Solution Manual for Modern Control Systems 13th Edition Dorf Bishop 0134407628 9780134407623

CHAPTER 2

Mathematical Models of Systems

Exercises

E2.1 We have for the open-loop

$$y = r^2$$

and for the closed-loop

$$e = r - y \quad \text{and} \quad y = e^2 .$$

So, $e = r - e^2$ and $e^2 + e - r = 0$.

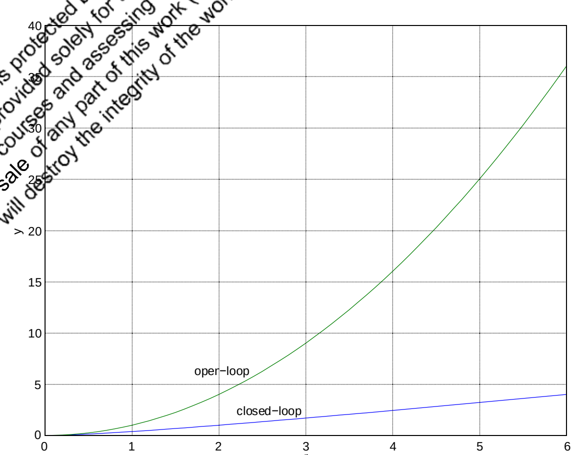


FIGURE E2.1

Plot of open-loop versus closed-loop.

For example, if $r = 1$, then $e^2 + e - 1 = 0$ implies that $e = 0.618$. Thus, $y = 0.382$. A plot y versus r is shown in Figure E2.1.

E2.2 Define

$$f(T) = R = R_0 e^{-0.1T}$$

and

$$\Delta R = f(T) - f(T_0)\,, \Delta T = T - T_0\,.$$

Then,

$$\Delta R = f(T) - f(T_0) = \left.\frac{\partial f}{\partial T}\right|_{T=T_0=20^\circ} \Delta T + \cdots$$

where

$$\left.\frac{\partial f}{\partial T}\right|_{T=T_0=20^\circ} = -0.1 R_0 e^{-0.1T_0} = -135,$$

when $R_0 = 10,000\Omega$. Thus, the linear approximation is computed by considering only the first-order terms in the Taylor series expansion, and is given by

$$\Delta R = -135\Delta T\,.$$

E2.3 The spring constant for the equilibrium point is found graphically by estimating the slope of a line tangent to the force versus displacement curve at the point $y = 0.5$cm, see Figure E2.3. The slope of the line is $K \approx 1$.

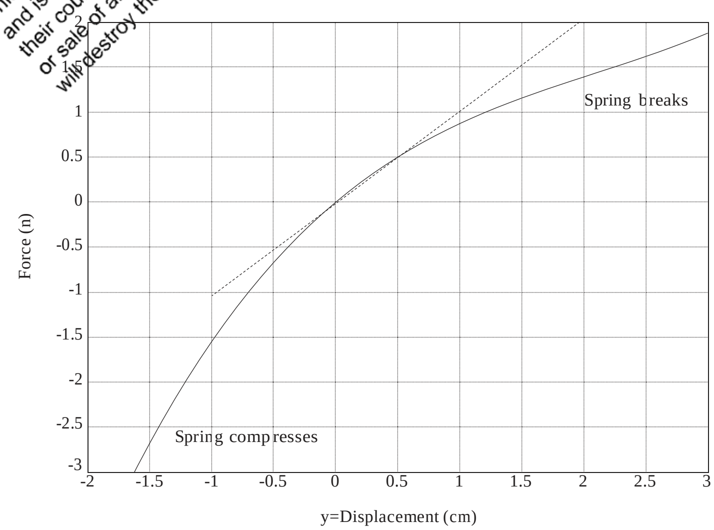


FIGURE E2.3
Spring force as a function of displacement.

E2.4 Since

$$R(s) = \frac{1}{s}$$

we have

$$Y(s) = \frac{6(s+50)}{s(s+30)(s+10)} .$$

The partial fraction expansion of $Y(s)$ is given by

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s+30} + \frac{A_3}{s+10}$$

where

$$A_1 = 1 , \quad A_2 = 0.2 \text{ and } A_3 = -1.2 .$$

Using the Laplace transform table, we find that

$$y(t) = 1 + 0.2e^{-30t} - 1.2e^{-10t} .$$

The final value is computed using the final value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s \left[\frac{6(s+50)}{s(s^2+40s+300)} \right] = 1 .$$

E2.5 The circuit diagram is shown in Figure E2.5.

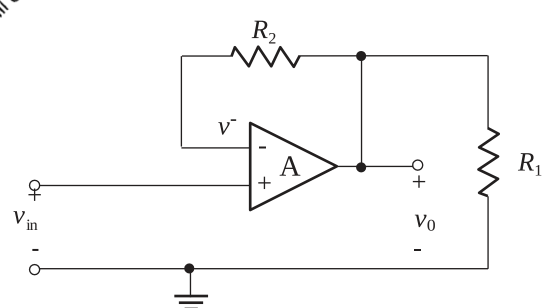


FIGURE E2.5
Noninverting op-amp circuit.

With an ideal op-amp, we have

$$v_o = A(v_{in} - v^-),$$

where A is very large. We have the relationship

$$v^- = \frac{R_1}{R_1 + R_2} v_o.$$

Therefore,

$$v_o = A\left(v_{in} - \frac{R_1}{R_1 + R_2} v_o\right),$$

and solving for v_o yields

$$v_o = \frac{A}{1 + \frac{AR_1}{R_1+R_2}} v_{in}.$$

Since $A \gg 1$, it follows that $1 + \frac{AR_1}{R_1+R_2} \approx \frac{AR_1}{R_1+R_2}$. Then the expression for v_o simplifies to

$$v_o = \frac{R_1 + R_2}{R_1} v_{in}.$$

E2.6 Given

$$y = f(x) = e^x$$

and the operating point $x_o = 1$, we have the linear approximation

$$y = f(x) = f(x_o) + \left.\frac{\partial f}{\partial x}\right|_{x=x_o} (x - x_o) + \cdots$$

where

$$f(x_o) = e, \quad \left.\frac{df}{dx}\right|_{x=x_o=1} = e, \quad \text{and} \quad x - x_o = x - 1.$$

Therefore, we obtain the linear approximation $y = ex$.

E2.7 The block diagram is shown in Figure E2.7.

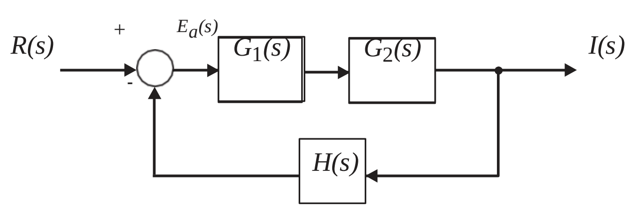


FIGURE E2.7
Block diagram model.

Starting at the output we obtain

$$I(s) = tt_1(s)tt_2(s)E(s).$$

But $E(s) = R(s) - H(s)I(s)$, so

$$I(s) = tt_1(s)tt_2(s)\left[R(s) - H(s)I(s)\right].$$

Solving for $I(s)$ yields the closed-loop transfer function

$$\frac{I(s)}{R(s)} = \frac{tt_1(s)tt_2(s)}{1 + tt_1(s)tt_2(s)H(s)}.$$

E2.8 The block diagram is shown in Figure E2.8.

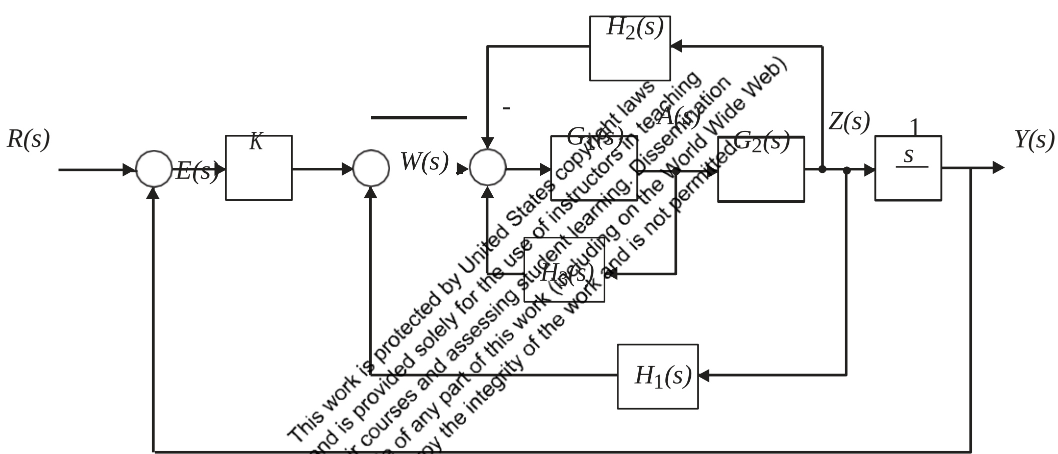


FIGURE E2.8
Block diagram model.

Starting at the output we obtain

$$Y(s) = \frac{1}{s}Z(s) = \frac{1}{s}tt_2(s)A(s).$$

But $A(s) = tt_1(s)\left[-H_2(s)Z(s) - H_3(s)A(s) + W(s)\right]$ and $Z(s) = sY(s)$, so

$$Y(s) = -tt_1(s)tt_2(s)H_2(s)Y(s) - tt_1(s)H_3(s)Y(s) + \frac{1}{s}tt_1(s)tt_2(s)W(s).$$

Substituting $W(s) = KE(s) - H_1(s)Z(s)$ into the above equation yields

$$Y(s) = -tt_1(s)tt_2(s)H_2(s)Y(s) - tt_1(s)H_3(s)Y(s) + \frac{1}{s}tt_1(s)tt_2(s)\left[KE(s) - H_1(s)Z(s)\right]$$

and with $E(s) = R(s) - Y(s)$ and $Z(s) = sY(s)$ this reduces to

$$Y(s) = [-tt_1(s)tt_2(s)\,(H_2(s) + H_1(s)) - tt_1(s)H_3(s) - \frac{1}{s}tt_1(s)tt_2(s)K]Y(s) + \frac{1}{s}tt_1(s)tt_2(s)KR(s).$$

Solving for $Y(s)$ yields the transfer function

$$Y(s) = T(s)R(s),$$

where

$$T(s) = \frac{Ktt_1(s)tt_2(s)/s}{1 + tt_1(s)tt_2(s)[(H_2(s) + H_1(s)] + tt_1(s)H_3(s) + Ktt_1(s)tt_2(s)/s} .$$

E2.9 From Figure E2.9, we observe that

$$F_f(s) = tt_2(s)U(s)$$

and

$$F_R(s) = tt_3(s)U(s) .$$

Then, solving for $U(s)$ yields

$$U(s) = \frac{1}{tt_2(s)} F_f(s)$$

and it follows that

$$F_R(s) = \frac{tt_3(s)}{tt_2(s)} U(s) .$$

Again, considering the block diagram in Figure E2.9 we determine

$$F_f(s) = tt_1(s)tt_2(s)[R(s) - H_2(s)F_f(s) - H_2(s)F_R(s)] .$$

But, from the previous result, we substitute for $F_R(s)$ resulting in

$$F_f(s) = tt_1(s)tt_2(s)R(s) - tt_1(s)tt_2(s)H_2(s)F_f(s) - tt_1(s)H_2(s)tt_3(s)F_f(s) .$$

Solving for $F_f(s)$ yields

$$F_f(s) = \left[\frac{tt_1(s)tt_2(s)}{1 + tt_1(s)tt_2(s)H_2(s) + tt_1(s)tt_3(s)H_2(s)}\right] R(s) .$$

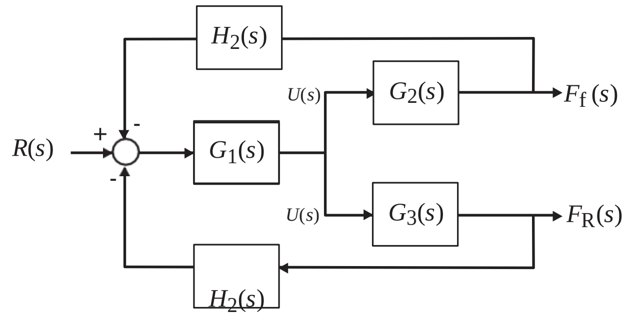


FIGURE E2.9
Block diagram model.

E2.10 The shock absorber block diagram is shown in Figure E2.10. The closed-loop transfer function model is

$$T(s) = \frac{G_c(s)G_p(s)G(s)}{1 + H(s)G_c(s)G_p(s)G(s)} .$$

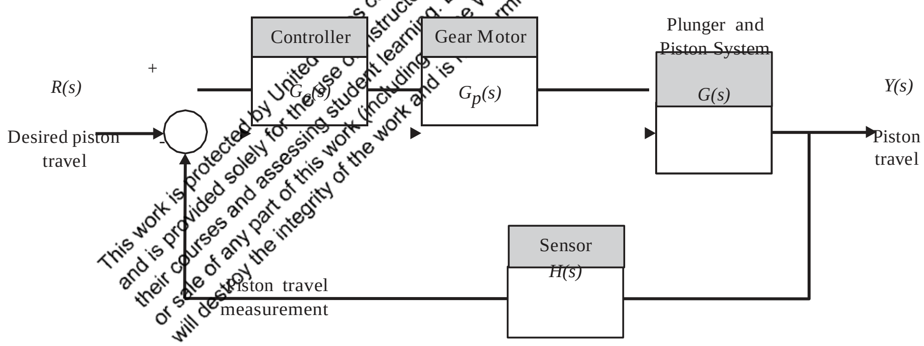


FIGURE E2.10
Shock absorber block diagram.

E2.11 Let f denote the spring force (n) and x denote the deflection (m). Then

$$K = \frac{\Delta f}{\Delta x} .$$

Computing the slope from the graph yields:

(a) $x_o = -0.14\text{m} \rightarrow K = \Delta f/\Delta x = 10$ n / 0.04 m = 250 n/m
(b) $x_o = 0\text{m} \rightarrow K = \Delta f/\Delta x = 10$ n / 0.05 m = 200 n/m
(c) $x_o = 0.35\text{m} \rightarrow K = \Delta f/\Delta x = 3$n / 0.05 m = 60 n/m

E2.12 The signal flow graph is shown in Fig. E2 12. Find $Y(s)$ when $R(s) = 0$.

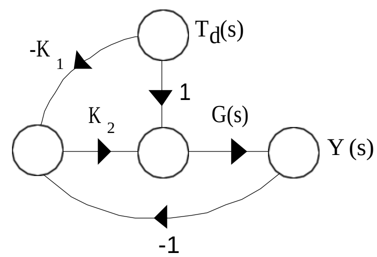


FIGURE E2.12
Signal flow graph.

The transfer function from $T_d(s)$ to $Y(s)$ is

$$Y(s) = \frac{tt(s)T_d(s) - K_1K_2tt(s)T_d(s)}{1 - (-K_2tt(s))} = \frac{tt(s)(1 - K_1K_2)T_d(s)}{1 + K_2tt(s)} .$$

If we set

$$K_1K_2 = 1 ,$$

then $Y(s) = 0$ for any $T_d(s)$.

E2.13 The transfer function from $R(s)$, $T_d(s)$, and $N(s)$ to $Y(s)$ is

$$Y(s) = \left(\frac{K}{s^2 + 10s + K}\right) R(s) + \left(\frac{1}{s^2 + 10s + K}\right) T_d(s) - \left(\frac{K}{s^2 + 10s + K}\right) N(s)$$

Therefore, we find that

$$Y(s)/T_d(s) = \frac{1}{s^2 + 10s + K} \quad \text{and} \quad Y(s)/N(s) = -\frac{K}{s^2 + 10s + K}$$

E2.14 Since we want to compute the transfer function from $R_2(s)$ to $Y_1(s)$, we can assume that $R_1 = 0$ (application of the principle of superposition). Then, starting at the output $Y_1(s)$ we obtain

$$Y_1(s) = tt_3(s)\left[-H_1(s)Y_1(s) + tt_2(s)tt_8(s)W(s) + tt_9(s)W(s)\right] ,$$

or

$$[1 + tt_3(s)H_1(s)]\, Y_1(s) = [tt_3(s)tt_2(s)tt_8(s)W(s) + tt_3(s)tt_9(s)]\, W(s).$$

Considering the signal $W(s)$ (see Figure E2.14), we determine that

$$W(s) = tt_5(s)\left[tt_4(s)R_2(s) - H_2(s)W(s)\right] ,$$

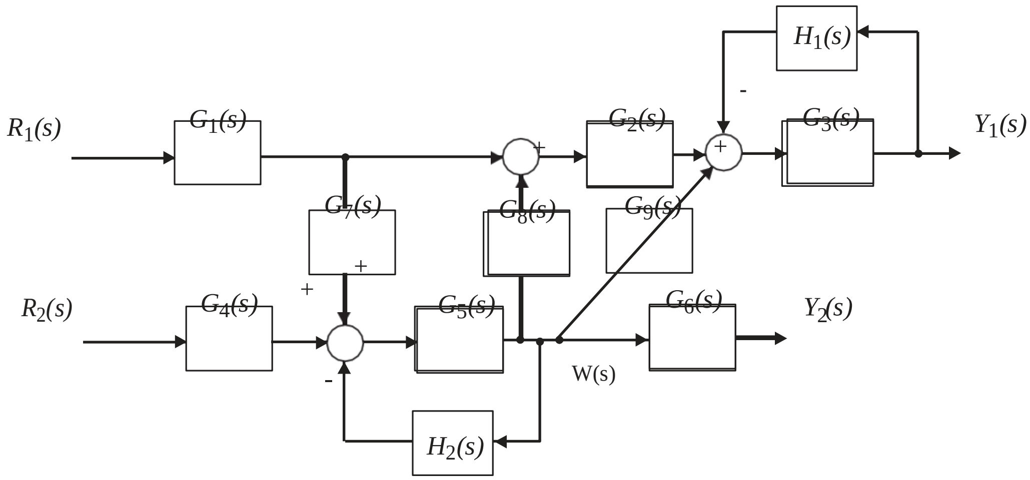


FIGURE E2.14
Block diagram model.

or

$$[1 + G_5(s)H_2(s)]W(s) = G_5(s)G_4(s)R_2(s).$$

Substituting the expression for $W(s)$ into the above equation for $Y_1(s)$ yields

$$\frac{Y_1(s)}{R_2(s)} = \frac{G_2(s)G_3(s)G_4(s)G_5(s)G_8(s) + G_3(s)G_4(s)G_5(s)G_9(s)}{1 + G_3(s)H_1(s) + G_5(s)H_2(s) + G_3(s)G_5(s)H_1(s)H_2(s)}.$$

E2.15 For loop 1, we have

$$R_1 i_1 + L_1 \frac{di_1}{dt} + \frac{1}{C_1}\int (i_1 - i_2)dt + R_2(i_1 - i_2) = v(t) .$$

And for loop 2, we have

$$\frac{1}{C_2}\int i_2 dt + L_2 \frac{di_2}{dt} + R_2(i_2 - i_1) + \frac{1}{C_1}\int (i_2 - i_1)dt = 0 .$$

E2.16 The transfer function from $R(s)$ to $P(s)$ is

$$\frac{P(s)}{R(s)} = \frac{4.2}{s^3 + 2s^2 + 4s + 4.2} .$$

The block diagram is shown in Figure E2.16a. The corresponding signal flow graph is shown in Figure E2.16b for

$$P(s)/R(s) = \frac{4.2}{s^3 + 2s^2 + 4s + 4.2} .$$

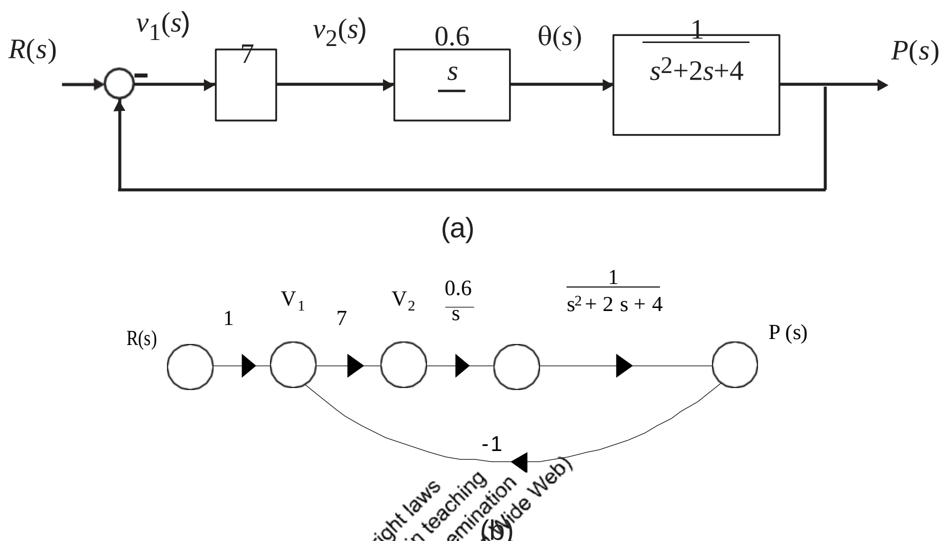


FIGURE E2.16
(a) Block diagram, (b) Signal flow graph.

E2.17 A linear approximation for f is given by

$$\Delta f = \left.\frac{\partial f}{\partial x}\right|_{x=x_o} \Delta x = 2kx_o\Delta x = k\Delta x$$

where $x_o = 1/2$, $\Delta f = f(x) - f(x_o)$, and $\Delta x = x - x_o$.

E2.18 The linear approximation is given by

$$\Delta y = m\Delta x$$

where

$$m = \left.\frac{\partial y}{\partial x}\right|_{x=x_o} .$$

(a) When $x_o = 1$, we find that $y_o = 2.4$, and $y_o = 13.2$ when $x_o = 2$.

(b) The slope m is computed as follows:

$$m = \left.\frac{\partial y}{\partial x}\right|_{x=x_o} = 1 + 4.2x_o^2 .$$

Therefore, $m = 5.2$ at $x_o = 1$, and $m = 18.8$ at $x_o = 2$.

E2.19 The output (with a step input) is

$$Y(s) = \frac{28(s+1)}{s(s+7)(s+2)} .$$

The partial fraction expansion is

$$Y(s) = \frac{2}{s} - \frac{4.8}{s+7} + \frac{2.8}{s+2} .$$

Taking the inverse Laplace transform yields

$$y(t) = 2 - 4.8e^{-7t} + 2.8e^{-2t} .$$

E2.20 The input-output relationship is

$$\frac{V_o}{V} = \frac{A(K-1)}{1+AK}$$

where

$$K = \frac{Z_1}{Z_1 + Z_2} .$$

Assume $A \gg 1$. Then,

$$\frac{V_o}{V} = \frac{K-1}{K} = -\frac{Z_2}{Z_1}$$

where

$$Z_1 = \frac{R_1}{R_1C_1s+1} \quad \text{and} \quad Z_2 = \frac{R_2}{R_2C_2s+1} .$$

Therefore,

$$\frac{V_o(s)}{V(s)} = -\frac{R_2(R_1C_1s+1)}{R_1(R_2C_2s+1)} = -\frac{2(s+1)}{s+2} .$$

E2.21 The equation of motion of the mass m_c is

$$m_c\ddot{x}_p + (b_d + b_s)\dot{x}_p + k_dx_p = b_d\dot{x}_{in} + k_dx_{in} .$$

Taking the Laplace transform with zero initial conditions yields

$$[m_cs^2 + (b_d + b_s)s + k_d]X_p(s) = [b_ds + k_d]X_{in}(s) .$$

So, the transfer function is

$$\frac{X_p(s)}{X_{in}(s)} = \frac{b_ds + k_d}{m_cs^2 + (b_d + b_s)s + k_d} = \frac{0.65s + 1.8}{s^2 + 1.55s + 1.8} .$$

E2.22 The rotational velocity is

$$\omega(s) = \frac{2(s+4)}{(s+5)(s+1)^2}\frac{1}{s} .$$

Expanding in a partial fraction expansion yields

$$\omega(s) = \frac{8}{5}\frac{1}{s} + \frac{1}{40}\frac{1}{s+5} - \frac{3}{2}\frac{1}{(s+1)^2} - \frac{13}{8}\frac{1}{s+1} .$$

Taking the inverse Laplace transform yields

$$\omega(t) = \frac{8}{5} + \frac{1}{40}e^{-5t} - \frac{3}{2}te^{-t} - \frac{13}{8}e^{-t} .$$

E2.23 The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = T(s) = \frac{K_1K_2}{s^2 + (K_1 + K_2K_3 + K_1K_2)s + K_1K_2K_3} .$$

E2.24 Let $x = 0.6$ and $y = 0.8$. Then, with $y = ax^3$, we have

$$0.8 = a(0.6)^3 .$$

Solving for a yields $a = 3.704$. A linear approximation is

$$y - y_o = 3ax_o^2(x - x_o)$$

or $y = 4x - 1.6$, where $y_o = 0.8$ and $x_o = 0.6$.

E2.25 The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = T(s) = \frac{10}{s^2 + 21s + 10} .$$

E2.26 The equations of motion are

$$m_1\ddot{x}_1 + k(x_1 - x_2) = F$$
$$m_2\ddot{x}_2 + k(x_2 - x_1) = 0 .$$

Taking the Laplace transform (with zero initial conditions) and solving for $X_2(s)$ yields

$$X_2(s) = \frac{k}{(m_2s^2 + k)(m_1s^2 + k) - k^2}F(s) .$$

Then, with $m_1 = m_2 = k = 1$, we have

$$X_2(s)/F(s) = \frac{1}{s^2(s^2 + 2)} .$$

E2.27 The transfer function from $T_d(s)$ to $Y(s)$ is

$$Y(s)/T_d(s) = \frac{tt_2(s)}{1 + tt_1 tt_2 H(s)} .$$

E2.28 The transfer function is

$$\frac{V_o(s)}{V(s)} = \frac{R_2 R_4 C}{R_3} s + \frac{R_2 R_4}{R_1 R_3} = 46.08s + 344.91 .$$

E2.29 (a) If

$$tt(s) = \frac{1}{s^2 + 15s + 50} \quad \text{and} \quad H(s) = 2s + 15 ,$$

then the closed-loop transfer function of Figure E2.28(a) and (b) (in Dorf & Bishop) are equivalent.

(b) The closed-loop transfer function is

$$T(s) = \frac{1}{s^2 + 17s + 65} .$$

E2.30 (a) The closed-loop transfer function is

$$T(s) = \frac{tt(s)}{1 + tt(s)} \frac{1}{s} = \frac{15}{s(s^2 + 5s + 30)} \quad \text{where} \quad tt(s) = \frac{15}{s^2 + 5s + 15} .$$

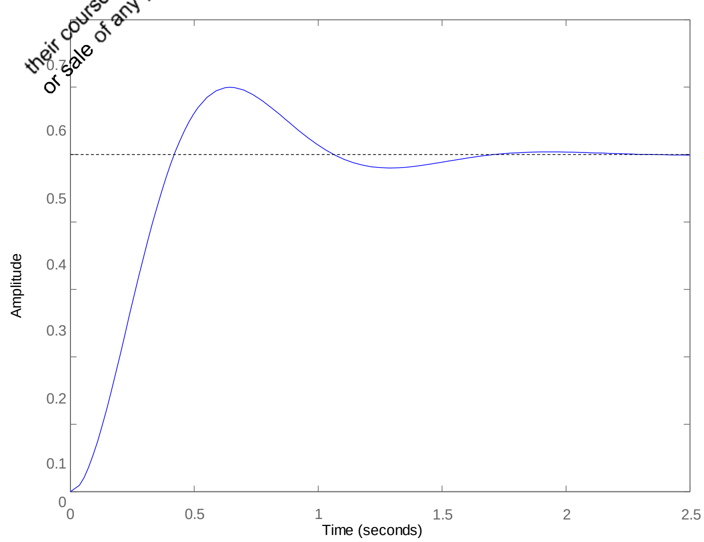


FIGURE E2.30
Step response.

(b) The output $Y(s)$ (when $R(s) = 1/s$) is

$$Y(s) = \frac{0.5}{s} + \frac{-0.25 + 0.1282j}{s + 2.5 - 4.8734} + \frac{-0.25 - 0.1282j}{s + 2.5 + 4.8734j}$$

or

$$Y(s) = \frac{1}{2}\left[\frac{1}{s} - \frac{s+5}{s^2 + 5s + 30}\right]$$

(c) The plot of $y(t)$ is shown in Figure E2.30. The output is given by

$$y(t) = 0.5(1 - 1.1239e^{-2.5t}\sin(4.8734t + 1.0968));$$

E2.31 The partial fraction expansion is

$$V(s) = \frac{a}{s + p_1} + \frac{b}{s + p_2}$$

where $p_1 = 4 - 22j$ and $p_2 = 4 + 22j$. Then, the residues are

$$a = -11.37j \quad b = 11.37j\ .$$

The inverse Laplace transform is

$$v(t) = -11.37je^{(-4+22j)t} + 11.37je^{(-4-22j)t} = 22.73e^{-4t}\sin 22t\ .$$

Problems

P2.1 The integrodifferential equations, obtained by Kirchoff's voltage law to each loop, are as follows:

$$R_2 i_1 + \frac{1}{C_1}\int i_1\,dt + L_1\frac{d(i_1 - i_2)}{dt} + R_1(i_1 - i_2) = v(t) \quad \text{(loop 1)}$$

and

$$R_3 i_2 + \frac{1}{C_2}\int i_2\,dt + R_1(i_2 - i_1) + L_1\frac{d(i_2 - i_1)}{dt} = 0 \quad \text{(loop 2)}.$$

P2.2 The differential equations describing the system can be obtained by using a free-body diagram analysis of each mass. For mass 1 and 2 we have

$$M_1\ddot{y}_1 + k_{12}(y_1 - y_2) + b\dot{y}_1 + k_1 y_1 = F(t)$$
$$M_2\ddot{y}_2 + k_{12}(y_2 - y_1) = 0\,.$$

Using a force-current analogy, the analagous electric circuit is shown in Figure P2.2, where $C_i \to M_i$, $L_1 \to 1/k_1$, $L_{12} \to 1/k_{12}$, and $R \to 1/b$.

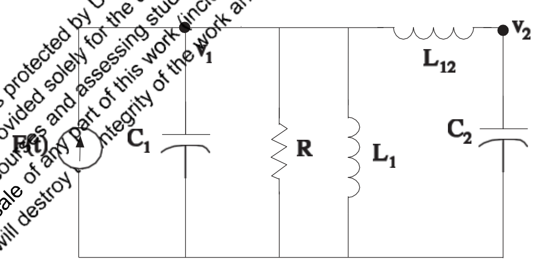


FIGURE P2.2
Analagous electric circuit.

P2.3 The differential equations describing the system can be obtained by using a free-body diagram analysis of each mass. For mass 1 and 2 we have

$$M\ddot{x}_1 + kx_1 + k(x_1 - x_2) = F(t)$$
$$M\ddot{x}_2 + k(x_2 - x_1) + b\dot{x}_2 = 0\,.$$

Using a force-current analogy, the analagous electric circuit is shown in Figure P2.3, where

$$C \to M \qquad L \to 1/k \qquad R \to 1/b\,.$$

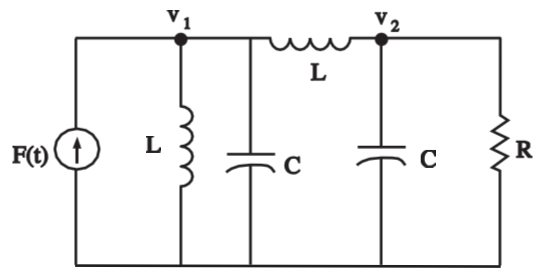


FIGURE P2.3
Analagous electric circuit.

P2.4 (a) The linear approximation around $v_{in} = 0$ is $v_o = 0v_{in}$, see Figure P2.4(a).

(b) The linear approximation around $v_{in} = 1$ is $v_o = 2v_{in} - 1$, see Figure P2.4(b).

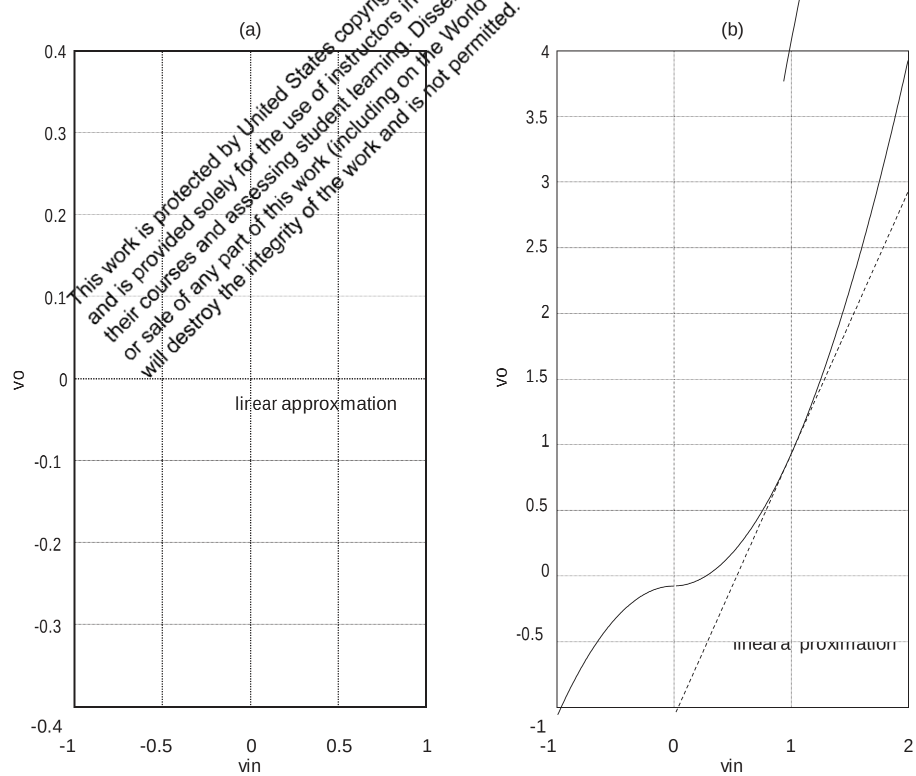


FIGURE P2.4
Nonlinear functions and approximations.

P2.5 Given

$$Q = K(P_1 - P_2)^{1/2} .$$

Let $\delta P = P_1 - P_2$ and δP_o = operating point. Using a Taylor series expansion of Q, we have

$$Q = Q_o + \frac{\partial Q}{\partial \delta P}\bigg|_{\delta P = \delta P_o} (\delta P - \delta P_o) + \cdots$$

where

$$Q_o = K\delta P_o^{1/2} \quad \text{and} \quad \frac{\partial Q}{\partial \delta P}\bigg|_{\delta P = \delta P_o} = \frac{K}{2}\delta P_o^{-1/2} .$$

Define $\Delta Q = Q - Q_o$ and $\Delta P = \delta P - \delta P_o$. Then, dropping higher-order terms in the Taylor series expansion yields

$$\Delta Q = m\Delta P$$

where

$$m = \frac{K}{2\delta P_o^{1/2}} .$$

P2.6 From P2.1 we have

$$R_2 i_1 + \frac{1}{C_1}\int i_1 dt + L_1 \frac{d(i_1 - i_2)}{dt} + R_1(i_1 - i_2) = v(t)$$

and

$$R_3 i_2 + \frac{1}{C_2}\int i_2 dt + R_1(i_2 - i_1) + L_1 \frac{d(i_2 - i_1)}{dt} = 0 .$$

Taking the Laplace transform and using the fact that the initial voltage across C_2 is 10v yields

$$[R_2 + \frac{1}{C_1 s} + L_1 s + R_1]I_1(s) + [-R_1 - L_1 s]I_2(s) = 0$$

and

$$[-R_1 - L_1 s]I_1(s) + [L_1 s + R_3 + \frac{1}{C_2 s} + R_1]I_2(s) = -\frac{10}{s} .$$

Rewriting in matrix form we have

$$\begin{bmatrix} R_2 + \frac{1}{C_1 s} + L_1 s + R_1 & -R_1 - L_1 s \\ -R_1 - L_1 s & L_1 s + R_3 + \frac{1}{C_2 s} + R_1 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ -10/s \end{bmatrix} .$$

Solving for I_2 yields

$$\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} L_1 s + R_3 + \frac{1}{C_2 s} + R_1 & R_1 + L_1 s \\ R_1 + L_1 s & R_2 + \frac{1}{C_1 s} + L_1 s + R_1 \end{bmatrix} \begin{bmatrix} 0 \\ -10/s \end{bmatrix}.$$

or

$$I_2(s) = \frac{-10(R_2 + 1/C_1 s + L_1 s + R_1)}{s\Delta}$$

where

$$\Delta = \left(R_2 + \frac{1}{C_1 s} + L_1 s + R_1\right)\left(L_1 s + R_3 + \frac{1}{C_2 s} + R_1\right) - (R_1 + L_1 s)^2 .$$

P2.7 Consider the differentiating op-amp circuit in Figure P2.7. For an ideal op-amp, the voltage gain (as a function of frequency) is

$$V_2(s) = -\frac{Z_2(s)}{Z_1(s)} V_1(s),$$

where

$$Z_1 = \frac{R_1}{1 + R_1 C s}$$

and $Z_2 = R_2$ are the respective circuit impedances. Therefore, we obtain

$$V_2(s) = -\left[\frac{R_2(1 + R_1 C s)}{R_1}\right] V_1(s).$$

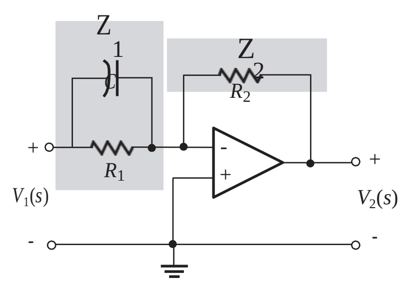


FIGURE P2.7
Differentiating op-amp circuit.

P2.8 Let

$$\Delta = \begin{vmatrix} tt_1 + Cs & -Cs & -tt_1 \\ -Cs & tt_2 + 2Cs & -Cs \\ -tt_1 & -Cs & Cs + tt_1 \end{vmatrix} .$$

Then,

$$V_j = \frac{\Delta_{ij}}{\Delta} I_1 \quad \text{or} \quad \frac{V_3}{V_1} = \frac{\Delta_{13} I_1 / \Delta}{\Delta_{11} I_1 / \Delta} .$$

Therefore, the transfer function is

$$T(s) = \frac{V_3}{V_1} = \frac{\Delta_{13}}{\Delta_{11}} = \frac{\begin{vmatrix} -Cs & 2Cs + tt_2 \\ -tt_1 & -Cs \end{vmatrix}}{\begin{vmatrix} 2Cs + tt_2 & -Cs \\ -Cs & Cs + tt_1 \end{vmatrix}} .$$

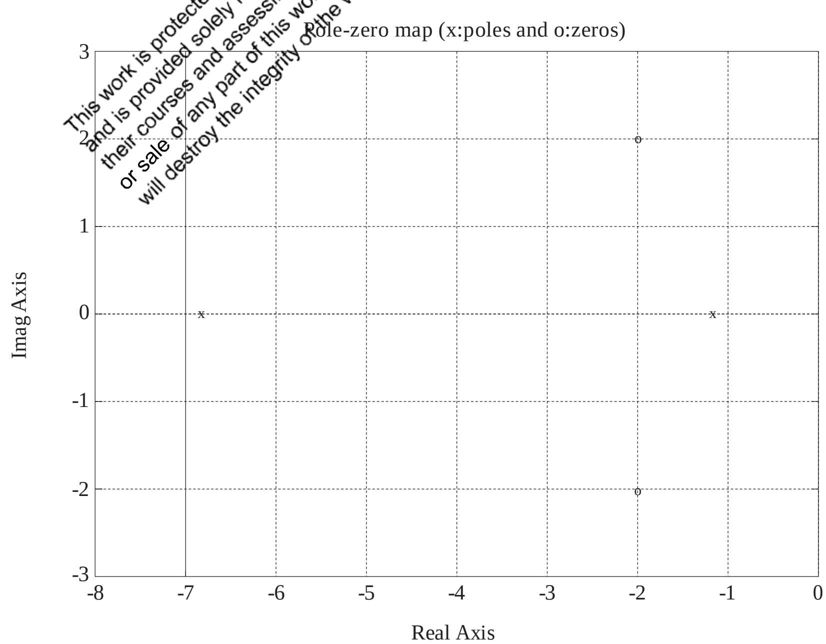


FIGURE P2.8
Pole-zero map.

$$= \frac{C^2R_1R_2s^2 + 2CR_2s + 1}{C^2R_1R_2s^2 + (2R_2 + R_1)Cs + 1} .$$

Using $R_1 = 1.0$, $R_2 = 0.5$, and $C = 0.5$, we have

$$T(s) = \frac{s^2 + 4s + 8}{s^2 + 8s + 8} = \frac{(s + 2 + 2j)(s + 2 - 2j)}{(s + 4 + \sqrt{8})(s + 4 - \sqrt{8})} .$$

The pole-zero map is shown in Figure P2.8.

P2.9 From P2.3 we have

$$M\ddot{x}_1 + kx_1 + k(x_1 - x_2) = F(t)$$
$$M\ddot{x}_2 + k(x_2 - x_1) + b\dot{x}_2 = 0 .$$

Taking the Laplace transform of both equations and writing the result in matrix form, it follows that

$$\begin{bmatrix} Ms^2 + 2k & -k \\ -k & Ms^2 + bs + k \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix} ,$$

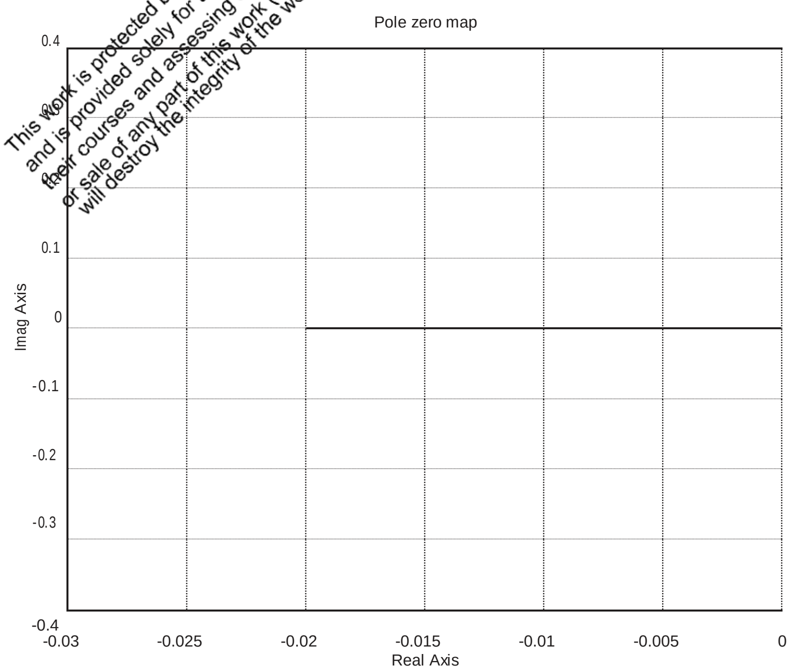


FIGURE P2.9
Pole-zero map.

or

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} Ms^2 + bs + k & k \\ k & Ms^2 + 2k \end{bmatrix}\begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

where $\Delta = (Ms^2 + bs + k)(Ms^2 + 2k) - k^2$. So,

$$tt(s) = \frac{X_1(s)}{F(s)} = \frac{Ms^2 + bs + k}{\Delta} \ .$$

When $b/k = 1$, $M = 1$, $b^2/Mk = 0.04$, we have

$$tt(s) = \frac{s^2 + 0.04s + 0.04}{s^4 + 0.04s^3 + 0.12s^2 + 0.0032s + 0.0016} \ .$$

The pole-zero map is shown in Figure P2.9.

P2.10 From P2.2 we have

$$M_1\ddot{y}_1 + k_{12}(y_1 - y_2) + b\dot{y}_1 + k_1y_1 = F(t)$$
$$M_2\ddot{y}_2 + k_{12}(y_2 - y_1) = 0 \ .$$

Taking the Laplace transform of both equations and writing the result in matrix form, it follows that

$$\begin{bmatrix} M_1s^2 + bs + k_1 + k_{12} & -k_{12} \\ -k_{12} & M_2s^2 + k_{12} \end{bmatrix}\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} M_2s^2 + k_{12} & k_{12} \\ k_{12} & M_1s^2 + bs + k_1 + k_{12} \end{bmatrix}\begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

where

$$\Delta = (M_2s^2 + k_{12})(M_1s^2 + bs + k_1 + k_{12}) - k_{12}^2 \ .$$

So, when $f(t) = a \sin \omega_o t$, we have that $Y_1(s)$ is given by

$$Y_1(s) = \frac{aM_2\omega_o(s^2 + k_{12}/M_2)}{(s^2 + \omega_o^2)\Delta(s)} \ .$$

For motionless response (in the steady-state), set the zero of the transfer function so that

$$(s^2 + \frac{k_{12}}{M_2}) = s^2 + \omega_o^2 \quad \text{or} \quad \omega_o^2 = \frac{k_{12}}{M_2} \ .$$

P2.11 The transfer functions from $V_c(s)$ to $V_d(s)$ and from $V_d(s)$ to $\theta(s)$ are:

$$V_d(s)/V_c(s) = \frac{K_1K_2}{(L_qs + R_q)(L_cs + R_c)} \text{, and}$$

$$\theta(s)/V_d(s) = \frac{K_m}{(Js^2 + fs)((L_d + L_a)s + R_d + R_a) + K_3K_ms} .$$

The block diagram for $\theta(s)/V_c(s)$ is shown in Figure P2.11, where

$$\theta(s)/V_c(s) = \frac{\theta(s)}{V_d(s)}\frac{V_d(s)}{V_c(s)} = \frac{K_1K_2K_m}{\Delta(s)} ,$$

where

$$\Delta(s) = s(L_cs + R_c)(L_qs + R_q)((Js + b)((L_d + L_a)s + R_d + R_a) + K_mK_3) .$$

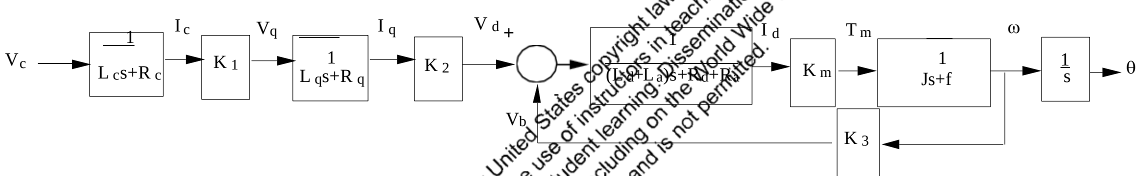


FIGURE P2.11
Block diagram.

P2.12 The open-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K}{s + 50} .$$

With $R(s) = 1/s$, we have

$$Y(s) = \frac{K}{s(s + 50)} .$$

The partial fraction expansion is

$$Y(s) = \frac{K}{50}\left[\frac{1}{s} - \frac{1}{s + 50}\right] ,$$

and the inverse Laplace transform is

$$y(t) = \frac{K}{50}\left[1 - e^{-50t}\right] ,$$

As $t \to \infty$, it follows that $y(t) \to K/50$. So we choose $K = 50$ so that $y(t)$

approaches 1. Alternatively we can use the final value theorem to obtain

$$y(t)_{t\to\infty} = \lim_{s\to 0} sY(s) = \frac{K}{50} .$$

It follows that choosing $K = 50$ leads to $y(t) \to 1$ as $t \to \infty$.

P2.13 The motor torque is given by

$$\begin{aligned} T_m(s) &= (J_m s^2 + b_m s)\theta_m(s) + (J_L s^2 + b_L s)n\theta_L(s) \\ &= n((J_m s^2 + b_m s)/n^2 + J_L s^2 + b_L s)\theta_L(s) \end{aligned}$$

where

$$n = \theta_L(s)/\theta_m(s) = \text{gear ratio} .$$

But

$$T_m(s) = K_m I_g(s)$$

and

$$I_g(s) = \frac{1}{(L_g + L_f)s + R_g + R_f} V_g(s) ,$$

and

$$V_g(s) = K_g I_f(s) = \frac{K_g}{R_f + L_f s} V_f(s) .$$

Combining the above expressions yields

$$\frac{\theta_L(s)}{V_f(s)} = \frac{K_g K_m}{n\Delta_1(s)\Delta_2(s)} .$$

where

$$\Delta_1(s) = J_L s^2 + b_L s + \frac{J_m s^2 + b_m s}{n^2}$$

and

$$\Delta_2(s) = (L_g s + L_f s + R_g + R_f)(R_f + L_f s) .$$

P2.14 For a field-controlled dc electric motor we have

$$\omega(s)/V_f(s) = \frac{K_m/R_f}{Js + b} .$$

With a step input of $V_f(s) = 80/s$, the final value of $\omega(t)$ is

$$\omega(t)_{t\to\infty} = \lim_{s\to 0} s\omega(s) = \frac{80K_m}{R_f\, b} = 2.4 \quad \text{or} \quad \frac{K_m}{R_f\, b} = 0.03 \ .$$

Solving for $\omega(t)$ yields

$$\omega(t) = \frac{80K_m}{R_f J} \mathcal{L}^{-1}\left[\frac{1}{s(s+b/J)}\right] = \frac{80K_m}{R_f\, b}\left(1 - e^{-(b/J)t}\right) = 2.4\left(1 - e^{-(b/J)t}\right) .$$

At $t = 1/2$, $\omega(t) = 1$, so

$$\omega(1/2) = 2.4(1 - e^{-(b/J)t}) = 1 \quad \text{implies} \quad b/J = 1.08 \text{ sec} .$$

Therefore,

$$\omega(s)/V_f(s) = \frac{0.0324}{s(s+1.08)} .$$

P2.15 Summing the forces in the vertical direction and using Newton's Second Law we obtain

$$\ddot{x} + \frac{k}{m} x = 0 .$$

The system has no damping and no external inputs. Taking the Laplace transform yields

$$X(s) = \frac{x_0 s}{s^2 + k/m} ,$$

where we used the fact that $x(0) = x_0$ and $\dot{x}(0) = 0$. Then taking the inverse Laplace transform yields

$$x(t) = x_0 \cos \sqrt{\frac{k}{m}}\, t .$$

P2.16 (a) For mass 1 and 2, we have

$$M_1\ddot{x}_1 + K_1(x_1 - x_2) + b_1(\dot{x}_3 - \dot{x}_1) = 0$$
$$M_2\ddot{x}_2 + K_2(x_2 - x_3) + b_2(\dot{x}_3 - \dot{x}_2) + K_1(x_2 - x_1) = 0 .$$

(b) Taking the Laplace transform yields

$$(M_1s^2 + b_1s + K_1)X_1(s) - K_1X_2(s) = b_1sX_3(s)$$
$$-K_1X_1(s) + (M_2s^2 + b_2s + K_1 + K_2)X_2(s) = (b_2s + K_2)X_3(s) .$$

(c) Let

$$G_1(s) = K_2 + b_2s$$

$$tt_2(s) = 1/p(s)$$
$$tt_3(s) = 1/q(s)$$
$$tt_4(s) = sb_1\ ,$$

where

$$p(s) = s^2M_2 + sf_2 + K_1 + K_2$$

and

$$q(s) = s^2M_1 + sf_1 + K_1\ .$$

The signal flow graph is shown in Figure P2.16.

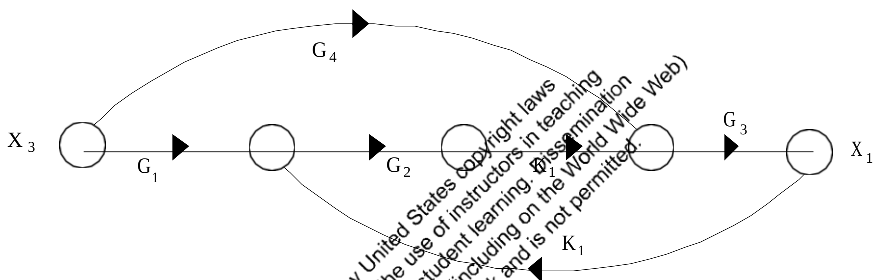


FIGURE P2.16
Signal flow graph.

(d) The transfer function from $X_3(s)$ to $X_1(s)$ is

$$\frac{X_1(s)}{X_3(s)} = \frac{K_1tt_1(s)tt_2(s)tt_3(s) + tt_4(s)tt_3(s)}{1 - K_1^2tt_2(s)tt_3(s)}\ .$$

P2.17 Using Cramer's rule, we have

$$\begin{bmatrix} 1 & 1.5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} 4 & -1.5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

where $\Delta = 4(1) - 2(1.5) = 1$. Therefore,

$$x_1 = \frac{4(6) - 1.5(11)}{1} = 7.5 \quad \text{and} \quad x_2 = \frac{-2(6) + 1(11)}{1} = -1\ .$$

The signal flow graph is shown in Figure P2.17.

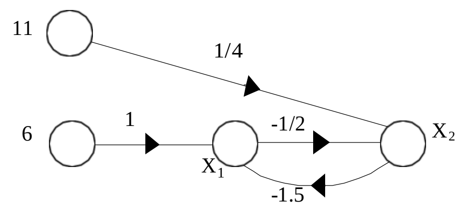


FIGURE P2.17
Signal flow graph.

So,

$$x_1 = \frac{6(1) - 1.5\left(\frac{11}{4}\right)}{1 - \frac{3}{4}} = 7.5 \quad \text{and} \quad x_2 = \frac{11\left(\frac{1}{4}\right) + \frac{-1}{2}(6)}{1 - \frac{3}{4}} = -1 \; .$$

P2.18 The signal flow graph is shown in Figure P2.18.

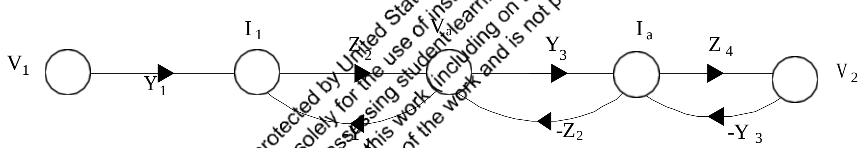


FIGURE P2.18
Signal flow graph.

The transfer function is

$$\frac{V_2(s)}{V_1(s)} = \frac{Y_1Z_2Y_3Z_4}{1 + Y_1Z_2 + Y_3Z_2 + Y_3Z_4 + Y_1Z_2Z_4Y_3} \; .$$

P2.19 (a) Assume $R_g \gg R_s$ and $R_s \gg R_1$. Then $R_s = R_1 + R_2 \approx R_2$, and

$$v_{gs} = v_{in} - v_o \, ,$$

where we neglect i_{in}, since $R_g \gg R_s$. At node S, we have

$$\frac{v_o}{R_s} = g_m v_{gs} = g_m(v_{in} - v_o) \quad \text{or} \quad \frac{v_o}{v_{in}} = \frac{g_mR_s}{1 + g_mR_s} \; .$$

(b) With $g_mR_s = 20$, we have

$$\frac{v_o}{v_{in}} = \frac{20}{21} = 0.95 \; .$$

(c) The block diagram is shown in Figure P2.19.

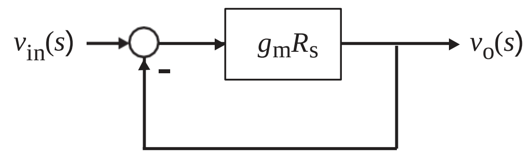


FIGURE P2.19
Block diagram model.

P2.20 From the geometry we find that

$$\Delta z = k\frac{l_1 - l_2}{l_1}(x - y) - \frac{l_2}{l_1}y .$$

The flow rate balance yields

$$A\frac{dy}{dt} = p\Delta z \quad \text{which implies} \quad Y(s) = \frac{p\Delta Z(s)}{As} .$$

By combining the above results it follows that

$$Y(s) = \frac{p}{As}\left[k\frac{l_1 - l_2}{l_1}(X(s) - Y(s)) - \frac{l_2}{l_1}Y(s)\right] .$$

Therefore, the signal flow graph is shown in Figure P2.20. Using Mason's

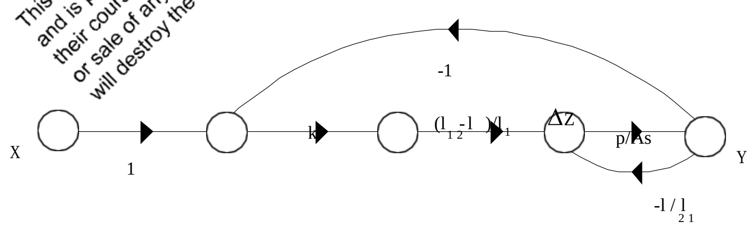


FIGURE P2.20
Signal flow graph.

gain formula we find that the transfer function is given by

$$\frac{Y(s)}{X(s)} = \frac{\frac{k(l_1 - l_2)p}{l_1 As}}{1 + \frac{l_2 p}{l_1 As} + \frac{k(l_1 - l_2)p}{l_1 As}} = \frac{K_1}{s + K_2 + K_1} ,$$

where

$$K_1 = \frac{k(l_1 - l_2)p}{l_1 A}p \quad \text{and} \quad K_2 = \frac{l_2 p}{l_1 A} .$$

P2.21 (a) The equations of motion for the two masses are

$$ML^2\ddot{\theta}_1 + MgL\theta_1 + k\left(\frac{L}{2}\right)^2(\theta_1 - \theta_2) = \frac{L}{2}f(t)$$

$$ML^2\ddot{\theta}_2 + MgL\theta_2 + k\left(\frac{L}{2}\right)^2(\theta_2 - \theta_1) = 0\,.$$

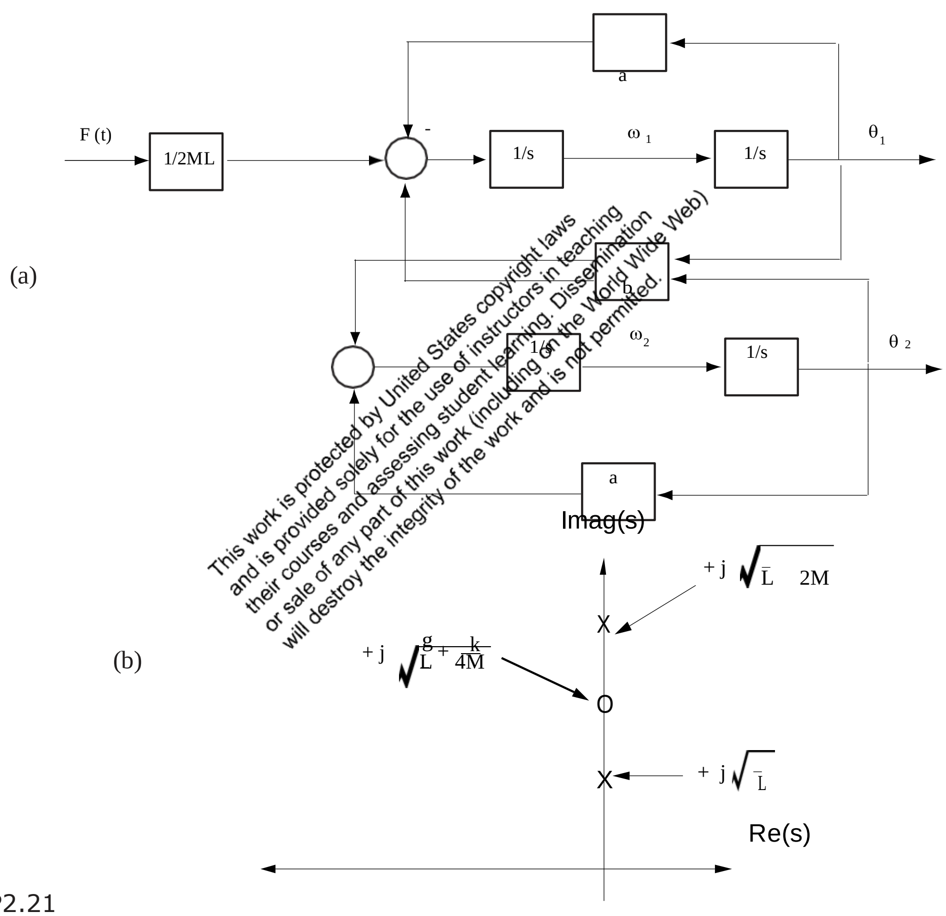


FIGURE P2.21
(a) Block diagram. (b) Pole-zero map.

With $\dot{\theta}_1 = \omega_1$ and $\dot{\theta}_2 = \omega_2$, we have

$$\dot{\omega}_1 = -\left(\frac{g}{L} + \frac{k}{4M}\right)\theta_1 + \frac{k}{4M}\theta_2 + \frac{f(t)}{2ML}$$

$$\dot{\omega}_2 = \frac{k}{4M}\theta_1 - \left(\frac{g}{L} + \frac{k}{4M}\right)\theta_2 .$$

(b) Define $a = g/L + k/4M$ and $b = k/4M$. Then

$$\frac{\theta_1(s)}{F(s)} = \frac{1}{2ML}\frac{s^2 + a}{(s^2 + a)^2 - b^2} .$$

(c) The block diagram and pole-zero map are shown in Figure P2.21.

P2.22 For a noninverting op-amp circuit, depicted in Figure P2.22a, the voltage gain (as a function of frequency) is

$$V_o(s) = \frac{Z_1(s) + Z_2(s)}{Z_1(s)} V_{in}(s),$$

where $Z_1(s)$ and $Z_2(s)$ are the impedances of the respective circuits. In

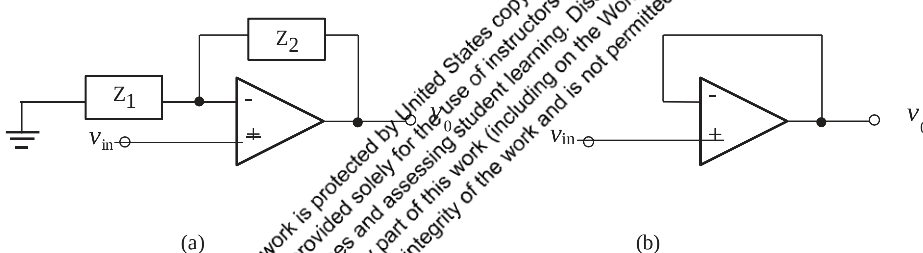


FIGURE P2.22
(a) Noninverting op-amp circuit. (b) Voltage follower circuit.

the case of the voltage follower circuit, shown in Figure P2.22b, we have $Z_1 = \infty$ (open circuit) and $Z_2 = 0$. Therefore, the transfer function is

$$\frac{V_o(s)}{V_{in}(s)} = \frac{Z_1}{Z_1} = 1.$$

P2.23 The input-output ratio, V_{ce}/V_{in}, is found to be

$$\frac{V_{ce}}{V_{in}} = \frac{\beta(R - 1) + h_{ie}R_f}{-\beta h_{re} + h_{ie}(-h_{oe} + R_f)} .$$

P2.24 (a) The voltage gain is given by

$$\frac{v_o}{v_{in}} = \frac{R_L\beta_1\beta_2(R_1 + R_2)}{(R_1 + R_2)(R_g + h_{ie1}) + R_1(R_1 + R_2)(1 + \beta_1) + R_1R_L\beta_1\beta_2} .$$

(b) The current gain is found to be

$$\frac{i_{c2}}{i_{b1}} = \beta_1 \beta_2 .$$

(c) The input impedance is

$$\frac{v_{in}}{i_{b1}} = \frac{(R_1 + R_2)(R_g + h_{ie1}) + R_1(R_1 + R_2)(1 + \beta_1) + R_1 R_L \beta_1 \beta_2}{R_1 + R_2} ,$$

and when $\beta_1\beta_2$ is very large, we have the approximation

$$\frac{v_{in}}{i_{b1}} \approx \frac{R_L R_1 \beta_1 \beta_2}{R_1 + R_2} .$$

P2.25 The transfer function from $R(s)$ and $T_d(s)$ to $Y(s)$ is given by

$$Y(s) = G(s)\left[R(s) - \frac{1}{G(s)}\left(G(s)R(s) + T_d(s)\right)\right] + T_d(s) + G(s)R(s)$$

$$= G(s)R(s) .$$

Thus,

$$Y(s)/R(s) = G(s) .$$

Also, we have that

$$Y(s) = 0 .$$

when $R(s) = 0$. Therefore, the effect of the disturbance, $T_d(s)$, is eliminated.

P2.26 The equations of motion for the two mass model of the robot are

$$M\ddot{x} + b(\dot{x} - \dot{y}) + k(x - y) = F(t)$$
$$m\ddot{y} + b(\dot{y} - \dot{x}) + k(y - x) = 0 .$$

Taking the Laplace transform and writing the result in matrix form yields

$$\begin{bmatrix} Ms^2 + bs + k & -(bs + k) \\ -(bs + k) & ms^2 + bs + k \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix} .$$

Solving for $Y(s)$ we find that

$$\frac{Y(s)}{F(s)} = \frac{\frac{1}{mM}(bs + k)}{s^2\left[s^2 + \left(1 + \frac{m}{M}\right)\left(\frac{b}{m}s + \frac{k}{m}\right)\right]} .$$

P2.27 The describing equation of motion is

$$m\ddot{z} = mg - k\frac{i^2}{z^2} .$$

Defining

$$f(z,i) = g - \frac{ki^2}{mz^2}$$

leads to

$$\ddot{z} = f(z, i) .$$

The equilibrium condition for i_o and z_o, found by solving the equation of motion when

$$\dot{z} = \ddot{z} = 0 ,$$

is

$$\frac{ki_o^2}{mg} = z_o^2 .$$

We linearize the equation of motion using a Taylor series approximation. With the definitions

$$\Delta z = z - z_o \quad \text{and} \quad \Delta i = i - i_o ,$$

we have $\Delta\dot{z} = \dot{z}$ and $\Delta\ddot{z} = \ddot{z}$. Therefore,

$$\Delta\ddot{z} = f(z,i) = f(z_o, i_o) + \left.\frac{\partial f}{\partial z}\right|_{\substack{z=z_o \\ i=i_o}} \Delta z + \left.\frac{\partial f}{\partial i}\right|_{\substack{z=z_o \\ i=i_o}} \Delta i + \cdots$$

But $f(z_o, i_o) = 0$, and neglecting higher-order terms in the expansion yields

$$\Delta\ddot{z} = \frac{2ki_o^2}{mz_o^3}\Delta z - \frac{2ki_o}{mz_o^2}\Delta i .$$

Using the equilibrium condition which relates z_o to i_o, we determine that

$$\Delta\ddot{z} = \frac{2g}{z_o}\Delta z - \frac{g}{i_o}\Delta i .$$

Taking the Laplace transform yields the transfer function (valid around the equilibrium point)

$$\frac{\Delta Z(s)}{\Delta I(s)} = \frac{-g/i_o}{s^2 - 2g/z_o} .$$

P2.28 The signal flow graph is shown in Figure P2.28.

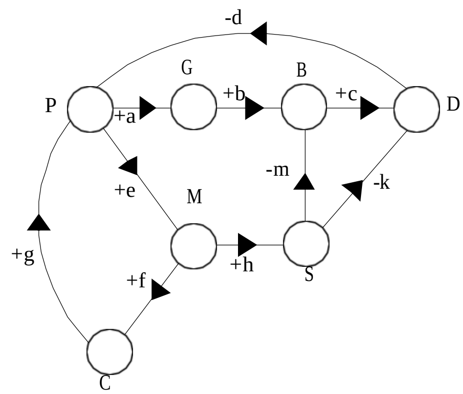


FIGURE P2.28
Signal flow graph.

(a) The PGBDP loop gain is equal to -abcd. This is a negative transmission since the population produces garbage which increases bacteria and leads to diseases, thus reducing the population.

(b) The PMCP loop gain is equal to +efg. This is a positive transmission since the population leads to modernization which encourages immigration, thus increasing the population.

(c) The PMSDP loop gain is equal to +ehkd. This is a positive transmission since the population leads to modernization and an increase in sanitation facilities which reduces diseases, thus reducing the rate of decreasing population.

(d) The PMSBDP loop gain is equal to +ehmcd. This is a positive transmission by similar argument as in (3).

P2.29 Assume the motor torque is proportional to the input current

$$T_m = ki \,.$$

Then, the equation of motion of the beam is

$$J\ddot{\varphi} = ki \,,$$

where J is the moment of inertia of the beam and shaft (neglecting the inertia of the ball). We assume that forces acting on the ball are due to gravity and friction. Hence, the motion of the ball is described by

$$m\ddot{x} = mg\varphi - b\dot{x}$$

where m is the mass of the ball, b is the coefficient of friction, and we have assumed small angles, so that $\sin\varphi \approx \varphi$. Taking the Laplace transfor of both equations of motion and solving for $X(s)$ yields

$$X(s)/I(s) = \frac{gk/J}{s^2(s^2 + b/m)} .$$

P2.30 Given

$$H(s) = \frac{k}{\tau s + 1}$$

where $\tau = 5\mu s = 4 \times 10^{-6}$ seconds and $0.999 \leq k < 1.001$. The step response is

$$Y(s) = \frac{k}{\tau s + 1} \cdot \frac{1}{s} = \frac{k}{s} - \frac{k}{s + 1/\tau} .$$

Taking the inverse Laplace transform yields

$$y(t) = k - ke^{-t/\tau} = k(1 - e^{-t/\tau}) .$$

The final value is k. The time it takes to reach 98% of the final value is $t = 19.57\mu s$ independent of k.

P2.31 From the block diagram we have

$$Y_1(s) = tt_2(s)[tt_1(s)E_1(s) + tt_3(s)E_2(s)]$$
$$= tt_2(s)tt_1(s)[R_1(s) - H_1(s)Y_1(s)] + tt_2(s)tt_3(s)E_2(s) .$$

Therefore,

$$Y_1(s) = \frac{tt_1(s)tt_2(s)}{1 + tt_1(s)tt_2(s)H_1(s)} R_1(s) + \frac{tt_2(s)tt_3(s)}{1 + tt_1(s)tt_2(s)H_1(s)} E_2(s) .$$

And, computing $E_2(s)$ (with $R_2(s) = 0$) we find

$$E_2(s) = H_2(s)Y_2(s) = H_2(s)tt_6(s) \left[\frac{tt_4(s)}{tt_2(s)} Y_1(s) + tt_5(s)E_2(s) \right]$$

or

$$E_2(s) = \frac{tt_4(s)tt_6(s)H_2(s)}{tt_2(s)(1 - tt_5(s)tt_6(s)H_2(s))} Y_1(s) .$$

Substituting $E_2(s)$ into equation for $Y_1(s)$ yields

$$Y_1(s) = \frac{tt_1(s)tt_2(s)}{1 + tt_1(s)tt_2(s)H_1(s)} R_1(s)$$

$$+ \frac{tt_3(s)tt_4(s)tt_6(s)H_2(s)}{(1 + tt_1(s)tt_2(s)H_1(s))(1 - tt_5(s)tt_6(s)H_2(s))} \quad Y_1(s) \ .$$

Finally, solving for $Y_1(s)$ yields

$$Y_1(s) = T_1(s)R_1(s)$$

where

$$T_1(s) = \frac{tt_1(s)tt_2(s)(1 - tt_5(s)tt_6(s)H_2(s))}{(1 + tt_1(s)tt_2(s)H_1(s))(1 - tt_5(s)tt_6(s)H_2(s)) - tt_3(s)tt_4(s)tt_6(s)H_2(s)} \ .$$

Similarly, for $Y_2(s)$ we obtain

$$Y_2(s) = T_2(s)R_1(s) \ .$$

where

$$T_2(s) = \frac{tt_1(s)tt_4(s)tt_6(s)}{(1 + tt_1(s)tt_2(s)H_1(s))(1 - tt_5(s)tt_6(s)H_2(s)) - tt_3(s)tt_4(s)tt_6(s)H_2(s)} \ .$$

P2.32 The signal flow graph shows three loops:

$$L_1 = -tt_1tt_3tt_4H_2$$
$$L_2 = -tt_2tt_5tt_6H_1$$
$$L_3 = -H_1tt_8tt_6tt_2tt_7tt_4H_2tt_1 \ .$$

The transfer function Y_2/R_1 is found to be

$$\frac{Y_2(s)}{R_1(s)} = \frac{tt_1tt_8tt_6\Delta_1 - tt_2tt_5tt_6\Delta_2}{1 - (L_1 + L_2 + L_3) + (L_1L_2)} \ ,$$

where for path 1

$$\Delta_1 = 1$$

and for path 2

$$\Delta_2 = 1 - L_1 \ .$$

Since we want Y_2 to be independent of R_1, we need $Y_2/R_1 = 0$. Therefore, we require

$$tt_1tt_8tt_6 - tt_2tt_5tt_6(1 + tt_1tt_3tt_4H_2) = 0 \ .$$

P2.33 The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G_3(s)G_1(s)(G_2(s) + K_5K_6)}{1 - G_3(s)(H_1(s) + K_6) + G_3(s)G_1(s)(G_2(s) + K_5K_6)(H_2(s) + K_4)} .$$

P2.34 The equations of motion are

$$m_1\ddot{y}_1 + b(\dot{y}_1 - \dot{y}_2) + k_1(y_1 - y_2) = 0$$
$$m_2\ddot{y}_2 + b(\dot{y}_2 - \dot{y}_1) + k_1(y_2 - y_1) + k_2y_2 = k_2x$$

Taking the Laplace transform yields

$$(m_1s^2 + bs + k_1)Y_1(s) - (bs + k_1)Y_2(s) = 0$$
$$(m_2s^2 + bs + k_1 + k_2)Y_2(s) - (bs + k_1)Y_1(s) = k_2X(s)$$

Therefore, after solving for $Y_1(s)/X(s)$, we have

$$\frac{Y_2(s)}{X(s)} = \frac{k_2(bs + k_1)}{(m_1s^2 + bs + k_1)(m_2s^2 + bs + k_1 + k_2) - (bs + k_1)^2} .$$

P2.35 (a) We can redraw the block diagram as shown in Figure P2.35. Then,

$$T(s) = \frac{K_1/s(s + 1)}{1 + K_1(1 + K_2s)/s(s + 1)} = \frac{K_1}{s^2 + (1 + K_2K_1)s + K_2} .$$

(b) The signal flow graph reveals two loops (both touching):

$$L_1 = \frac{-K_1}{s(s + 1)} \quad \text{and} \quad L_2 = \frac{-K_1K_2}{s + 1} .$$

Therefore,

$$T(s) = \frac{K_1/s(s + 1)}{1 + K_1/s(s + 1) + K_1K_2/(s + 1)} = \frac{K_1}{s^2 + (1 + K_2K_1)s + K_1} .$$

(c) We want to choose K_1 and K_2 such that

$$s^2 + (1 + K_2K_1)s + K_1 = s^2 + 20s + 100 = (s + 10)^2 .$$

Therefore, $K_1 = 100$ and $1 + K_2K_1 = 20$ or $K_2 = 0.19$.

(d) The step response is shown in Figure P2.35.

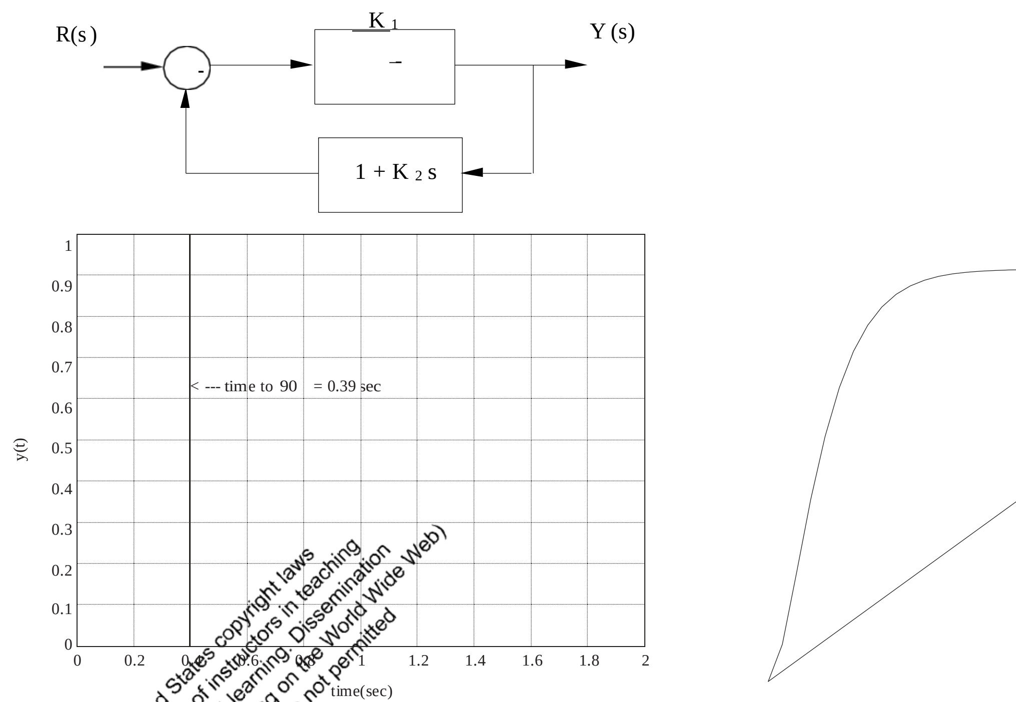


FIGURE P2.35
The equivalent block diagram and the system step response.

P2.36 (a) Given $R(s) = 1/s^2$, the partial fraction expansion is

$$Y(s) = \frac{30}{s^2(s+5)(s^2+4s+6)} = \frac{0.1091}{s+5} - \frac{0.7576(s+3.4)}{s^2+4s+6} + \frac{1}{s^2} - \frac{0.8667}{s}.$$

Therefore, using the Laplace transform table, we determine that the ramp response for $t \geq 0$ is

$$y(t) = \frac{6}{55}e^{-5t} + \frac{25}{33}e^{-2t}\left(\cos\sqrt{2}t + \frac{7}{10}\sqrt{2}\sin\sqrt{2}t\right) + t - \frac{13}{15}.$$

(b) For the ramp input, $y(t) \approx 0.25$ at $t = 1$ second (see Figure P2.36a).

(c) Given $R(s) = 1$, the partial fraction expansion is

$$Y(s) = \frac{30}{(s+5)(s^2+4s+6)} = \frac{30}{11}\frac{1}{s+5} - \frac{30}{11}\frac{s-1}{s^2+4s+6}.$$

Therefore, using the Laplace transform table, we determine that the impulse response for $t \geq 0$ is

$$y(t) = \frac{30}{11}e^{-5t} - \frac{30}{11}e^{-t}\left(\cos\sqrt{2}t - \frac{3\sqrt{2}}{2}\sin\sqrt{2}t\right).$$

(d) For the impulse input, $y(t) \approx 0.73$ at $t = 1$ seconds (see Figure P2.36b).

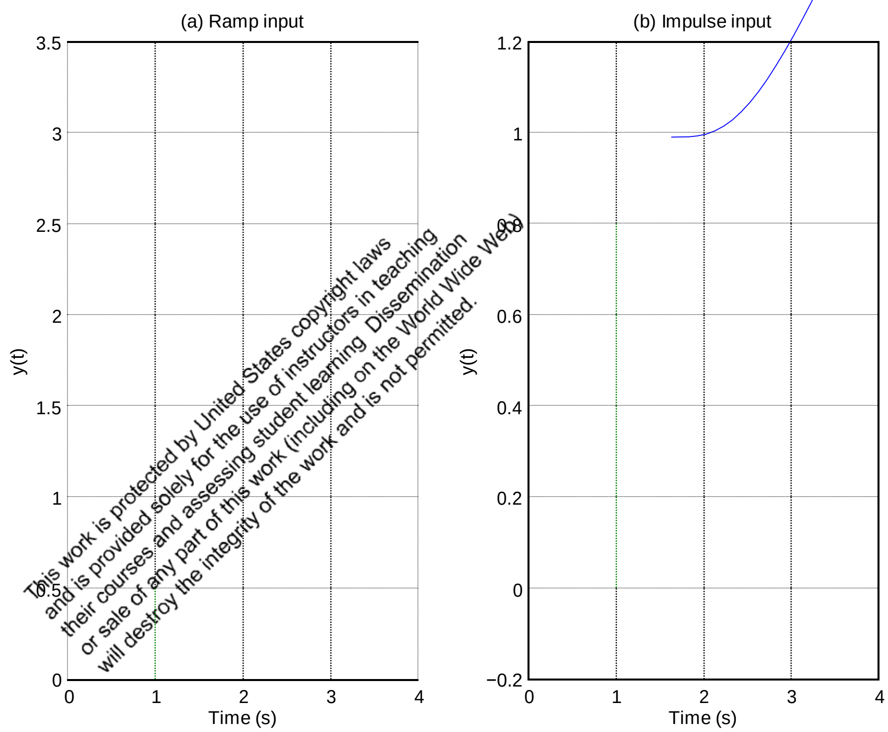


FIGURE P2.36
(a) Ramp input response. (b) Impulse input response.

P2.37 The equations of motion are

$$m_1\frac{d^2x}{dt^2} = -(k_1 + k_2)x + k_2y \quad \text{and} \quad m_2\frac{d^2y}{dt^2} = k_2(x - y) + u .$$

When $m_1 = m_2 = 1$ and $k_1 = k_2 = 1$, we have

$$\frac{d^2x}{dt^2} = -2x + y \quad \text{and} \quad \frac{d^2y}{dt^2} = x - y + u .$$

P2.38 The equation of motion for the system is

$$J\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + k\theta = 0 ,$$

where k is the rotational spring constant and b is the viscous friction coefficient. The initial conditions are $\theta(0) = \theta_o$ and $\dot{\theta}(0) = 0$. Taking the Laplace transform yields

$$J(s^2\theta(s) - s\theta_o) + b(s\theta(s) - \theta_o) + k\theta(s) = 0 .$$

Therefore,

$$\theta(s) = \frac{(s + \frac{b}{J}\theta_o)}{(s^2 + \frac{b}{J}s + \frac{K}{J})} = \frac{(s + 2\zeta\omega_n)\theta_o}{s^2 + 2\zeta\omega_n s + \omega_n^2} .$$

Neglecting the mass of the rod, the moment of inertia is detemined to be

$$J = 2Ml^2 = 0.5 \text{ kg} \cdot \text{m}^2 .$$

Also,

$$\omega_n = \sqrt{\frac{k}{J}} = 0.02 \text{ rad/s} \quad \text{and} \quad \zeta = \frac{b}{2J\omega_n} = 0.01 .$$

Solving for $\theta(t)$, we find that

$$\theta(t) = \frac{\theta_o}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\, t + \varphi) ,$$

where $\tan\varphi = \sqrt{1-\zeta^2}/\zeta)$. Therefore, the envelope decay is

$$\theta_e = \frac{\theta_o}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} .$$

So, with $\zeta\omega_n = 2 \times 10^{-4}$, $\theta_o = 4000^o$ and $\theta_f = 10^o$, the elapsed time is computed as

$$t = \frac{1}{\zeta\omega_n} ln \frac{\theta_o}{\sqrt{1-\zeta^2}\,\theta_f} = 8.32 \text{ hours} .$$

P2.39 When $t < 0$, we have the steady-state conditions

$$i_1(0) = \frac{6}{7}A , \quad v_a(0) = \frac{12}{7}V \quad \text{and} \quad v_c(0) = \frac{36}{7}V ,$$

where $v_c(0)$ is associated with the 0.75F capacitor. After $t \geq 0$, we have

1.5

$$\frac{di_1}{dt} + 2i_1 + 5(i_1 - i_2) = 10e^{-2t}$$

and

$$0.75 \int i_2 dt + 10 i_2 + 5(i_2 - i_1) - i_1 = 0 \,.$$

Taking the Laplace transform (using the initial conditions) yields

$$1.5(sI_1(s) - i_1(0)) + 2I_1(s) + 5I_1(s) - 5I_2(s) = \frac{10}{s+2}$$

or

$$\left(s + \frac{14}{3}\right) I_1(s) - \frac{10}{3} I_2(s) = \frac{18s + 176}{21(s+2)}$$

and

$$\frac{3}{4}\left(\frac{1}{s} I_2(s) - v_c(0)\right) + 10 I_2(s) + 5(I_2(s) - I_1(s)) = I_1(s)$$

or

$$-24sI_1(s) + (60s + 3)I_2(s) = \frac{108}{7} s \,.$$

Solving for $I_2(s)$ yields

$$I_2(s) = \frac{4s(27s^2 + 216s + 604)}{7(s+2)(60s^2 + 203s + 14)} \,.$$

Then, $V_o(s) = 10 I_2(s)$.

P2.40 The equations of motion are

$$J_1 \ddot{\theta}_1 = K(\theta_2 - \theta_1) - b(\dot{\theta}_1 - \dot{\theta}_2) + T \qquad \text{and} \qquad J_2 \ddot{\theta}_2 = b(\dot{\theta}_1 - \dot{\theta}_2) \,.$$

Taking the Laplace transform yields

$$(J_1 s^2 + bs + K)\theta_1(s) - bs\theta_2(s) = K\theta_2(s) + T(s)$$

and $(J_2 s^2 + bs)\theta_2(s) - bs\theta_1(s) = 0$. Solving for $\theta_1(s)$ and $\theta_2(s)$, we find that

$$\theta_1(s) = \frac{(K\theta_2(s) + T(s))(J_2 s + b)}{\Delta(s)} \qquad \text{and} \qquad \theta_2(s) = \frac{b(K\theta_2(s) + T(s))}{\Delta(s)} \,,$$

where $\Delta(s) = J_1 J_2 s^3 + b(J_1 + J_2)s^2 + J_2 K s + bK$.

P2.41 Assume that the only external torques acting on the rocket are control torques, T_c and disturbance torques, T_d, and assume small angles, $\theta(t)$. Using the small angle approximation, we have

$$\dot{h} = V\theta$$

$$J\ddot{\theta} = T_c + T_d ,$$

where J is the moment of inertia of the rocket and V is the rocket velocity (assumed constant). Now, suppose that the control torque is proportional to the lateral displacement, as

$$T_c(s) = -KH(s) ,$$

where the negative sign denotes a negative feedback system. The corresponding block diagram is shown in Figure P2.41.

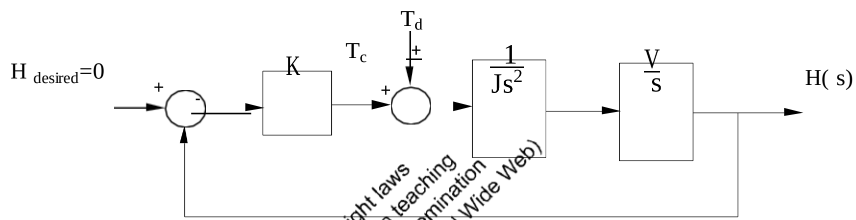


FIGURE P2.41
Block diagram.

P2.42 (a) The equation of motion of the motor is

$$J\frac{d\omega}{dt} = T_m - b\omega ,$$

where $J = 0.1$, $b = 0.06$, and T_m is the motor input torque.

(b) Given $T_m(s) = 1/s$, and $\omega(0) = 0.7$, we take the Laplace transform of the equation of motion yielding

$$s\omega(s) - \omega(0) + 0.6\omega(s) = 10T_m$$

or

$$\omega(s) = \frac{0.7s + 10}{s(s+0.6)} .$$

Then, computing the partial fraction expansion, we find that

$$\omega(s) = \frac{A}{s} + \frac{B}{s+0.6} = \frac{16.67}{s} - \frac{15.97}{s+0.6} .$$

The step response, determined by taking the inverse Laplace transform, is

$$\omega(t) = 16.67 - 15.97e^{-0.6t} , \qquad t \geq 0 .$$

P2.43 The work done by each gear is equal to that of the other, therefore

$$T_m\theta_m = T_L\theta_L .$$

Also, the travel distance is the same for each gear, so

$$r_1\theta_m = r_2\theta_L .$$

The number of teeth on each gear is proportional to the radius, or

$$r_1N_2 = r_2N_1 .$$

So,

$$\frac{\theta_m}{\theta_L} = \frac{r_2}{r_1} = \frac{N_2}{N_1} ,$$

and

$$\theta_L = \frac{N_1}{N_2}\theta_m = n\theta_m ,$$

where

$$n = N_1/N_2 .$$

Finally,

$$\frac{T_m}{T_L} = \frac{\theta_L}{\theta_m} = \frac{N_1}{N_2} = n .$$

P2.44 The inertia of the load is

$$J_L = \frac{\pi\rho L r^4}{2} .$$

Also, from the dynamics we have

$$T_2 = J_L\dot{\omega}_2 + b_L\omega_2$$

and

$$T_1 = nT_2 = n(J_L\dot{\omega}_2 + b_L\omega_2) .$$

So,

$$T_1 = n^2(J_L\dot{\omega}_1 + b_L\omega_1) ,$$

since

$$\omega_2 = n\omega_1 .$$

Therefore, the torque at the motor shaft is

$$T = T_1 + T_m = n^2(J_L\dot{\omega}_1 + b_L\omega_1) + J_m\dot{\omega}_1 + b_m\omega_1 .$$

P2.45 Let $U(s)$ denote the human input and $F(s)$ the load input. The transfer function is

$$P(s) = \frac{tt(s) + Ktt_1(s)}{\Delta(s)} U(s) + \frac{tt_c(s) + Ktt_1(s)}{\Delta(s)} F(s) ,$$

where

$$\Delta = 1 + ttH(s) + tt_1KBH(s) + tt_cE(s) + tt_1KE(s) .$$

P2.46 Consider the application of Newton's law ($\sum F = m\ddot{x}$). From the mass m_v we obtain

$$m_v\ddot{x}_1 = F - k_1(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2).$$

Taking the Laplace transform, and solving for $X_1(s)$ yields

$$X_1(s) = \frac{1}{\Delta_1(s)} F(s) + \frac{b_1 s + k_1}{\Delta_1(s)} X_2(s),$$

where

$$\Delta_1 := m_v s^2 + b_1 s + k_1.$$

From the mass m_t we obtain

$$m_t\ddot{x}_2 = -k_2x_2 - b_2\dot{x}_2 + k_1(x_1 - x_2) + b_1(\dot{x}_1 - \dot{x}_2).$$

Taking the Laplace transform, and solving for $X_2(s)$ yields

$$X_2(s) = \frac{b_1 s + k_1}{\Delta_2(s)} X_1(s),$$

where

$$\Delta_2 := m_t s^2 + (b_1 + b_2)s + k_1 + k_2.$$

Substituting $X_2(s)$ above into the relationship fpr $X_1(s)$ yields the transfer function

$$\frac{X_1(s)}{F(s)} = \frac{\Delta_2(s)}{\Delta_1(s)\Delta_2(s) - (b_1 s + k_1)^2} .$$

P2.47 Using the following relationships

$$h(t) = \int (1.6\theta(t) - h(t))dt$$

$$\omega(t) = \dot{\theta}(t)$$

$$J\dot{\omega}(t) = K_m i_a(t)$$

$$v_a(t) = 50v_i(t) = 10i_a(t) + v_b(t)$$

$$\dot{\theta} = K v_b$$

we find the differential equation is

$$\frac{d^3h}{dt^3} + \left[1 + \frac{K_m}{10JK}\right]\frac{d^2h}{dt^2} + \frac{K_m}{10JK}\frac{dh}{dt} = \frac{8K_m}{J}v_i .$$

P2.48 (a) The transfer function is

$$\frac{V_2(s)}{V_1(s)} = \frac{(1 + sR_1C_1)(1 + sR_2C_2)}{R_1C_2s} .$$

(b) When $R_1 = 250\ k\Omega$, $R_2 = 200\ k\Omega$, $C_1 = 2\ \mu F$ and $C_2 = 0.1\ \mu F$, we have

$$\frac{V_2(s)}{V_1(s)} = \frac{0.4s^2 + 20.8s + 40}{s} .$$

(c) The partial fraction expansion is

$$\frac{V_2(s)}{V_1(s)} = 20.8 + \frac{40}{s} + 0.4s .$$

P2.49 (a) The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{5000}{s^3 + 20s^2 + 1000s + 5000} .$$

(b) The poles of $T(s)$ are $s_1 = -5.43$ and $s_{2,3} = -7.28 \pm j29.46$.

(c) The partial fraction expansion (with a step input) is

$$Y(s) = \frac{1}{s} - \frac{1.06}{s + 5.43} + \frac{0.0285 + 0.0904j}{s + 7.28 - j29.46} + \frac{\overline{0.0285 - 0.0904j}}{s + 7.28 + j29.46} ,$$

and

$$y(t) = 1 - 1.06e^{-5.43t} + 0.06e^{-7.28t}(\cos 29.46t - 3.17 \sin 29.46t) ;$$

(d) The step response is shown in Figure P2.49. The real and complex roots are close together and by looking at the poles in the s-plane we

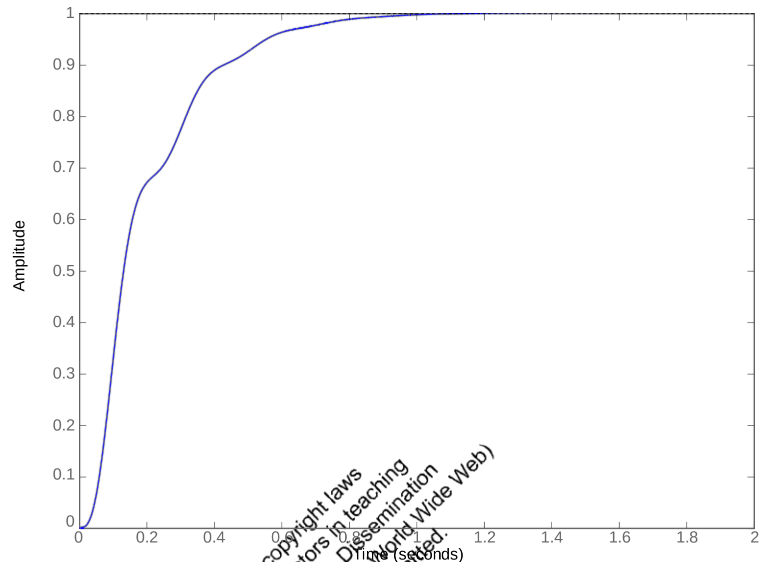


FIGURE P2.49
Step response.

have difficulty deciding which is dominant. However, the residue at the real pole is much larger and thus dominates the response.

P2.50 (a) The closed-loop transfer function is

$$T(s) = \frac{14000}{s^3 + 45s^2 + 3100s + 14500} .$$

(b) The poles of $T(s)$ are

$$s_1 = -5 \quad \text{and} \quad s_{2,3} = -20 \pm j50.$$

(c) The partial fraction expansion (with a step input) is

$$Y(s) = \frac{0.9655}{s} - \frac{1.0275}{s+5} + \frac{0.0310 - 0.0390j}{s + 20 + j50} + \frac{0.0310 + 0.0390j}{s + 20 - j50} .$$

(d) The step response is shown in Figure P2.50. The real root dominates the response.

(e) The final value of $y(t)$ is

$$y_{ss} = \lim_{s \to 0} sY(s) = 0.9655 .$$

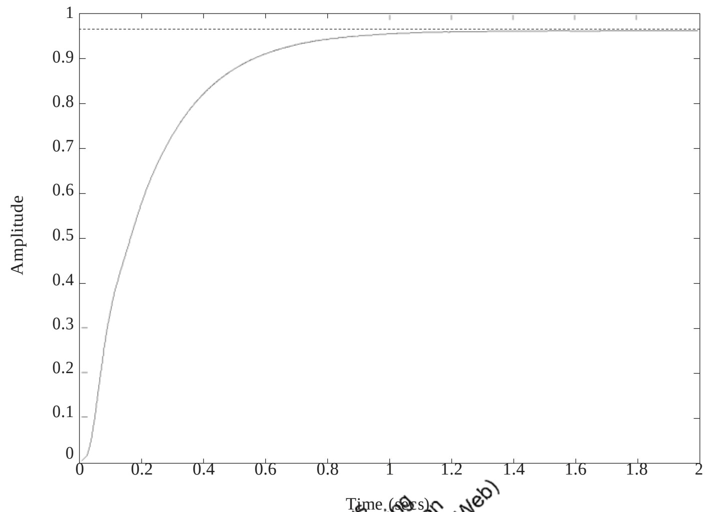


FIGURE P2.50
Step response.

P2.51 Consider the free body diagram in Figure P2.51. Using Newton's Law and summing the forces on the two masses yields

$$M_1\ddot{x}(t) + b_1\dot{x}(t) + k_1x(t) = b_1\dot{y}(t)$$
$$M_2\ddot{y}(t) + b_1\dot{y}(t) + k_2y(t) = b_1\dot{x}(t) + u(t)$$

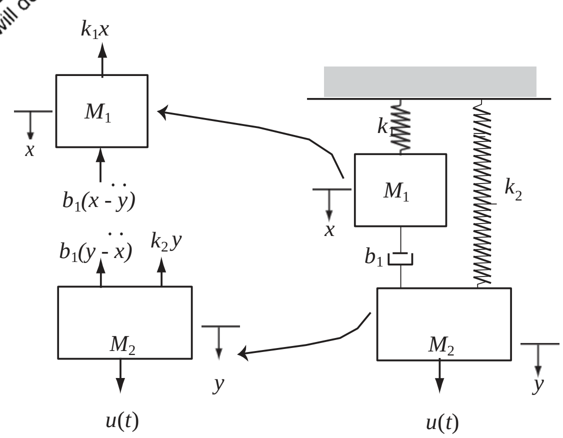


FIGURE P2.51
Free body diagram.

Advanced Problems

AP2.1 The transfer function from $V(s)$ to $\omega(s)$ has the form

$$\frac{\omega(s)}{V(s)} = \frac{K_m}{\tau_m s + 1} .$$

In the steady-state,

$$\omega_{ss} = \lim_{s\to 0} s \left(\frac{K_m}{\tau_m s + 1} \right) \left(\frac{5}{s} \right) = 5K_m .$$

So,

$$K_m = 70/5 = 14 .$$

Also,

$$\omega(t) = V_m K_m (1 - e^{-t/\tau_m})$$

where $V(s) = V_m/s$. Solving for τ_m yields

$$\tau_m = \frac{-t}{\ln(1 - \omega(t)/\omega_{ss})} .$$

When $t = 2$, we have

$$\tau_m = \frac{-2}{\ln(1 - 30/70)} = 3.57 .$$

Therefore, the transfer function is

$$\frac{\omega(s)}{V(s)} = \frac{14}{3.57s + 1} .$$

AP2.2 The closed-loop transfer function form $R_1(s)$ to $Y_2(s)$ is

$$\frac{Y_2(s)}{R_1(s)} = \frac{tt_1 tt_4 tt_5(s) + tt_1 tt_2 tt_3 tt_4 tt_6(s)}{\Delta}$$

where

$$\Delta = [1 + tt_3 tt_4 H_2(s)][1 + tt_1 tt_2 H_3(s)] .$$

If we select

$$tt_5(s) = -tt_2 tt_3 tt_6(s)$$

then the numerator is zero, and $Y_2(s)/R_1(s) = 0$. The system is now decoupled.

AP2.3 (a) Computing the closed-loop transfer function:

$$Y(s) = \left[\frac{tt(s)tt_c(s)}{1 + tt_c(s)tt(s)H(s)}\right] R(s)\,.$$

Then, with $E(s) = R(s) - Y(s)$ we obtain

$$E(s) = \frac{1 + tt_c(s)tt(s)(H(s) - 1)}{1 + tt_c(s)tt(s)H(s)} R(s)\,.$$

If we require that $E(s) \equiv 0$ for any input, we need $1 + tt_c(s)tt(s)(H(s) - 1) = 0$ or

$$H(s) = \frac{tt_c(s)tt(s) - 1}{tt_c(s)tt(s)} = \frac{n(s)}{d(s)}\,.$$

Since we require $H(s)$ to be a causal system, the order of the numerator polynomial, $n(s)$, must be less than or equal to the order of the denominator polynomial, $d(s)$. This will be true, in general, only if both $tt_c(s)$ and $tt(s)$ are proper rational functions (that is, the numerator and denominator polynomials have the same order). Therefore, making $E \equiv 0$ for any input $R(s)$ is possible only in certain circumstances.

(b) The transfer function from $T_d(s)$ to $Y(s)$ is

$$Y(s) = \left[\frac{tt_d(s)tt(s)}{1 + tt_c(s)tt(s)H(s)}\right] T_d(s)\,.$$

With $H(s)$ as in part (a) we have

$$Y(s) = \left[\frac{tt_d(s)}{tt_c(s)}\right] T_d(s)\,.$$

(c) No. Since

$$Y(s) = \left[\frac{tt_d(s)tt(s)}{1 + tt_c(s)tt(s)H(s)}\right] T_d(s) = T(s)T_d(s)\,,$$

the only way to have $Y(s) \equiv 0$ for any $T_d(s)$ is for the transfer function $T(s) \equiv 0$ which is not possible in general (since $tt(s) \neq 0$).

AP2.4 (a) With $q(s) = 1/s$ we obtain

$$\tau(s) = \frac{1/C_t}{s + \frac{QS+1/R}{C_t}} \cdot \frac{1}{s}\,.$$

Define

$$\alpha := \frac{QS + 1/R}{C_t} \quad \text{and} \quad \beta := 1/C_t\,.$$

Then, it follows that

$$\tau(s) = \frac{\beta}{s+\alpha} \cdot \frac{1}{s} = \frac{-\beta/\alpha}{s+\alpha} + \frac{\beta/\alpha}{s} .$$

Taking the inverse Laplace transform yields

$$\tau(t) = \frac{-\beta}{\alpha} e^{-\alpha t} + \frac{\beta}{\alpha} = \frac{\beta}{\alpha}\left[1 - e^{-\alpha t}\right] .$$

(b) As $t \to \infty$, $\tau(t) \to \frac{\beta}{\alpha} = \frac{1}{Qs+1/R}$.

(c) To increase the speed of response, you want to choose C_t, Q, S and R such that

$$\alpha := \frac{Qs + 1/R}{C_t}$$

is "large."

AP2.5 Considering the motion of each mass, we have

$$M_3\ddot{x}_3 + b_3\dot{x}_3 + k_3x_3 = b_3\dot{x}_2 + k_3x_2$$
$$M_2\ddot{x}_2 + (b_2 + b_3)\dot{x}_2 + (k_2 + k_3)x_2 = u_2 + b_3\dot{x}_3 + k_3x_3 + b_2\dot{x}_1 + k_2x_1$$
$$M_1\ddot{x}_1 + (b_1 + b_2)\dot{x}_1 + (k_1 + k_2)x_1 = u_1 + b_2\dot{x}_2 + k_2x_2$$

In matrix form the three equations can be written as

$$\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} b_1+b_2 & -b_2 & 0 \\ -b_2 & b_2+b_3 & -b_3 \\ 0 & -b_3 & b_3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$
$$+ \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} .$$

AP2.6 Considering the cart mass and using Newton's Law we obtain

$$M\ddot{x} = u - b\dot{x} - F\sin\phi$$

where F is the reaction force between the cart and the pendulum. Considering the pendulum we obtain

$$m\frac{d^2(x + L\sin\phi)}{dt^2} = F\sin\phi$$

$$m \frac{d^2(L\cos\phi)}{dt^2} = F\cos\phi + mg$$

Eliminating the reaction force F yields the two equations

$$(m+M)\ddot{x} + b\dot{x} + mL\ddot{\phi}\cos\phi - mL\dot{\phi}^2\sin\phi = u$$
$$mL^2\ddot{\phi} + mgL\sin\phi + mL\ddot{x}\cos\phi = 0$$

If we assume that the angle $\phi \approx 0$, then we have the linear model

$$(m+M)\ddot{x} + b\dot{x} + mL\ddot{\phi} = u$$
$$mL^2\ddot{\phi} + mgL\phi = -mL\ddot{x}$$

AP2.7 The transfer function from the disturbance input to the output is

$$Y(s) = \frac{1}{s + 40 + K} T_d(s) .$$

When $T_d(s) = 1$, we obtain

$$y(t) = e^{-(40+K)t} .$$

Solving for t when $y(t) < 0.1$ yields

$$t > \frac{2.3}{40 + K} .$$

When $t = 0.05$ and $y(0.05) = 0.1$, we find $K = 6.05$.

AP2.8 The closed-loop transfer function is

$$T(s) = \frac{200K(0.25s + 1)}{(0.25s + 1)(s + 1)(s + 8) + 200K}$$

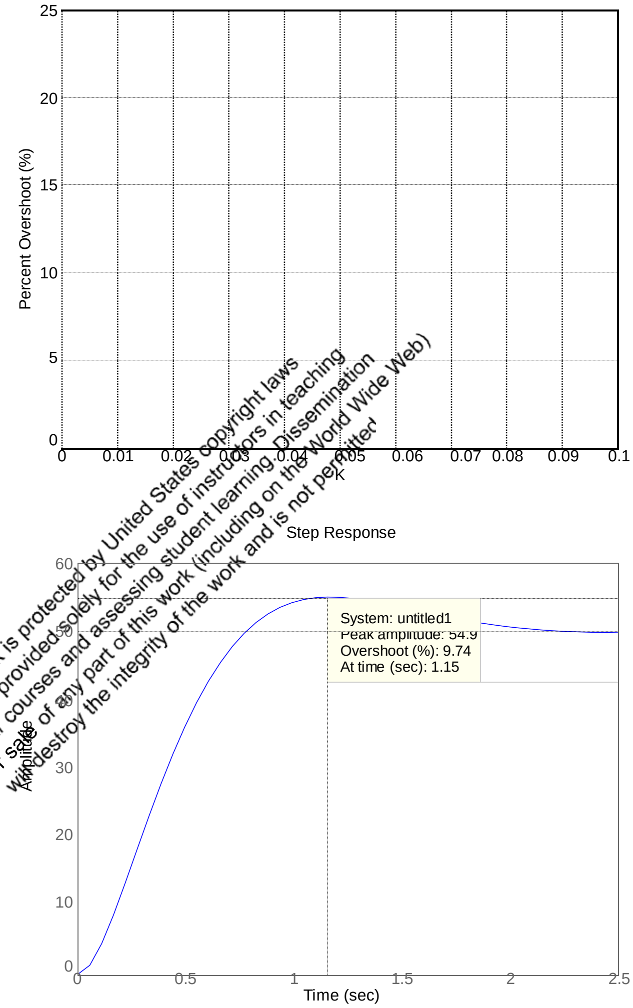
The final value due to a step input of $R(s) = A/s$ is

$$v(t) \rightarrow A\frac{200K}{200K + 8} .$$

We need to select K so that $v(t) \rightarrow 50$. However, to keep the percent overshoot to less than 10%, we need to limit the magnitude of K. Figure AP2.8a shows the percent overshoot as a function of K. Let $K = 0.06$ and select the magnitude of the input to be $A = 83.3$. The inverse Laplace transform of the closed-loop response with $R(s) = 83.3/s$ is

$$v(t) = 50 + 9.85e^{-9.15t} - e^{-1.93t}(59.85\cos(2.24t) + 11.27\sin(2.24t))$$

The result is $P.O. = 9.74\%$ and the steady-state value of the output is approximately 50 m/s, as shown in Figure AP2.8b.



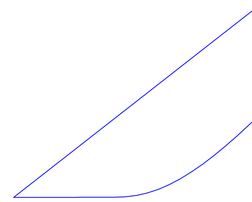


FIGURE AP2.8
(a) Percent overshoot versus the gain K. (b) Step response.

AP2.9 The transfer function is

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)},$$

where

$$Z_1(s) = \frac{R_1}{R_1C_1s + 1} \quad \text{and} \quad Z_2(s) = \frac{R_2C_2s + 1}{C_2s} .$$

Then we can write

$$\frac{V_o(s)}{V_i(s)} = K_p + \frac{K_I}{s} + K_D s$$

where

$$K_P = -\left(\frac{R_1C_1}{R_2C_2} + 1\right), \quad K_I = -\frac{1}{R_1C_2}, \quad K_D = -R_2C_1$$

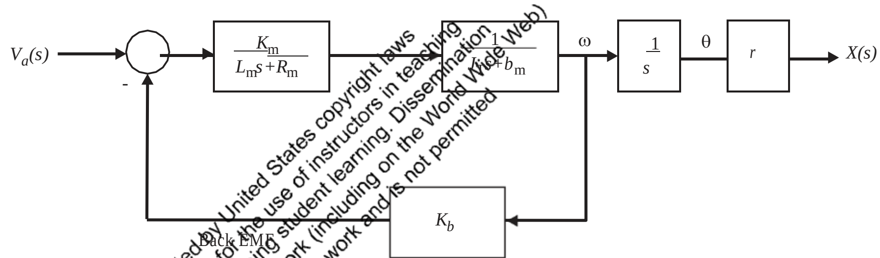
Design Problems

CDP2.1 The model of the traction drive, capstan roller, and linear slide follows closely the armature-controlled dc motor model depicted in Figure 2.18 in Dorf and Bishop. The transfer function is

$$T(s) = \frac{rK_m}{s\left[(L_m s + R_m)(J_T\, s + b_m) + K_b K_m\right]} \quad ,$$

where

$$J_T = J_m + r^2(M_s + M_b) \ .$$



DP2.1 The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{tt_1(s)tt_2(s)}{1 + tt_1(s)H_1(s) - tt_2(s)H_2(s)} \ .$$

When $tt_1H_1 = tt_2H_2$ and $tt_1tt_2 = 1$, then $Y(s)/R(s) = 1$. Therefore, select

$$tt_1(s) = \frac{1}{tt_2(s)} \quad \text{and} \quad H_1(s) = \frac{tt_2(s)H_2(s)}{tt_1(s)} = tt_2^2(s)H_2(s) \ .$$

DP2.2 At the lower node we have

$$v\left(\frac{1}{4} + \frac{1}{3} + tt\right) + 2i_2 - 20 = 0 \ .$$

Also, we have $v = 24$ and $i_2 = ttv$. So

$$v\left(\frac{1}{4} + \frac{1}{3} + tt\right) + 2ttv - 20 = 0$$

and

$$tt = \frac{20 - v\left(\frac{1}{4} + \frac{1}{3}\right)}{3v} = \frac{1}{12}\, S\ .$$

DP2.3 Taking the Laplace transform of

$$y(t) = e^{-t} - \frac{1}{4}e^{-2t} - \frac{3}{4} + \frac{1}{2}t$$

yields

$$Y(s) = \frac{1}{s+1} - \frac{1}{4(s+2)} - \frac{3}{4s} + \frac{1}{2s^2}\ .$$

Similarly, taking the Laplace transform of the ramp input yields

$$R(s) = \frac{1}{s^2}\ .$$

Therefore

$$tt(s) = \frac{Y(s)}{R(s)} = \frac{1}{(s+1)(s+2)}\ .$$

DP2.4 For an ideal op-amp, at node a we have

$$\frac{v_{in} - v_a}{R_1} + \frac{v_o - v_a}{R_1} = 0\ ,$$

and at node b

$$\frac{v_{in} - v_b}{R_2} = C\dot{v}_b\ ,$$

from it follows that

$$\left(\frac{1}{R_2} + Cs\right)V_b = \frac{1}{R_2}V_{in}\ .$$

Also, for an ideal op-amp, $V_b - V_a = 0$. Then solving for V_b in the above equation and substituting the result into the node a equation for V_a yields

$$\frac{V_o}{V_{in}} = \frac{2}{\frac{1}{R_2} + Cs}\left[\frac{1}{R_2} - \frac{\frac{1}{R_2} + Cs}{2}\right]$$

or

$$\frac{V_o(s)}{V_{in}(s)} = -\frac{R_2Cs - 1}{R_2Cs + 1}\ .$$

For $v_{in}(t) = At$, we have $V_{in}(s) = A/s^2$, therefore

$$v_o(t) = A\left[\frac{2}{\beta}e^{-\beta t} + t - \frac{2}{\beta}\right]$$

where $\beta = 1/R_2C$.

DP2.5 The equation of motion describing the motion of the inverted pendulum (assuming small angles) is

$$\ddot{\phi} + \frac{g}{L}\phi = 0 .$$

Assuming a solution of the form $\phi = k \cos \phi$, taking the appropriate derivatives and substituting the result into the equation of motion yields the relationship

$$\dot{\phi} = \sqrt{\frac{g}{L}} .$$

If the period is $T = 2$ seconds, we compute $\dot{\phi} = 2\pi/T$. Then solving for L yields $L = 0.99$ meters when $g = 9.81$ m/s^2. So, to fit the pendulum into the grandfather clock, the dimensions are generally about 1.5 meters or more.

Computer Problems

CP2.1 The m-file script is shown in Figure CP2.1.

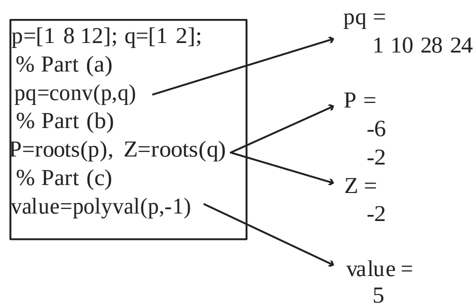


FIGURE CP2.1
Script for various polynomial evaluations.

CP2.2 The m-file script and step response is shown in Figure CP2.2.

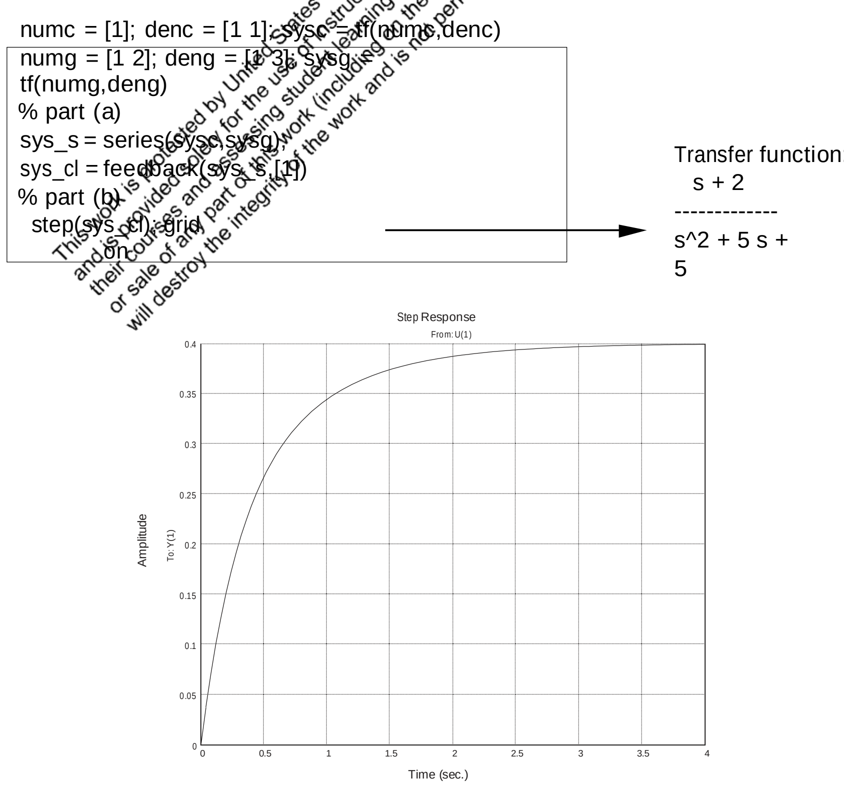


FIGURE CP2.2
Step response.

CP2.3 Given

$$\ddot{y} + 6\dot{y} + 5y = u$$

with $y(0) = \dot{y} = 0$ and $U(s) = 1/s$, we obtain (via Laplace transform)

$$Y(s) = \frac{1}{s(s^2 + 6s + 5)} = \frac{1}{s(s + 5)(s + 1)} .$$

Expanding in a partial fraction expansion yields

$$Y(s) = \frac{1}{5s} - \frac{1}{20(s + 5)} - \frac{1}{4(s + 1)} .$$

Taking the inverse Laplace transform we obtain the solution

$$y(t) = 0.2 + 0.05e^{-5t} - 0.25e^{-t} .$$

The m-file script and step response is shown in Figure CP2.3.

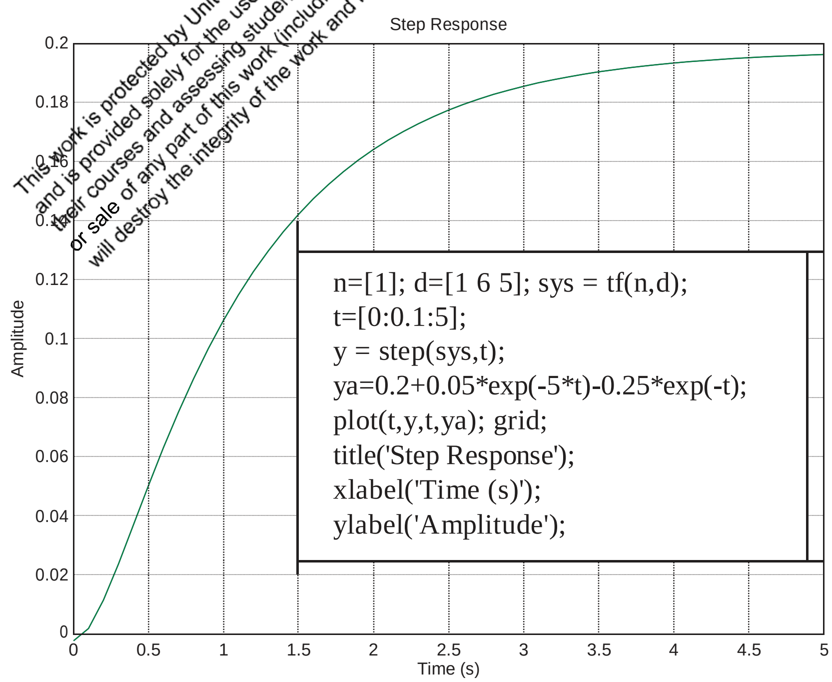


FIGURE CP2.3
Step response.

CP2.4 The mass-spring-damper system is represented by

$$m\ddot{x} + b\dot{x} + kx = f \, .$$

Taking the Laplace transform (with zero initial conditions) yields the transfer function

$$X(s)/F(s) = \frac{1/m}{s^2 + bs/m + k/m} \, .$$

The m-file script and step response is shown in Figure CP2.4.

```
m=10; k=1; b=0.5;
num=[1/m]; den=[1 b/m
k/m]; sys = tf(num,den);
t=[0:0.1:150];
step(sys,t)
```

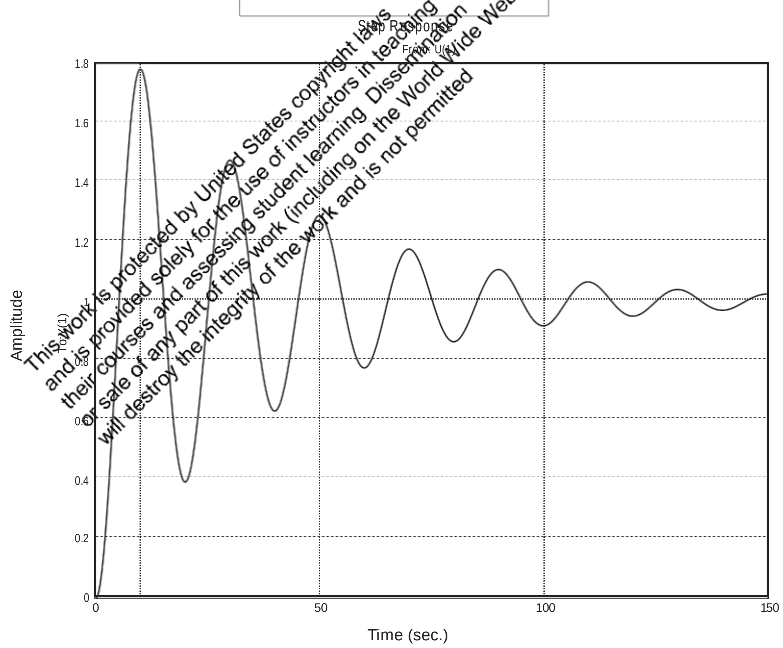
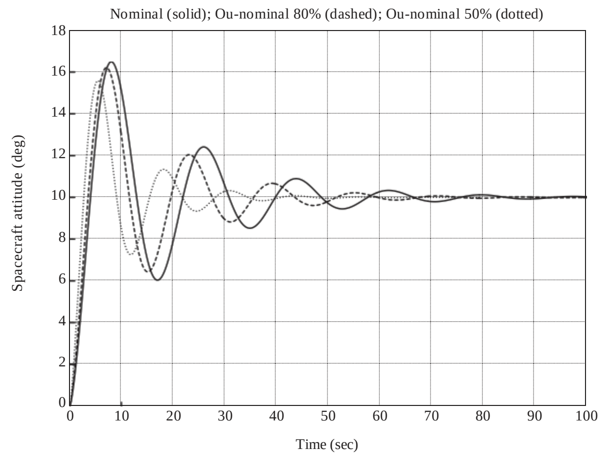


FIGURE CP2.4
Step response.

CP2.5 The spacecraft simulations are shown in Figure CP2.5. We see that as J is decreased, the time to settle down decreases. Also, the overhoot from 10^o decreases as J decreases. Thus, the performance seems to get better (in some sense) as J decreases.




```
%Part (a)
a=1; b=8; k=10.8e+08; J=10.8e+08;
num=k*[1 a];
den=J*[1 b 0 0]; sys=tf(num,den);
sys_cl=feedback(sys,[1]);
%
% Part (b) and (c)
t=[0:0.1: 100];
%
% Nominal case
f=10*pi/180; sysf=sys_cl*f;
y=step(sysf,t);
%
% Ou-nominal case 80%
J=10.8e+08*0.8; den=J*[1 b 0 0];
sys=tf(num,den);
sys_cl=feedback(sys,[1]);
sysf=sys_cl*f;
y1=step(sysf,t);
%
% Ou-nominal case 50%
J=10.8e+08*0.5; den=J*[1 b 0 0];
sys=tf(num,den); sys_cl=feedback(sys,[1]);
sysf=sys_cl*f;
y2=step(sysf,t);
%
plot(t,y*180/pi,t,y1*180/pi,'--',t,y2*180/pi,':'),grid
xlabel('Time (sec)')
ylabel('Spacecraft attitude (deg)')
title('Nominal (solid); Ou-nominal 80% (dashed); Ou-nominal 50% (dotted)')
```


FIGURE CP2.5
Step responses for the nominal and off-nominal spacecraft parameters.

CP2.6 The closed-loop transfer function is

$$T(s) = \frac{4s^6 + 8s^5 + 4s^4 + 56s^3 + 112s^2 + 56s}{\Delta(s)},$$

```
num1=[4]; den1=[1]; sys1 = tf(num1,den1);
num2=[1]; den2=[1 1]; sys2 = tf(num2,den2);
num3=[1 0]; den3=[1 0 2]; sys3 =
tf(num3,den3); num4=[1]; den4=[1 0 0]; sys4 =
tf(num4,den4); num5=[4 2]; den5=[1 2 1]; sys5
= tf(num5,den5); num6=[50]; den6=[1]; sys6 =
tf(num6,den6); num7=[1 0 2]; den7=[1 0 0 14];
sys7 = tf(num7,den7); sysa
=feedback(sys4,sys6,+1);
sysb = series(sys2,sys3);
sysc =
feedback(sysb,sys5); sysd
= series(sysc,sysa); syse
= feedback(sysd,sys7);
sys = series(sys1,syse)
%
pzmap(sys)
%
p=pole(sy
s)
z=zero(sy
s)
```

poles

```
p =
  7.0709
 -7.0713
  1.2051 + 2.0863i
  1.2051 - 2.0863i
  0.1219 + 1.8374i
  0.1219 - 1.8374i
 -2.3933
 -2.3333
 -0.4635 + 0.1997i
 -0.4635 - 0.1997i

z =
     0
  1.2051 + 2.0872i
  1.2051 - 2.0872i
 -2.4101
 -1.0000 + 0.0000i
 -1.0000 - 0.0000i
```

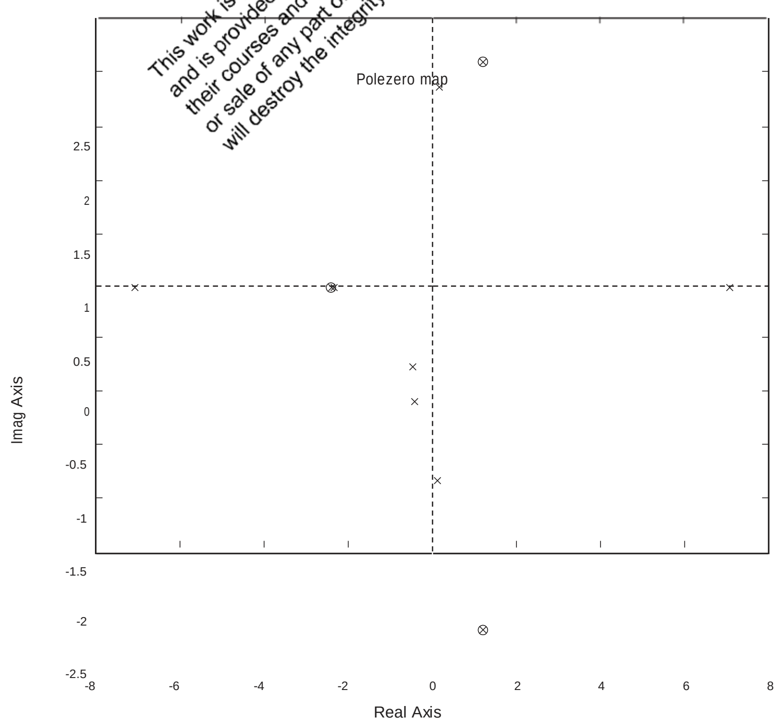



FIGURE CP2.6
Pole-zero map.

where

$$\Delta(s) = s^{10} + 3s^9 - 45s^8 - 125s^7 - 200s^6 - 1177s^5 - 2344s^4 - 3485s^3 - 7668s^2 - 5598s - 1400\,.$$

CP2.7 The m-file script and plot of the pendulum angle is shown in Figure CP2.7. With the initial conditions, the Laplace transform of the linear system is

$$\theta(s) = \frac{\theta_0 s}{s^2 + g/L}\,.$$

To use the step function with the m-file, we can multiply the transfer function as follows:

$$\theta(s) = \frac{s^2}{s^2 + g/L}\frac{\theta_0}{s}\,,$$

which is equivalent to the original transfer function except that we can use the step function input with magnitude θ_0. The nonlinear response is shown as the solid line and the linear response is shown as the dashed line. The difference between the two responses is not great since the initial condition of $\theta_0 = 30^\circ$ is not that large.

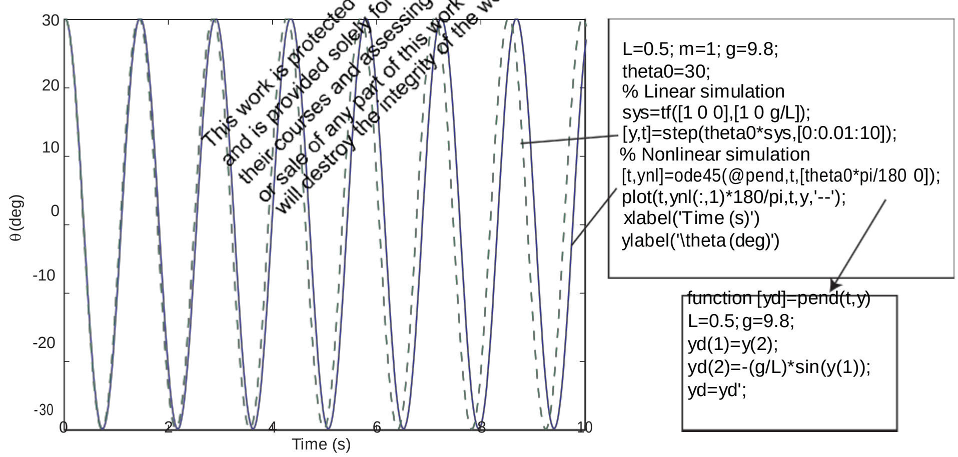


FIGURE CP2.7
Plot of θ versus xt when $\theta_0 = 30^\circ$.

CP2.8 The system step responses for $z = 5, 10,$ and 15 are shown in Figure CP2.8.

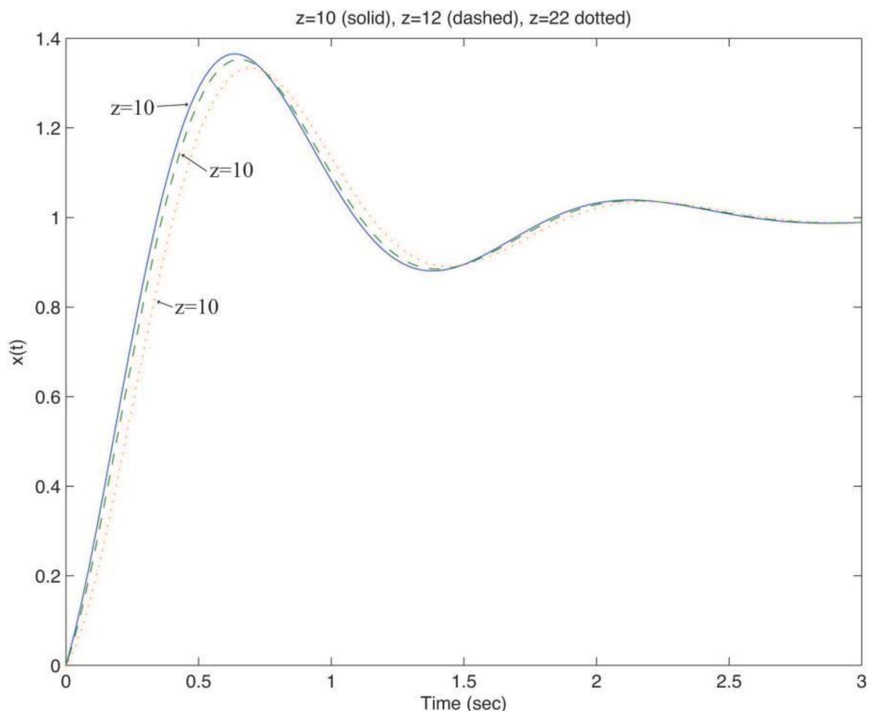


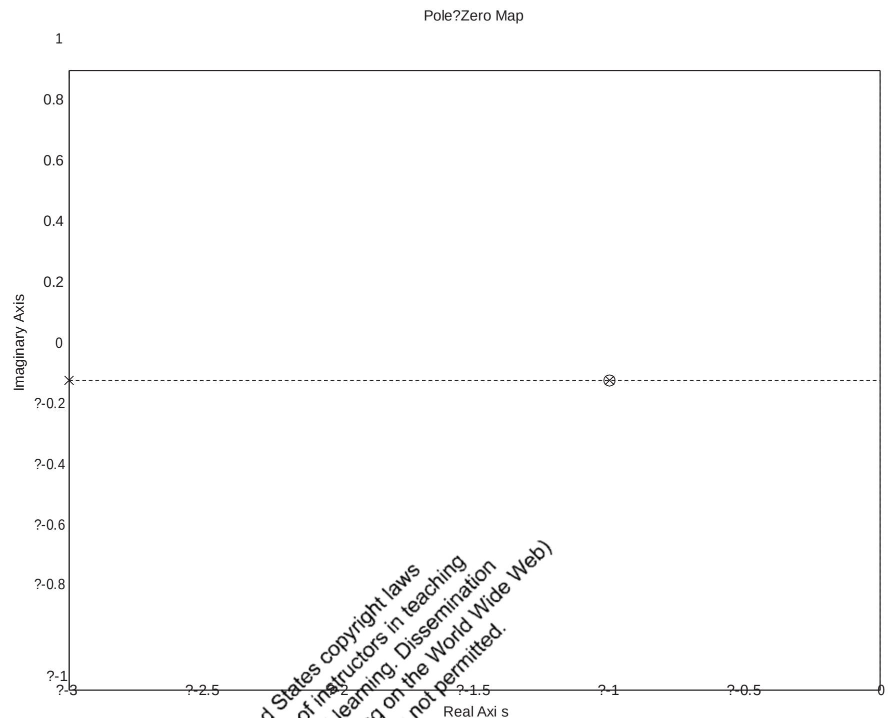
FIGURE CP2.8
The system response.

CP2.9 (a,b) Computing the closed-loop transfer function yields

$$T(s) = \frac{tt(s)}{1 + tt(s)H(s)} = \frac{s^2 + 2s + 1}{s^2 + 4s + 3} .$$

The poles are $s = -3, -1$ and the zeros are $s = -1, -1$.
(c) Yes, there is one pole-zero cancellation. The transfer function (after pole-zero cancellation) is

$$T(s) = \frac{s + 1}{s + 3} .$$




```
ng=[1 1];
dg=[1 2]; sysg=tf(ng,dg); nh=[1];
dh=[1 1]; sysh = tf(nh,dh);
sys=feedback(sysg,sysh)
%
pzmap(sys)
%
pole(sys)
zero(sys)
```

poles

zeros

```
>>
Transfer function:
s^2 + 2 s + 1
-------------
s^2 + 4 s +
3

p =

  -3
  -1

z =

  -1
  -1
```


FIGURE CP2.9
Pole-zero map.

CP2.10 Figure CP2.10 shows the steady-state response to a unit step input and a unit step disturbance. We see that $K = 1$ leads to the same steady-state response.

```
K=[0.1:0.1:10];
sysg=tf([1],[1 20
20]); for
i=1:length(K)
  nc=K(i);
  dc=[1];sysc=tf(nc,dc);
  syscl=feedback(sysc*sysg,1
  );
  systd=feedback(sysg,sysc);
  y1=step(syscl);
  Tf1(i)=y1(end);
  y2=step(systd);
  Tf2(i)=y2(end);
end
plot(K,Tf1,K,Tf2,'--
') xlabel('K')
ylabel('Steady-state
response')
```

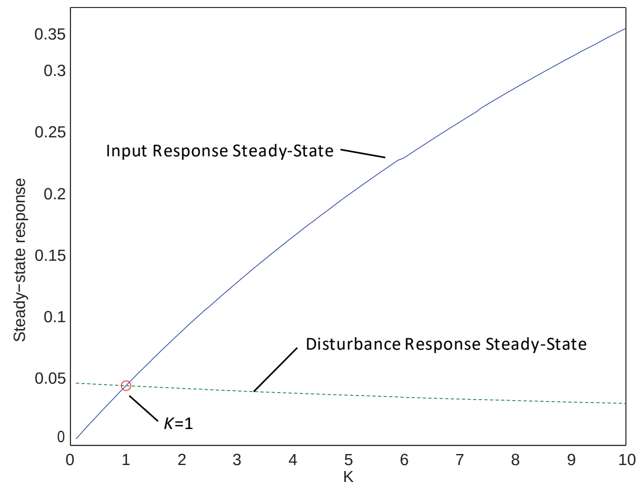


FIGURE CP2.10
Gain K versus steady-state value.