

Solution Manual for Adaptive Filter Theory 5th Edition

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Chapter 2

Problem 2.1

a)

Let

$$w_k = x + j y$$

$$p(-k) = a + j b$$

We may then write

$$\begin{aligned} f &= w_k p^*(-k) \\ &= (x + j y)(a - j b) \\ &= (ax + by) + j(ay - bx) \end{aligned}$$

Letting

$$f = u + jv$$

where

$$u = ax + by$$

$$v = ay - bx$$

Hence,

$$\begin{aligned} \frac{\partial u}{\partial x} &= a & \frac{\partial u}{\partial y} &= b \\ \frac{\partial v}{\partial y} &= a & \frac{\partial v}{\partial x} &= -b \end{aligned}$$

From these results we can immediately see that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

In other words, the product term $w_k p^*(-k)$ satisfies the Cauchy-Riemann equations, and so this term is analytic.

b)

Let

$$\begin{aligned} \mathbf{f} &= w_k p^*(-k) \\ &= (x - j y)(a + j b) \\ &= (ax + by) + j(bx - ay) \end{aligned}$$

Let

$$\mathbf{f} = u + j v$$

with

$$u = ax + by$$

$$v = bx - ay$$

Hence,

$$\begin{aligned} \frac{\partial u}{\partial x} &= a & \frac{\partial u}{\partial y} &= b \\ \frac{\partial v}{\partial x} &= b & \frac{\partial v}{\partial y} &= -a \end{aligned}$$

From these results we immediately see that

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}$$

In other words, the product term $w^* p(\overline{-k})$ does not satisfy the Cauchy-Riemann equations, and so this term is not analytic.

Problem 2.2

a)

From the Wiener-Hopf equation, we have

$$\mathbf{w}_0 = \mathbf{R}^{-1} \mathbf{p}$$

(1) We are given that

$$\mathbf{R} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

Hence the inverse of \mathbf{R} is

$$\begin{aligned} \mathbf{R}^{-1} &= \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}^{-1} \\ &= \frac{1}{0.75 - 0.25} \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}^{-1} \end{aligned}$$

Using Equation (1), we therefore get

$$\begin{aligned} \mathbf{w}_0 &= \frac{1}{0.75 - 0.25} \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix} \\ &= \frac{1}{0.75} \begin{bmatrix} 0.375 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \end{aligned}$$

b)

The minimum mean-square error is

$$\begin{aligned} \mathbf{J}_{\min} &= \sigma_d^2 - \mathbf{p}^H \mathbf{w}_0 \\ &= \sigma_d^2 - \begin{bmatrix} 0.5 & 0.25 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \\ &= \sigma_d^2 - 0.25 \end{aligned}$$

c)

The eigenvalues of the matrix R are roots of the characteristic equation:

$$(1 - \lambda)^2 - (0.5)^2 = 0$$

That is, the two roots are

$$\lambda_1 = 0.5 \quad \text{and} \quad \lambda_2 = 1.5$$

The associated eigenvectors are defined by

$$Rq = \lambda q$$

For $\lambda_1 = 0.5$, we have

$$\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} = 0.5 \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix}$$

Expanded this becomes

$$q_{11} + 0.5q_{12} = 0.5q_{11}$$

$$0.5q_{11} + q_{12} = 0.5q_{12}$$

Therefore,

$$q_{11} = -q_{12}$$

Normalizing the eigenvector q_1 to unit length, we therefore have

$$q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Similarly, for the eigenvalue $\lambda_2 = 1.5$, we may show that

$$q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Accordingly, we may express the Wiener filter in terms of its eigenvalues and eigenvectors as follows:

$$\begin{aligned}
 w_0 &= \sum_{i=1}^2 \frac{1}{\lambda_i} q_i q_i^H p \\
 &= \frac{1}{\lambda_1} q_1 q_1^H + \frac{1}{\lambda_2} q_2 q_2^H p \\
 &= \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \cdot 0.25 \\
 &= \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot 0.25 \\
 &= \frac{1}{3} \begin{bmatrix} 4 & 2 & 0 \\ -2 & 4 & 0 \\ 3 & 3 & 1 \end{bmatrix} \cdot 0.25 \\
 &= \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix} \cdot 0.25 \\
 &= \begin{bmatrix} 0.33 & 0 & 0 \\ -0.33 & 0.33 & 0 \\ 1 & 1 & 0.33 \end{bmatrix} \cdot 0.25 \\
 &= \begin{bmatrix} 0.083 & 0 & 0 \\ -0.083 & 0.083 & 0 \\ 0.25 & 0.25 & 0.083 \end{bmatrix} \\
 &= \begin{bmatrix} 0.083 & 0 & 0 \\ -0.083 & 0.083 & 0 \\ 0.25 & 0.25 & 0.083 \end{bmatrix}
 \end{aligned}$$

Problem 2.3

a)

From the Wiener-Hopf equation we have

$$w_0 = R^{-1} p \tag{1}$$

We are given

$$R = \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 1 \end{bmatrix}$$

and

$$p = [0.5 \quad 0.25 \quad 0.125]^T$$

Hence, the use of these values in Equation (1) yields

$$\begin{aligned}
 \mathbf{w}_0 &= \mathbf{R}^{-1} \mathbf{p} \\
 &= \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 1 & 0.25 \\ 0.25 & 0.25 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 1 \\ 0.25 & 0.25 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.25 \\ 0 \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} 1.33 & -0.67 & 0 \\ -0.67 & 1.67 & -0.67 \\ 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0 \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} -0.67 & 1.67 & -0.67 \\ 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} \\
 \mathbf{w}_0 &= \begin{bmatrix} 0.5 & 0 & 0 \end{bmatrix}^T
 \end{aligned}$$

b)

The Minimum mean-square error is

$$\begin{aligned}
 \mathbf{J}_{\min} &= \mathbf{o}_d^2 - \mathbf{p}^H \mathbf{w}_0 \\
 &= \mathbf{o}_d^2 - \begin{bmatrix} 0.5 & 0.25 & 0.125 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix} \\
 &= \mathbf{o}_d^2 - 0.25
 \end{aligned}$$

c)

The eigenvalues of the matrix \mathbf{R} are

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0.4069 & 0.75 & 1.8431 \end{bmatrix}$$

The corresponding eigenvectors constitute the orthogonal matrix:

$$\mathbf{Q} = \begin{bmatrix} -0.4544 & -0.7071 & 0.5418 \\ 0.7662 & 0 & 0.6426 \\ -0.4544 & 0.7071 & 0.5418 \end{bmatrix}$$

PROBLEM 2.3.

Accordingly, we may express the Wiener filter in terms of its eigenvalues and eigenvectors as follows:

$$w_0 = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} q_i q_i^H p$$

$$\begin{aligned}
 w_0 = & \frac{1}{0.4069} \begin{bmatrix} -0.4544 & & \\ & 0.7662 & \\ & & -0.4544 \end{bmatrix} \begin{bmatrix} -0.4544 & 0.7662 & -0.4544 \end{bmatrix} \\
 & + \frac{1}{0.75} \begin{bmatrix} -0.7071 & & \\ & 0 & \\ & & 0.7071 \end{bmatrix} \begin{bmatrix} -0.7071 & 0 & -0.7071 \end{bmatrix} \\
 & + \frac{1}{1.8431} \begin{bmatrix} 0.5418 & & \\ & 0.6426 & \\ & & 0.5418 \end{bmatrix} \begin{bmatrix} 0.418 & 0.6426 & 0.5418 \end{bmatrix} \times \begin{bmatrix} 0.5 \\ 0.25 \\ 0.125 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 w_0 = & \frac{1}{0.4069} \begin{bmatrix} 0.2065 & -0.3482 & 0.2065 \\ -0.3482 & 0.5871 & -0.3482 \\ 0.2065 & -0.3482 & 0.2065 \end{bmatrix} \\
 & + \frac{1}{0.75} \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 0 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix} \\
 & + \frac{1}{1.8431} \begin{bmatrix} 0.2935 & 0.3482 & 0.2935 \\ 0.3482 & 0.4129 & 0.3482 \\ 0.2935 & 0.3482 & 0.2935 \end{bmatrix} \times \begin{bmatrix} 0.5 \\ 0.25 \\ 0.125 \end{bmatrix} \\
 = & \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}
 \end{aligned}$$

Problem 2.4

By definition, the correlation matrix

$$R = E[u(n)u^H(n)]$$

Where

$$u(n) = \begin{bmatrix} u(n) \\ u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$$

PROBLEM 2.4.

Invoking the ergodicity theorem,

$$R(N) = \frac{1}{N+1} \sum_{n=0}^N \mathbf{u}(n)\mathbf{u}^H(n)$$

Likewise, we may compute the cross-correlation vector

$$\mathbf{p} = E[\mathbf{u}(n)d^*(n)]$$

as the time average

$$\mathbf{p}(N) = \frac{1}{N+1} \sum_{n=0}^N \mathbf{u}(n)d^*(n)$$

The tap-weight vector of the wiener filter is thus defined by the matrix product

$$\mathbf{w}_0(N) = \left(\sum_{n=0}^N \mathbf{u}(n)\mathbf{u}^H(n) \right)^{-1} \sum_{n=0}^N \mathbf{u}(n)d^*(n)$$

Problem 2.5

a)

$$\begin{aligned} \mathbf{R} &= E[\mathbf{u}(n)\mathbf{u}^H(n)] \\ &= E[(\alpha(n)s(n) + v(n))(\alpha^*(n)s^H(n) + v^H(n))] \end{aligned}$$

With $\alpha(n)$ uncorrelated with $v(n)$, we have

$$\begin{aligned} \mathbf{R} &= E[|\alpha(n)|^2]s(n)s^H(n) + E[v(n)v^H(n)] \\ &= \sigma_\alpha^2 s(n)s^H(n) + \mathbf{R}_v \end{aligned} \tag{1}$$

where \mathbf{R}_v is the correlation matrix of v

b)

The cross-correlation vector between the input vector $\mathbf{u}(n)$ and the desired response $d(n)$ is

$$\mathbf{p} = E[\mathbf{u}(n)d^*(n)] \tag{2}$$

If $d(n)$ is uncorrelated with $\mathbf{u}(n)$, we have

$$\mathbf{p} = \mathbf{0}$$

Hence, the tap-weight of the wiener filter is

$$\begin{aligned} \mathbf{w}_0 &= \mathbf{R}^{-1}\mathbf{p} \\ &= \mathbf{0} \end{aligned}$$

c)

With $\sigma_\alpha^2 = 0$, Equation (1) reduces to

$$\mathbf{R} = \mathbf{R}_v$$

with the desired response

$$d(n) = v(n - k)$$

Equation (2) yields

$$\begin{aligned} \mathbf{p} &= E[(\alpha(n)s(n) + v(n)v^*(n - k))] \\ &= E[(v(n)v^*(n - k))] \\ &= E \begin{bmatrix} v(n) \\ v(n - 1) \\ \vdots \\ v(n - M + 1) \end{bmatrix} \begin{bmatrix} v^*(n - k) \\ v^*(n - k - 1) \\ \vdots \\ v^*(n - k - M + 1) \end{bmatrix} \\ &= E \begin{bmatrix} r_v(n) \\ r_v(n - 1) \\ \vdots \\ r_v(n - M + 1) \end{bmatrix} \begin{bmatrix} r_v(k) \\ r_v(k - 1) \\ \vdots \\ r_v(k - M + 1) \end{bmatrix}, \quad 0 \leq k \leq M - 1 \end{aligned} \tag{3}$$

where $r_v(k)$ is the autocorrelation of $v(n)$ for lag k . Accordingly, the tap-weight vector of the (optimum) wiener filter is

$$\begin{aligned} \mathbf{w}_0 &= \mathbf{R}^{-1} \mathbf{p} \\ &= \mathbf{R}_v^{-1} \mathbf{p} \end{aligned}$$

where \mathbf{p} is defined in Equation (3).

d)

For a desired response

$$d(n) = \alpha(n) \exp(-j \omega \tau)$$

The cross-correlation vector p is

$$\begin{aligned}
 p &= E[u(n)(d^*n)] \\
 &= E[(\alpha(n)s(n) + v(n)) \alpha^*(n) \exp(-j\omega\tau)] \\
 &= s(n) \exp(j\omega\tau) E[|\alpha(n)|^2] \\
 &= \sigma_\alpha^2 s(n) \exp(j\omega\tau) \\
 &= \sigma_\alpha^2 \begin{bmatrix} 1 \exp(-j\omega\tau) \\ \exp((-j\omega)(M-1)) \\ \vdots \\ \exp(j\omega\tau) \\ \exp(j\omega(\tau-1)) \\ \vdots \\ \exp(j\omega(\tau-M+1)) \end{bmatrix} \exp(j\omega\tau)
 \end{aligned}$$

The corresponding value of the tap-weight vector of the Wiener filter is

$$\begin{aligned}
 w_0 &= \sigma_\alpha^2 (\sigma_\alpha^2 s(n)s^H(n) + R_v)^{-1} \begin{bmatrix} \exp(j\omega\tau) \\ \exp(j\omega(\tau-1)) \\ \vdots \\ \exp(j\omega(\tau-M+1)) \end{bmatrix} \\
 &= s(n)s^H(n) + \frac{1}{\sigma_\alpha^2} R_v \begin{bmatrix} \exp(j\omega\tau) \\ \exp(j\omega(\tau-1)) \\ \vdots \\ \exp(j\omega(\tau-M+1)) \end{bmatrix}
 \end{aligned}$$

Problem 2.6

The optimum filtering solution is defined by the Wiener-Hopf equation

$$R w_0 = p \tag{1}$$

for which the minimum mean-square error is

$$\mathbf{J}_{\min} = \mathbf{o}_d^2 - \mathbf{p}^H \mathbf{w}_0$$

(2)

Combine Equations (1) and Equation(2) into a single relation:

$$\begin{bmatrix} \sigma_d^2 & \mathbf{p}^H \\ \mathbf{p} & \mathbf{R} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{w}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{\min} \\ \mathbf{0} \end{bmatrix}$$

Define

$$\mathbf{A} = \begin{bmatrix} \sigma_d^2 & \mathbf{p}^H \\ \mathbf{p} & \mathbf{R} \end{bmatrix} \tag{3}$$

Since

$$\begin{aligned} \sigma_d^2 &= E[d(n)d^*(n)] \\ \mathbf{p} &= E[\mathbf{u}(n)d^*(n)] \\ \mathbf{R} &= E[\mathbf{u}(n)\mathbf{u}^H(n)] \end{aligned}$$

we may rewrite Equation (3) as

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} E[d(n)d^*(n)] & E[d(n)\mathbf{u}^H(n)] \\ E[\mathbf{u}(n)d^*(n)] & E[\mathbf{u}(n)\mathbf{u}^H(n)] \end{bmatrix} \\ &= E \begin{bmatrix} d(n) & \mathbf{u}^H(n) \\ \mathbf{u}(n) & d^*(n) \end{bmatrix} \end{aligned}$$

The minimum mean-square error equals

$$\mathbf{J}_{\min} = \sigma_d^2 - \mathbf{p}^H \mathbf{w}_0 \tag{4}$$

Eliminating σ_d^2 between Equation (1) and Equation (4):

$$\mathbf{J}(\mathbf{w}) = \mathbf{J}_{\min} + \mathbf{p}^H \mathbf{w}_0 - \mathbf{p}^H \mathbf{R} \mathbf{w} - \mathbf{w}^H \mathbf{R} \mathbf{w}_0 + \mathbf{w}^H \mathbf{R} \mathbf{w} \tag{5}$$

Eliminating \mathbf{p} between Equation (2) and Equation (5)

$$\mathbf{J}(\mathbf{w}) = \mathbf{J}_{\min} + \mathbf{w}_0^H \mathbf{R} \mathbf{w}_0 - \mathbf{w}_0^H \mathbf{R} \mathbf{w} - \mathbf{w}^H \mathbf{R} \mathbf{w}_0 + \mathbf{w}^H \mathbf{R} \mathbf{w} \tag{6}$$

where we have used the property $\mathbf{R}^H = \mathbf{R}$. We may rewrite Equation (6) as

$$\mathbf{J}(\mathbf{w}) = \mathbf{J}_{\min} + (\mathbf{w} - \mathbf{w}_0)^H \mathbf{R} (\mathbf{w} - \mathbf{w}_0)$$

which clearly shows that $\mathbf{J}(\mathbf{w}_0) = \mathbf{J}_{\min}$

Problem 2.7

The minimum mean-square error is

$$J_{\min} = \sigma_d^2 - \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p} \tag{1}$$

Using the spectral theorem, we may express the correlation matrix \mathbf{R} as

$$\begin{aligned} \mathbf{R} &= \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^H \\ \mathbf{R} &= \sum_{k=1}^M \lambda_k \mathbf{q}_k \mathbf{q}_k^H \end{aligned} \tag{2}$$

Substituting Equation (2) into Equation (1)

$$\begin{aligned} J_{\min} &= \sigma_d^2 - \sum_{k=1}^M \frac{1}{\lambda_k} \mathbf{p}^H \mathbf{q}_k \mathbf{p}^H \mathbf{q}_k \\ &= \sigma_d^2 - \sum_{k=1}^M \frac{1}{\lambda_k} |\mathbf{p}^H \mathbf{q}_k|^2 \end{aligned}$$

Problem 2.8

When the length of the Wiener filter is greater than the model order m , the tail end of the tap-weight vector of the Wiener filter is zero; thus,

$$\mathbf{w}_0 = \begin{bmatrix} \mathbf{a}_m \\ \mathbf{0} \end{bmatrix}$$

Therefore, the only possible solution for the case of an over-fitted model is

$$\mathbf{w}_0 = \begin{bmatrix} \mathbf{a}_m \\ \mathbf{0} \end{bmatrix}$$

Problem 2.9

a)

The Wiener solution is defined by

$$\mathbf{R}_M \mathbf{a}_M = \mathbf{p}_M$$

$$\begin{bmatrix} \mathbf{R}_M & \mathbf{r}_{M-m} \\ \mathbf{r}_{M-m}^H & \mathbf{R}_{M-m, M-m} \end{bmatrix} \begin{bmatrix} \mathbf{a}_m \\ 0_{M-m} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_m \\ \mathbf{p}_{M-m} \end{bmatrix}$$

$$\mathbf{R}_M \mathbf{a}_m = \mathbf{p}_m$$

$$\mathbf{r}_{M-m}^H \mathbf{a}_m = \mathbf{p}_{M-m}$$

$$\mathbf{p}_{M-m} = \mathbf{r}_{M-m}^H \mathbf{a}_m = \mathbf{r}_{M-m}^H \mathbf{R}_M^{-1} \mathbf{p}_m \tag{1}$$

b)

Applying the conditions of Equation (1) to the example in Section 2.7 in the textbook

$$\mathbf{r}_{M-m}^H = \begin{bmatrix} -0.05 & 0.1 & 0.15 \end{bmatrix}$$

$$\mathbf{a}_m = \begin{bmatrix} 0.8719 \\ -0.9129 \\ 0.2444 \end{bmatrix}$$

The last entry in the 4-by-1 vector \mathbf{p} is therefore

$$\begin{aligned} \mathbf{r}_{M-m}^H \mathbf{a}_m &= -0.0436 - 0.0912 + 0.1222 \\ &= -0.0126 \end{aligned}$$

Problem 2.10

$$\begin{aligned} \mathbf{J}_{\min} &= \mathbf{o}_d^2 - \mathbf{p}^H \mathbf{w}_0 \\ &= \mathbf{o}_d^2 - \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p} \end{aligned}$$

when $m = 0$,

$$\begin{aligned} \mathbf{J}_{\min} &= \mathbf{o}_d^2 \\ &= 1.0 \end{aligned}$$

When $m = 1$,

$$\begin{aligned} \mathbf{J}_{\min} &= 1 - 0.5 \times \frac{1}{1.1} \times 0.5 \\ &= 0.9773 \end{aligned}$$

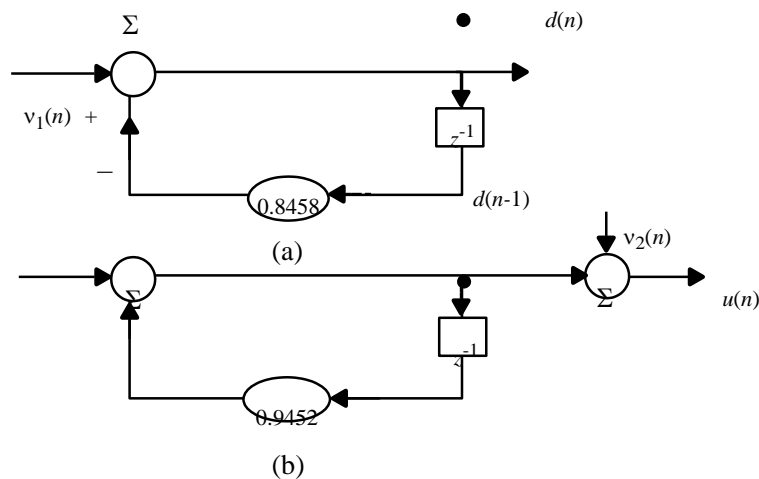
when $m = 2$

$$\mathbf{J}_{\min} = 1 - \begin{bmatrix} 0.5 & -0.4 \\ 0.5 & 1.1 \end{bmatrix}^{-1} \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}$$

$$\mathbf{R}_x = \begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix}$$

$$r_x(0) = \sigma_x^2$$

$$= \frac{1 + a_2}{1 - a_2} \cdot \frac{\sigma_1^2}{(1 + a_2)^2 - a_1^2} = 1$$



$$r_x(1) = \frac{-a_1}{1 + a_2} = 0.5$$

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

a)

$$u(n) = x(n) + v_2(n) \tag{1}$$

$$d(n) = -d(n-1) \times 0.8458 + v_1(n) \tag{2}$$

$$x(n) = d(n) + 0.9458x(n-1) \tag{3}$$

Equation (3) rearranged to solve for d(n) is

$$d(n) = x(n) - 0.9458x(n-1)$$

Using Equation (2) and Equation (3):

$$x(n) - 0.9458x(n-1) = 0.8458[-x(n-1) + 0.9458x(n-2)] + v_1(n)$$

Rearranging the terms this produces:

$$\begin{aligned} x(n) &= (0.9458 - 8.8458)x(n-1) + 0.8x(n-2) + v_1(n) \\ &= (0.1)x(n-1) + 0.8x(n-2) + v_1(n) \end{aligned}$$

b)

$$u(n) = x(n) + v_2(n)$$

where x(n) and v₂(n) are uncorrelated, therefore

$$\begin{aligned} \mathbf{R} &= \mathbf{R}_x + \mathbf{R}_{v_2} \\ \mathbf{R}_x &= \begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} r_x(0) &= \sigma_x^2 \\ &= \frac{1 + a_2}{1 - a_2} \frac{\sigma_v^2}{(1 + a_2)^2 - a_1^2} = 1 \end{aligned}$$

$$r_x(1) = \frac{-a_1}{1 + a_2}$$

$$r_x(1) = 0.5$$

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$\mathbf{R}_{v_2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{R}_x + \mathbf{R}_{v_2} = \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}$$

$$\begin{bmatrix} p(0) \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} p(1) \end{bmatrix}$$

$$p(k) = E[u(n-k)d(n)], \quad k = 0, 1$$

$$\begin{aligned} p(0) &= r_x(0) + b_1 r_x(-1) \\ &= 1 - 0.9458 \times 0.5 \\ &= 0.5272 \end{aligned}$$

$$\begin{aligned} p(1) &= r_x(1) + b_1 r_x(0) \\ &= 0.5 - 0.9458 \\ &= -0.4458 \end{aligned}$$

Therefore,

$$\mathbf{p} = \begin{bmatrix} 0.5272 \\ -0.4458 \end{bmatrix}$$

c)

The optimal weight vector is given by the equation $\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{p}$; hence,

$$\begin{aligned} \mathbf{w}_0 &= \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5272 \\ -0.4458 \end{bmatrix} \\ &= \begin{bmatrix} 0.8363 \\ -0.7853 \end{bmatrix} \end{aligned}$$

Problem 2.12

a)

For $M = 3$ taps, the correlation matrix of the tap inputs is

$$\mathbf{R} = \begin{bmatrix} 1.1 & 0.5 & 0.85 \\ 0.5 & 1.1 & 0.5 \\ 0.85 & 0.5 & 1.1 \end{bmatrix}$$

The cross-correlation vector between the tap inputs and the desired response is

$$\mathbf{p} = \begin{bmatrix} 0.527 \\ -0.446 \\ 0.377 \end{bmatrix}$$

b)

The inverse of the correlation matrix is

$$\mathbf{R}^{-1} = \begin{bmatrix} 2.234 & -0.304 & -1.666 \\ -0.304 & 1.186 & -0.304 \\ -1.66 & -0.304 & 2.234 \end{bmatrix}$$

Hence, the optimum weight vector is

$$\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{p} = \begin{bmatrix} 0.738 \\ -0.803 \\ 0.138 \end{bmatrix}$$

The minimum mean-square error is

$$\mathbf{J}_{\min} = 0.15$$

Problem 2.13

a)

The correlation matrix \mathbf{R} is

$$\begin{aligned} \mathbf{R} &= \mathbf{E}[\mathbf{u}(n)\mathbf{u}^H(n)] \\ &= \mathbf{E}[|A_1|^2] \begin{bmatrix} e^{-j\omega_1 n} & & & \\ e^{-j\omega_1(n-1)} & & & \\ \vdots & & & \\ e^{-j\omega_1(n-M+1)} & & & \end{bmatrix} \begin{bmatrix} e^{+j\omega_1 n} & e^{+j\omega_1(n-1)} & \dots & e^{+j\omega_1(n-M+1)} \end{bmatrix} \\ &= \mathbf{E}[|A_1|^2] \mathbf{s}(\omega_1) \mathbf{s}^H(\omega_1) + \mathbf{IE}[|v(n)|^2] \\ &= \sigma_1^2 \mathbf{s}(\omega_1) \mathbf{s}^H(\omega_1) + \sigma_v^2 \mathbf{I} \end{aligned}$$

where \mathbf{I} is the identity matrix.

b)

The tap-weights vector of the Wiener filter is

$$\mathbf{w}_0 = \mathbf{R}^{-1} \mathbf{p}$$

From part a),

$$\mathbf{R} = \sigma_1^2 \mathbf{s}(\omega_1) \mathbf{s}^H(\omega_1) + \sigma_v^2 \mathbf{I}$$

We are given

$$\mathbf{p} = \sigma_0^2 \mathbf{s}(\omega_0)$$

To invert the matrix \mathbf{R} , we use the matrix inversion lemma (see Chapter 10), as described here:

If:

$$\mathbf{A} = \mathbf{B}^{-1} + \mathbf{C} \mathbf{D}^{-1} \mathbf{C}^H$$

then:

$$\mathbf{A}^{-1} = \mathbf{B} - \mathbf{B} \mathbf{C} (\mathbf{D} + \mathbf{C}^H \mathbf{B} \mathbf{C})^{-1} \mathbf{C}^H \mathbf{B}$$

In our case:

$$\mathbf{A} = \sigma_v^2 \mathbf{I}$$

$$\mathbf{B}^{-1} = \sigma_v^2 \mathbf{I}$$

$$\mathbf{D}^{-1} = \sigma_1^2$$

$$\mathbf{C} = s(\omega_1)$$

Hence,

$$\mathbf{R}^{-1} = \frac{1}{\sigma_v^2} \mathbf{I}_2 - \frac{\frac{1}{2} s(\omega_1) s^H(\omega_1)}{\frac{\sigma_v^2}{\sigma_1^2} + s^H(\omega_1) s(\omega_1)}$$

The corresponding value of the Wiener tap-weight vector is

$$\mathbf{w}_0 = \mathbf{R}^{-1} \mathbf{p}$$

$$\mathbf{w}_0 = \frac{\sigma_0^2}{\sigma_v^2} s(\omega_0) - \frac{\frac{\sigma_0^2}{\sigma_v^2} s(\omega_1) s^H(\omega_1)}{\frac{\sigma_v^2}{\sigma_1^2} + s^H(\omega_1) s(\omega_1)} s(\omega_0)$$

we note that

$$s^H(\omega_1) s(\omega_1) = \mathbf{M}$$

which is a scalar hence,

$$\mathbf{w}_0 = \frac{\sigma_0^2}{\sigma_v^2} s(\omega_0) - \frac{\frac{\sigma_0^2}{\sigma_v^2} s^H(\omega_1) s(\omega_1)}{\frac{\sigma_v^2}{\sigma_1^2} + \mathbf{M}} s(\omega_0)$$

Problem 2.14

The output of the array processor equals

$$e(n) = u(1, n) - w u(2, n)$$

The mean-square error equals

$$\begin{aligned} \mathbf{J}(\mathbf{w}) &= E[|e(n)|^2] \\ &= E[(u(1, n) - w u(2, n))(u^*(1, n) - w^* u^*(2, n))] \end{aligned}$$

$$=E[|u(1, n)|^2] + |w|^2E[|u(2, n)|^2] - wE[u(2, n)u^*(1, n)] - wE[u(1, n)u^*(2, n)]$$

Differentiating $\mathbf{J}(w)$ with respect to w :

$$\frac{\partial \mathbf{J}}{\partial w} = -2\mathbf{E}[\mathbf{u}(1, n)\mathbf{u}^*(2, n)] + 2w\mathbf{E}[|\mathbf{u}(2, n)|^2]$$

Putting $\frac{\partial \mathbf{J}}{\partial w} = 0$ and solving for the optimum value of w :

$$w_0 = \frac{\mathbf{E}[\mathbf{u}(1, n)\mathbf{u}^*(2, n)]}{\mathbf{E}[|\mathbf{u}(2, n)|^2]}$$

Problem 2.15

Define the index of the performance (i.e., cost function)

$$\mathbf{J}(w) = \mathbf{E}[|e(n)|^2] + c^H s^H w + w^H s c - 2c^H \mathbf{D}^{1/2} \mathbf{1}$$

$$\mathbf{J}(w) = w^H \mathbf{R} w + c^H s^H w + w^H s c - 2c^H \mathbf{D}^{1/2} \mathbf{1}$$

Differentiate $\mathbf{J}(w)$ with respect to w and set the result equal to zero:

$$\frac{\partial \mathbf{J}}{\partial w} = 2\mathbf{R}w + 2sc = \mathbf{0}$$

Hence,

$$w_0 = -\mathbf{R}^{-1}sc$$

But, we must constrain w_0 as

$$s^H w_0 = \mathbf{D}^{1/2} \mathbf{1}$$

therefore, the vector c equals

$$c = -(s^H \mathbf{R}^{-1} s)^{-1} \mathbf{D}^{1/2} \mathbf{1}$$

Correspondingly, the optimum weight vector equals

$$w_0 = \mathbf{R}^{-1} s (s^H \mathbf{R}^{-1} s)^{-1} \mathbf{D}^{1/2} \mathbf{1}$$

Problem 2.16

The weight vector w of the beamformer that maximizes the output signal-to-noise ratio:

$$(\text{SNR})_0 = \frac{w^H \mathbf{R}_s w}{w^H \mathbf{R}_v w}$$

is derived in part b) of the problem 2.18; there it is shown that the optimum weight vector w_{SN} so defined is given by

$$w_{\text{SN}} = \mathbf{R}_v^{-1} s \quad (1)$$

where s is the signal component and \mathbf{R}_v is the correlation matrix of the noise $v(n)$. On the other hand, the optimum weight vector of the LCMV beamformer is defined by

$$w_0 = g^* \frac{\mathbf{R}^{-1} s(\varphi)}{s^H(\varphi) \mathbf{R}^{-1} s(\varphi)} \quad (2)$$

where $s(\varphi)$ is the steering vector. In general, the formulas (1) and (2) yield different values for the weight vector of the beamformer.

Problem 2.17

Let τ_i be the propagation delay, measured from the zero-time reference to the i th element of a nonuniformly spaced array, for a plane wave arriving from a direction defined by angle θ with respect to the perpendicular to the array. For a signal of angular frequency ω , this delay amounts to a phase shift equal to $-\omega\tau_i$. Let the phase shifts for all elements of the array be collected together in a column vector denoted by $d(\omega, \theta)$. The response of a beamformer with weight vector w to a signal (with angular frequency ω) originates from angle $\theta = w^H d(\omega, \theta)$. Hence, constraining the response of the array at ω and θ to some value g involves the linear constraint

$$w^H d(\omega, \theta) = g$$

Thus, the constraint vector $d(\omega, \theta)$ serves the purpose of generalizing the idea of an LCMV beamformer beyond simply the case of a uniformly spaced array. Everything else is the same as before, except for the fact that the correlation matrix of the received signal is no longer Toeplitz for the case of a nonuniformly spaced array

Problem 2.18

a)

Under hypothesis H_1 , we have

$$\mathbf{u} = \mathbf{s} + \mathbf{v}$$

The correlation matrix of \mathbf{u} equals

$$\mathbf{R} = E[\mathbf{u}\mathbf{u}^T]$$

$$\mathbf{R} = \mathbf{s}\mathbf{s}^T + \mathbf{R}_N, \quad \text{where } \mathbf{R}_N = E[\mathbf{v}\mathbf{v}^T]$$

The tap-weight vector w_k is chosen so that $w^T \mathbf{u}$ yields an optimum estimate of the k th element of \mathbf{s} . Thus, with $s(k)$ treated as the desired response, the cross-correlation vector between \mathbf{u} and $s(k)$ equals

$$\begin{aligned} p_k &= E[\mathbf{u}s(k)] \\ &= \mathbf{s}s(k), \quad k = 1, 2, \dots, m \end{aligned}$$

Hence, the Wiener-Hopf equation yields the optimum value of w_k as

$$\begin{aligned} w_{k0} &= \mathbf{R}^{-1} p_k \\ w_{k0} &= (\mathbf{s}\mathbf{s}^T + \mathbf{R}_N)^{-1} \mathbf{s}s(k), \quad k = 1, 2, \dots, M \end{aligned} \tag{1}$$

To apply the matrix inversion lemma (introduced in Problem 2.13), we let

$$\mathbf{A} = \mathbf{R}$$

$$\mathbf{B}^{-1} = \mathbf{R}_N$$

$$\mathbf{C} = \mathbf{s}$$

$$\mathbf{D} = 1$$

Hence,

$$\mathbf{R}^{-1} = \mathbf{R}_N^{-1} - \frac{\mathbf{R}_N^{-1} \mathbf{s} \mathbf{s}^T \mathbf{R}_N^{-1}}{1 + \mathbf{s}^T \mathbf{R}_N^{-1} \mathbf{s}} \tag{2}$$

Substituting Equation (2) into Equation (1) yields:

$$w_{k0} = \mathbf{R}_N^{-1} - \frac{\mathbf{R}_N^{-1} \mathbf{s} \mathbf{s}^T \mathbf{R}_N^{-1}}{1 + \mathbf{s}^T \mathbf{R}_N^{-1} \mathbf{s}} \mathbf{s}s(k)$$

$$w_{k0} = \frac{\mathbf{R}_N^{-1} \mathbf{s} (1 + \mathbf{s}^T \mathbf{R}_N^{-1} \mathbf{s}) - \mathbf{R}_N^{-1} \mathbf{s} \mathbf{s}^T \mathbf{R}_N^{-1} \mathbf{s}}{1 + \mathbf{s}^T \mathbf{R}_N^{-1} \mathbf{s}} s(k)$$

$$w_{k0} = \frac{s(k)}{1 + \mathbf{s}^T \mathbf{R}_N^{-1} \mathbf{s}} \mathbf{R}_N^{-1} \mathbf{s}$$

b)

The output signal-to-noise ratio is

$$\begin{aligned} \text{SNR} &= \frac{E[(w^T \mathbf{s})^2]}{E[(w^T \mathbf{v})^2]} \\ &= \frac{\mathbf{w}^T \mathbf{s} \mathbf{s}^T \mathbf{w}}{\mathbf{w}^T E[\mathbf{v} \mathbf{v}^T] \mathbf{w}} \\ &= \frac{\mathbf{w}^T \mathbf{s} \mathbf{s}^T \mathbf{w}}{\mathbf{w}^T \mathbf{R}_N \mathbf{w}} \end{aligned} \quad (3)$$

Since \mathbf{R}_N is positive definite, we may write,

$$\mathbf{R}_N = \mathbf{R}_N^{1/2} \mathbf{R}_N^{1/2}$$

Define the vector

$$\mathbf{a} = \mathbf{R}_N^{1/2} \mathbf{w}$$

or equivalently,

$$\mathbf{w} = \mathbf{R}_N^{-1/2} \mathbf{a} \quad (4)$$

Accordingly, we may rewrite Equation (3) as follows

$$\text{SNR} = \frac{\mathbf{a}^T \mathbf{R}_N^{-1/2} \mathbf{s} \mathbf{s}^T \mathbf{R}_N^{-1/2} \mathbf{a}}{\mathbf{a}^T \mathbf{a}} \quad (5)$$

where we have used the symmetric property of \mathbf{R}_N . Define the normalized vector

$$\bar{\mathbf{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

where $\|\mathbf{a}\|$ is the norm of \mathbf{a} . Equation (5) may be rewritten as:

$$\text{SNR} = \bar{\mathbf{a}}^T \mathbf{R}_N^{-1/2} \mathbf{s} \mathbf{s}^T \mathbf{R}_N^{-1/2} \bar{\mathbf{a}}$$

$$\text{SNR} = \bar{\mathbf{a}}^T \mathbf{R}_N^{-1/2} \mathbf{s}$$

Thus the output signal-to-noise ratio SNR equals the squared magnitude of the inner product of the two vectors $\bar{\mathbf{a}}$ and $\mathbf{R}_N^{-1/2} \mathbf{s}$. This inner product is maximized when $\bar{\mathbf{a}}$ equals $\mathbf{R}_N^{-1/2} \mathbf{s}$. That is,

$$\mathbf{a}_{\text{SN}} = \mathbf{R}_N^{-1/2} \mathbf{s} \tag{6}$$

Let \mathbf{w}_{SN} denote the value of the tap-weight vector that corresponds to Equation (6). Hence, the use of Equation (4) in Equation (6) yields

$$\begin{aligned} \mathbf{w}_{\text{SN}} &= \mathbf{R}_N^{-1/2} (\mathbf{R}_N^{-1/2} \mathbf{s}) \\ \mathbf{w}_{\text{SN}} &= \mathbf{R}_N^{-1} \mathbf{s} \end{aligned}$$

c)

Since the noise vector $\mathbf{v}(n)$ is Gaussian, its joint probability density function equals

$$f_{\mathbf{v}}(\mathbf{v}) = \frac{1}{(2\pi)^{M/2} (\det(\mathbf{R}_N))^{1/2}} \exp \left\{ -\frac{1}{2} \mathbf{v}^T \mathbf{R}_N^{-1} \mathbf{v} \right\}$$

Under the hypothesis H_0 we have

$$\mathbf{u} = \mathbf{v}$$

and

$$f_{\mathbf{u}}(\mathbf{u}|H_0) = \frac{1}{(2\pi)^{M/2} (\det \mathbf{R}_N)^{1/2}} \exp \left\{ -\frac{1}{2} \mathbf{u}^T \mathbf{R}_N^{-1} \mathbf{u} \right\}$$

Under hypothesis H_1 we have

$$\mathbf{u} = \mathbf{s} + \mathbf{v}$$

and

$$f_{\mathbf{u}}(\mathbf{u}|H_1) = \frac{1}{(2\pi)^{M/2} (\det \mathbf{R}_N)^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{u} - \mathbf{s})^T \mathbf{R}_N^{-1} (\mathbf{u} - \mathbf{s}) \right\}$$

Hence, the likelihood ratio is defined by

$$\begin{aligned}\Lambda &= \frac{f_u(\mathbf{u}|H_1)}{f_u(\mathbf{u}|H_0)} \\ &= \exp \left[-\frac{1}{2} \mathbf{s}^T \mathbf{R}_N^{-1} \mathbf{s} + \mathbf{s}^T \mathbf{R}_N^{-1} \mathbf{u} \right]\end{aligned}$$

The natural logarithm of the likelihood ratio equals

$$\ln \Lambda = -\frac{1}{2} \mathbf{s}^T \mathbf{R}_N^{-1} \mathbf{s} + \mathbf{s}^T \mathbf{R}_N^{-1} \mathbf{u} \quad (7)$$

The first term in (7) represents a constant. Hence, testing $\ln \Lambda$ against a threshold is equivalent to the test

$$\underset{H_0}{\overset{H_1}{\mathbf{s}^T \mathbf{R}_N^{-1} \mathbf{u}}} \underset{\geq}{\geq} \lambda$$

where λ is some threshold. Equivalently, we may write

$$\mathbf{w}_{ML} = \mathbf{R}_N^{-1} \mathbf{s}$$

where \mathbf{w}_{ML} is the maximum likelihood weight vector.

The results of parts a), b), and c) show that the three criteria discussed here yield the same optimum value for the weight vector, except for a scaling factor.

Problem 2.19

a)

Assuming the use of a noncausal Wiener filter, we write

$$\sum_{i=-\infty}^{\infty} w_{0i} r(i-k) = p(-k), \quad k = 0, \pm 1, \pm 2, \dots, \pm \infty \quad (1)$$

where the sum now extends from $i = -\infty$ to $i = \infty$. Define the z-transforms:

$$S(z) = \sum_{k=-\infty}^{\infty} r(k)z^{-k}, \quad H_u(z) = \sum_{k=-\infty}^{\infty} w_{0,k}z^{-k}$$

$$P(z) = \sum_{k=-\infty}^{\infty} p(-k)z^{-k} = P(z^{-1})$$

Hence, applying the z-transform to Equation (1):

$$H_u(z)S(z) = P(z^{-1})$$

$$H_u(z) = \frac{P(1/z)}{S(z)} \quad (2)$$

b)

$$P(z) = \frac{0.36}{1 - \frac{0.2}{z} (1 - 0.2z)}$$

$$P(1/z) = \frac{0.36}{(1 - 0.2z) 1 - \frac{0.2}{z}}$$

$$S(z) = 1.37 \frac{(1 - 0.146z^{-1})(1 - 0.146z)}{(1 - 0.2z^{-1})(1 - 0.2z)}$$

Thus, applying Equation (2) yields

$$\begin{aligned} H_u(z) &= \frac{0.36}{1.37(1 - 0.146z^{-1})(1 - 0.146z)} \\ &= \frac{0.36z^{-1}}{1.37(1 - 0.146z^{-1})(z^{-1} - 0.146)} \\ &= \frac{0.2685}{1 - 0.146z^{-1}} + \frac{0.0392}{z^{-1} - 0.146} \end{aligned}$$

Clearly, this system is noncausal. Its impulse response is $h(n)$ = inverse z-transform of $H_u(z)$ is given by

$$h(n) = 0.2685(0.146)^n u_{\text{step}}(n) - \frac{0.0392}{0.146} \frac{1}{0.146} \overset{\setminus n}{u_{\text{step}}(-n)}$$

where $u_{\text{step}}(n)$ is the unit-step function:

$$u_{\text{step}}(n) = \begin{cases} 1 & \text{for } n = 0, 1, 2, \dots \\ 0 & \text{for } n = -1, -2, \dots \end{cases}$$

and $u_{\text{step}}(-n)$ is its mirror image:

$$u_{\text{step}}(-n) = \begin{cases} 1 & \text{for } n = 0, -1, -2, \dots \\ 0 & \text{for } n = 1, 2, \dots \end{cases}$$

Simplifying,

$$h_u(n) = 0.2685 \times (0.146)^n u_{\text{step}}(n) - 0.2685 \times (6.849)^{-n} u_{\text{step}}(-n)$$

Simplifying,

$$h_u(n) = 0.2685 \times (0.146)^n u_{\text{step}}(+n) - 0.2685(6.849)^{-n} u_{\text{step}}(-n)$$

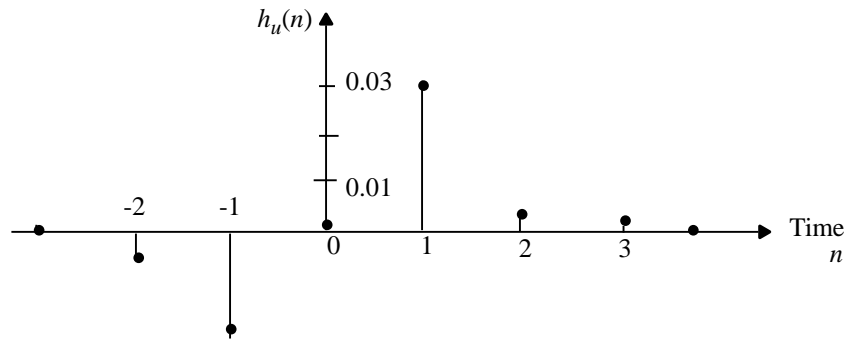
Evaluating $h_u(n)$ for varying n :

$$h_u(0) = 0, \text{ and}$$

$$h_u(1) = 0.03, \quad h_u(2) = 0.005, \quad h_u(3) = 0.0008$$

$$h_u(-1) = -0.03, \quad h_u(-2) = -0.005, \quad h_u(-3) = -0.0008$$

These are plotted in the following figure:



(c) A delay by 3 time units applied to the impulse response will make the system causal and therefore realizable.
A delay of 3 time units applied to the impulse response will make the system causal and therefore realizable.