2

Signals and Systems

SIGNALS AND THEIR PROPERTIES

Solution 2.1

(a) $\delta_s(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n) = \sum_{m=-\infty}^{\infty} \delta(x - m) \cdot \sum_{n=-\infty}^{\infty} \delta(y - n)$, therefore it is a separable signal.

(b) $\delta_l(x, y)$ is separable if $\sin(2\theta) = 0$. In this case, either $\sin\theta = 0$ or $\cos\theta = 0$, $\delta_l(x, y)$ is a product of a constant function in one axis and a 1-D delta function in another. But in general, $\delta_l(x, y)$ is not separable.

(c) $e(x, y) = \exp[j2\pi(u_0x+v_0y)] = \exp(j2\pi u_0x)\cdot\exp(j2\pi v_0y) = e_{1D}(x; u_0)\cdot e_{1D}(y; v_0)$, where $e_{1D}(t; \omega) = \exp(j2\pi\omega t)$. Therefore, $e(x, y)$ is a separable signal.

(d) $s(x, y)$ is a separable signal when $u_0v_0 = 0$. For example, if $u_0 = 0$, $s(x, y) = \sin(2\pi v_0y)$ is the product of a constant signal in x and a 1-D sinusoidal signal in y. But in general, when both u_0 and v_0 are nonzero, $s(x, y)$ is not separable.

Solution 2.2

(a) Not periodic. $\delta(x, y)$ is non-zero only when $x = y = 0$.

(b) Periodic. By definition

$$\text{comb}(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n) .$$

For arbitrary integers M and N, we have

$$\text{comb}(x + M, y + N) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m + M, y - n + N)$$

$$= \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta(x - p, y - q) \; [\text{let } p = m - M,\ q = n - N\,]$$

$$= \text{comb}(x, y)\,.$$

So the smallest period is 1 in both x and y directions.

(c) Periodic. Let $f(x + T_x, y) = f(x, y)$, we have

$$\sin(2\pi x)\cos(4\pi y) = \sin(2\pi(x + T_x))\cos(4\pi y)\,.$$

Solving the above equation, we have $2\pi T_x = 2k\pi$ for arbitrary integer k. So the smallest period for x is $T_{x0} = 1$. Similarly, we find that the smallest period for y is $T_{y0} = 1/2$.

(d) Periodic. Let $f(x + T_x, y) = f(x, y)$, we have

$$\sin(2\pi(x + y)) = \sin(2\pi(x + T_x + y)).$$

So the smallest period for x is $T_{x0} = 1$ and the smallest period for y is $T_{y0} = 1$.

(e) Not periodic. We can see this by contradiction. Suppose $f(x, y) = \sin(2\pi(x^2 + y^2))$ is periodic; then there exists some T_x such that $f(x + T_x, y) = f(x, y)$, and

$$\begin{aligned}\sin(2\pi(x^2 + y^2)) &= \sin(2\pi((x + T_x)^2 + y^2)) \\ &= \sin(2\pi(x^2 + y^2 + 2xT_x + T_x^2))\,.\end{aligned}$$

In order for the above equation to hold, we must have that $2xT_x + T_x^2 = k$ for some integer k. The solution for T_x depends on x. So $f(x, y) = \sin(2\pi(x^2 + y^2))$ is not periodic.

(f) Periodic. Let $f_d(m + M, n) = f_d(m, n)$. Then

$$\sin\left(\frac{\pi}{5}m\right)\cos\left(\frac{\pi}{5}n\right) = \sin\left(\frac{\pi}{5}(m + M)\right)\cos\left(\frac{\pi}{5}n\right).$$

Solving for M, we find that $M = 10k$ for any integer k. The smallest period for both m and n is therefore 10.

(g) Not periodic. Following the same strategy as in (f), we let $f_d(m + M, n) = f_d(m, n)$, and then

$$\sin\left(\frac{1}{5}m\right)\cos\left(\frac{1}{5}n\right) = \sin\left(\frac{1}{5}(m + M)\right)\cos\left(\frac{1}{5}n\right).$$

The solution for M is $M = 10k\pi$. Since $f_d(m, n)$ is a discrete signal, its period must be an integer if it is to be periodic. There is no integer k that solves the equality for $M = 10k\pi$ for some M. So, $f_d(m, n) = \sin\left(\frac{1}{5}m\right)\cos\left(\frac{1}{5}n\right)$ is not periodic.

Solution 2.3

(a) We have

$$\begin{aligned}E_\infty(\delta_s) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta_s^2(x, y)\,dx\,dy \\ &= \lim_{X\to\infty}\lim_{Y\to\infty}\int_{-X}^{X}\int_{-Y}^{Y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x - m, y - n)\,dx\,dy \\ &= \lim_{X\to\infty}\lim_{Y\to\infty}(2\lfloor X\rfloor + 1)(2\lfloor Y\rfloor + 1) \\ &= \infty\,,\end{aligned}$$

where $\lfloor X \rfloor$ is the greatest integer that is smaller than or equal to X. We also have

$$\begin{aligned}
P_\infty(\delta_s) &= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\delta_s^2(x,y)\,dx\,dy\\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m,y-n)\,dx\,dy\\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{(2\lfloor X\rfloor+1)(2\lfloor Y\rfloor+1)}{4XY}\\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{4\lfloor X\rfloor\lfloor Y\rfloor}{4XY}+\frac{2\lfloor X\rfloor+2\lfloor Y\rfloor}{4XY}+\frac{1}{4XY}\\
&= 1\,.
\end{aligned}$$

(b) We have

$$\begin{aligned}
E_\infty(\delta_l) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\delta(x\cos\theta+y\sin\theta-l)|^2\,dx\,dy\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x\cos\theta+y\sin\theta-l)\,dx\,dy\\
&\overset{1}{=} \begin{cases}\displaystyle\int_{-\infty}^{\infty}\frac{1}{|\sin\theta|}\,dx, & \sin\theta=0\\[2ex] \displaystyle\int_{-\infty}^{\infty}\frac{1}{|\cos\theta|}\,dy, & \cos\theta=0\end{cases}\\
E_\infty(\delta_l) &= \infty\,.
\end{aligned}$$

Equality 1 comes from the scaling property of the point impulse. The 1-D version of Eq. (2.8) in the text is $\delta(ax) = \frac{1}{|a|}\delta(x)$. Suppose $\cos\theta = 0$. Then

$$\delta(x\cos\theta+y\sin\theta-l) = \frac{1}{|\cos\theta|}\,\delta\left(x+y\frac{\sin\theta}{\cos\theta}-\frac{l}{\cos\theta}\right).$$

Therefore,

$$\int_{-\infty}^{\infty}\delta(x\cos\theta+y\sin\theta-l)dx = \frac{1}{|\cos\theta|}\,.$$

We also have

$$\begin{aligned}
P_\infty(\delta_l) &= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}|\delta(x\cos\theta+y\sin\theta-l)|^2dx\,dy\\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\delta(x\cos\theta+y\sin\theta-l)dx\,dy\,.
\end{aligned}$$

Without loss of generality, assume $\theta = 0$ and $l = 0$, so that we have $\sin\theta = 0$ and $\cos\theta = 1$. Then it follows that

$$\begin{aligned}
P_\infty(\delta_l) &= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\delta(x)\,dx\,dy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-Y}^{Y}\left(\int_{-X}^{X}\delta(x)dx\right)dy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-Y}^{Y}1dx \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{2Y}{4XY} \\
&= \lim_{X\to\infty}\frac{1}{2X} \\
&= 0\,.
\end{aligned}$$

(c) We have

$$\begin{aligned}
E_\infty(e) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\exp[j2\pi(u_0x+v_0y)]|^2\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}1\,dx\,dy \\
&= \infty\,.
\end{aligned}$$

And also

$$\begin{aligned}
P_\infty(e) &= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}|\exp[j2\pi(u_0x+v_0y)]|^2\,dx\,dy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}1\,dx\,dy \\
&= 1\,.
\end{aligned}$$

(d) We have

$$\begin{aligned}
E_\infty(s) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin^2[2\pi(u_0x+v_0y)]\,dx\,dy \\
&\stackrel{2}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1-\cos[4\pi(u_0x+v_0y)]}{2}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}\,dx\,dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\cos[4\pi(u_0x+v_0y)]}{2}\,dx\,dy \\
&\stackrel{3}{=} \infty\,.
\end{aligned}$$

Equality 2 comes from the trigonometric identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$. Equality 3 holds because the first integral goes to infinity. The absolute value of the second integral is bounded, although it does not converge as X and Y go to infinity. We also have

$$
\begin{aligned}
P_\infty(s) &= \lim_{X\to\infty}\lim_{Y\to\infty} \frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y} \sin^2[2\pi(u_0x+v_0y)]\,dx\,dy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty} \frac{1}{4XY}\int_{-Y}^{Y}\left(\int_{-X}^{X}\frac{1-\cos[4\pi(u_0x+v_0y)]}{2}\,dx\right)dy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty} \frac{1}{4XY}\int_{-Y}^{Y}\left[X+\frac{\sin[4\pi(u_0X+v_0y)]-\sin[4\pi(-u_0X+v_0y)]}{8\pi u_0}\right]dy \\
&\stackrel{4}{=} \lim_{X\to\infty}\lim_{Y\to\infty} \frac{1}{4XY}\int_{-Y}^{Y}\left[X-\frac{\sin(4\pi u_0X)\cos(4\pi v_0y)}{4\pi u_0}\right]dy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty} \frac{1}{4XY}\left[2XY-\frac{2\sin(4\pi u_0X)\sin(4\pi v_0Y)}{(4\pi)^2u_0v_0}\right] \\
&= \frac{1}{2}.
\end{aligned}
$$

In order to get 4, we have used the trigonometric identity $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$. The rest of the steps are straightforward.

Since $s(x, y)$ is a periodic signal with periods $X_0 = 1/u_0$ and $Y_0 = 1/v_0$, we have an alternative way to compute P_∞ by considering only one period in each dimension. Accordingly,

$$
\begin{aligned}
P_\infty(s) &= \frac{1}{4X_0Y_0}\int_{-X_0}^{X_0}\int_{-Y_0}^{Y_0} \sin^2[2\pi(u_0x+v_0y)]\,dx\,dy \\
&= \frac{1}{4X_0Y_0}\left[2X_0Y_0-\frac{2\sin(4\pi u_0X_0)\sin(4\pi v_0Y_0)}{(4\pi)^2u_0v_0}\right] \\
&= \frac{1}{4X_0Y_0}\left[2X_0Y_0-\frac{2\sin(4\pi)\sin(4\pi)}{(4\pi)^2u_0v_0}\right] \\
&= \frac{1}{2}.
\end{aligned}
$$

SYSTEMS AND THEIR PROPERTIES

Solution 2.4

Suppose two LSI systems S_1 and S_2 are connected in cascade. For any two input signals $f_1(x, y)$, $f_2(x, y)$, and two constants a_1 and a_2, we have the following:

$$
\begin{aligned}
S_2[S_1[a_1f_1(x,y)+a_2f_2(x,y)]] &= S_2[a_1S_1[f_1(x,y)]+a_2S_1[f_2(x,y)]] \\
&= a_1S_2[S_1[f_1(x,y)]]+a_2S_2[S_1[f_2(x,y)]].
\end{aligned}
$$

So the cascade of two LSI systems is also linear. Now suppose for a given signal $f(x, y)$ we have $S_1[f(x, y)] = g(x, y)$, and $S_2[g(x, y)] = h(x, y)$. By using the shift-invariance of the systems, we can prove that the cascade of two LSI systems is also shift invariant:

$$S_2[S_1[f(x-\xi, y-\eta)]] = S_2[g(x-\xi, y-\eta)] = h(x-\xi, y-\eta) \, .$$

This proves that two LSI systems in cascade is an LSI system

To prove Eq. (2.46) we carry out the following:

$$
\begin{aligned}
g(x,y) &= h_2(x,y) * [h_1(x,y) * f(x,y)] \\
&= h_2(x,y) * \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\xi,\eta) f(x-\xi, y-\eta)\, d\xi\, d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(u,v) \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\xi,\eta) f(x-u-\xi, y-v-\eta)\, d\xi\, d\eta\right] du\, dv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(u,v) h_1(\xi,\eta) f(x-u-\xi, y-v-\eta)\, d\xi\, d\eta du\, dv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\xi,\eta) \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(u,v) f(x-\xi-u, y-\eta-v)\, du\, dv\right] d\xi\, d\eta \\
&= h_1(x,y) * [h_2(x,y) * f(x,y)]\,.
\end{aligned}
$$

This proves the second equality in (2.46). By letting $\alpha = u + \xi$, and $\beta = v + \eta$, we have

$$
\begin{aligned}
g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(u,v) h_1(\xi,\eta) f(x-u-\xi, y-v-\eta)\, d\xi\, d\eta du\, dv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\alpha-\xi, \beta-\eta) h_1(\xi,\eta)\, d\xi\, d\eta\right] f(x-\alpha, y-\beta)\, d\alpha\, d\beta \\
&= [h_1(x,y) * h_2(x,y)] * f(x,y)\,,
\end{aligned}
$$

which proves the second equality in (2.46).

To prove (2.47) we start with the definition of convolution

$$
\begin{aligned}
g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\xi,\eta) h_1(x-\xi, y-\eta) d\xi\, d\eta \\
&= h_1(x,y) * h_2(x,y)\,.
\end{aligned}
$$

We then make the substitution $\alpha = x - \xi$ and $\beta = y - \eta$ and manipulate the result

$$
\begin{aligned}
g(x,y) &= \int_{+\infty}^{-\infty}\int_{+\infty}^{-\infty} h_2(x-\alpha, y-\beta) h_1(\alpha,\beta)(-d\alpha)\,(-d\beta) \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_1(\alpha,\beta) h_2(x-\alpha, y-\beta) d\alpha\, d\beta \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_1(\xi,\eta) h_2(x-\xi, y-\eta) d\xi\, d\eta \\
&= h_2(x,y) * h_1(x,y)\,,
\end{aligned}
$$

where the next to last equality follows since α and β are just dummy variables in the integral.

Solution 2.5

1. Suppose the PSF of an LSI system is absolutely integrable.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x, y)|\, dx\, dy \le C < \infty \tag{S2.1}$$

where C is a finite constant. For a bounded input signal $f(x, y)$

$$|f(x, y)| \le B < \infty\,, \quad \text{for every } (x, y)\,, \tag{S2.2}$$

for some finite B, we have

$$\begin{aligned}
|g(x, y)| &= |h(x, y) * f(x, y)| \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\xi, y-\eta) f(\xi, \eta) d\xi d\eta \\
&\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x-\xi, y-\eta)| \cdot |f(\xi, \eta)|\, d\xi d\eta \\
&\le B \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x, y)|\, dx\, dy \\
&\le BC < \infty\,, \quad \text{for every } (x, y)
\end{aligned} \tag{S2.3}$$

So $g(x, y)$ is also bounded. The system is BIBO stable.

2. We use contradiction to show that if the LSI system is BIBO stable, its PSF must be absolutely integrable. Suppose the PSF of a BIBO stable LSI system is $h(x, y)$, which is not absolutely integrable, that is,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x, y)|\, dx\, dy$$

is not bounded. Then for a bounded input signal $f(x, y) = 1$, the output is

$$|g(x, y)| = |h(x, y) * f(x, y)| = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x, y)|\, dx\, dy,$$

which is also not bounded. So the system can not be BIBO stable. This shows that if the LSI system is BIBO stable, its PSF must be absolutely integrable.

Solution 2.6

(a) If $g'(x, y)$ is the response of the system to input $\sum_{k=1}^{K} w_k f_k(x, y)$, then

$$\begin{aligned}
g'(x, y) &= \sum_{k=1}^{K} w_k f_k(x, -1) + \sum_{k=1}^{K} w_k f_k(0, y) \\
&= \sum_{k=1}^{K} w_k [f_k(x, -1) + f_k(0, y)] \\
&= \sum_{k=1}^{K} w_k g_k(x, y)
\end{aligned}$$

where $g_k(x, y)$ is the response of the system to input $f_k(x, y)$. Therefore, the system is linear.

(b) If $g'(x, y)$ is the response of the system to input $f(x - x_0, y - y_0)$, then

$$g'(x, y) = f(x - x_0, -1 - y_0) + f(-x_0, y - y_0);$$

while

$$g(x - x_0, y - y_0) = f(x - x_0, -1) + f(0, y - y_0).$$

Since $g'(x, y) = g(x - x_0, y - y_0)$, the system is not shift-invariant.

Solution 2.7

(a) If $g'(x, y)$ is the response of the system to input $\sum_{k=1}^{K} w_k f_k(x, y)$, then

$$\begin{aligned} g'(x, y) &= \left(\sum_{k=1}^{K} w_k f_k(x, y)\right) \left(\sum_{k=1}^{K} w_k f_k(x - x_0, y - y_0)\right) \\ &= \sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j f_i(x, y) f_j(x - x_0, y - y_0), \end{aligned}$$

while

$$\sum_{k=1}^{K} w_k g_k(x, y) = \sum_{k=1}^{K} w_k f_k(x, y) f_k(x - x_0, y - y_0).$$

Since $g'(x, y) = \sum_{k=1}^{K} g_k(x, y)$, the system is nonlinear.

On the other hand, if $g'(x, y)$ is the response of the system to input $f(x - a, y - b)$, then

$$\begin{aligned} g'(x, y) &= f(x - a, y - b) f(x - a - x_0, y - b - y_0) \\ &= g(x - a, y - b) \end{aligned}$$

and the system is thus shift-invariant.

(b) If $g'(x, y)$ is the response of the system to input $\sum_{k=1}^{K} w_k f_k(x, y)$, then

$$\begin{aligned} g'(x, y) &= \int_{-\infty}^{\infty} \sum_{k=1}^{K} w_k f_k(x, \eta)\, d\eta \\ &= \sum_{k=1}^{K} w_k \int_{-\infty}^{\infty} f_k(x, \eta)\, d\eta \\ &= \sum_{k=1}^{K} w_k g_k(x, y), \end{aligned}$$

where $g_k(x, y)$ is the response of the system to input $f_k(x, y)$. Therefore, the system is linear.

On the other hand, if $g'(x, y)$ is the response of the system to input $f(x - x_0, y - y_0)$, then

$$\begin{aligned} g'(x, y) &= \int_{-\infty}^{\infty} f(x - x_0, \eta - y_0)\, d\eta \\ &= \int_{-\infty}^{\infty} f(x - x_0, \eta - y_0)\, d(\eta - y_0) \\ &= \int_{-\infty}^{\infty} f(x - x_0, \eta)\, d\eta. \end{aligned}$$

Since $g(x - x_0, y - y_0) = \int_{-\infty}^{\infty} f(x - x_0, \eta)\, d\eta$, the system is shift-invariant.

Solution 2.8

From the results in Problem 2.5, we know that an LSI system is BIBO stable if and only if its PSF is absolutely integrable.

(a) Not stable. The PSF $h(x, y)$ goes to infinite when x and/or y go to infinity. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x, y)|\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2 + y^2) dx\, dy = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x^2 dx\right] dy + \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} y^2 dy\right] dx$. Since $\int_{-\infty}^{\infty} x^2 dx = \int_{-\infty}^{\infty} y^2 dy$ is not bounded, then $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2 + y^2) dx\, dy$ is not bounded.

(b) Stable. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x, y)|\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\exp\{-(x^2 + y^2)\}) dx\, dy = \left[\int_{-\infty}^{\infty} e^{-x^2} dx\right]^2 = \pi$, which is bounded. So the system is stable.

(c) Not stable. The absolute integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 e^{-y^2} dx\, dy = \int_{-\infty}^{\infty} x^2 \left[\int_{-\infty}^{\infty} e^{-y^2} dy\right] dx = \int_{-\infty}^{\infty} \sqrt{\pi} x^2 dx$ is unbounded. So the system is not stable.

Solution 2.9

(a) $g(x) = \int_{-\infty}^{\infty} f(x - t) f(t) dt$.

(b) Given an input as $af_1(x) + bf_2(x)$, where a, b are some constant, the output is

$$\begin{aligned} g'(x) &= [af_1(x) + bf_2(x)] * [af_1(x) + bf_2(x)] \\ &= a^2 f_1(x) * f_1(x) + 2ab f_1(x) * f_2(x) + b^2 f_2(x) * f_2(x) \\ &= ag_1(x) + bg_2(x), \end{aligned}$$

where $g_1(x)$ and $g_2(x)$ are the output corresponding to an input of $f_1(x)$ and $f_2(x)$ respectively. Hence, the system is nonlinear.

(c) Given a shifted input $f_1(x) = f(x - x_0)$, the corresponding output is

$$\begin{aligned} g_1(x) &= f_1(x) * f_1(x) \\ &= \int_{-\infty}^{\infty} f_1(x - t) f_1(t) dt \\ &= \int_{-\infty}^{\infty} f(x - t - x_0) f_1(t - x_0) dt. \end{aligned}$$

Changing variable $t' = t - x_0$ in the above integration, we get

$$\begin{aligned} g_1(x) &= \int_{-\infty}^{\infty} f(x - 2x_0 - t') f_1(t') dt' \\ &= g(x - 2x_0). \end{aligned}$$

Thus, if the input is shifted by x_0, the output is shifted by $2x_0$. Hence, the system is not shift-invariant.

CONVOLUTION OF SIGNALS

Solution 2.10

(a)

$$\begin{aligned} f(x, y)\delta(x - 1, y - 2) &= f(1, 2)\delta(x - 1, y - 2) \\ &= (1 + 2^2)\delta(x - 1, y - 2) \\ &= 5\delta(x - 1, y - 2) \end{aligned}$$

(b)

$$\begin{aligned} f(x, y) * \delta(x - 1, y - 2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)\delta(x - \xi - 1, y - \eta - 2)\, d\xi\, d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x - 1, y - 2)\delta(x - \xi - 1, y - \eta - 2)\, d\xi\, d\eta \\ &= f(x - 1, y - 2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - \xi - 1, y - \eta - 2)\, d\xi\, d\eta \\ &= f(x - 1, y - 2) \\ &= (x - 1) + (y - 2)^2 \end{aligned}$$

(c)

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - 1, y - 2) f(x, 3) dx\, dy &\overset{1}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - 1, y - 2) f(1, 3) dx\, dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - 1, y - 2)(1 + 3^2) dx\, dy \\ &= 10\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - 1, y - 2) dx\, dy \\ &\overset{2}{=} 10 \end{aligned}$$

Equality 1 comes from the Eq. (2.7) in the text. Equality 2 comes from the fact:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - 1, y - 2) dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x, y) dx\, dy = 1.$$

(d)

$$
\begin{aligned}
\delta(x-1, y-2) * f(x+1, y+2) &\overset{3}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-\xi-1, y-\eta-2) f(\xi+1, \eta+2) d\xi\, d\eta \\
&\overset{4}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-\xi-1, y-\eta-2) f((x-1)+1, (y-2)+2) d\xi\, d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-\xi-1, y-\eta-2) f(x, y) d\xi\, d\eta \\
&\overset{5}{=} f(x, y) = x + y^2
\end{aligned}
$$

3 comes from the definition of convolution; 4 comes from the Eq. (2.7) in text; 5 is the same as 2 in part (c). Alternatively, by using the sifting property of $\delta(x, y)$ and defining $g(x, y) = f(x+1, y+2)$, we have

$$
\begin{aligned}
\delta(x-1, y-2) * g(x, y) &= g(x-1, y-2) \\
&= f(x-1+1, y-2+2) \\
&= f(x, y) \\
&= x + y^2 .
\end{aligned}
$$

Solution 2.11

(a)

$$
\begin{aligned}
f(x, y) * g(x, y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta) g(x-\xi, y-\eta)\, d\xi\, d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(\xi) f_2(\eta) g_1(x-\xi) g_2(y-\eta)\, d\xi\, d\eta \\
f(x, y) * g(x, y) &= \int_{-\infty}^{\infty} f_1(\xi) g_1(x-\xi)\, d\xi \int_{-\infty}^{\infty} f_2(\eta) g_2(y-\eta)\, d\eta \quad .
\end{aligned}
$$

Hence, their convolution is also separable.

(b)

$$
f(x, y) * g(x, y) = (f_1(x) * g_1(x))\, (f_2(y) * g_2(y)) .
$$

Solution 2.12

$$
\begin{aligned}
g(x, y) &= f(x, y) * h(x, y) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi, \eta)d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\xi+y-\eta)\exp\{-(\xi^2+\eta^2)\}d\xi d\eta \\
&= (x+y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\xi^2-\eta^2} d\xi d\eta - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi e^{-\xi^2-\eta^2} d\xi d\eta - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta e^{-\xi^2-\eta^2} d\xi d\eta \\
&= (x+y)\left(\int_{-\infty}^{\infty} e^{-\xi^2}d\xi\right)^2 - \int_{-\infty}^{\infty} e^{-\eta^2}\left(\int_{-\infty}^{\infty}\xi e^{-\xi^2}d\xi\right)d\eta - \int_{-\infty}^{\infty} e^{-\xi^2}\left(\int_{-\infty}^{\infty}\eta e^{-\eta^2}d\eta\right)d\xi \\
&= \pi(x+y) \qquad \text{(S2.4)}
\end{aligned}
$$

We get (S2.4) by noticing that since ξ is an odd function and $e^{-\xi^2}$ is an even function, we must have

$$\int_{-\infty}^{\infty} \xi e^{-\xi^2} d\xi = 0 .$$

Also,

$$\int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi} .$$

FOURIER TRANSFORMS AND THEIR PROPERTIES

Solution 2.13

(a) See the solution to part (b) below. The Fourier transform is

$$\mathcal{F}_2\{\delta_s(x, y)\} = \delta_s(u, v)$$

(b)

$$\mathcal{F}_2\{\delta_s(x, y; \Delta x, \Delta y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_s(x, y; \Delta x, \Delta y)e^{-j2\pi(ux+vy)}\, dx\, dy$$

$\delta_s(x, y; \Delta x, \Delta y)$ is a periodic signal with periods Δx and Δy in x and y axes. Therefore it can be written as a Fourier series expansion. (Please review Oppenheim, Willsky, and Nawad, Signals and Systems for the definition of Fourier series expansion of periodic signals.)

$$\delta_s(x, y; \Delta x, \Delta y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} C_{mn}e^{j2\pi(\frac{mx}{\Delta x}+\frac{ny}{\Delta y})} ,$$

where

$$
\begin{aligned}
C_{mn} &= \frac{1}{\Delta x\Delta y}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}}\int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}} \delta_s(x,y;\Delta x,\Delta y)\, e^{-j2\pi\left(\frac{mx}{\Delta x}+\frac{ny}{\Delta y}\right)}\, dx\, dy \\
&= \frac{1}{\Delta x\Delta y}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}}\int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \delta(x-m\Delta x, y-n\Delta y)\, e^{-j2\pi\left(\frac{mx}{\Delta x}+\frac{ny}{\Delta y}\right)}\, dx\, dy.
\end{aligned}
$$

In the integration region $-\frac{\Delta x}{2} < x < \frac{\Delta x}{2}$ and $-\frac{\Delta y}{2} < y < \frac{\Delta y}{2}$ there is only one impulse corresponding to $m = 0, n = 0$. Therefore, we have

$$
\begin{aligned}
C_{mn} &= \frac{1}{\Delta x\Delta y}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}}\int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}} \delta(x,y)\, e^{-j2\pi\left(\frac{0\cdot x}{\Delta x}+\frac{0\cdot y}{\Delta y}\right)}\, dx\, dy \\
&= \frac{1}{\Delta x\Delta y}.
\end{aligned}
$$

We have:

$$
\delta_s(x,y;\Delta x,\Delta y) = \frac{1}{\Delta x\Delta y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{j2\pi\left(\frac{mx}{\Delta x}+\frac{ny}{\Delta y}\right)}.
$$

Therefore,

$$
\begin{aligned}
F_2\{\delta_s\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_s(x,y;\Delta x,\Delta y)\, e^{-j2\pi(ux+vy)}\, dx\, dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\Delta x\Delta y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{j2\pi\left(\frac{mx}{\Delta x}+\frac{ny}{\Delta y}\right)} e^{-j2\pi(ux+vy)}\, dx\, dy \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{\Delta x\Delta y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j2\pi\left(\frac{mx}{\Delta x}+\frac{ny}{\Delta y}\right)} e^{-j2\pi(ux+vy)}\, dx\, dy \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{\Delta x\Delta y} F_2\left\{ e^{j2\pi\left(\frac{mx}{\Delta x}+\frac{ny}{\Delta y}\right)} \right\} \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{\Delta x\Delta y}\, \delta\left(u-\frac{m}{\Delta x}, v-\frac{n}{\Delta y}\right) \\
&\overset{5}{=} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{\Delta x\Delta y}\cdot \Delta x\Delta y\,\delta(u\Delta x-m, v\Delta y-n)
\end{aligned}
$$

$$m=-\infty \; n=-\infty$$

$$F_2\{\delta_s\} = \delta_s(u\Delta x, v\Delta y)$$

Equality 5 comes from the property $\delta(ax) = \frac{1}{|a|}\delta(x)$.

(c)

$$
\begin{aligned}
\mathcal{F}_2\{s(x,y)\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin[2\pi(u_0x+v_0y)]e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2j}\left[e^{j2\pi(u_0x+v_0y)} - e^{-j2\pi(u_0x+v_0y)}\right]e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \frac{1}{2j}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j2\pi(u_0x+v_0y)}e^{-j2\pi(ux+vy)}\,dx\,dy \\
&\quad - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(u_0x+v_0y)}e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \frac{1}{2j}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi[(u-u_0)x+(v-v_0)y]}\,dx\,dy \\
&\quad - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi[(u+u_0)x+(v+v_0)y]}\,dx\,dy \\
\mathcal{F}_2\{s(x,y)\} &= \frac{1}{2j}\left[\delta(u-u_0,v-v_0) - \delta(u+u_0,v+v_0)\right].
\end{aligned}
$$

We used Eq. (2.69) twice to get the last equality.

(d)

$$
\begin{aligned}
\mathcal{F}_2(c)(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos[2\pi(u_0x+v_0y)]e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}[e^{j2\pi(u_0x+v_0y)} + e^{-j2\pi(u_0x+v_0y)}]e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j2\pi(u_0x+v_0y)}e^{-j2\pi(ux+vy)}\,dx\,dy \\
&\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(u_0x+v_0y)}e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi[(u-u_0)x+(v-v_0)y]}\,dx\,dy \\
&\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi[(u+u_0)x+(v+v_0)y]}\,dx\,dy \\
\mathcal{F}_2(c)(u,v) &= \frac{1}{2}[\delta(u-u_0,v-v_0) + \delta(u+u_0,v+v_0)].
\end{aligned}
$$

We used Eq. (2.69) twice to get the last equality.

(e)

$$
\begin{aligned}
\mathcal{F}_2(f)(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-(x^2+j4\pi\sigma^2ux)/2\sigma^2} e^{-(y^2+j4\pi\sigma^2vy)/2\sigma^2}\,dx\,dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x^2+j4\pi\sigma^2ux)/2\sigma^2}\,dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y^2+j4\pi\sigma^2vy)/2\sigma^2}\,dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x+j2\pi\sigma^2u)^2/2\sigma^2} e^{(j2\pi\sigma^2u)^2/2\sigma^2}\,dx \cdot \\
&\quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y+j2\pi\sigma^2v)^2/2\sigma^2} e^{(j2\pi\sigma^2v)^2/2\sigma^2}\,dy \\
&= e^{-2\pi^2\sigma^2u^2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x+j2\pi\sigma^2u)^2/2\sigma^2}\,dx \cdot \\
&\quad e^{-2\pi^2\sigma^2v^2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y+j2\pi\sigma^2v)^2/2\sigma^2}\,dy \\
&= e^{-2\pi^2\sigma^2u^2}\cdot e^{-2\pi^2\sigma^2v^2} \\
\mathcal{F}_2(f)(u,v) &= e^{-2\pi^2\sigma^2(u^2+v^2)}.
\end{aligned}
$$

Solution 2.14

The Fourier transform of $f(x)$ is

$$F(u) = \int_{-\infty}^{\infty} f(x)e^{-j2\pi ux}dx.$$

(a) Assuming $f(x)$ is real and $f(x) = f(-x)$,

$$
\begin{aligned}
F^*(u) &= \int_{-\infty}^{\infty} \left(f(x)e^{-j2\pi ux}\right)^* dx \\
&= \int_{-\infty}^{\infty} f^*(x)e^{j2\pi ux}dx \\
&= \int_{-\infty}^{\infty} f^*(-\xi)e^{-j2\pi u\xi}d\xi, \text{ let } \xi = -x \\
&= \int_{-\infty}^{\infty} f(\xi)e^{-j2\pi u\xi}d\xi, \text{ since } f(-x) = f(x) \text{ and } f(x) \text{ is real} \\
&= F(u).
\end{aligned}
$$

(b) Similarly, assuming $f(x)$ is real and $f(x) = -f(-x)$,

$$
\begin{aligned}
F^*(u) &= \int_{-\infty}^{\infty} f^*(-\xi)e^{-j2\pi u\xi}d\xi \\
&= \int_{-\infty}^{\infty} -f(\xi)e^{-j2\pi u\xi}d\xi, \text{ since } f(-x) = -f(x) \\
&= -F(u)\,.
\end{aligned}
$$

Solution 2.15

In deriving the symmetric property $F^*(u) = F(u)$, we used the fact that $f(x)$ is real. If $f(x)$ is a complex signal, we have $f^*(-\xi) = f^*(\xi)$ instead of $f^*(-\xi) = f(\xi)$. Therefore,

$$
\begin{aligned}
F^*(u) &= \left[\int_{-\infty}^{\infty} f(x)e^{-j2\pi ux}\right]^* dx \\
&= \int_{-\infty}^{\infty} f^*(-\xi)e^{-j2\pi u\xi}d\xi, \text{ let } \xi = -x \\
&= \int_{-\infty}^{\infty} f^*(\xi)e^{-j2\pi u\xi}d\xi, \\
&= \mathcal{F}\{f^*(x)\}
\end{aligned}
$$

Solution 2.16

(a) Conjugate property: $F_2(f^*)(u,v) = F^*(-u,-v)$.

$$
\begin{aligned}
F_2(f^*)(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{j2\pi(ux+vy)}\,dx\,dy\right]^* \\
&= \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi[(-u)x+(-v)y]}\,dx\,dy\right]^* \\
&= [F(-u,-v)]^* \\
&= F^*(-u,-v)\,.
\end{aligned}
$$

Conjugate symmetry property: If $f(x,y)$ is real, $F(u,v) = F^*(-u,-v)$. Since $f(x,y)$ is real, $f^*(x,y) = f(x,y)$. Therefore,

$$F^*(-u,-v) = F_2\{f^*(x,y)\} = F_2\{f(x,y)\} = F(u,v)\,.$$

(b) Scaling property: $\mathcal{F}_2(f^{ab})(u,v) = \frac{1}{|ab|}\mathcal{F}_2(f)\left(\frac{u}{a}, \frac{v}{b}\right)$.

$$
\begin{aligned}
\mathcal{F}_2(f^{ab})(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(ax, by)e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(ax, by)e^{-j2\pi[u(ax)/a+v(by)/b]}\,\frac{1}{ab}\,d(ax)\,d(by) \\
&= \frac{1}{|ab|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p,q)e^{-j2\pi[(u/a)p+(v/b)q]}\,dp\,dq \\
&= \frac{1}{|ab|}\mathcal{F}_2(f)\left(\frac{u}{a}, \frac{v}{b}\right).
\end{aligned}
$$

(c) Convolution property: $\mathcal{F}_2(f * g)(u,v) = \mathcal{F}_2(g)(u,v)\cdot\mathcal{F}_2(f)(u,v)$.

$$
\mathcal{F}_2(f * g)(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)g(x-\xi, y-\eta)\,d\xi\,d\eta\right)e^{-j2\pi(ux+vy)}\,dx\,dy.
$$

Interchange the order of integration to yield

$$
\begin{aligned}
\mathcal{F}_2(f * g)(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-\xi, y-\eta)e^{-j2\pi(ux+vy)}\,dx\,dy\right)d\xi\,d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-\xi, y-\eta)\right. \\
&\qquad \left. e^{-j2\pi[u(x-\xi)+v(y-\eta)]}e^{-j2\pi(u\xi+v\eta)}\,dx\,dy\right)d\xi\,d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-j2\pi(u\xi+v\eta)}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-\xi, y-\eta)\right. \\
&\qquad \left. e^{-j2\pi[u(x-\xi)+v(y-\eta)]}\,dx\,dy\right)d\xi\,d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-j2\pi(u\xi+v\eta)}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(p,q)e^{-j2\pi[up+vq]}\,dp\,dq\right)d\xi\,d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-j2\pi(u\xi+v\eta)}\mathcal{F}_2(g)(u,v)\,d\xi\,d\eta \\
&= \mathcal{F}_2(g)(u,v)\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}
\end{aligned}
$$

$f(\xi, \eta) e^{-j 2\pi (u\xi + v\eta)}\, d\xi\, d\eta$

$$F_2(f * g)(u, v) = F_2(g)(u, v) \cdot F_2(f)(u, v).$$

(d) Product property: $\mathcal{F}_2(fg)(u,v) = F(u,v) * G(u,v)$.

$$\begin{aligned}
\mathcal{F}_2(fg)(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)g(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\xi,\eta)e^{j2\pi(x\xi+y\eta)}\,d\xi\,d\eta\right] f(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\xi,\eta)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{j2\pi(x\xi+y\eta)}e^{-j2\pi(ux+vy)}\,dxdy\right] d\xi\,d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\xi,\eta)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi[(u-\xi)x+(v-\eta)y]}\,dx\,dy\right] d\xi\,d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\xi,\eta)F(u-\xi,v-\eta)\,d\xi\,d\eta \\
&= F(u,v) * G(u,v).
\end{aligned}$$

Solution 2.17

Since both the rect and sinc functions are separable, it is sufficient to show the result for 1-D rect and sinc functions. A 1-D rect function is

$$\mathrm{rect}(x) = \begin{cases} 1, & \text{for } |x| < \frac{1}{2} \\ 0, & \text{for } |x| > \frac{1}{2} \end{cases}$$

$$\begin{aligned}
\mathcal{F}\{\mathrm{rect}(x)\} &= \int_{-\infty}^{\infty} \mathrm{rect}(x)e^{-j2\pi ux}dx \\
&= \int_{-1/2}^{1/2} e^{-j2\pi ux}dx \\
&= \int_{-1/2}^{1/2} \cos(2\pi ux)dx - j\int_{-1/2}^{1/2} \sin(2\pi ux)dx, \quad e^{j\theta} = \cos\theta + j\sin\theta \\
&= \int_{-1/2}^{1/2} \cos(2\pi ux)dx \\
&= \frac{\sin(\pi u)}{\pi u} \\
&= \mathrm{sinc}(u)\,.
\end{aligned}$$

Therefore, we have $\mathcal{F}\{\mathrm{sinc}(x)\} = \mathrm{rect}(u)$. Using Parseval's Theorem, we have

$$\begin{aligned}
E_\infty &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \|\mathrm{rect}(x,y)\|^2 dx\,dy \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} dx\,dy \\
&= 1
\end{aligned}$$

For the sinc function, $P_\infty = 0$, because E_∞ is finite.

Solution 2.18

Since the signal is separable, we have

$$\mathbf{F}[f(x,y)] = \mathbf{F}_{1D}[\sin(2\pi ax)]\mathbf{F}_{1D}[\cos(2\pi by)]\,,$$
$$\mathbf{F}_{1D}[\sin(2\pi ax)] = \frac{1}{2j}[\delta(u-a) - \delta(u+a)]\,,$$
$$\mathbf{F}_{1D}[\cos(2\pi by)] = \frac{1}{2}[\delta(v-b) + \delta(v+b)]\,.$$

So,

$$\mathbf{F}[f(x,y)] = \frac{1}{4j}[\delta(u-a)\delta(v-b) - \delta(u+a)\delta(v-b) + \delta(u-a)\delta(v+b) - \delta(u+a)\delta(v+b)]\,.$$

Now we need to show that $\delta(u)\delta(v) = \delta(u,v)$ (in a generalized way):

$$\delta(u)\delta(v) = 0, \quad \text{for } u = 0, \text{ or } v = 0$$

Therefore,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\delta(u)\delta(v)du\,dv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\delta(u)du\;\delta(v)dv = \int_{-\infty}^{\infty} f(0,v)\delta(v)dv = f(0,0)\,.$$

Based on the argument above $\delta(u)\delta(v) = \delta(u,v)$, and

$$\mathbf{F}[f(x,y)] = \frac{1}{4j}[\delta(u-a,v-b) - \delta(u+a,v-b) + \delta(u-a,v+b) - \delta(u+a,v+b)]\,.$$

The above solution can also be obtained by using the relationship:

$$\sin(2\pi ax)\cos(2\pi by) = \frac{1}{2}[\sin(2\pi(ax-by)) + \sin(2\pi(ax+by))]\,.$$

Solution 2.19

A function $f(x,y)$ can be expressed in polar coordinates as:

$$f(x,y) = f(r\cos\theta, r\sin\theta) = f_p(r,\theta)\,.$$

If it is circularly symmetric, we have $f_p(r,\theta)$ is constant for fixed r. The Fourier transform of $f(x,y)$ is defined as:

$$\begin{aligned}
F(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dx\,dy \\
&= \int_0^{\infty}\int_0^{2\pi} f_p(r,\theta)e^{-j2\pi(ur\cos\theta+vr\sin\theta)}r\,dr\,d\theta \\
&= \int_0^{\infty} f_p(r,\theta)\int_0^{2\pi} e^{-j2\pi(ur\cos\theta+vr\sin\theta)}d\theta\;r\,dr\,.
\end{aligned}$$

Letting $u = q\cos\varphi$ and $v = q\sin\varphi$, the above equation becomes:

$$F(u, v) = \int_0^\infty f_p(r,\theta)\left[\int_0^{2\pi} e^{-j2\pi qr\cos(\varphi-\theta)}d\theta\right] r\,dr\,.$$

Since $F(u, v)$ is also circularly symmetric, it can be written as $F_q(q,\varphi)$ and is constant for fixed q. In particular, $F_q(q,\varphi) = F_q(q,\pi/2)$, and therefore

$$F_q(q,\varphi) = F_q(q,\pi/2) = \int_0^\infty f_p(r,\theta)\left[\int_0^{2\pi} e^{-j2\pi qr\sin\theta}d\theta\right] r\,dr\,.$$

Now we will show that (2.108) holds.

$$\begin{aligned}\int_0^{2\pi} e^{-j2\pi qr\sin\theta}d\theta &= \int_0^{2\pi}\cos(2\pi qr\sin\theta)d\theta - j\int_0^{2\pi}\sin(2\pi qr\sin\theta)\\ &\overset{1}{=} 2\int_0^{\pi}\cos(2\pi qr\sin\theta)d\theta\\ &= 2\pi J_0(2\pi qr)\,.\end{aligned}$$

Equality 1 holds because $\cos(-\theta) = \cos(\theta)$, and $\sin(\theta) = -\sin(\theta)$.

Based on the above derivation, we have proven (2.108).

Solution 2.20

The unit disk is expressed as $f(r) = \mathrm{rect}(r)$ and its Hankel transform is

$$\begin{aligned}F(q) &= 2\pi\int_0^\infty f(r)J_0(2\pi qr)r\,dr\\ &= 2\pi\int_0^\infty \mathrm{rect}(r)J_0(2\pi qr)r\,dr\\ &= 2\pi\int_0^{1/2} J_0(2\pi qr)r\,dr\,.\end{aligned}$$

Now apply the following change of variables

$$\begin{aligned}s &= 2\pi qr\,,\\ r &= \frac{s}{2\pi q}\,,\\ dr &= \frac{ds}{2\pi q}\,,\end{aligned}$$

to yield

$$F(q) = \frac{1}{2\pi q^2}\int_0^{\pi q} J_0(s)s\,ds\,.$$

From mathematical tables, we note that

$$\int_0^x J_0(\ )\ d \quad = \quad xJ_1(x)\,.$$

Therefore,

$$\begin{aligned} F(q) &= \frac{J_1(\pi q)}{2q} \\ &= \mathrm{jinc}(q)\,. \end{aligned}$$

TRANSFER FUNCTION

Solution 2.21

(a) The impulse response function is shown in Figure S2.1.

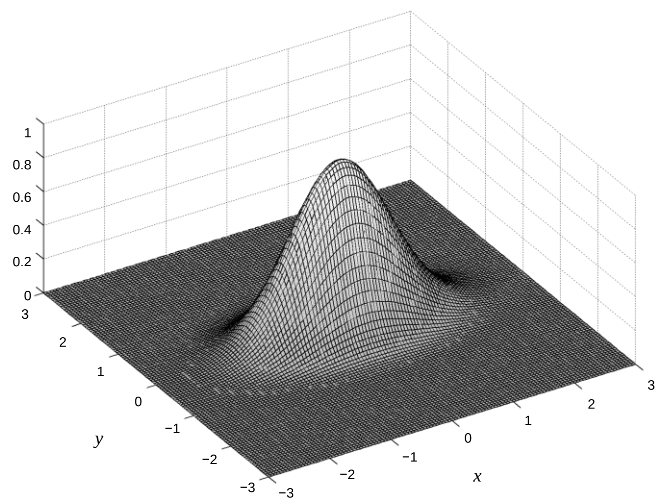

Figure S2.1 Impulse response function of the system. See Problem 2.21(a).

(b) The transfer function of the function is the Fourier transform of the impulse response function:

$$\begin{aligned} H(u,\ v) &= F\{h(x,y)\} \\ &= F\{e^{-\pi x^2}\}F\{e^{-\pi y^2/4}\}\,, \text{ since } h(x,\ y) \text{ is separable} \\ &= 2e^{-\pi(u^2+4v^2)}\,. \end{aligned}$$

Solution 2.22

(a) The 1D profile of the bar phantom is:

$$f(x) = \begin{cases} 1, & \frac{k-1}{2} w \le x \le \frac{k+1}{2} w \\ 0, & \frac{k+1}{2} w \le x \le \frac{k+3}{2} w \end{cases},$$

where k is an integer. The response of the system to the bar phantom is:

$$g(x) = f(x) * l(x) = \int_{-\infty}^{\infty} f(x-\xi) l(\xi) d\xi \,.$$

At the center of the bar, we have

$$\begin{aligned} g(0) &= \int_{-\infty}^{\infty} f(0-\xi) l(\xi) d\xi \\ &= \int_{-w/2}^{w/2} \cos(\alpha\xi) d\xi \\ &= \frac{2}{\alpha} \sin\left(\frac{\alpha w}{2}\right) . \end{aligned}$$

At the point halfway between two adjacent bars, we have

$$\begin{aligned} g(w) &= \int_{-\infty}^{\infty} f(w-\xi) l(\xi) d\xi \\ &= \int_{w-\pi/2\alpha}^{w/2} \cos(\alpha\xi) d\xi + \int_{3w/2}^{w+\pi/2\alpha} \cos(\alpha\xi) d\xi \\ &= 2 \int_{w-\pi/2\alpha}^{w/2} \cos(\alpha\xi) d\xi \\ &= \frac{2}{\alpha} \left[\sin\frac{\alpha w}{2} - \sin\left(\alpha w - \frac{\pi}{2}\right) \right] . \end{aligned}$$

(b) From the line spread function alone, we cannot tell whether the system is isotropic. The line spread function is a "projection" of the PSF. During the projection, the information along the y direction is lost.

(c) Since the system is separable with $h(x, y) = h_{1D}(x) h_{1D}(y)$, we know that

$$\begin{aligned} l(x) &= \int_{-\infty}^{\infty} h(x, y) dy \\ &= h_{1D}(x) \int_{-\infty}^{\infty} h_{1D}(y) dy \,. \end{aligned}$$

Therefore $h_{1D}(x) = cl(x)$ where $1/c = \int_{-\infty}^{\infty} h_{1D}(y)dy$. Hence,

$$\begin{aligned} 1/c &= \int_{-\infty}^{\infty} cl(y)dy\,, \\ 1/c^2 &= \int_{-\pi/2\alpha}^{\pi/2\alpha} \cos(\alpha y)dy\,, \\ 1/c^2 &= 2/\alpha\,. \end{aligned}$$

Therefore,

$$h(x,\,y) = \begin{cases} \frac{\alpha}{2}\cos(\alpha x)\,\cos(\alpha y) & |\alpha x| \le \pi/2 \text{ and } |\alpha y| \le \pi/2 \\ 0 & \text{otherwise} \end{cases}.$$

The transfer function is

$$\begin{aligned} H(u,\,v) &= F_{2D}\{h(x,y)\} \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(x,\,y)e^{j2\pi ux}dx\right) e^{j2\pi uy}dy \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h_{1D}(x)h_{1D}(y)e^{j2\pi ux}dx\right) e^{j2\pi uy}dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{1D}(x)e^{j2\pi ux}dx\; h_{1D}(y)e^{j2\pi uy}dy \\ &= \int_{-\infty}^{\infty} h_{1D}(x)e^{j2\pi ux}dx \int_{-\infty}^{\infty} h_{1D}(y)e^{j2\pi uy}dy \\ &= H_{1D}(u)H_{1D}(v)\,, \end{aligned}$$

which is also separable with $H(u,\,v) = H_{1D}(u)H_{1D}(v)$. We have

$$\begin{aligned} H_{1D} &= \sqrt{\frac{\alpha}{2}}F_{1D}\{l(x)\} \\ &= \sqrt{\frac{\alpha}{2}}F_{1D}\{\cos(\alpha x)\} * F_{1D}\left\{\mathrm{rect}\left(\frac{\alpha x}{\pi}\right)\right\} \\ &= \sqrt{\frac{\pi}{2}}\left[\mathrm{sinc}\left(\frac{\pi}{\alpha}(u-\alpha/2\pi)\right) + \mathrm{sinc}\left(\frac{\pi}{\alpha}(u+\alpha/2\pi)\right)\right]. \end{aligned}$$

Therefore, the transfer function is

$$\begin{aligned} H(u,\,v) = \frac{\pi}{2}&\left[\mathrm{sinc}\left(\frac{\pi}{\alpha}(u-\alpha/2\pi)\right) + \mathrm{sinc}\left(\frac{\pi}{\alpha}(u+\alpha/2\pi)\right)\right] \\ &\left[\mathrm{sinc}\left(\frac{\pi}{\alpha}(v-\alpha/2\pi)\right) + \mathrm{sinc}\left(\frac{\pi}{\alpha}(v+\alpha/2\pi)\right)\right]. \end{aligned}$$

APPLICATIONS, EXTENSIONS AND ADVANCED TOPICS

Solution 2.23

(a) The system is separable because $h(x, y) = e^{-(|x|+|y|)} = e^{-|x|}e^{-|y|}$.

(b) The system is not isotropic since $h(x, y)$ is not a function of $r = \sqrt{x^2 + y^2}$.

Additional comments: An easy check is to plug in $x = 1$, $y = 1$ and $x = 0$, $y = \sqrt{2}$ into $h(x, y)$. By noticing that $h(1, 1) = h(0, \sqrt{2})$, we can conclude that $h(x, y)$ is not rotationally invariant, and hence not isotropic.

Isotropy is rotational symmetry around the origin, not just symmetry about a few axes, e.g., the x- and y-axes. $h(x, y) = e^{-(|x|+|y|)}$ is symmetric about a few lines, but it is not rotationally invariant.

When we studied the properties of Fourier transform, we learned that if a signal is isotropic then its Fourier transform has a certain symmetry. Note that the symmetry of the Fourier transform is only a necessary, but not sufficient, condition for the signal to be isotropic.

(c) The response is

$$
\begin{aligned}
g(x, y) &= h(x, y) * f(x, y) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi, \eta) f(x - \xi, y - \eta) d\xi\, d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(|\xi|+|\eta|)} \delta(x - \xi) d\xi\, d\eta \\
&= \int_{-\infty}^{\infty} e^{-(|x|+|\eta|)} d\eta \\
&= e^{-|x|} \int_{-\infty}^{\infty} e^{-|\eta|} d\eta \\
&= e^{-|x|} \left(\int_{-\infty}^{0} e^{\eta} d\eta + \int_{0}^{\infty} e^{-\eta} d\eta \right) \\
&= 2e^{-|x|} .
\end{aligned}
$$

(d) The response is

$$
\begin{aligned}
g(x, y) &= h(x, y) * f(x, y) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi, \eta) f(x - \xi, y - \eta) d\xi\, d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(|\xi|+|\eta|)} \delta(x - \xi - y + \eta) d\xi\, d\eta \\
&= \int_{-\infty}^{\infty} e^{-|\eta|} \left(\int_{-\infty}^{\infty} e^{-|\xi|} \delta(x - \xi - y + \eta) d\xi \right) d\eta \\
&= \int_{-\infty}^{\infty} e^{-|\eta|} e^{-|x-y+\eta|} d\eta .
\end{aligned}
$$

1. Now assume $x - y < 0$, then $x - y + \eta < \eta$. The range of integration in the above can be divided into three parts (see Fig. S2.2):

I. $\eta \in (-\infty, 0)$. In this interval, $x - y + \eta < \eta < 0$. $|\eta| = -\eta$, $|x - y + \eta| = -(x - y + \eta)$;

II. $\eta \in [0, -(x - y))$. In this interval, $x - y + \eta < 0 \le \eta$. $|\eta| = \eta$, $|x - y + \eta| = -(x - y + \eta)$;

III. $\eta \in [-(x - y), \infty)$. In this interval, $0 \le x - y + \eta < \eta$. $|\eta| = \eta$, $|x - y + \eta| = x - y + \eta$.

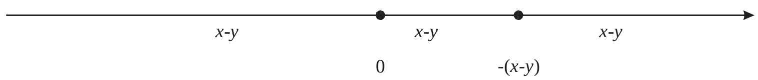

Figure S2.2 For $x - y < 0$ the integration interval $(-\infty, \infty)$ can be partitioned into three segments. See Problem 2.23(d).

Based on the above analysis, we have:

$$\begin{aligned}
g(x, y) &= \int_{-\infty}^{\infty} e^{-|\eta|} e^{-|x-y+\eta|} d\eta \\
&= \int_{-\infty}^{0} e^{-(|\eta|+|x-y+\eta|)} d\eta + \int_{0}^{-(x-y)} e^{-(|\eta|+|x-y+\eta|)} + \int_{-(x-y)}^{\infty} e^{-(|\eta|+|x-y+\eta|)} \\
&= \int_{-\infty}^{0} e^{x-y+2\eta} d\eta + \int_{0}^{-(x-y)} e^{x-y} d\eta + \int_{-(x-y)}^{\infty} e^{-(x-y+2\eta)} d\eta \\
&= \frac{1}{2} e^{x-y} - (x-y) e^{x-y} + \frac{1}{2} e^{x-y} \\
&= [1 - (x-y)] e^{x-y}.
\end{aligned}$$

2. For $x - y \geq 0$, $\eta < x - y + \eta$. The range of integration in the above can be divided into three parts (see Fig. S2.3):

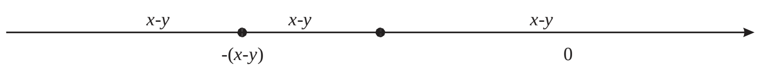

Figure S2.3 For $x - y > 0$ the integration interval $(-\infty, \infty)$ can be partitioned into three segments. See Problem 2.23(d).

I. $\eta \in (-\infty, -(x-y))$. In this interval, $\eta < x - y + \eta < 0$. $|\eta| = -\eta$, $|x - y + \eta| = -(x - y + \eta)$;
II. $\eta \in [-(x-y), 0)$. In this interval, $\eta < 0 \leq x - y + \eta$. $|\eta| = -\eta$, $|x - y + \eta| = x - y + \eta$;
III. $\eta \in [0, \infty)$. In this interval, $0 \leq \eta < x - y + \eta$. $|\eta| = \eta$, $|x - y + \eta| = x - y + \eta$.
Based on the above analysis, we have:

$$\begin{aligned}
g(x, y) &= \int_{-\infty}^{\infty} e^{-|\eta|} e^{-|x-y+\eta|} d\eta \\
&= \int_{-\infty}^{-(x-y)} e^{-(|\eta|+|x-y+\eta|)} d\eta + \int_{-(x-y)}^{0} e^{-(|\eta|+|x-y+\eta|)} + \int_{0}^{\infty} e^{-(|\eta|+|x-y+\eta|)} \\
&= \int_{-\infty}^{-(x-y)} e^{x-y+2\eta} d\eta + \int_{-(x-y)}^{0} e^{-(x-y)} d\eta + \int_{0}^{\infty} e^{-(x-y+2\eta)} d\eta \\
&= \frac{1}{2} e^{-(x-y)} + (x-y) e^{-(x-y)} + \frac{1}{2} e^{-(x-y)} \\
&= [1 + (x-y)] e^{-(x-y)}.
\end{aligned}$$

Based on the above two steps, we have:

$$g(x, y) = (1 + |x - y|)e^{-|x-y|} .$$

Solution 2.24

(a) Yes, it is shift invariant because its impulse response depends on $x - \xi$.

(b) By linearity, the output is

$$g(x) = e^{\frac{-(x+1)^2}{2}} + e^{\frac{-(x)^2}{2}} + e^{\frac{-(x-1)^2}{2}} .$$

Solution 2.25

(a) The impulse response of the filter is the inverse Fourier transform of $H(u)$, which can be written as

$$H(u) = 1 - \mathrm{rect}\left(\frac{u}{2U_0}\right) .$$

Using the linearity of the Fourier transform and the Fourier transform pairs

$$\begin{aligned} \mathcal{F}\{\delta(t)\} &= 1 , \\ \mathcal{F}\{\mathrm{sinc}(t)\} &= \mathrm{rect}(u) , \end{aligned}$$

we have

$$\begin{aligned} h(t) &= \mathcal{F}^{-1}\{H(u)\} \\ &= \delta(t) - 2U_0 \,\mathrm{sinc}(2U_0 t) . \end{aligned}$$

(b) The system response to $f(t) = c$ is 0, since $f(t)$ contains only a zero frequency component while $h(t)$ passes only high frequency components. Formal proof:

$$\begin{aligned} f(t) * h(t) &= f(t) * [\delta(t) - 2U_0 \,\mathrm{sinc}(2U_0 t)] \\ &= f(t) - 2U_0 f(t) * \mathrm{sinc}(2U_0 t) \\ &= c - c \int_{-\infty}^{\infty} 2U_0 \,\mathrm{sinc}(2U_0 t) dt \\ &= c - c \int_{-\infty}^{\infty} \mathrm{sinc}(\tau) d\tau \\ &= 0 . \end{aligned}$$

The system response to $f(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$ is

$$\begin{aligned}
f(t) * h(t) &= f(t) * [\delta(t) - 2U_0 \operatorname{sinc}(2U_0 t)] \\
&= f(t) - 2U_0 f(t) * \operatorname{sinc}(2U_0 t) \\
&= f(t) - \int_{-\infty}^{\infty} f(x) 2U_0 \operatorname{sinc}(2U_0(t-x))dx \\
&= f(t) - \int_{0}^{\infty} 2U_0 \operatorname{sinc}(2U_0(t-x))dx \\
&= f(t) + \int_{t}^{-\infty} 2U_0 \operatorname{sinc}(2U_0(y))dy \\
&= f(t) - \int_{-\infty}^{t} 2U_0 \operatorname{sinc}(2U_0(y))dy \\
&= \begin{cases} 1 - \int_{-\infty}^{0} 2U_0 \operatorname{sinc}(2U_0(y))dy + \int_{t}^{0} 2U_0 \operatorname{sinc}(2U_0(y))dy & t < 0 \\ 1 - \int_{-\infty}^{0} 2U_0 \operatorname{sinc}(2U_0(y))dy - \int_{0}^{t} 2U_0 \operatorname{sinc}(2U_0(y))dy & t > 0 \end{cases} \\
&= \begin{cases} -\frac{1}{2} + \int_{t}^{0} 2U_0 \operatorname{sinc}(2U_0(y))dy & t < 0 \\ 1 - \frac{1}{2} - \int_{0}^{t} 2U_0 \operatorname{sinc}(2U_0(y))dy & t > 0 \end{cases} \\
&= \begin{cases} -\frac{1}{2} + \int_{t}^{0} 2U_0 \operatorname{sinc}(2U_0(y))dy & t < 0 \\ \frac{1}{2} - \int_{0}^{t} 2U_0 \operatorname{sinc}(2U_0(y))dy & t > 0 \end{cases}.
\end{aligned}$$

Solution 2.26

(a) The rect function is defined as

$$\mathrm{rect}(t) = \begin{cases} 1, & |t| \le 1/2 \\ 0, & \text{otherwise} \end{cases}.$$

So we have

$$\mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} 1, & |t| \le T/2 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathrm{rect}\left(\frac{t+0.75T}{0.5T}\right) = \begin{cases} 1, & |t+0.75T| \le T/4 \\ 0, & \text{otherwise} \end{cases}.$$

Therefore,

$$h(t) = \begin{cases} -1/T, & -T < t < -T/2 \\ 1/T, & -T/2 < t < T/2 \\ -1/T, & T/2 < t < T \\ 0, & \text{otherwise} \end{cases}.$$

The impulse response is plotted in Fig. S2.4.

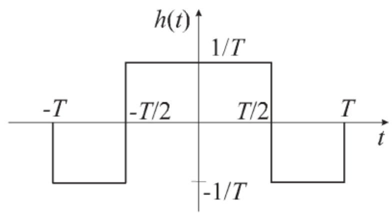

Figure S2.4 The impulse response h(t). See Problem 2.26(a).

The absolute integral of $h(t)$ is $\int_{-\infty}^{\infty} |h(t)|^2 dt = 2/T$. So The system is stable when $T > 0$. The system is not causal, since $h(t) = 0$ for $-T < t < 0$.

(b) The response of the system to a constant signal $f(t) = c$ is

$$g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau = c\int_{-\infty}^{\infty} h(\tau)d\tau = 0\,.$$

(c) The response of the system to the unit step function is

$$g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

$$g(t) = \begin{cases} 0, & t < -T \\ -t/T - 1, & -T < t < -T/2 \\ t/T, & -T/2 < t < T/2 \\ -t/T + 1, & T/2 < t < T \\ 0, & t > T \end{cases}$$

The response of the system to the unit step signal is plotted in Figure S2.5.

Solution 2.26

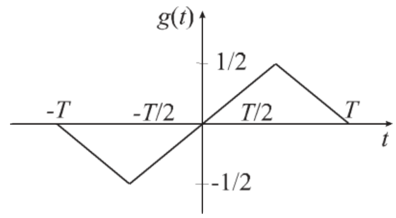

Figure S2.5 The response of the system to the unit step signal. See Problem 2.26(c).

(d) The Fourier transform of a rect function is a sinc function (see Problem 2.17). By using the properties of the Fourier transform (scaling, shifting, and linearity), we have

$$\begin{aligned} H(u) &= \mathbf{F}\{h(t)\} \\ &= -0.5e^{-j2\pi u(-0.75T)}\operatorname{sinc}(0.5uT) + \operatorname{sinc}(uT) - 0.5e^{-j2\pi u(0.75T)}\operatorname{sinc}(0.5uT) \\ &= \operatorname{sinc}(uT) - \cos(1.5\pi uT)\operatorname{sinc}(0.5uT)\,. \end{aligned}$$

(e) The magnitude spectrum of $h(t)$ is plotted in Figure S2.6.

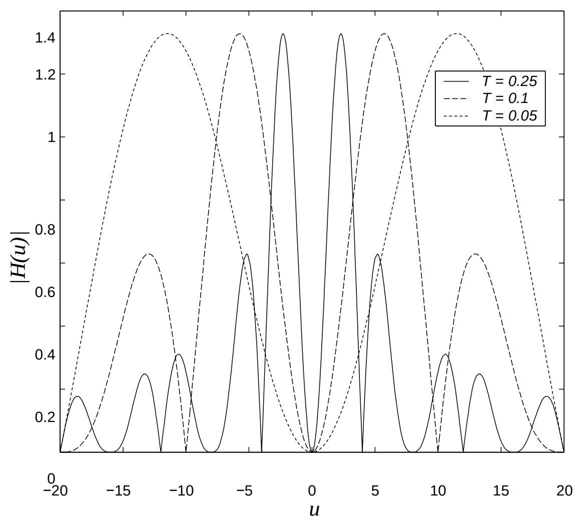

Figure S2.6 The magnitude spectrum of $h(t)$. See Problem 2.26(e).

(f) From the calculation in part (d) and the plot in part (c), it can be seen that $|H(0)| = 0$. So the output of the system does not have a DC component. The system is not a low pass filter. The system is not a high-pass filter since it also filters out high frequency components. As $T \to 0$, the pass band of the system moves to higher frequencies, and the system tends toward a high-pass filter.

Solution 2.27

(a) The inverse Fourier transform of $\hat{H}(\%)$ is

$$
\begin{aligned}
\hat{h}(r) &= \mathcal{F}^{-1}\{\hat{H}(\%)\} \\
&= \int_{-\infty}^{\infty} \hat{H}(\%) e^{j2\pi r\%} d\% \\
&= \int_{-\%_0}^{\%_0} |\%| e^{j2\pi r\%} d\% \\
&= \int_0^{\%_0} \% e^{j2\pi r\%} d\% - \int_{-\%_0}^{0} \% e^{j2\pi r\%} d\% \\
&= \int_0^{\%_0} \% e^{j2\pi r\%} d\% + \int_0^{-\%_0} \% e^{j2\pi r\%} d\% \\
&= \int_0^{\%_0} \% e^{j2\pi r\%} d\% + \int_0^{\%_0} \% e^{-j2\pi r\%} d\% \\
&= \int_0^{\%_0} \% \left(e^{j2\pi r\%} + e^{-j2\pi r\%} \right) d\% \\
&= 2\int_0^{\%_0} \% \cos(2\pi r\%) d\% \\
&= 2\left[\left. \frac{\% \sin(2\pi r\%)}{2\pi r} \right|_{\%=0}^{\%_0} - \int_0^{\%_0} \frac{\sin(2\pi r\%)}{2\pi r} d\% \right] \\
&= 2\left[\frac{\% \sin(2\pi r\%_0)}{2\pi r} + \left. \frac{\cos(2\pi r\%)}{4\pi^2 r^2} \right|_{\%=0}^{\%_0} \right] \\
&= \frac{1}{2\pi^2 r^2}\left[\cos(2\pi r\%_0) + 2\pi r\%_0 \sin(2\pi r\%_0) - 1 \right].
\end{aligned}
$$

(b) The response of the filter is $g(r) = f(r) * \hat{h}(r)$, hence $G(\%) = F(\%)\hat{H}(\%)$. i) A constant function $f(r) = c$ has the Fourier transform

$$F(\%) = c\delta(\%).$$

The transfer function of a ramp filter has a value zero at $\% = 0$. So the system response has the Fourier transform

$$G(\%) = 0.$$

Therefore, the responses of a ramp filter to a constant function is $g(r) = 0$. ii) The Fourier transform of a sinusoid function $f(r) = \sin(\omega r)$ is

$$F(\%) = \frac{1}{2j}\left[\delta(\% - \frac{\omega}{2\pi}) - \delta(\% + \frac{\omega}{2\pi}) \right].$$

Hence,

$$
G(\%) = \begin{cases} \frac{\omega}{4\pi j}\left[\delta\left(\% - \frac{\omega}{2\pi}\right) - \delta\left(\% + \frac{\omega}{2\pi}\right) \right] & \%_0 = \omega \\ 0 & \text{otherwise} \end{cases}.
$$

Solution 2.27

Therefore, the response of a ramp filter to a sinusoid function is

$$g(r) = \begin{cases} \dfrac{\omega}{2\pi}\sin(\omega r) & \omega_0 = \omega \\ 0 & \text{otherwise} \end{cases} .$$

Solution 2.28

Suppose the Fourier transform of $f(x, y)$ is $F(u, v)$. Using the scaling properties, we have that the Fourier transform of $f(ax, by)$ is $\frac{1}{|ab|}F\left(\frac{u}{a}, \frac{v}{b}\right)$. The output of the system is

$$\begin{aligned} g(x, y) &= \mathcal{F}\left\{\frac{1}{|ab|}F\left(\frac{u}{a}, \frac{v}{b}\right)\right\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{|ab|}F\left(\frac{u}{a}, \frac{v}{b}\right)e^{-j2\pi(ux+vy)}du\,dv \\ &= \frac{1}{|ab|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\xi, \eta)e^{j2\pi(a\xi(-x)+b\eta(-y))}|ab|d\xi\, d\eta . \end{aligned}$$

Given the inverse Fourier transform

$$f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u, v)e^{j2\pi(ux+vy)}du\,dv$$

we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\xi, \eta)e^{j2\pi(a\xi(-x)+b\eta(-y))}|ab|d\xi d\eta = |ab|f(-ax, -by) .$$

Therefore, $g(x, y) = f(-ax, -by)$ is a scaled and inverted replica of the input.

Solution 2.29

The Fourier transform of the signal $f(x, y)$ and the noise $\eta(x, y)$ are:

$$\begin{aligned} F(u, v) &= \mathcal{F}\{f(x, y)\} \\ &= |ab|\mathcal{F}\{\text{sinc}(ax, by)\} \\ &= |ab|\frac{1}{|ab|}\text{rect}\left(\frac{u}{a}, \frac{v}{b}\right) \\ &= \text{rect}\left(\frac{u}{a}, \frac{v}{b}\right) \\ &= \begin{cases} 1, & |x| < |a|/2 \text{ and } |y| < |b|/2 \\ 0, & \text{otherwise} \end{cases} , \\ E(u, v) &= \mathcal{F}\{\eta(x, y)\} \\ &= \frac{1}{2}[\delta(u - A, v - B) + \delta(u + A, v + B)] . \end{aligned}$$

Using the linearity of Fourier transform, the Fourier transform of the measurements $g(x, y)$ is

$$G(u, v) = \text{rect}\left(\frac{u}{a}, \frac{v}{b}\right) + \frac{1}{2}[\delta(u - A, v - B) + \delta(u + A, v + B)] ,$$

which is plotted in Figure S2.7. In order for an ideal low pass filter to recover $f(x, y)$, the cutoff frequencies of the

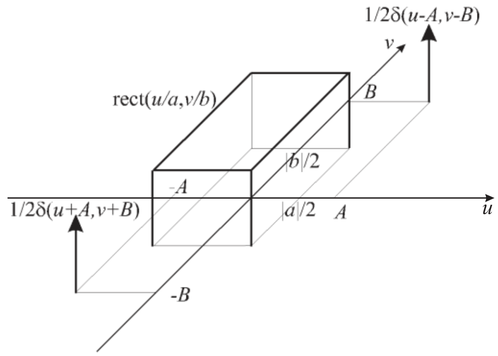

Figure S2.7 The Fourier transform of g(x, y). See Problem 2.29.

filter must satisfy

$$|a|/2 < U < A \text{ and } |b|/2 < V < B\,.$$

The Fourier transform of h(x, y) is rect $\left(\frac{u}{2U}, \frac{v}{2V}\right)$; therefore, the impulse response is

$$h(x, y) = \mathcal{F}^{-1}\left\{\text{rect}\left(\frac{u}{2U}, \frac{v}{2V}\right)\right\} = 4UV\ \text{sinc}(2U\,x)\ \text{sinc}(2V\,y)\,.$$

For given a and b, we need $A > |a|/2$ and $B > |b|/2$. Otherwise we cannot find an ideal low pass filter to exactly recover $f(x, y)$.

Solution 2.30

(a) The continuous Fourier transform of a rect function is a sinc function. Using the scaling property of the Fourier transform, we have:

$$G(u) = \mathcal{F}_{1D}\{g(x)\} = 2\,\text{sinc}(2u).$$

A sinc function, sinc(x), is shown in Figure 2.4(b).

(b) If the sampling period is $\Delta x_1 = 1/2$, we have

$$g_1(m) = g(m/2) = \begin{cases} 1, & -2 \le m \le 2 \\ 0, & \text{otherwise} \end{cases}.$$

Its DTFT is

$$\begin{aligned} G_1(\omega) &= \mathcal{F}_{DTFT}\{g_1(m)\} \\ &= e^{j2\omega} + e^{j\omega} + 1e^{j0\omega} + e^{-j\omega} + 2e^{-j2\omega} \\ &= 1 + 2\cos(\omega) + 2\cos(2\omega)\,. \end{aligned}$$

The DTFT of $g_1(m)$ is shown in Figure S2.8.

which is plotted in Figure S2.7. In order for an ideal low pass filter to recover $f(x, y)$, the cutoff frequencies of the

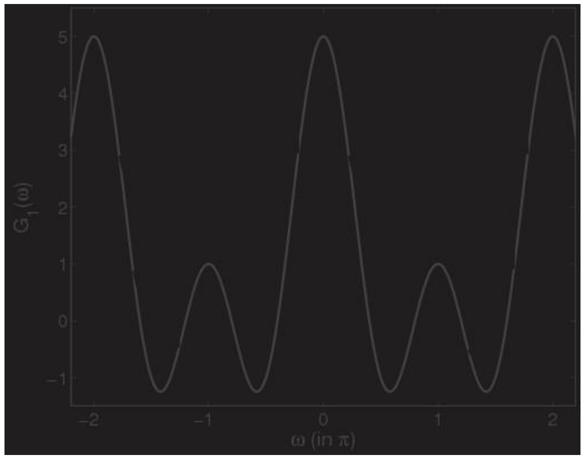

Figure S2.8 The DTFT $g_1(m)$. See Problem 2.30(b).

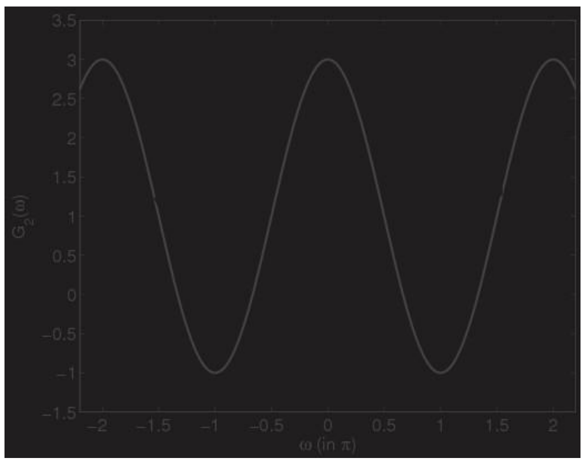

Figure S2.9 The DTFT $g_2(m)$. See Problem 2.30(c).

(c) If the sampling period is $\Delta x_2 = 1$, we have

$$g_2(m) = g(m) = \begin{cases} 1, & -1 \le m \le 1 \\ 0, & \text{otherwise} \end{cases} .$$

Its DTFT is

$$\begin{aligned} G_2(\omega) &= F_{DTFT}\{g_2(m)\} \\ &= e^{j\omega} + 1e^{j0\omega} + e^{-j\omega} \\ &= 1 + 2\cos(\omega) . \end{aligned}$$

The DTFT of $g_2(m)$ is shown in Figure S2.9.

(d) The discrete version of signal $g(x)$ can be written as

$$g_1(m) = g(x - m\Delta x_1), \quad m = -\infty, \cdots, -1, 0, 1, \cdots, +\infty .$$

The DTFT of $g_1(m)$ is

$$\begin{aligned} G_1(\omega) &= F_{DTFT}\{g_1(m)\} \\ &= \sum_{m=-\infty}^{\infty} g_1(m)e^{-j\omega m} \\ &= \sum_{m=-\infty}^{\infty} g(m\Delta x_1)e^{-j\omega m} \\ &= \int_{-\infty}^{\infty} g(x)\delta_s(x;\Delta x_1)e^{-j\omega \frac{x}{\Delta x_1}} dx . \end{aligned}$$

In the above, $\delta_s(x;\Delta x_1)$ is the sampling function with the space between impulses equal to Δx_1. Because of the sampling function, we are able to convert the summation into integration. The last equation in the above is the continuous Fourier transform of the product of $g(x)$ and $\delta_s(x;\Delta x_1)$ evaluated as $u = \omega/(2\pi\Delta x_1)$. Using the product property of the continuous Fourier transform, we have:

$$\begin{aligned} G_1(\omega) &= F\{g(x)\} * F\{\delta_s(x;\Delta x_1)\}|_{u=\omega/(2\pi\Delta x_1)} \\ &= G(u) * \mathrm{comb}(u\Delta x_1)|_{u=\omega/(2\pi\Delta x_1)} . \end{aligned}$$

The convolution of $G(u)$ and $\mathrm{comb}(u\Delta x_1)$ is to replicate $G(u)$ to $u = k/\Delta x_1$. Since $u = \omega/(2\pi\Delta x_1)$, $G_1(\omega)$ is periodic with period $\Omega = 2\pi$.

(e) The proof is similar to that for the continuous Fourier transform:

$$\begin{aligned} F_{DTFT}\{x(m) * y(m)\} &= F_{DTFT}\{x(m) * y(m)\} \\ &= F_{DTFT}\left\{\sum_{n=-\infty}^{\infty} x(m-n)y(n)\right\} \\ &= \sum_{m=-\infty}^{\infty} e^{-j\omega m} \sum_{n=-\infty}^{\infty} x(m-n)y(n) \\ &= \sum_{n=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} e^{-j\omega m}x(m-n)\right] y(n) \\ &= \sum_{n=-\infty}^{\infty} e^{-j\omega n}\left[\sum_{k=-\infty}^{\infty} e^{-j\omega k}x(k)\right] y(n) \\ &\quad (\text{let } k = m - n) \\ &= \sum_{n=-\infty}^{\infty} e^{-j\omega n}F_{DTFT}\{x(m)\}y(n) \\ &= F_{DTFT}\{x(m)\}F_{DTFT}\{y(m)\} . \end{aligned}$$

(f) First we evaluate the convolution of $g_1(m)$ with $g_2(m)$:

$$g_1(m) * g_2(m) = \begin{cases} 3, & -1 \le m \le 1 \\ 2, & m = \pm 2 \\ 1, & m = \pm 3 \\ 0, & \text{otherwise} \end{cases} .$$

Then by direct computation, we have

$$\begin{aligned} \mathcal{F}_{\mathrm{DTFT}}\{g_1(m) * g_2(m)\} &= 3 + 3 \times 2\cos(\omega) + 2 \times 2\cos(2\omega) + 2\cos(3\omega) \\ &= 3 + 6\cos(\omega) + 4\cos(2\omega) + 2\cos(3\omega) . \end{aligned}$$

On the other hand, we have

$$\mathcal{F}_{\mathrm{DTFT}}\{g_1(m)\} = 1 + 2\cos(\omega) + 2\cos(2\omega)$$

and

$$\mathcal{F}_{\mathrm{DTFT}}\{g_2(m)\} = 1 + 2\cos(\omega) .$$

So, the product of the DTFT's of $g_1(m)$ and $g_2(m)$ is

$$\begin{aligned} \mathcal{F}_{\mathrm{DTFT}}\{g_1(m)\}\mathcal{F}_{\mathrm{DTFT}}\{g_2(m)\} &= [1 + 2\cos(\omega)][1 + 2\cos(\omega) + 2\cos(2\omega)] \\ &= 1 + 4\cos(\omega) + 2\cos(2\omega) \\ &\quad + 4\cos^2(\omega) + 4\cos(\omega)\cos(2\omega) \\ &= 1 + 4\cos(\omega) + 2\cos(2\omega) \\ &\quad + 4\frac{1 + \cos(2\omega)}{2} + 4\frac{\cos(\omega) + \cos(3\omega)}{2} \\ &= 3 + 6\cos(\omega) + 4\cos(2\omega) + 2\cos(3\omega) . \end{aligned}$$

Therefore,

$$\mathcal{F}_{\mathrm{DTFT}}\{g_1(m) * g_2(m)\} = \mathcal{F}_{\mathrm{DTFT}}\{g_1(m)\}\mathcal{F}_{\mathrm{DTFT}}\{g_2(m)\} .$$