F \cap S$
Females who smoke.

e. $F \cap (S \cap P)$
Males who smoke and have a health problem.

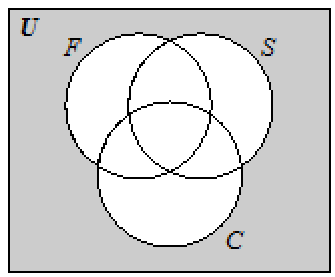
8. **a.** Students who are members of at least two of the film, science, and computer clubs.
$(F \cap S) \cup (S \cap C) \cup (C \cap F)$

b. Students who are members of both the science and computer clubs, but not the film club.
$\overline{F} \cap (S \cap C)$

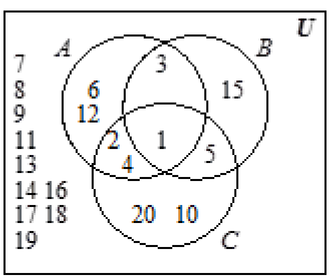
c. $\overline{C} \cap (S \cap F)$



d. $\overline{F \cup S \cup C}$



9. **a.**



b. All numbers that don't evenly divide 12, 15, or 20; $\overline{A \cup B \cup C}$ or $\overline{A} \cap \overline{B} \cap \overline{C}$

c. All numbers that evenly divide 12 and 20, but not 15; $\overline{B} \cap (A \cap C)$

d. All numbers except 1 and 3.

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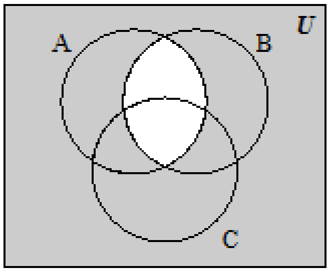

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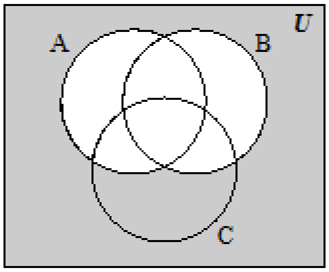
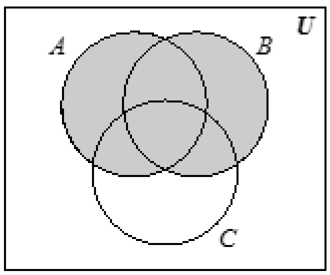
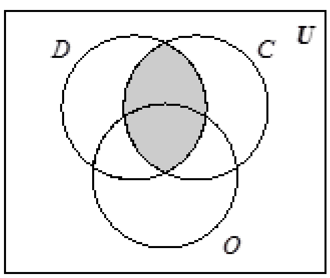
e. All numbers from 1 to 20, except those that divide 12 or 15 evenly.



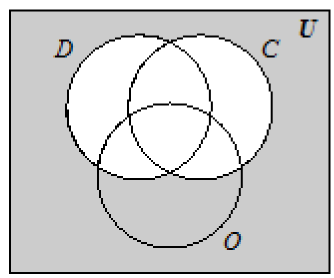
f. Note: This description is ambiguous; it depends on how one interprets "or." $A \cup B$



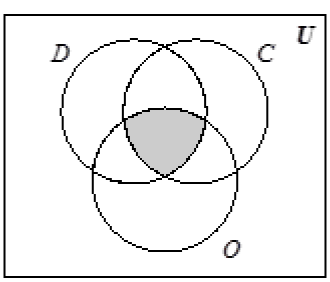
10. a. Students who have at least one cat and at least one dog.



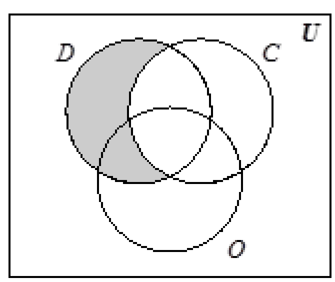
b. Students who have neither cats nor dogs.



c. Students who have at least one cat, at least one dog, and at least one other pet.



d. $D \cap \overline{C}$



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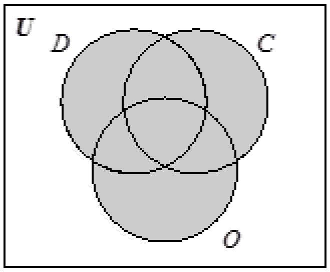
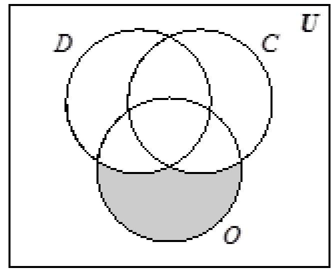
e. $D \cup C \cup O$



f. $O \cap \left(\overline{D \cup C}\right)$



g. $\overline{D \cup C \cup O}$ or $\overline{D} \cap \overline{C} \cap \overline{O}$

Students who have no pets.

h. $C \cap (\overline{D \cup O})$

Students who have at least one cat and no other pets.

11. 15 possible committees. Label the members with *A*, *B*, *C*, *D*, *E*, and *F*.
The committees could be: *AB, AC, AD, AE, AF, BC, BD, BE, BF, CD, CE, CF, DE, DF, EF*.

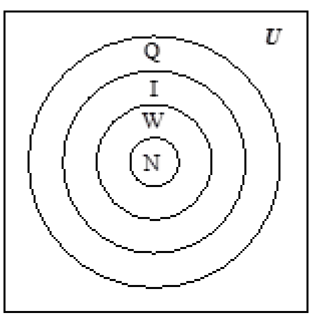
12. **a. and b.** Answers will vary.

13. Answers will vary.

14. Answers will vary.

15. The circles enable us to easily represent visually all the possible subsets.
The diagram is not equivalent because there is no region corresponding to elements that are in all three sets.

16.



17. **a.** 6 + 8 + 12 + 3 = 29%

b. 6 + 25 + 15 = 46%

c. Those people who agree with his foreign policy and those people who agree with his economic and his social policy.

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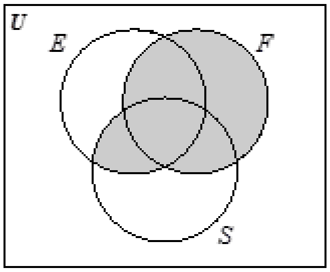

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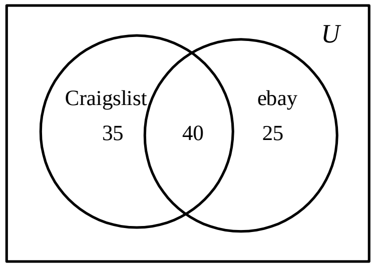
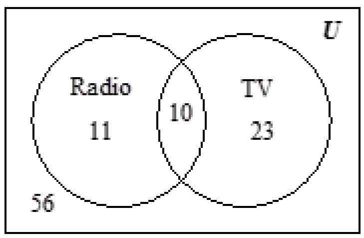
18. a.



b. Yes, they are well defined.

19. a. Construct a Venn diagram. 100 – 11 – 10 – 23 = 56%



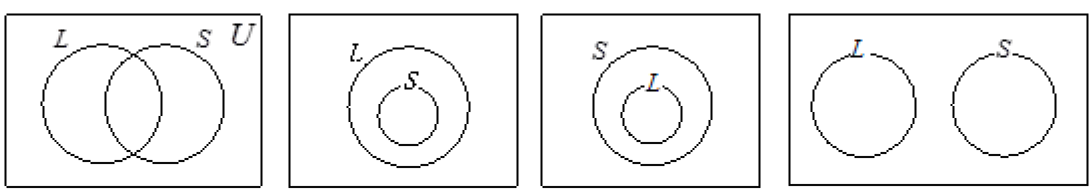
b. 11 + 23 = 34%

20. a. A lesson in which the teacher would be using a lab approach with small groups.

b. Lessons that use a lab approach and concrete materials and/or small groups.

21. Answers will vary.

22. a. Theoretically, there are four possibilities. I would pick the one at the left, because I think there can be successful people who are not very intelligent, intelligent people who are not successful, people who are successful and intelligent, and people who are neither.



b. Answers will vary.

23. Answers will vary.

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SECTION 2.2 Numeration

1. **a.**

	Maya	Luli	South American
7		lokep moile tamlip	
8			teyente toazumba
12			caya-ente-cayupa
13		is yaoum moile tamlip	caya-ente-toazumba
15		is yaoum is alapea	
16	uac-lahun	is yaoum moile lokep moile tamop	toazumba-ente-tey
21	hun hunkal	is eln yaoum moile alapea	cajesa-ente-tey
22	ca huncal	is eln yaoum moile tamop	cajesa-ente-cayupa

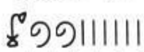
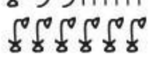
b. Answers will vary.

c. Answers will vary.

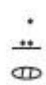
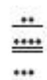
2. **a.** 3031 **b.** 230,012 **c.** 1666 **d.** 1519

e. 109 **f.** 75,602 **g.** 133 **h.** 23

3.

	Egyptian	Roman	Babylonian
a. 312		CCCXII	
b. 1206		MCCIIIIII or MCCVI	
c. 6000		MMMMMM	
d. 10,000		MMMMMMMMMM	
e. 123,456		Can't do	

4 **a.** 87 **b.** 360 **c.** 5407

d.  **e.**  **f.** 

5. **a.** 26 **b.** 240 **c.** 25 **d.** 450

e. three thousand four hundred **f.** 3450

6. **a.** –400 = **b.** –7770 = **c.** +80,000 = **d.** +4040 =

e. –346,733 = **f.** –111,111 = **g.** Answers will vary.

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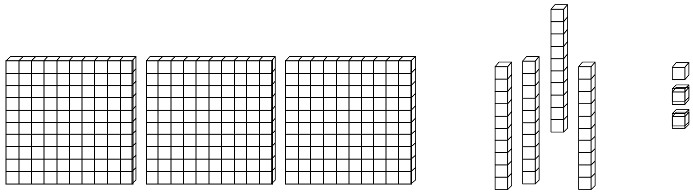
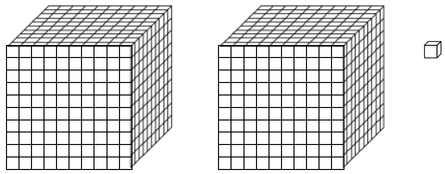
7. **a.** 

b. 

8. **a.** $3 \cdot 10^2 + 4 \cdot 10 + 5$ **b.** $2 \cdot 10^3 + 1$ **c.** $1 \cdot 10^4 + 1 \cdot 10^2 + 1$

9. **a.** 4859 b. 30,240 c. 750,003

10. **a.** 40_{five} **b.** 1100_{two} **c.** $9a_{sixteen}$ **d.** $709a_{sixteen}$
e. 110_{two} **f.** 112_{twelve} **g.** 130_{five} **h.** 410_{six}

11. **a.** 1004_{five} **b.** 334_{five} **c.** $0ff_{sixteen}$ **d.** 1101_{two}
e. 1001_{two} **f.** $10f_{sixteen}$ **g.** 113_{four} **h.** 56_{seven}

12. **a.** 1009

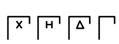
b. $\text{MIMIC} = 1000 + 1 + 1000 + 1 + 100 = 2102$

c. Answers will vary.

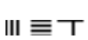
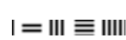
13. **a.** $500 + 10 + 10 + 5 + 1 + 1 = 527$

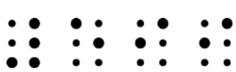
b. $100 + 50 + 10 + 10 + 5 + 1 = 176$

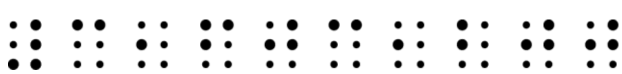
c. 

d. 

14. **a.** 32,570 b. 646

c.  **d.** 

15. **a.** 

b. 

c. 460,859

d. 135,246

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16. **a.** (1) No: Needs a new symbol for each new power of ten.
(2) Sort of: The value of each numeral is 10 times the value of the previous numeral.
(3) Sort of: By decorating each basic symbol, you now have one basic symbol for each place, the number of dots on the symbol varies.
(4) No.
(5) Sort of: though, given the origin of this system, it would be more likely to be counted. For example, 2 thousands, 8 hundreds, etc. However, technically, you would multiply the value of each basic symbol by the number of dots on the symbol.
(6) No zero.

b. It has characteristic 2: The value of each place is 10 times the value of the previous place. It "sort of" has characteristic 3, with the modification that each "place" contains two symbols. Some might say that it has characteristics 4 and 5, but the order of the numerals is still a matter of convention – unlike base 10, where changing the order changes the value.

c. This system has all characteristics.

17. **a.** 585 cartons of milk

b. It has all 6 characteristics because this system is essentially base 6. The places are called cartons, boxes, crates, flats, and pallets. The value of each place is 6 times that of the previous place.

18. 1:0:58:4 or 1 hour, 58.04 seconds

19. The child does not realize that every ten numbers you need a new prefix. At “twenty-ten” the ones place is filled up, but the child does not realize this. Alternatively, the child does not realize the cycle, so that after nine comes a new prefix.

20. The child skipped thirty, because he or she does not think of the zero in the ones place as a number. The child counts from one to nine and starts over. In this case, the child has internalized the natural numbers (N), but not the whole numbers (W).

21. Yes, 5 is the middle number between 0 and 10

22. Because the Hindu-Arabic system has place value and a place holder (zero, 0), it allows extremely large numbers to be represented with only 10 symbols. It is also much less cumbersome, since it only takes six digits to represent one hundred thousand.

23. We mark our years, in retrospect, with respect to the approximate birth year of Jesus Christ—this is why they are denoted 1996 A.D.; A.D. stands for Anno Domini, Latin for “in the year of our Lord.” Because we are marking in retrospect from a fixed point, we call the first hundred years after that point the first century, the second hundred years the second century, and so on. The first hundred years are numbered zero (for the period less than a year after Jesus’ birth) through ninety-nine. This continues until we find that the twentieth century is numbered 1900 A.D. through 1999 A.D.

24. In our numeration system every three digits have a different name, such as thousands, millions, and billions.

25. Place value is the idea of assigning different *number values* to *digits* depending on their position in a number. This means that the numeral 4 (four) would have a different value in the “ones” place than in the “hundreds” place, because 4 ones are very different from 4 hundreds. (That’s why 4 isn’t equal to 400.)

26. If we use 2 feet as our average shoulder width, and we use 25,000 miles as the circumference of Earth, we have 25,000 miles × 5280 feet per mile divided by 2 feet per person = 66,000,000.

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27. **a.** 11.57 days

b. 11,570 days, or 31.7 years.

28. **a.** 94.7 miles. Depending on the value you use for the length of a dollar bill, you might get a slightly different amount.

b. 94,700 miles long, or almost 4 times around Earth!

29.

a. 21	**b.** 35	**c.** 55	**d.** 279
e. 26	**f.** 259	**g.** 51	**h.** 300
i. 13	**j.** 17	**k.** 153	**l.** 2313

30.

a. 134_{five}	**b.** 1102_{five}	**c.** 1011100_{two}	**d.** 11001110_{two}
e. 60_{twelve}	**f.** $192_{sixteen}$	**g.** 112_{six}	**h.** 5444_{six}
i. $90_{sixteen}$	**j.** 400_{five}	**k.** $63_{sixteen}$	**l.** 13202_{five}

m. 1120000000_{five}

n. 100 110 001 001 011 010 000 0 (The spaces are only for readability.)

31. base 9

32. base 9

33. $x = 9$

34. 50_x candy bars, $x = 6$

35. This has to do with dimensions. The base 10 long is 2 times the length of the base 5 long. When we go to the next place, we now have a new dimension, so the value will be 2×2 as much. This links to measurement. If we compare two cubes, one of whose sides is double the length of the other, the ratio of lengths of sides is 2:1, the ratio of the surface area is 4:1, the ratio of the volumes is 8:1.

36. **a.** Just as each of the places in a base 10 numeral has a specific value that is a power of 10, each of the places in a base 5 numeral also has a value, but in a base 5 numeral these values are powers of 5. Let's look at this diagram:

___	___	___	___	5
125	25	5	1	
5^3	5^2	5^1	5^0	

We can see that the places of a base 5 number, starting from the right, have the values 1, 5, 25, and 125. Now we ask ourselves how many times these go into $234_{ten}\cdot$ 125 goes into 234 once, leaving 109; there are four 25s in 109, leaving 9; and, finally, the 9 can be written as one 5 and four 1s, so our number is 1214_5.

1	4	1	4	five
125	25	5	1	
5^3	5^2	5^1	5^0	

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b. A similar chart can be created to show that $405_{eight} = (4 \times 64) + (0 \times 8) + (5 \times 1) = 261_{ten}$

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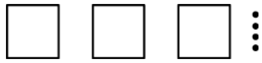
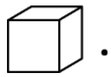
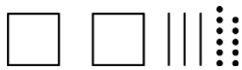
37. **a.** 

b. 

c. 

38. **a.** 7777

b. f f f

39. Answers will vary.

40. 1_{four}, 2_{four}, 3_{four}, 10_{four}, 11_{four}, 12_{four}, 13_{four}, 20_{four}, 21_{four}, 22_{four}, 23_{four}, 30_{four}, 31_{four}, 32_{four}, 33_{four}, 100_{four}, …

41. 835

42. (9 × 10) + (5 × 1) + (8 × 100) = 90 + 5 + 800 = 895; the correct answer is b.

43. (12 × 10) + 30,605 = 120 + 30,605 = 30,725; the correct answer is b.

44. (6 × 100,000) + (23 × 100) = 600,000 + 2300 = 602,300; the correct answer is c.

45. The digit being replaced is in the tens place. So, if the digit 1 is replaced by the digit 5, the number is increased by (5 × 10) – (1 × 10) = 50 – 10 = 40. The correct answers is b.

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CHAPTER 2 REVIEW EXERCISES

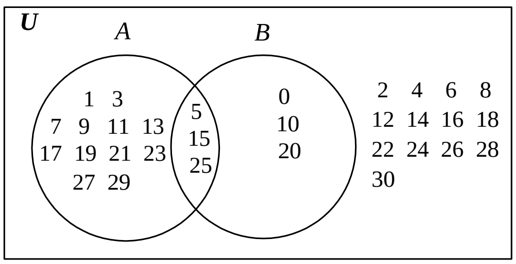
1. **a.** $\{x \mid x = 10^n, n = 0, 1, 2, 3, \ldots\}$

b. $\{10, 100, 1000, 10{,}000, \ldots\}$ or $\{10^1, 10^2, 10^3, \ldots\}$

2. **a.** $\in$ **b.** $\subset$ **c.** $\subset$ **d.** $\not\subset$

3. **a.** $D \not\subset E$ **b.** $0 \notin \{\ \}$

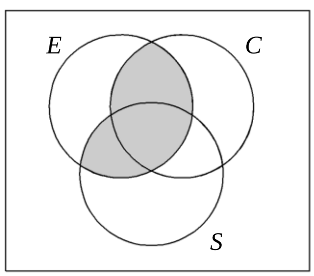
4. **a.**



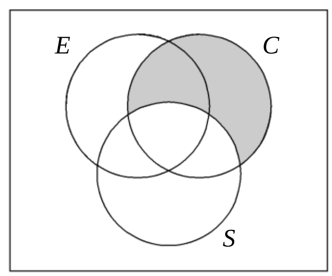
b. 5, 15, 25

c. The set of even numbers between 0 and 30.

5. **a.**



b.



6. 50 have both.

7. The former means the same elements, and the latter means the same number of elements.

8.

		Egyptian	Roman	Babylonian
a.	47	∩∩∩∩ |||||||	XLVII	<<<< ▼▼▼▼▼▼▼
b.	95	∩∩∩∩∩∩∩∩∩ |||||	XCV	▼ <<< ▼▼▼▼▼
c.	203	𓍢𓍢 |||	CCIII	▼▼▼ << ▼▼▼
d.	3210	𓆼𓆼𓆼 𓍢𓍢 ∩	MMMCCX	<<<<< ▼▼▼ <<<

9. **a.** 410_{five} **b.** 1300_{five} **c.** 1010_{two}

10. **a.** 4314_{five} **b.** 30034_{five} **c.** 1011_{two}

11. $25 \times 2 + 4 \times 5 + 3 = 73$

 CHAPTER 2 Fundamental Concepts 25

12. Because $1000_{five} = 125_{ten}$, I would rather have $\$200_{ten}$.

 CHAPTER 2 Fundamental Concepts 26

13. They both have the value of 3 flats, 2 longs, and 1 single. Because base 6 flats and longs have greater value than base 5 flats and longs, the two numbers do not have the same value.

14. Because we are dealing with powers. Thus, the value of a base 10 flat is $2 \times 2 = 4$ times the value of a base 5 flat.

15. The value of the 5th place in base 10 is $10^4 = 10,000$. The value of the 5th place in base 5 is $5^4 = 625$. $10000 \div 625 = 16$.

16. There are many possible responses. Here are three: "One-zero" is the amount obtained when the first place is full. It means you have used up all the single digits in your base. It is the first two-digit number.

17. There are several equivalent representations:

$2000 + 60 + 8$

$2 \times 1000 + 0 \times 100 + 6 \times 10 + 8 \times 1$

$2 \times 1000 + 6 \times 10 + 8 \times 1$

$2 \times 10^3 + 0 \times 10^2 + 6 \times 10^1 + 8 \times 10^0$

$2 \times 10^3 + 6 \times 10^1 + 8 \times 10^0$

18. Answers will need to include all six characteristics described in the section.

Fundamental Concepts

2

Section 2.2

Numeration

- Origins of Numbers and Counting
- Numeration Systems
- Base 10
- Units and Other Bases

Origins of Numbers and Counting

Origin of Numbers and Counting

The Common Core State Standards have kindergarten students gaining a foundation for place value by using objects and drawings to understand that the numbers 11 to 19 are composed of one ten and a certain number of ones. Developing a deeper understanding of place value continues through the elementary school years and supports understanding the operations as well.

Origin of Numbers and Counting

Did you know that people had to invent counting?

The earliest systems must have been quite simple, probably tallies. The oldest archaeological evidence of such thinking is a wolf bone over 30,000 years old, discovered in the former Czechoslovakia.

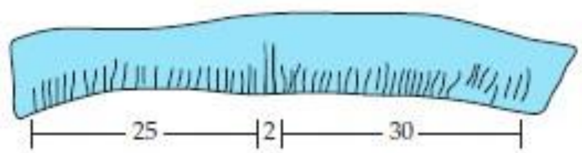


Figure 2.17

On the bone are 55 notches in two rows, divided into groups of five. We can only guess what the notches represent—how many animals the hunter had killed or how many people there were in the tribe.

Origin of Numbers and Counting

Other anthropologists have discovered how shepherds were able to keep track of their sheep without using numbers to count them. Each morning as the sheep left the pen, the shepherds made a notch on a piece of wood or on some other object.

In the evening, when the sheep returned, they would again make a notch for each sheep. Looking at the two tallies, they could quickly see whether any sheep were missing.

Origin of Numbers and Counting

Anthropologists also have discovered several tribes in the twentieth century that did not have any counting systems!

The beginnings of what we call civilization were laid when humans made the transition from being hunter-gatherers to being farmers.

Archaeologists generally agree that this transition took place almost simultaneously in many parts of the world some 10,000 to 12,000 years ago.

Origin of Numbers and Counting

It was probably during this transition that the need for more sophisticated numeration systems developed.

Origin of Numbers and Counting

For example, a tribe need kill only a few animals, but one crop of corn will yield many hundreds of ears of corn.

The invention of numeration systems was not as simple as you might think.

The ancient Sumerian words for one, two, and three were the words for man, woman, and many.

Origin of Numbers and Counting

The Aranda tribe in Australia used the word *ninta* for one and *tara* for two. Their words for three and four were *tara-ma-ninta* and *tara-ma-tara*.

Origin of Numbers and Counting

Requirements for counting In order to have a counting system, people first needed to realize that the number of objects is independent of the objects themselves. Look at the figure below. What do you see?

Figure 2.18

There are three objects in each of the sets. However, the number three is an abstraction that represents an amount.

Origin of Numbers and Counting

Archaeologists have found that people didn’t always understand this.

For example, the Thimshians, a tribe in British Columbia, had seven sets of words in their language for each number they knew, depending on whether the word referred to

1. animals and flat objects,
2. time and round objects,
3. humans,
4. trees and long objects,
5. canoes,
6. measures, and

Origin of Numbers and Counting

7. miscellaneous objects.

Origin of Numbers and Counting

Whereas we would say three people, three beavers, three days, and so on, they would use a different word for “three” in each case.

There is another aspect of counting that needs to be noted.

Most people think of numbers in terms of counting discrete objects.

Origin of Numbers and Counting

However, this is only one of the contexts in which numbers occur. For example, in Figure 2.24, there are 3 balls, there are 3 ounces of water in the jar, and the length of the line
is 3 centimeters.

Figure 2.20

In the first case, the 3 tells us how many objects we have.

Origin of Numbers and Counting

However, in the two latter cases, the number tells how many of the units we have. In this example, the units are ounces and centimeters.

Working with numbers that represent discrete amounts is more concrete than working with numbers that represent measures.

We distinguish between **number,** which is an abstract idea that represents a quantity, and **numeral,** which refers to the symbol(s) used to designate the quantity.

Origin of Numbers and Counting

As humans developed names for amounts larger than the number of fingers on one or two hands, the names for the larger amounts were often combinations of names for smaller amounts.

People who have investigated the development of numeration systems, from prehistoric tallies to the Hindu Arabic system, have discovered that most of the numeration systems had patterns, both in the symbols and in the words, around the amounts we call 5 and 10.

Origin of Numbers and Counting

However, a surprising number of systems also show patterns around 2, 20, and 60. For example, the French word for eighty, *quatre-vingts*, literally means "four twenties."

As time went on, people developed increasingly elaborate numeration systems so that they could have words and symbols for larger and larger amounts.

We will examine three different numeration systems—Egyptian, Roman, and Babylonian—before we examine our own base ten system.

Numeration Systems

Numeration Systems

Egyptian System

The earliest known written numbers are from about 5000 years ago in Egypt. The Egyptians made their paper from a water plant called papyrus that grew in the marshes.

They found that if they cut this plant into thin strips, placed the strips very close together, placed another layer crosswise, and finally let it dry, they could write on the substance that resulted.

Our word *paper* derives from their word *papyrus*.

Numeration Systems

Symbols in the Egyptian system

The Egyptians developed a numeration system that combined picture symbols (hieroglyphics) with tally marks to represent amounts. The table below gives the primary symbols in the Egyptian system.

TABLE 2.4

1,000,000	100,000	10,000	1000	100	10	1
Astonished person	Polliwog or burbot fish	Pointing finger	Lotus flower	Scroll	Heelbone	Staff, stroke

The Egyptians could represent numerals using combinations of these basic symbols.

Numeration Systems

Egyptian System

Take a few minutes to think about the following questions.

1. What do you notice about the Egyptian system? Do you see any patterns?

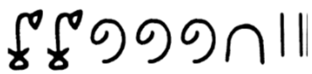
2. What similarities do you see between this and the more primitive systems we have discussed?

3. What limitations or disadvantages do you find in this system?

Numeration Systems

Egyptian System

The Egyptian numeration system resembles many earlier counting systems in that it uses tallies and pictures. In this sense, it is called an *additive system*.

Look at the way this system represents the amount 2312. In one sense, the Egyptians saw this amount as 1000 + 1000 + 100 + 100 + 100 + 10 + 1+ 1 and wrote it as . In an **additive system,** the value of a number is literally the sum of the digits.

However, this system represents a powerful advance: The

Numeration Systems

Egyptians created a new digit for every *power of 10*.

Numeration Systems

Egyptian System

They had a digit for the amount 1. To represent amounts between 1 and 10, they simply repeated the digit. For the amount 10, they created a new digit.

All amounts between 10 and 100 can now be expressed using combinations of these two digits. For the amount
100, they created a new digit, and so on.

These amounts for which they created digits are called **powers of ten.** From your work with exponents

Numeration Systems

from algebra, that we can express 10 as 10^1 and 1 as 10^0.

Numeration Systems

Egyptian System

Thus we can express the value of each of the Egyptian digits as a power of 10:

1,000,000	100,000	10,000	1000	100	10	1
$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$	$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$	$10 \cdot 10 \cdot 10 \cdot 10$	$10 \cdot 10 \cdot 10$	$10 \cdot 10$	10	1
10^6	10^5	10^4	10^3	10^2	10^1	10^0

The Egyptian system was a remarkable achievement for its time. Egyptian rulers could represent very large numbers. One of the primary limitations of this system was that computation was extremely cumbersome.

It was so difficult, in fact, that the few who could compute enjoyed very high status in the society.

Numeration Systems

Roman System

The Roman system is of historical importance because it was the numeration system used in Europe from the time of the Roman Empire until after the Renaissance.

In fact, several remote areas of Europe continued to use it well into the twentieth century.

Some film makers still list the copyright year of their films in Roman numerals.

Numeration Systems

Symbols in the Roman System

The table below gives the primary symbols used by early Romans and later Romans.

TABLE 2.5

Amount	Early Roman	Later Roman
1	I	I
5	V or Λ	V
10	X	X
50	↓	L
100	⊙	C
500	ᗡ and ↁ	D
1000	⦶ or ∞	M

Numeration Systems

Roman System

Like the Egyptians, the Romans created new digits with each power of 10, that is, 1, 10, 100, 1000, etc. However, the Romans also created new digits at "halfway" amounts— that is, 5, 50, 500, etc.

This invention reduced some of the repetitiveness that encumbered the Egyptian system. For example, 55 is not XXXXXIIIII but LV.

Basically, the Roman system, like the Egyptian system, was an additive system. However, the Later Roman system introduced a *subtractive* aspect.

Numeration Systems

Roman System

For example, IV can be seen as “one before five.” This invention further reduced the length of many large numbers.

As in the Egyptian system, computation in the Roman system was complicated and cumbersome, and neither system had anything resembling our zero.

Numeration Systems

Babylonian System

The Babylonian numeration system is a refinement of a system developed by the Sumerians several thousand years ago. Both the Sumerian and Babylonian empires were located in the region occupied by modern Iraq.

The Sumerians did not have papyrus, but clay was abundant. Thus they kept records by writing on clay tablets with a pointed stick called a stylus.

Numeration Systems

Thousands of clay tablets with their writing and numbers have survived to the present time; the earliest of these tablets were written almost 5000 years ago.

Numeration Systems

Symbols in the Babylonian System

Because the Babylonians had to make their numerals by pressing into clay instead of writing on papyrus, their symbols could not be as fancy as the Egyptian symbols.

Amount	Symbol
1	𒁹
10	𒌋

They had only two symbols, an upright wedge that symbolized "one" and a sideways wedge that symbolized "ten." In fact, the Babylonian writing system is called *cuneiform*, which means "wedge-shaped."

Numeration Systems

Babylonian System

Amounts could be expressed using combinations of these numerals; for example, 23 was written as 
.

However, being restricted to two digits creates a problem with large amounts. The Babylonians' solution to this problem was to choose the amount 60 as an important number.

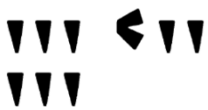
Unlike the Egyptians and the Romans, they did not create a new digit for this amount. Rather, they decided that they would have a new *place*. For example, the amount 73 was

Numeration Systems

represented as .

Babylonian System

That is, the  at the left represented 60 and the  to the right represented 13. In other words, they saw 73 as
60 + 13.

Similarly,  was seen as six 60s plus 12, or

372. We consider the Babylonian system to be a

positional

Numeration Systems

system because the value of a numeral depends on its position (place) in the number.

Numeration Systems

Babylonian System

To represent larger amounts, the Babylonians invented the idea of the value of a digit being a function of its place in the numeral.

This is the earliest occurrence of the concept of **place value** in recorded history. With this idea of place value, they could represent any amount using only two digits, 𒁹 and 𒌋.

Numeration Systems

We can understand the value of their system by examining their numerals with expanded notation. Look at the following Babylonian number: 

Numeration Systems

Babylonian System

Because the  occurs in the first (or rightmost) place, its value is simply the sum of the values of the digits—that is, $10 + 10 + 1 = 21$. However, the value of the  in the second place is determined by multiplying the face value of the digits by 60—that is, 60×23.

The value of the  in the third place is determined by multiplying the face value of the digits by 60^2—that is,
$60^2 \times 2$.

The value of this amount is

Numeration Systems

$(60^2 \times 2) + (60 \times 23) + 21 = 7200 + 1380 + 21 = 8601.$

Numeration Systems

Babylonian System

Thus, in order to understand the Babylonian system, you have to look at the face value of the digits *and* the place of the digits in the numeral.

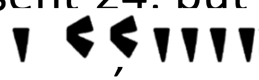
The value of a numeral is no longer determined simply by adding the values of the digits. One must take into account the place of each digit in the numeral.

The Babylonian system is more sophisticated than the Egyptian and Roman systems. However, there were some “glitches” associated with this invention.

Numeration Systems

Babylonian System

If we represent this amount from the Babylonian perspective, we note that $60^2 = 3600$. Thus the Babylonians saw 3624 as 3600 + 24.

They would use  in the third place to represent 3600, and they would use  in the first place to represent 24, but the second place is empty. Thus, if they wrote  how was the reader to know that this was not
60 + 24 = 84?

Numeration Systems

A Babylonian mathematical table from about 300 B.C. contains a new symbol  that acts like a zero.

Numeration Systems

Babylonian System

Using this convention, they could represent 3624 as

The slightly sideways wedges indicate that the second place is empty, and thus we can unambiguously interpret this numeral as representing

$$60^2 + 0 + 24 = 3624$$

This later Babylonian system is thus considered by many scholars to be the first place value system[3] because the value of every symbol depends on its place in the

Numeration Systems

numeral and there is a symbol to designate when a place is empty.

Numeration Systems

Mayan System

One of the most impressive of the ancient numeration systems comes from the Mayans, who lived in the Yucatan Peninsula in Mexico, around the fourth century A.D.

Many mathematics historians credit the Mayans as being the first civilization to develop a numeration system with a fully functioning zero.

Numeration Systems

Mayan System

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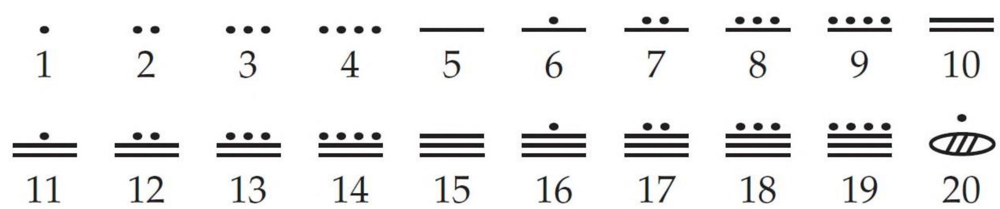
Numeration Systems

Mayan System

The table below shows their symbols for the amounts 1 through 20. Note that they wrote their numerals vertically.



Their numeral for 20 consisted of one dot and their symbol for zero. Thus, their numeral for 20 represents 1 group of

Numeration Systems

20 and 0 ones, just as our symbol for 10 represents 1 group of 10 and 0 ones.

Numeration Systems

Mayan System

Theirs was not a pure base twenty system because the value of their third place was not 20 x 20 but 18 x 20.

The value of each succeeding place was 20 times the value of the previous places. The values of their first five places were 1, 20, 360, 7200, and 14,400.

Numeration Systems

Hindu-Arabic System

The numeration system that we use was developed in India around A.D. 600. By A.D. 800, news of this system came to Baghdad, which had been founded in A.D. 762.

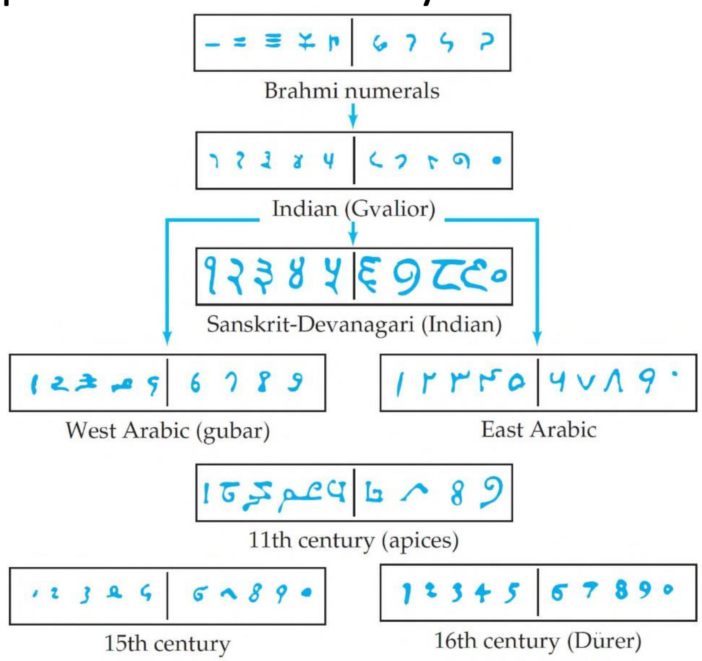
Leonardo of Pisa traveled throughout the Mediterranean and the Middle East, where he first heard of the new system.

In his book *Liber Abaci* (translated as *Book of Computations*), published in 1202, he argued the merits of this new system.

Numeration Systems

Hindu-Arabic System

The figure below traces the development of the ten digits that make up our numeration system.



Numeration Systems

Hindu-Arabic System

The development of numeration systems from the most primitive (tally) to the most efficient (base ten) has taken tens of thousands of years.

Although the base ten system is the one you grew up with, it is also the most abstract of the systems and possibly the most difficult initially for children.

Stop and reflect on what you have learned thus far in your own investigations.

Section 2.2

Base 10

Base 10

Our base ten numeration system has several characteristics that make it so powerful.

No tallies The base ten system has no vestiges of tallies. Any amount can be expressed using only 10 **digits:** 0, 1, 2,
3, 4,5, 6, 7, 8, and 9. In fact, the word *digit* literally means "finger."

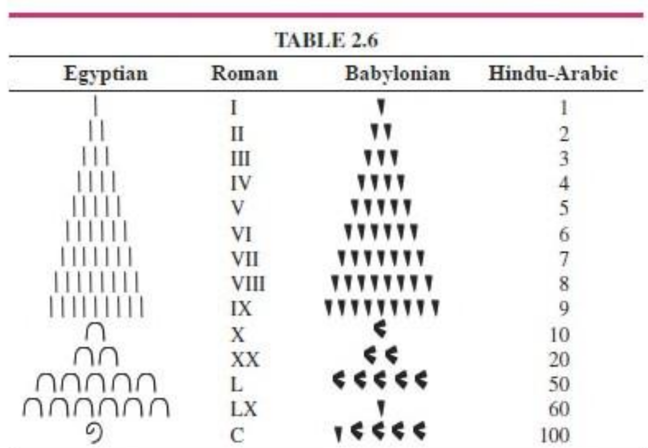


TABLE 2.6

Egyptian	Roman	Babylonian	Hindu-Arabic
|	I	▼	1
||	II	▼▼	2
|||	III	▼▼▼	3
||||	IV	▼▼▼▼	4
|||||	V	▼▼▼▼▼	5
||||||	VI	▼▼▼▼▼▼	6
|||||||	VII	▼▼▼▼▼▼▼	7
||||||||	VIII	▼▼▼▼▼▼▼▼	8
|||||||||	IX	▼▼▼▼▼▼▼▼▼	9
∩	X	◀	10
∩∩	XX	◀◀	20
∩∩∩∩∩	L	◀◀◀◀◀	50
∩∩∩∩∩∩	LX	▼	60
[illegible]	C	▼◀◀◀◀	100

Base 10

Decimal system The base ten system is a **decimal** system, because it is based on groupings (powers) of 10. The value of each successive place to the left is 10 times the value of the previous place:

100,000 10,000 1000 100 10 1

Ten ones make one ten.

Ten tens make one hundred.

Ten hundreds make one thousand.

Ten thousands make ten thousand.

Base 10

Expanded form When we represent a number by decomposing it into the sum of the values from each place, we are using **expanded form.** There are different variations of expanded form.

For example, all of the expressions below emphasize the structure of the numeral, 234—some more simply and some using exponents.

$$234 = 200 + 30 + 4$$

$$= 2 \times 100 + 3 \times 10 + 4 \times 1$$

$$= 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$$

Note: $10^1 = 10$ and $10^0 = 1$.

Base 10

The concept of zero "The invention of zero marks one of the most important developments in the whole history of mathematics."[6] This is the feature that moves us beyond the Babylonian system.

Recall the Babylonians' attempts to deal with the confusion when a place was empty.

Base 10

It was the genius of some person or persons in ancient India to develop this idea, which made for the most efficient system of representing amounts and also made computation much easier.

One of the most difficult aspects of this system is that the symbol 0 has two related meanings: In one sense, it works just like any other digit (it can be seen as the number 0), and at the same time, it also acts as a place holder.

Section 2.2

Units and Other Bases

Units and Other Bases

There is an old parable that says a journey of 1000 miles begins with a single step. The same can be said for counting. We always begin with 1. However, unlike the phrase “a rose is a rose is a rose,” a 1 is not always the same.

For example, a 1 in the millions place represents 1 million. This is the power of our numeration system, but it is very abstract.

When counting objects, one is our key term. When asked to count a pile of objects, for example, 240 pennies, children will count one at a time.

Units and Other Bases

However, if they lose their count, they have to start all over. Some children realize that they can put the pennies into piles of 10. Now if they lose count, they can go back and count by tens, for example, 10, 20, 30, 40, etc.

In this case, 10 is a key term, that is, it is composed of a number of smaller units. Some children can see that 1 pile is also 10 pennies. To be able to hold these two amounts simultaneously is a challenge for young children, and it is an essential milestone along the way.

Units and Other Bases

We have composite units everywhere: 100 is equivalent to ten 10s, 1000 is equivalent to ten 100s. In fact, our language shows this: some people will say thirty-four hundred for
3400.

We talk about 1 dozen eggs, a case of soda (24 cans), and a pound (16 ounces).

When we say that we will need 6 dozen eggs for a pancake breakfast fundraiser, we can see 6 dozen and we also know that this is 72 individual eggs.

Units and Other Bases

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Units and Other Bases

INVESTIGATION 2.2b

What If Our System Was Based on One Hand?

The people who developed base ten decided to base it on two hands. What if they had decided to base it on one hand? That is, what if one-zero had come not after we counted two hands but after we counted one hand? Our counting would look like this: 1, 2, 3, 4, 10, ... The manipulatives (see the figure below) would have the same basic shape as the base ten blocks.

Units and Other Bases

INVESTIGATION 2.2c

How Well Do You Understand Base Five?

One way to assess your understanding of this new base is to figure out what numbers come after and before given numbers.

A. What number comes after 234_{five}?

B. What number comes after 1024_{five}?

C. What number comes before 210_{five}?

D. What number comes before 3040_{five}?

Units and Other Bases

Any base—base two, base five, base ten, base twelve, etc…will have the following fundamental characteristics:

1. Any base has the same number of symbols as the number of base. In a base ten system, we have ten symbols (0-9) and in a base five system we have five symbols (0-4). With those symbols we can represent any amount.
2. The value of each place is the base times the previous place. In base ten, the value of the places are ones, ten, ten^2, ten^3, etc. Similarly, in base five the value of the places are ones, five, $five^2$, $five^3$, etc.

Units and Other Bases

Any base—base two, base five, base ten, base twelve, etc…will have the following fundamental characteristics:

3. Each place can contain only one symbol. When a place is full, we "move" to the next place by trading (regrouping) to the next higher place.

4. The value of a digit depends on its place in the numeral.

5. The value of a numeral is determined by multiplying each digit by its place value and then adding these products.

6. Zero represents an empty place and 0 represents an actual amount, with a place value on the number line.

Units and Other Bases

INVESTIGATION 2.2e

Relative Magnitude of Numbers

Our modern society deals with large numbers all the time.

Politicians talk about a war costing $100 billion a year.

The federal deficit is more than $17 trillion at the writing of this book.

The closest star to us is about 24,600,000,000,000 miles from earth.

The cleanup from Hurricane Katrina involved the removal of 500 million cubic yards of debris.

More than 25 million people have died of AIDS.

Explorations Manual 2.6

2

Fundamental Concepts

Section 2.1

Sets

- The Language and Notation of Sets
- Venn Diagrams
- Operations on Sets
- Relationships between Sets

The Language and Notation of Sets

The Language and Notation of Sets

Sets as a Classification tool

- Children use set ideas in everyday life as they look for similarities and differences between sets; for example, they want to know why lions are in the cat family and wolves are in the dog family.
- Children also look for similarities and differences within sets; for example, they look within a set of blocks for the blocks that they can stack and the blocks that they can't.
- Whether we realize it or not, we are classifying many times each day, and our lives are shaped by classifications that we and others have made.

The Language and Notation of Sets

INVESTIGATION 2.1a Classifying Quadrilaterals

Without classifying various objects and ideas, it would not be possible to have mathematics. This introductory investigation will help you to connect set language and concepts to other, more concrete mathematical ideas. Look at the eight shapes below. How can you classify these shapes into two groups so that each group has a common characteristic? Give a name to each group if you can. Work and then read on. . . .

The Language and Notation of Sets

Definitions

- A set is a well defined collection of objects, and classified that set into smaller groups (*subsets*) having certain common features. In general, a **subset** is a set that is part of some other set.
- We speak of individual objects in a given set as **members** or **elements** of the set. The symbol ∈ means "is a member of."
- The symbol ∉ means "is not a member of." For example, if E is the set of even numbers, then $4 \in E$ but $3 \notin E$.

The Language and Notation of Sets

Describing sets

There are three different ways to describe sets:

1. We can use words.

2. We can make a list.

3. We can use *set-builder notation*.

The Language and Notation of Sets

Describing sets: Using words and lists

The first set of numbers that young children learn is called the set of **natural numbers.**

N is the set of natural numbers or counting numbers. (words)

N = {1, 2, 3, . . .} (list)

We use braces to indicate a set. The three dots are referred to as an ellipsis and are used to indicate that the established pattern continues indefinitely.

The Language and Notation of Sets

Describing sets: Using words and lists

At some point, children realize that zero is also a number, and this leads to the next set: the set of **whole numbers** (W), which we can describe with words or with a list:

W is the set of natural numbers and zero. (words)

$W = \{0, 1, 2, 3, . . .\}$ (list)

The Language and Notation of Sets

Describing sets: Using words and lists

Later, children become aware of negative numbers, so we have the set of **integers** (*I*):

$I = \{. . . -3, -2, -1, 0, 1, 2, 3, . . .\}$

The Language and Notation of Sets

Describing sets: Using words and lists

Another important set is the set of **rational numbers** (*Q*), which we can describe in words:

Q is the set of all numbers that can be represented as the ratio of two integers as long as the denominator is not zero.

The Language and Notation of Sets

Describing sets: Using set-builder notation

Another way to express a set, such as the set of rational numbers, Q, is using set-builder notation.

$$Q = \left\{ \frac{a}{b} \,\middle|\, a \in I \text{ and } b \in I,\, b \neq 0 \right\}$$

This statement is read in English as, "*Q* is the set of all numbers of the form $\frac{a}{b}$ such that *a* and *b* are both integers, but *b* is not equal to zero."

Set-builder notation always takes the form

{*x* | *x* has a certain property}.

The Language and Notation of Sets

INVESTIGATION 2.1b Describing Sets

Consider the following set:

$$T = \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120\}$$

Try to describe this set with words and with set-builder notation. What do you see as advantages and disadvantages of each of the three ways to describe this set?

The Language and Notation of Sets

Types of Sets: Finite and Infinite Sets

If the number of elements in a set is a whole number, that set is said to be **finite.**

An **infinite set** has an unlimited number of members.

The Language and Notation of Sets

Types of Sets: Finite and Infinite Sets

Consider the following infinite set. Does each way of describing the set make sense?

Verbal description $\quad$ E is the set of positive even

numbers. List $\quad$ $E = \{2, 4, 6, 8, . . .\}$

Set-builder notation $\quad$ $E = \{x \mid x = 2n, n \in N\}$

The Language and Notation of Sets

Types of Sets: Subsets

A set X is a **subset** of a set Y if and only if every member of X is also a member of Y.

The symbol $\subseteq$ means "is a subset

of." Thus we say $X \subseteq Y$.

On the other hand, if a set X is not a subset of a set Y, we say $X \not\subseteq Y$.

The Language and Notation of Sets

Types of Sets: Subsets

There is another symbol that we can use when talking about subsets. This symbol (⊂) is used when we want to emphasize that the subset is a **proper subset.**

A subset *X* is a proper subset of set *Y* if, and only if, the two sets are not equal *and* every member of *X* is also a member of *Y*.

In the case of finite sets, this means that the proper subset has fewer elements than the given set.

The Language and Notation of Sets

INVESTIGATION 2.1c How Many Subsets?

This investigation opens the idea of families of subsets, shows how subsets might be useful, and provides an opportunity to develop problem-solving tools.

Let's say that you and your friends decide to go out and get a large pizza. Let T represent the set of toppings that this restaurant offers:

$$T = \{\text{onions, sausage, mushrooms, peppers}\}$$

List all the possible different combinations of pizza that you could order, such as a mushroom and onion pizza. Then read on. . . .

The Language and Notation of Sets

Types of Sets: Empty Set

We use the terms **empty set** and **null set** interchangeably to mean the set with no elements.

The following symbol is used to represent a set that is empty: $\varnothing$.

Using brackets, we would write { }.

In investigation 2.1c, the subset "plain pizza" is an empty set.

The Language and Notation of Sets

Types of Sets

Two mathematical statements worth noting:

- Every set is a subset of itself.
- The empty set is a subset of every set.

Types of Sets: Equal and Equivalent Sets

Two sets are said to be equal if they contain the same elements. For example, {1, 2, 3, 4, 5} = {5, 4, 3, 2, 1}.

Two sets are said to be equivalent if they have the same number of elements. More precisely, two sets are equivalent if their elements can be placed in a **one-to-one correspondence**. In such a correspondence, an element of either set if paired exactly with one element in the other set. We use the symbol ~ to designate set equivalence. For example,

{United States, Canada, Mexico} ~ {1, 2, 3}.

Venn Diagrams

Venn Diagrams

One way to represent sets is to use **Venn diagrams,** which are named after John Venn.

An elementary teacher explained how she had used the Venn diagram below to help her students understand the similarities and differences between butterflies and moths.

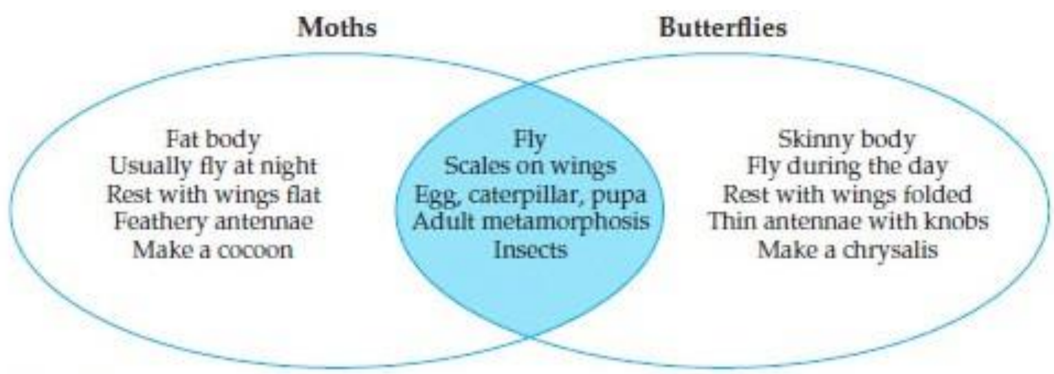


Figure 2.1

Venn Diagrams

In this Venn Diagram, one region represents the set of moths' characteristics, another region represents the set of butterflies' characteristics, and the overlapping region represents the set of characteristics common to both.

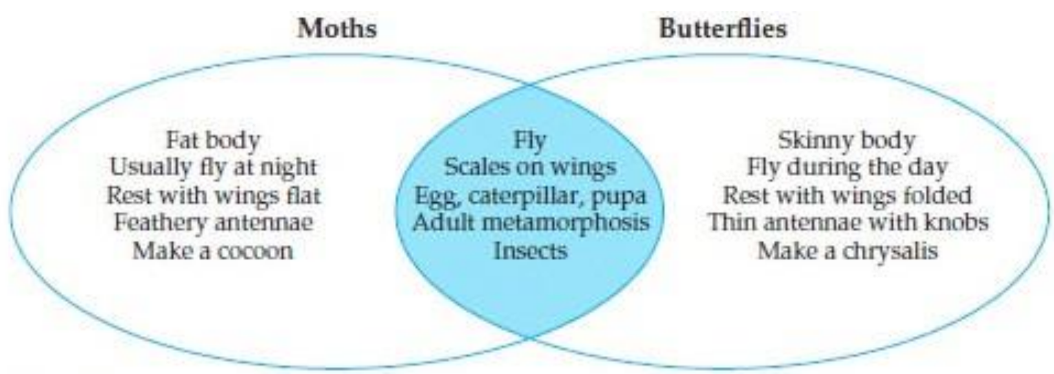


Figure 2.1

Operations on Sets

Operations on Sets

We will use Venn diagrams as we examine three operations on sets: intersection, union, and complement.

When we perform operations on sets of objects, it is often useful to refer to the set that consists of all the elements being considered as the **universal set,** or the **universe,** and to symbolize it as U.

We represent U in the Venn diagram with a rectangle.

Operations on Sets

In the following discussions, we will let *U* be the set of students in a small class.

U = {Amy, Uri, Tia, Eli, Pam, Sue, Tom,

Riki} We begin with two subsets of U:

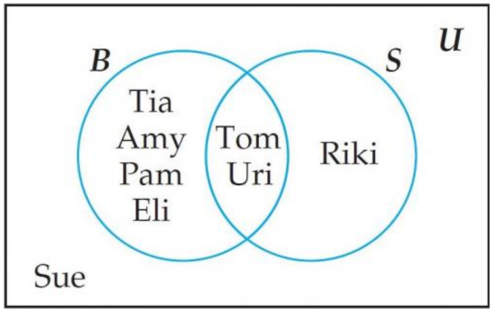
B = {students who have at least one brother}

S = {students who have at least one sister}

We represent *U* in the Venn diagram with a rectangle.

Operations on Sets

The figure below represents the students in this hypothetical class.



Operations on Sets

We can group this class of eight students into various subsets.

How would you describe the subset consisting of Tom and

Uri?

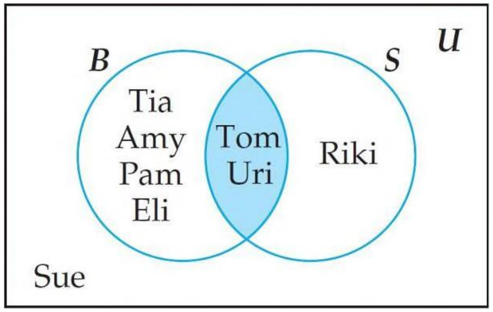
Operations on Sets

Intersection

One way to describe this subset is, "Those students who have at least one brother and at least one sister."

Mathematically, we call this subset the *intersection* of sets *B* and *S*. In mathematical language, we say that the **intersection** of two sets *B* and *S* consists of the set of all elements common to both *B* and *S*.

We represent the intersection of *B* and *S* by shading it.

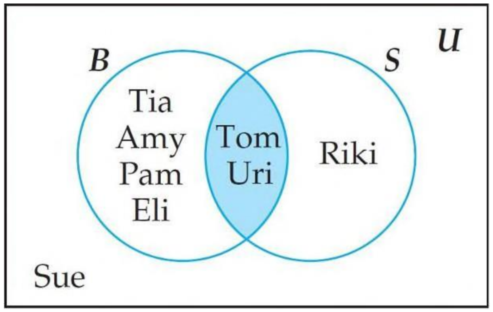


Operations on Sets

Intersection

Using set-builder notation, we write $B \cap S = \{x \mid x \in B \text{ and } x \in S\}$ The symbol ∩ is used to denote "intersection."

Connecting the concept of intersection to previous notation, we can say Tom $\in B \cap S$, and we can say $(B \cap S) \subset U$.

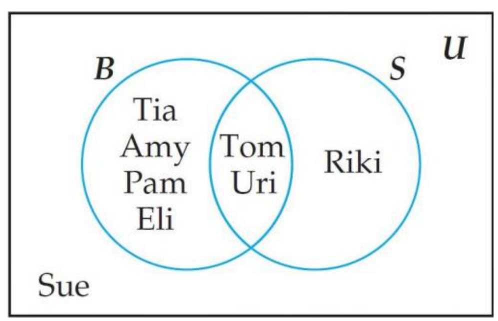


Operations on Sets

Let us examine another subset of the class:

{Tia, Amy, Pam, Eli, Tom, Uri, Riki}.

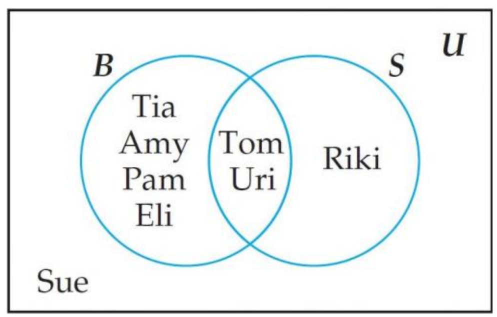
How would you describe this subset in everyday English?



Operations on Sets

There are actually several ways to describe this subset:

- Those students who have at least one brother or sister.
- Those students who have at least one sibling.
- Those students who are not an only child.



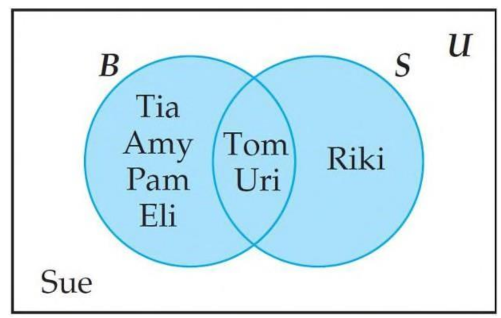
Operations on Sets

Union

Mathematically, we describe this subset as the *union* of sets *B* and *S*.

In mathematical language, we say that the **union** of two sets *B* and *S* consists of the set of all elements that are in set *B or* in set *S or* in both sets *B* and *S*.

We represent the union of *B* and *S* by shading it.



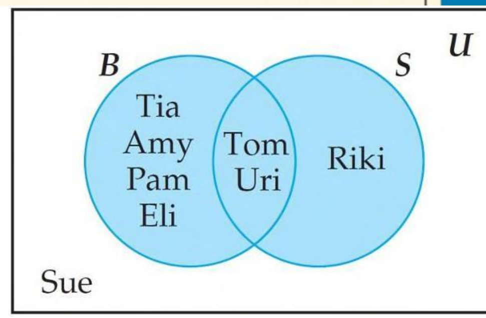
Operations on Sets

Union

Symbolically, we write $B \cup S = \{x \mid x \in B$ and/or $x \in S\}$ The symbol $\cup$ is used to denote "union."

Connecting the concept of union to previous notation, we can say Tom $\in B \cup S$, and we can also say $B \subset (B \cup S)$.

Operations on Sets

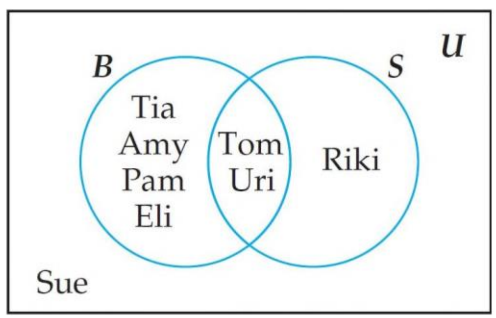


Operations on Sets

Let us examine another subset of the class. Consider this subset of the class:

{Tia, Amy, Pam, Eli, Sue}.

One way to describe this subset is, "Those students who have no sisters."



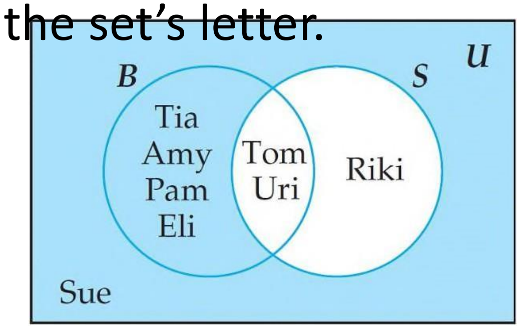
Operations on Sets

Complement

We describe this subset as the *complement* of set *S*. In mathematical language, the **complement** of set *S* consists of the set of all elements in *U* that are *not* in *S*.

We represent the complement of *S* by shading it.

In symbols, we write $S = \{x \mid x \notin S\}$. We represent the complement of a set by placing a line over the set's letter.



Operations on Sets

Complement

Some people understand complement better if they think of the complement of *S* as "not *S*"—that is, all elements that are not in set *S*.

Operations on Sets

Subtraction

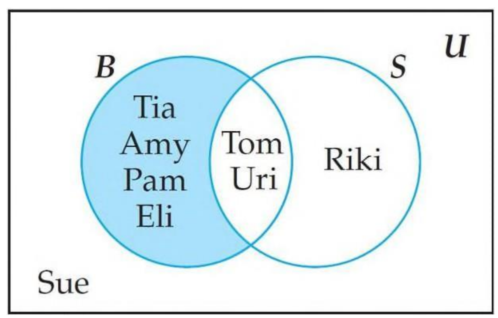
Subtraction Just as we have the operation of subtraction on whole numbers, we have the operation of subtraction on sets. Thinking of what subtraction means, what elements do you predict would be in $B - S$?

Operations on Sets

Subtraction

Verbally, we define set difference as the set of all elements that are in *B* that are not in *S*.

We represent *B* – *S* by shading it as shown below.



Operations on Sets

Symbolically, we write $B - S = \{x \mid x \in B \text{ and } x \notin S\}$

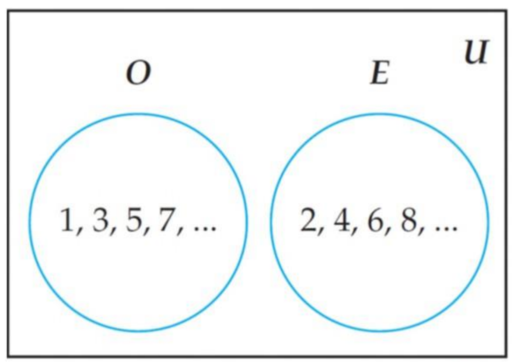
Relationships between Sets

Relationships between Sets

Disjoint

When we are considering two sets, there are three ways in which they might be related.

1. They can have nothing in common. In this case, we call them **disjoint sets,** as shown below.



Relationships between Sets

Disjoint

The set of odd and even numbers are disjoint.

$O = \{1, 3, 5, 7, \ldots\}$

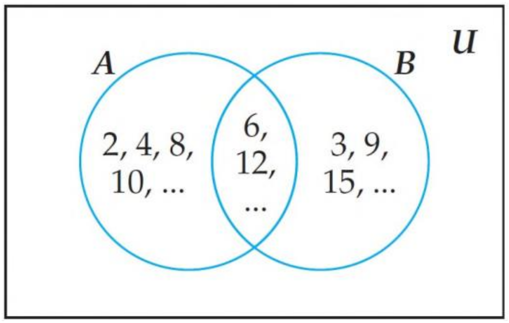
$E = \{0, 2, 4, 6, \ldots\}$

Relationships between Sets

2. They can have some elements in common (Figure 2.8).
   Multiples of 2 and 3 have some elements in common.

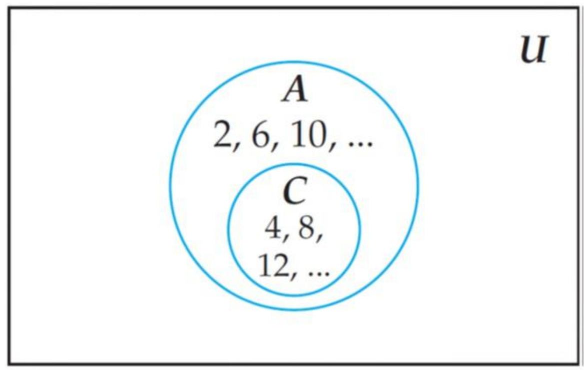
$A = \{2, 4, 6, 8, 10, 12, ...\}$

$B = \{3, 6, 9, 12, 15, ...\}$



Relationships between Sets

3. One set can be a subset of the other, as shown below.



The multiples of 4 are a subset of the multiples of 2.

A = {2, 4, 6, 8, 10, 12, ...}

C = {4, 8, 12, ...}

Relationships between Sets

INVESTIGATION 2.1d Translating Among Representations

Consider the following situation:

U = the set of natural numbers from 1 to 30

A = the set of multiples of 2

B = the set of multiples of 3

C = the set of multiples of 5

First, draw a Venn diagram showing these sets and their relationships. Then, translate the following situations into notation:

A. The numbers that are multiples of 2 and 3

B. The numbers that are multiples of 2 and 3 but not 5

C. The numbers that are only multiples of 2, 3, and 5

Relationships between Sets

INVESTIGATION 2.1e

Finding Information from Venn Diagrams

The owners of Marksalot (a company that makes all sorts of markers) believe that healthy, happy employees are productive employees, so they had a consultant conduct an anonymous survey about employees' exercise habits, their smoking habits, and one particular aspect of their eating habits. Figure 2.12 represents the survey results.

U = the employees of Marksalot

E = employees who exercise regularly

S = employees who do not smoke at all

F = employees who average at least five servings of fruits and vegetables each day

Answer the four questions below on your own and then check your answers. . . .

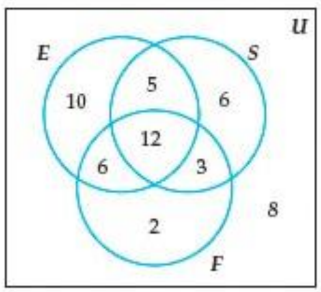


Figure 2.12

A. How many employees exercise regularly but don't average five daily servings of fruits and vegetables? Represent that subset visually and with symbols.

B. Describe the 8 employees who are outside all three circles (Figure 2.13), first in everyday English and then with symbols.

C. Describe the following subset both in everyday English and visually:

$(E \cap F) \cup (E \cap S)$