Solutions Manual for Multivariable Calculus 8th Edition Stewart 1305266641 9781305266643

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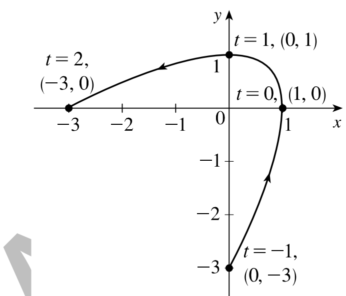
10 ☐ PARAMETRIC EQUATIONS AND POLAR COORDINATES

10.1 Curves Defined by Parametric Equations

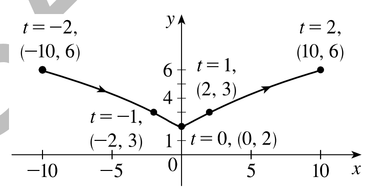
1. $x = 1 - t^2, \quad y = 2t - t^2, \quad -1 \le t \le 2$

t	-1	0	1	2
x	0	1	0	-3
y	-3	0	1	0



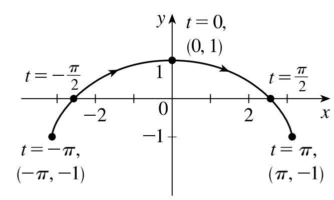
2. $x = t^3 + t, \quad y = t^2 + 2, \quad -2 \le t \le 2$

t	-2	-1	0	1	2
x	-10	-2	0	2	10
y	6	3	2	3	6



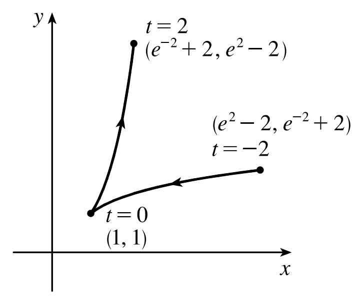
3. $x = t + \sin t, \quad y = \cos t, \quad -\pi \le t \le \pi$

t	$-\pi$	$-\pi/2$	0	$\pi/2$	π
x	$-\pi$	$-\pi/2 + 1$	0	$\pi/2 + 1$	π
y	-1	0	1	0	-1



4. $x = e^{-t} + t, \quad y = e^t - t, \quad -2 \le t \le 2$

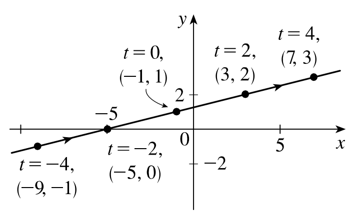
t	-2	-1	0	1	2
x	$e^2 - 2$ 5.39	$e - 1$ 1.72	1	$e^{-1} + 1$ 1.37	$e^{-2} + 2$ 2.14
y	$e^{-2} + 2$ 2.14	$e^{-1} + 1$ 1.37	1	$e - 1$ 1.72	$e^2 - 2$ 5.39



5. $x = 2t - 1, \quad y = \frac{1}{2}t + 1$

(a)

t	-4	-2	0	2	4
x	-9	-5	-1	3	7
y	-1	0	1	2	3



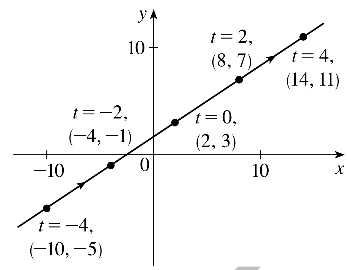
(b) $x = 2t - 1 \quad \Rightarrow \quad 2t = x + 1 \quad \Rightarrow \quad t = \frac{1}{2}x + \frac{1}{2}$, so

$$y = \tfrac{1}{2}t + 1 = \tfrac{1}{2}\left(\tfrac{1}{2}x + \tfrac{1}{2}\right) + 1 = \tfrac{1}{4}x + \tfrac{1}{4} + 1 \quad \Rightarrow \quad y = \tfrac{1}{4}x + \tfrac{5}{4}$$

6. $x = 3t + 2, \quad y = 2t + 3$

(a)

t	-4	-2	0	2	4
x	-10	-4	2	8	14
y	-5	-1	3	7	11



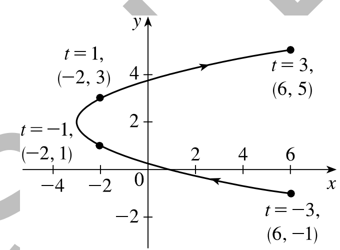
(b) $x = 3t + 2 \quad\Rightarrow\quad 3t = x - 2 \quad\Rightarrow\quad t = \frac{1}{3}x - \frac{2}{3}$, so

$y = 2t + 3 = 2\left(\frac{1}{3}x - \frac{2}{3}\right) + 3 = \frac{2}{3}x - \frac{4}{3} + 3 \quad\Rightarrow\quad y = \frac{2}{3}x + \frac{5}{3}$

7. $x = t^2 - 3, \quad y = t + 2, \quad -3 \le t \le 3$

(a)

t	-3	-1	1	3
x	6	-2	-2	6
y	-1	1	3	5



(b) $y = t + 2 \quad\Rightarrow\quad t = y - 2$, so

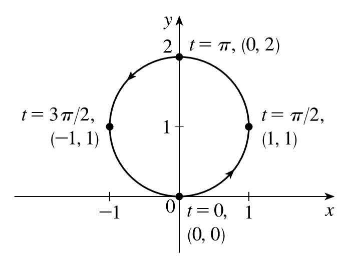
$x = t^2 - 3 = (y - 2)^2 - 3 = y^2 - 4y + 4 - 3 \quad\Rightarrow$

$x = y^2 - 4y + 1, -1 \le y \le 5$

8. $x = \sin t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi$

(a)

t	0	$\pi/2$	π	$3\pi/2$	2π
x	0	1	0	-1	0
y	0	1	2	1	0



(b) $x = \sin t, y = 1 - \cos t \quad [\text{or } y - 1 = -\cos t] \quad\Rightarrow$

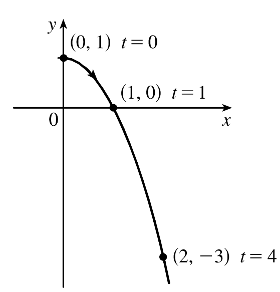
$x^2 + (y - 1)^2 = (\sin t)^2 + (-\cos t)^2 \quad\Rightarrow\quad x^2 + (y - 1)^2 = 1.$

As t varies from 0 to 2π, the circle with center $(0, 1)$ and radius 1 is traced out.

9. $x = \sqrt{t}, \; y = 1 - t$

(a)

t	0	1	2	3	4
x	0	1	1.414	1.732	2
y	1	0	-1	-2	-3



(b) $x = \sqrt{t} \quad\Rightarrow\quad t = x^2 \quad\Rightarrow\quad y = 1 - t = 1 - x^2$. Since $t \ge 0$, $x \ge 0$.

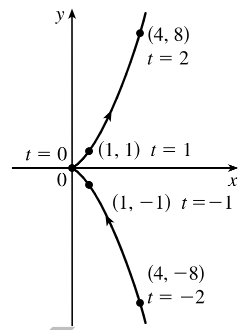
So the curve is the right half of the parabola $y = 1 - x^2$.

10. $x = t^2, y = t^3$

(a)

t	-2	-1	0	1	2
x	4	1	0	1	4
y	-8	-1	0	1	8

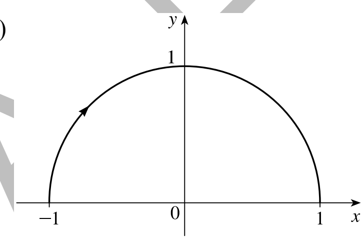
(b) $y = t^3 \;\Rightarrow\; t = \sqrt[3]{y} \;\Rightarrow\; x = t^2 = \left(\sqrt[3]{y}\right)^2 = y^{2/3}.\quad t \in \mathbb{R}, y \in \mathbb{R}, x \geq 0.$



11. (a) $x = \sin \frac{1}{2}\theta, y = \cos \frac{1}{2}\theta, -\pi \leq \theta \leq \pi.$

$x^2 + y^2 = \sin^2 \frac{1}{2}\theta + \cos^2 \frac{1}{2}\theta = 1$. For $-\pi \leq \theta \leq 0$, we have $-1 \leq x \leq 0$ and $0 \leq y \leq 1$. For $0 < \theta \leq \pi$, we have $0 < x \leq 1$ and $1 > y \geq 0$. The graph is a semicircle.

(b)

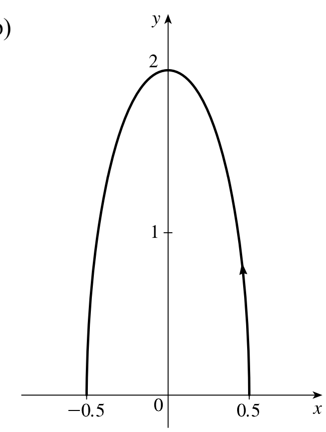


12. (a) $x = \frac{1}{2}\cos\theta, y = 2\sin\theta, 0 \leq \theta \leq \pi.$

$(2x)^2 + \left(\frac{1}{2}y\right)^2 = \cos^2\theta + \sin^2\theta = 1 \;\Rightarrow\; 4x^2 + \frac{1}{4}y^2 = 1 \;\Rightarrow$

$\dfrac{x^2}{(1/2)^2} + \dfrac{y^2}{2^2} = 1$, which is an equation of an ellipse with x-intercepts $\pm\frac{1}{2}$ and y-intercepts ± 2. For $0 \leq \theta \leq \pi/2$, we have $\frac{1}{2} \geq x \geq 0$ and $0 \leq y \leq 2$. For $\pi/2 < \theta \leq \pi$, we have $0 > x \geq -\frac{1}{2}$ and $2 > y \geq 0$. So the graph is the top half of the ellipse.

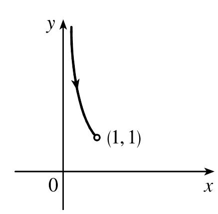
(b)



13. (a) $x = \sin t, y = \csc t, 0 < t < \frac{\pi}{2}$. $y = \csc t = \dfrac{1}{\sin t} = \dfrac{1}{x}$.

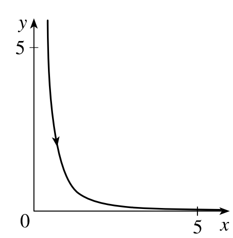
For $0 < t < \frac{\pi}{2}$, we have $0 < x < 1$ and $y > 1$. Thus, the curve is the portion of the hyperbola $y = 1/x$ with $y > 1$.

(b)



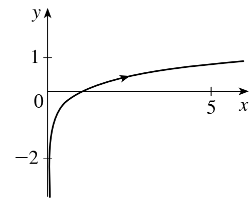
14. (a) $y = e^{-2t} = (e^t)^{-2} = x^{-2} = 1/x^2$ for $x > 0$ since $x = e^t$.

(b)



15. (a) $y = \ln t \quad \Rightarrow \quad t = e^y$, so $x = t^2 = (e^y)^2 = e^{2y}$.

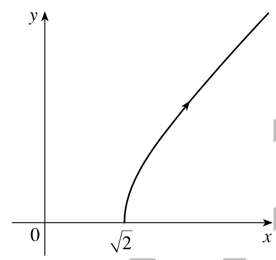
(b)



16. (a) $x = \sqrt{t+1} \quad \Rightarrow \quad x^2 = t + 1 \quad \Rightarrow \quad t = x^2 - 1$.

$y = \sqrt{t-1} = \sqrt{(x^2-1)-1} = \sqrt{x^2-2}$. The curve is the part of the hyperbola $x^2 - y^2 = 2$ with $x \geq \sqrt{2}$ and $y \geq 0$.

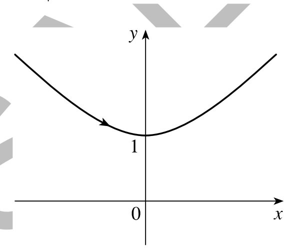
(b)



17. (a) $x = \sinh t$, $y = \cosh t \quad \Rightarrow \quad y^2 - x^2 = \cosh^2 t - \sinh^2 t = 1$.

Since $y = \cosh t \geq 1$, we have the upper branch of the hyperbola $y^2 - x^2 = 1$.

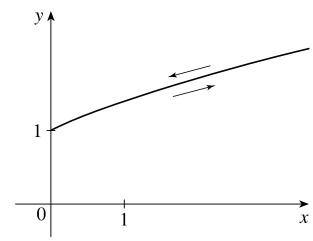
(b)



18. (a) $x = \tan^2 \theta$, $y = \sec \theta$, $-\pi/2 < \theta < \pi/2$.

$1 + \tan^2 \theta = \sec^2 \theta \quad \Rightarrow \quad 1 + x = y^2 \quad \Rightarrow \quad x = y^2 - 1$. For $-\pi/2 < \theta \leq 0$, we have $x \geq 0$ and $y \geq 1$. For $0 < \theta < \pi/2$, we have $0 < x$ and $1 < y$. Thus, the curve is the portion of the parabola $x = y^2 - 1$ in the first quadrant. As θ increases from $-\pi/2$ to 0, the point (x, y) approaches $(0, 1)$ along the parabola. As θ increases from 0 to $\pi/2$, the point (x, y) retreats from $(0, 1)$ along the parabola.

(b)



19. $x = 5 + 2\cos \pi t$, $y = 3 + 2\sin \pi t \quad \Rightarrow \quad \cos \pi t = \dfrac{x-5}{2}$, $\sin \pi t = \dfrac{y-3}{2}$. $\quad \cos^2(\pi t) + \sin^2(\pi t) = 1 \quad \Rightarrow$

$\left(\dfrac{x-5}{2}\right)^2 + \left(\dfrac{y-3}{2}\right)^2 = 1$. The motion of the particle takes place on a circle centered at $(5, 3)$ with a radius 2. As t goes from 1 to 2, the particle starts at the point $(3, 3)$ and moves counterclockwise along the circle $\left(\dfrac{x-5}{2}\right)^2 + \left(\dfrac{y-3}{2}\right)^2 = 1$ to $(7, 3)$ [one-half of a circle].

20. $x = 2 + \sin t$, $y = 1 + 3\cos t \quad \Rightarrow \quad \sin t = x - 2$, $\cos t = \dfrac{y-1}{3}$. $\quad \sin^2 t + \cos^2 t = 1 \quad \Rightarrow \quad (x-2)^2 + \left(\dfrac{y-1}{3}\right)^2 = 1$.

The motion of the particle takes place on an ellipse centered at $(2, 1)$. As t goes from $\pi/2$ to 2π, the particle starts at the point $(3, 1)$ and moves counterclockwise three-fourths of the way around the ellipse to $(2, 4)$.

21. $x = 5\sin t,\ y = 2\cos t \quad \Rightarrow \quad \sin t = \frac{x}{5},\ \cos t = \frac{y}{2}.\quad \sin^2 t + \cos^2 t = 1 \quad \Rightarrow \quad \left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$. The motion of the particle takes place on an ellipse centered at $(0, 0)$. As t goes from $-\pi$ to 5π, the particle starts at the point $(0, -2)$ and moves clockwise around the ellipse 3 times.

22. $y = \cos^2 t = 1 - \sin^2 t = 1 - x^2$. The motion of the particle takes place on the parabola $y = 1 - x^2$. As t goes from -2π to $-\pi$, the particle starts at the point $(0, 1)$, moves to $(1, 0)$, and goes back to $(0, 1)$. As t goes from $-\pi$ to 0, the particle moves to $(-1, 0)$ and goes back to $(0, 1)$. The particle repeats this motion as t goes from 0 to 2π.

23. We must have $1 \leq x \leq 4$ and $2 \leq y \leq 3$. So the graph of the curve must be contained in the rectangle $[1, 4]$ by $[2, 3]$.

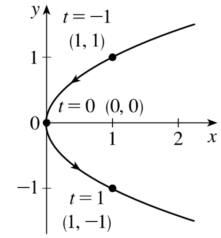
24. (a) From the first graph, we have $1 \leq x \leq 2$. From the second graph, we have $-1 \leq y \leq 1$. The only choice that satisfies either of those conditions is III.

(b) From the first graph, the values of x cycle through the values from -2 to 2 four times. From the second graph, the values of y cycle through the values from -2 to 2 six times. Choice I satisfies these conditions.

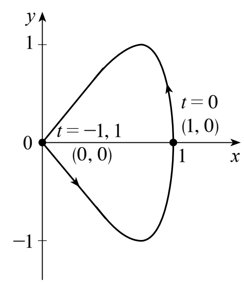
(c) From the first graph, the values of x cycle through the values from -2 to 2 three times. From the second graph, we have $0 \leq y \leq 2$. Choice IV satisfies these conditions.

(d) From the first graph, the values of x cycle through the values from -2 to 2 two times. From the second graph, the values of y do the same thing. Choice II satisfies these conditions.

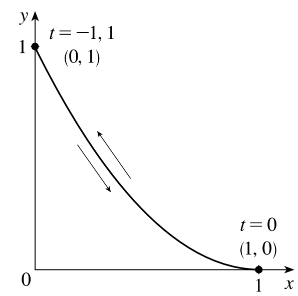
25. When $t = -1$, $(x, y) = (1, 1)$. As t increases to 0, x and y both decrease to 0. As t increases from 0 to 1, x increases from 0 to 1 and y decreases from 0 to -1. As t increases beyond 1, x continues to increase and y continues to decrease. For $t < -1$, x and y are both positive and decreasing. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



26. When $t = -1$, $(x, y) = (0, 0)$. As t increases to 0, x increases from 0 to 1, while y first decreases to -1 and then increases to 0. As t increases from 0 to 1, x decreases from 1 to 0, while y first increases to 1 and then decreases to 0. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



27. When $t = -1$, $(x, y) = (0, 1)$. As t increases to 0, x increases from 0 to 1 and y decreases from 1 to 0. As t increases from 0 to 1, the curve is retraced in the opposite direction with x decreasing from 1 to 0 and y increasing from 0 to 1. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



28. (a) $x = t^4 - t + 1 = (t^4 + 1) - t > 0$ [think of the graphs of $y = t^4 + 1$ and $y = t$] and $y = t^2 \geq 0$, so these equations are matched with graph V.

(b) $y = \sqrt{t} \geq 0$. $x = t^2 - 2t = t(t-2)$ is negative for $0 < t < 2$, so these equations are matched with graph I.

(c) $x = \sin 2t$ has period $2\pi/2 = \pi$. Note that

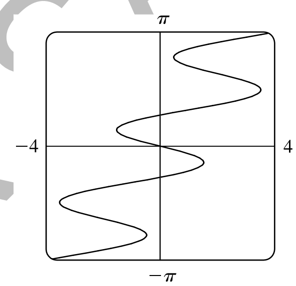
$y(t + 2\pi) = \sin[t + 2\pi + \sin 2(t + 2\pi)] = \sin(t + 2\pi + \sin 2t) = \sin(t + \sin 2t) = y(t)$, so y has period 2π. These equations match graph II since x cycles through the values -1 to 1 twice as y cycles through those values once.

(d) $x = \cos 5t$ has period $2\pi/5$ and $y = \sin 2t$ has period π, so x will take on the values -1 to 1, and then 1 to -1, before y takes on the values -1 to 1. Note that when $t = 0$, $(x, y) = (1, 0)$. These equations are matched with graph VI.

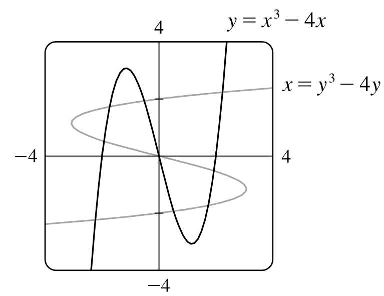
(e) $x = t + \sin 4t$, $y = t^2 + \cos 3t$. As t becomes large, t and t^2 become the dominant terms in the expressions for x and y, so the graph will look like the graph of $y = x^2$, but with oscillations. These equations are matched with graph IV.

(f) $x = \dfrac{\sin 2t}{4 + t^2}$, $y = \dfrac{\cos 2t}{4 + t^2}$. As $t \to \infty$, x and y both approach 0. These equations are matched with graph III.

29. Use $y = t$ and $x = t - 2\sin \pi t$ with a t-interval of $[-\pi, \pi]$.



30. Use $x_1 = t$, $y_1 = t^3 - 4t$ and $x_2 = t^3 - 4t$, $y_2 = t$ with a t-interval of $[-3, 3]$. There are 9 points of intersection; $(0, 0)$ is fairly obvious. The point in quadrant I is approximately $(2.2, 2.2)$, and by symmetry, the point in quadrant III is approximately $(-2.2, -2.2)$. The other six points are approximately $(\mp 1.9, \pm 0.5)$, $(\mp 1.7, \pm 1.7)$, and $(\mp 0.5, \pm 1.9)$.



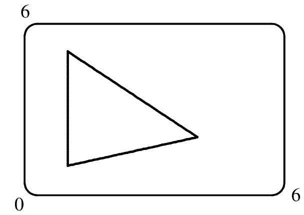
31. (a) $x = x_1 + (x_2 - x_1)t$, $y = y_1 + (y_2 - y_1)t$, $0 \leq t \leq 1$. Clearly the curve passes through $P_1(x_1, y_1)$ when $t = 0$ and through $P_2(x_2, y_2)$ when $t = 1$. For $0 < t < 1$, x is strictly between x_1 and x_2 and y is strictly between y_1 and y_2. For every value of t, x and y satisfy the relation $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, which is the equation of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Finally, any point (x, y) on that line satisfies $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$; if we call that common value t, then the given parametric equations yield the point (x, y); and any (x, y) on the line between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ yields a value of t in $[0, 1]$. So the given parametric equations exactly specify the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$.

(b) $x = -2 + [3 - (-2)]t = -2 + 5t$ and $y = 7 + (-1 - 7)t = 7 - 8t$ for $0 \leq t \leq 1$.

32. For the side of the triangle from A to B, use $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (4, 2)$. Hence, the equations are

$$x = x_1 + (x_2 - x_1)\,t = 1 + (4 - 1)\,t = 1 + 3t,$$
$$y = y_1 + (y_2 - y_1)\,t = 1 + (2 - 1)\,t = 1 + t.$$



Graphing $x = 1 + 3t$ and $y = 1 + t$ with $0 \le t \le 1$ gives us the side of the triangle from A to B. Similarly, for the side BC we use $x = 4 - 3t$ and $y = 2 + 3t$, and for the side AC we use $x = 1$ and $y = 1 + 4t$.

33. The circle $x^2 + (y - 1)^2 = 4$ has center $(0, 1)$ and radius 2, so by Example 4 it can be represented by $x = 2\cos t$, $y = 1 + 2\sin t$, $0 \le t \le 2\pi$. This representation gives us the circle with a counterclockwise orientation starting at $(2, 1)$.

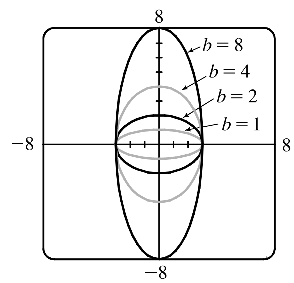
(a) To get a clockwise orientation, we could change the equations to $x = 2\cos t$, $y = 1 - 2\sin t$, $0 \le t \le 2\pi$.

(b) To get three times around in the counterclockwise direction, we use the original equations $x = 2\cos t$, $y = 1 + 2\sin t$ with the domain expanded to $0 \le t \le 6\pi$.

(c) To start at $(0, 3)$ using the original equations, we must have $x_1 = 0$; that is, $2\cos t = 0$. Hence, $t = \frac{\pi}{2}$. So we use $x = 2\cos t$, $y = 1 + 2\sin t$, $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

Alternatively, if we want t to start at 0, we could change the equations of the curve. For example, we could use $x = -2\sin t$, $y = 1 + 2\cos t$, $0 \le t \le \pi$.

34. (a) Let $x^2/a^2 = \sin^2 t$ and $y^2/b^2 = \cos^2 t$ to obtain $x = a\sin t$ and $y = b\cos t$ with $0 \le t \le 2\pi$ as possible parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$.



(b) The equations are $x = 3\sin t$ and $y = b\cos t$ for $b \in \{1, 2, 4, 8\}$.

(c) As b increases, the ellipse stretches vertically.

35. *Big circle:* It's centered at $(2, 2)$ with a radius of 2, so by Example 4, parametric equations are

$$x = 2 + 2\cos t, \qquad y = 2 + 2\sin t, \qquad 0 \le t \le 2\pi$$

Small circles: They are centered at $(1, 3)$ and $(3, 3)$ with a radius of 0.1. By Example 4, parametric equations are

	(left)	$x = 1 + 0.1\cos t,$	$y = 3 + 0.1\sin t,$	$0 \le t \le 2\pi$
and	*(right)*	$x = 3 + 0.1\cos t,$	$y = 3 + 0.1\sin t,$	$0 \le t \le 2\pi$

Semicircle: It's the lower half of a circle centered at $(2, 2)$ with radius 1. By Example 4, parametric equations are

$$x = 2 + 1\cos t, \qquad y = 2 + 1\sin t, \qquad \pi \le t \le 2\pi$$

To get all four graphs on the same screen with a typical graphing calculator, we need to change the last t-interval to$[0, 2\pi]$ in order to match the others. We can do this by changing t to $0.5t$. This change gives us the upper half. There are several ways to get the lower half—one is to change the "+" to a "−" in the y-assignment, giving us

$$x = 2 + 1\cos(0.5t), \qquad y = 2 - 1\sin(0.5t), \qquad 0 \le t \le 2\pi$$

36. If you are using a calculator or computer that can overlay graphs (using multiple t-intervals), the following is appropriate.

Left side: $x = 1$ and y goes from 1.5 to 4, so use

$$x = 1, \qquad y = t, \qquad 1.5 \le t \le 4$$

Right side: $x = 10$ and y goes from 1.5 to 4, so use

$$x = 10, \qquad y = t, \qquad 1.5 \le t \le 4$$

Bottom: x goes from 1 to 10 and $y = 1.5$, so use

$$x = t, \qquad y = 1.5, \qquad 1 \le t \le 10$$

Handle: It starts at $(10, 4)$ and ends at $(13, 7)$, so use

$$x = 10 + t, \qquad y = 4 + t, \qquad 0 \le t \le 3$$

Left wheel: It's centered at $(3, 1)$, has a radius of 1, and appears to go about 30° above the horizontal, so use

$$x = 3 + 1\cos t, \qquad y = 1 + 1\sin t, \qquad \tfrac{5\pi}{6} \le t \le \tfrac{13\pi}{6}$$

Right wheel: Similar to the left wheel with center $(8, 1)$, so use

$$x = 8 + 1\cos t, \qquad y = 1 + 1\sin t, \qquad \tfrac{5\pi}{6} \le t \le \tfrac{13\pi}{6}$$

If you are using a calculator or computer that cannot overlay graphs (using one t-interval), the following is appropriate. We'll start by picking the t-interval $[0, 2.5]$ since it easily matches the t-values for the two sides. We now need to find parametric equations for all graphs with $0 \le t \le 2.5$.

Left side: $x = 1$ and y goes from 1.5 to 4, so use

$$x = 1, \qquad y = 1.5 + t, \qquad 0 \le t \le 2.5$$

Right side: $x = 10$ and y goes from 1.5 to 4, so use

$$x = 10, \qquad y = 1.5 + t, \qquad 0 \le t \le 2.5$$

Bottom: x goes from 1 to 10 and $y = 1.5$, so use

$$x = 1 + 3.6t, \qquad y = 1.5, \qquad 0 \le t \le 2.5$$

To get the x-assignment, think of creating a linear function such that when $t = 0$, $x = 1$ and when $t = 2.5$, $x = 10$. We can use the point-slope form of a line with $(t_1, x_1) = (0, 1)$ and $(t_2, x_2) = (2.5, 10)$.

$x - 1 = \dfrac{10 - 1}{2.5 - 0}(t - 0) \quad \Rightarrow \quad x = 1 + 3.6t.$

Handle: It starts at $(10, 4)$ and ends at $(13, 7)$, so use

$$x = 10 + 1.2t, \qquad y = 4 + 1.2t, \qquad 0 \le t \le 2.5$$

$(t_1, x_1) = (0, 10)$ and $(t_2, x_2) = (2.5, 13)$ gives us $x - 10 = \dfrac{13 - 10}{2.5 - 0}(t - 0) \quad \Rightarrow \quad x = 10 + 1.2t.$

$(t_1, y_1) = (0, 4)$ and $(t_2, y_2) = (2.5, 7)$ gives us $y - 4 = \dfrac{7 - 4}{2.5 - 0}(t - 0) \quad \Rightarrow \quad y = 4 + 1.2t.$

Left wheel: It's centered at $(3,1)$, has a radius of 1, and appears to go about $30°$ above the horizontal, so use

$$x = 3 + 1\cos\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad y = 1 + 1\sin\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad 0 \le t \le 2.5$$

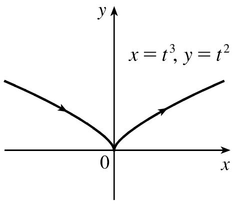
$(t_1, \theta_1) = \left(0, \tfrac{5\pi}{6}\right)$ and $(t_2, \theta_2) = \left(\tfrac{5}{2}, \tfrac{13\pi}{6}\right)$ gives us $\theta - \tfrac{5\pi}{6} = \dfrac{\frac{13\pi}{6} - \frac{5\pi}{6}}{\frac{5}{2} - 0}(t - 0) \;\Rightarrow\; \theta = \tfrac{5\pi}{6} + \tfrac{8\pi}{15}t.$

Right wheel: Similar to the left wheel with center $(8,1)$, so use

$$x = 8 + 1\cos\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad y = 1 + 1\sin\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad 0 \le t \le 2.5$$

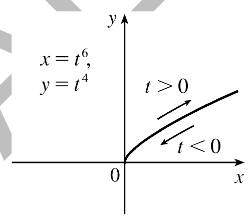
37. (a) $x = t^3 \;\Rightarrow\; t = x^{1/3}$, so $y = t^2 = x^{2/3}$.

We get the entire curve $y = x^{2/3}$ traversed in a left to right direction.



(b) $x = t^6 \;\Rightarrow\; t = x^{1/6}$, so $y = t^4 = x^{4/6} = x^{2/3}$.

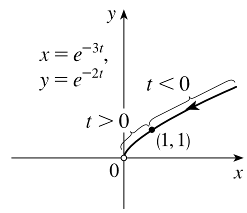
Since $x = t^6 \ge 0$, we only get the right half of the curve $y = x^{2/3}$.



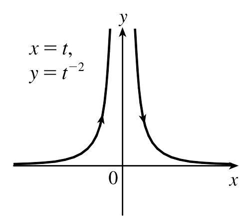
(c) $x = e^{-3t} = (e^{-t})^3$ [so $e^{-t} = x^{1/3}$],

$y = e^{-2t} = (e^{-t})^2 = (x^{1/3})^2 = x^{2/3}$.

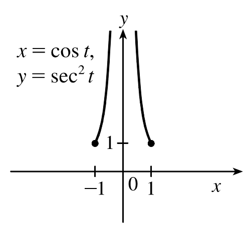
If $t < 0$, then x and y are both larger than 1. If $t > 0$, then x and y are between 0 and 1. Since $x > 0$ and $y > 0$, the curve never quite reaches the origin.



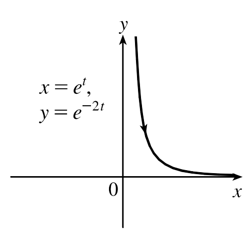
38. (a) $x = t$, so $y = t^{-2} = x^{-2}$. We get the entire curve $y = 1/x^2$ traversed in a left-to-right direction.



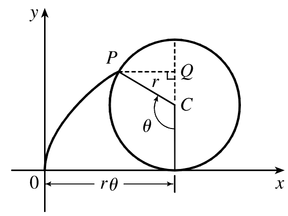
(b) $x = \cos t$, $y = \sec^2 t = \dfrac{1}{\cos^2 t} = \dfrac{1}{x^2}$. Since $\sec t \ge 1$, we only get the parts of the curve $y = 1/x^2$ with $y \ge 1$. We get the first quadrant portion of the curve when $x > 0$, that is, $\cos t > 0$, and we get the second quadrant portion of the curve when $x < 0$, that is, $\cos t < 0$.



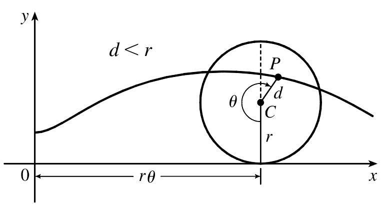
(c) $x = e^t$, $y = e^{-2t} = (e^t)^{-2} = x^{-2}$. Since e^t and e^{-2t} are both positive, we only get the first quadrant portion of the curve $y = 1/x^2$.

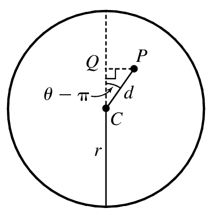


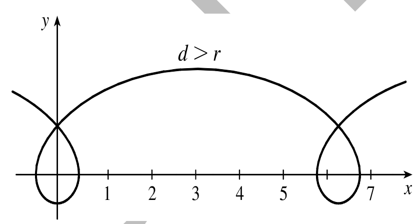
39. The case $\frac{\pi}{2} < \theta < \pi$ is illustrated. C has coordinates $(r\theta, r)$ as in Example 7, and Q has coordinates $(r\theta, r + r\cos(\pi - \theta)) = (r\theta, r(1 - \cos\theta))$ [since $\cos(\pi - \alpha) = \cos\pi\cos\alpha + \sin\pi\sin\alpha = -\cos\alpha$], so P has coordinates $(r\theta - r\sin(\pi - \theta), r(1 - \cos\theta)) = (r(\theta - \sin\theta), r(1 - \cos\,\theta))$ [since $\sin(\pi - \alpha) = \sin\pi\cos\alpha - \cos\pi\sin\alpha = \sin\alpha$]. Again we have the parametric equations $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



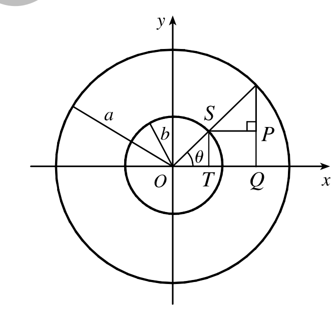
40. The first two diagrams depict the case $\pi < \theta < \frac{3\pi}{2}$, $d < r$. As in Example 7, C has coordinates $(r\theta, r)$. Now Q (in the second diagram) has coordinates $(r\theta, r + d\cos(\theta - \pi)) = (r\theta, r - d\cos\theta)$, so a typical point P of the trochoid has coordinates $(r\theta + d\sin(\theta - \pi), r - d\cos\theta)$. That is, P has coordinates (x, y), where $x = r\theta - d\sin\theta$ and $y = r - d\cos\theta$. When $d = r$, these equations agree with those of the cycloid.





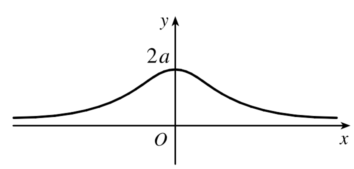


41. It is apparent that $x = |OQ|$ and $y = |QP| = |ST|$. From the diagram, $x = |OQ| = a\cos\theta$ and $y = |ST| = b\sin\theta$. Thus, the parametric equations are $x = a\cos\theta$ and $y = b\sin\theta$. To eliminate θ we rearrange: $\sin\theta = y/b \;\Rightarrow\; \sin^2\theta = (y/b)^2$ and $\cos\theta = x/a \;\Rightarrow\; \cos^2\theta = (x/a)^2$. Adding the two equations: $\sin^2\theta + \cos^2\theta = 1 = x^2/a^2 + y^2/b^2$. Thus, we have an ellipse.



42. A has coordinates $(a\cos\theta, a\sin\theta)$. Since OA is perpendicular to AB, ΔOAB is a right triangle and B has coordinates $(a\sec\theta, 0)$. It follows that P has coordinates $(a\sec\theta, b\sin\,\theta)$. Thus, the parametric equations are $x = a\sec\theta$, $y = b\sin\theta$.

43. $C = (2a\cot\theta, 2a)$, so the x-coordinate of P is $x = 2a\cot\theta$. Let $B = (0, 2a)$. Then $\angle OAB$ is a right angle and $\angle OBA = \theta$, so $|OA| = 2a\sin\theta$ and $A = ((2a\sin\theta)\cos\theta, (2a\sin\theta)\sin\theta)$. Thus, the y-coordinate of P is $y = 2a\sin^2\theta$.



44. (a) Let θ be the angle of inclination of segment OP. Then $|OB| = \dfrac{2a}{\cos\theta}$.

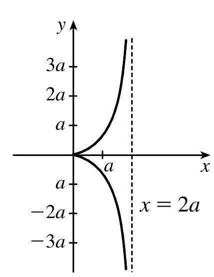
Let $C = (2a, 0)$. Then by use of right triangle OAC we see that $|OA| = 2a\cos\theta$.

Now

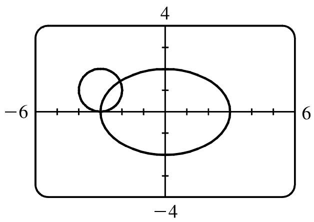
$$\begin{aligned} |OP| &= |AB| = |OB| - |OA| \\ &= 2a\left(\frac{1}{\cos\theta} - \cos\theta\right) = 2a\,\frac{1 - \cos^2\theta}{\cos\theta} = 2a\,\frac{\sin^2\theta}{\cos\theta} = 2a\sin\theta\,\tan\theta \end{aligned}$$

So P has coordinates $x = 2a\sin\theta\,\tan\theta \cdot \cos\theta = 2a\sin^2\theta$ and $y = 2a\sin\theta\,\tan\theta \cdot \sin\theta = 2a\sin^2\theta\,\tan\theta$.

(b)



45. (a)



There are 2 points of intersection:

$(-3, 0)$ and approximately $(-2.1, 1.4)$.

(b) A collision point occurs when $x_1 = x_2$ and $y_1 = y_2$ for the same t. So solve the equations:

$$3\sin t = -3 + \cos t \quad \textbf{(1)}$$
$$2\cos t = 1 + \sin t \quad \textbf{(2)}$$

From **(2)**, $\sin t = 2\cos t - 1$. Substituting into **(1)**, we get $3(2\cos t - 1) = -3 + \cos t \;\Rightarrow\; 5\cos t = 0 \;\; (\star) \;\Rightarrow$ $\cos t = 0 \;\Rightarrow\; t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. We check that $t = \frac{3\pi}{2}$ satisfies **(1)** and **(2)** but $t = \frac{\pi}{2}$ does not. So the only collision point occurs when $t = \frac{3\pi}{2}$, and this gives the point $(-3, 0)$. [We could check our work by graphing x_1 and x_2 together as functions of t and, on another plot, y_1 and y_2 as functions of t. If we do so, we see that the only value of t for which *both* pairs of graphs intersect is $t = \frac{3\pi}{2}$.]

(c) The circle is centered at $(3, 1)$ instead of $(-3, 1)$. There are still 2 intersection points: $(3, 0)$ and $(2.1, 1.4)$, but there are no collision points, since $(\star)$ in part (b) becomes $5\cos t = 6 \;\Rightarrow\; \cos t = \frac{6}{5} > 1$.

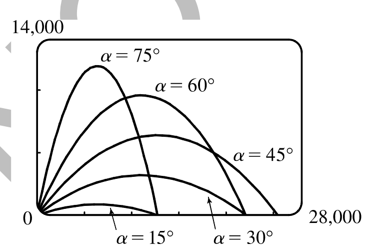
46. (a) If $\alpha = 30°$ and $v_0 = 500$ m/s, then the equations become $x = (500\cos 30°)t = 250\sqrt{3}t$ and $y = (500\sin 30°)t - \frac{1}{2}(9.8)t^2 = 250t - 4.9t^2$. $y = 0$ when $t = 0$ (when the gun is fired) and again when $t = \frac{250}{4.9} \approx 51$ s. Then $x = \left(250\sqrt{3}\right)\left(\frac{250}{4.9}\right) \approx 22{,}092$ m, so the bullet hits the ground about 22 km from the gun.

The formula for y is quadratic in t. To find the maximum y-value, we will complete the square:

$$y = -4.9\left(t^2 - \tfrac{250}{4.9}t\right) = -4.9\left[t^2 - \tfrac{250}{4.9}t + \left(\tfrac{125}{4.9}\right)^2\right] + \tfrac{125^2}{4.9} = -4.9\left(t - \tfrac{125}{4.9}\right)^2 + \tfrac{125^2}{4.9} \le \tfrac{125^2}{4.9}$$

with equality when $t = \frac{125}{4.9}$ s, so the maximum height attained is $\frac{125^2}{4.9} \approx 3189$ m.

(b)



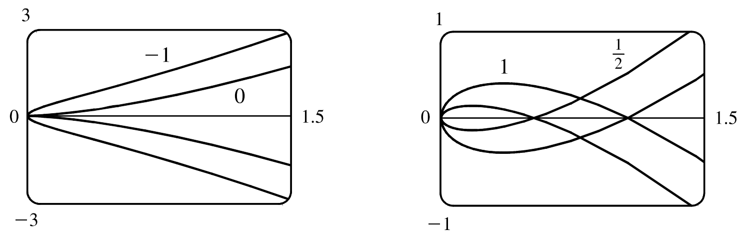
As α $(0° < \alpha < 90°)$ increases up to $45°$, the projectile attains a greater height and a greater range. As α increases past $45°$, the projectile attains a greater height, but its range decreases.

(c) $x = (v_0\cos\alpha)t \;\Rightarrow\; t = \dfrac{x}{v_0\cos\alpha}$.

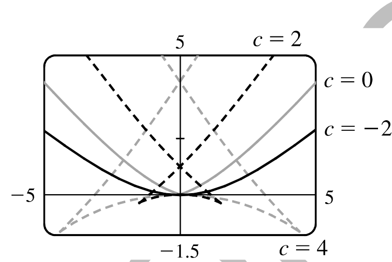
$$y = (v_0\sin\alpha)t - \tfrac{1}{2}gt^2 \;\Rightarrow\; y = (v_0\sin\alpha)\frac{x}{v_0\cos\alpha} - \frac{g}{2}\left(\frac{x}{v_0\cos\alpha}\right)^2 = (\tan\alpha)x - \left(\frac{g}{2v_0^2\cos^2\alpha}\right)x^2,$$

which is the equation of a parabola (quadratic in x).

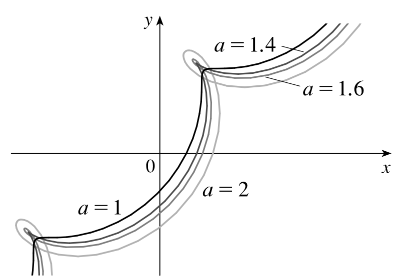
47. $x = t^2, y = t^3 - ct$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the x-axis. For $c < 0$, the graph does not cross itself, but for $c = 0$ it has a cusp at $(0, 0)$ and for $c > 0$ the graph crosses itself at $x = c$, so the loop grows larger as c increases.



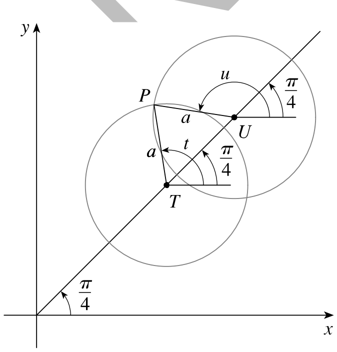
48. $x = 2ct - 4t^3, y = -ct^2 + 3t^4$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the y-axis. When $c < 0$, the graph resembles that of a polynomial of even degree, but when $c = 0$ there is a corner at the origin, and when $c > 0$, the graph crosses itself at the origin, and has two cusps below the x-axis. The size of the "swallowtail" increases as c increases.



49. $x = t + a\cos t, y = t + a\sin t, a > 0$. From the first figure, we see that curves roughly follow the line $y = x$, and they start having loops when a is between 1.4 and 1.6. The loops increase in size as a increases.



While not required, the following is a solution to determine the *exact* values for which the curve has a loop, that is, we seek the values of a for which there exist parameter values t and u such that $t < u$ and $(t + a\cos t, t + a\sin t) = (u + a\cos u, u + a\sin u)$.



In the diagram at the left, T denotes the point (t, t), U the point (u, u), and P the point $(t + a\cos t, t + a\sin t) = (u + a\cos u, u + a\sin u)$. Since $\overline{PT} = \overline{PU} = a$, the triangle PTU is isosceles. Therefore its base angles, $\alpha = \angle PTU$ and $\beta = \angle PUT$ are equal. Since $\alpha = t - \frac{\pi}{4}$ and $\beta = 2\pi - \frac{3\pi}{4} - u = \frac{5\pi}{4} - u$, the relation $\alpha = \beta$ implies that $u + t = \frac{3\pi}{2}$ **(1)**.

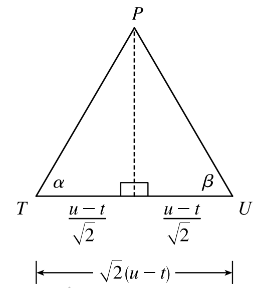
Since $\overline{TU} = \text{distance}((t,t),(u,u)) = \sqrt{2(u-t)^2} = \sqrt{2}\,(u-t)$, we see that

$\cos\alpha = \dfrac{\frac{1}{2}\overline{TU}}{\overline{PT}} = \dfrac{(u-t)/\sqrt{2}}{a}$, so $u - t = \sqrt{2}\,a\cos\alpha$, that is,

$u - t = \sqrt{2}\,a\cos\!\left(t - \frac{\pi}{4}\right)$ **(2)**. Now $\cos\!\left(t-\frac{\pi}{4}\right) = \sin\!\left[\frac{\pi}{2} - \left(t - \frac{\pi}{4}\right)\right] = \sin\!\left(\frac{3\pi}{4} - t\right)$,

so we can rewrite **(2)** as $u - t = \sqrt{2}\,a\sin\!\left(\frac{3\pi}{4} - t\right)$ **(2′)**. Subtracting **(2′)** from **(1)** and

dividing by 2, we obtain $t = \frac{3\pi}{4} - \frac{\sqrt{2}}{2}a\sin\!\left(\frac{3\pi}{4} - t\right)$, or $\frac{3\pi}{4} - t = \frac{a}{\sqrt{2}}\sin\!\left(\frac{3\pi}{4} - t\right)$ **(3)**.



Since $a > 0$ and $t < u$, it follows from **(2′)** that $\sin\!\left(\frac{3\pi}{4} - t\right) > 0$. Thus from **(3)** we see that $t < \frac{3\pi}{4}$. [We have implicitly assumed that $0 < t < \pi$ by the way we drew our diagram, but we lost no generality by doing so since replacing t by $t + 2\pi$ merely increases x and y by 2π. The curve's basic shape repeats every time we change t by 2π.] Solving for a in **(3)**, we get $a = \dfrac{\sqrt{2}\left(\frac{3\pi}{4} - t\right)}{\sin\left(\frac{3\pi}{4} - t\right)}$. Write $z = \frac{3\pi}{4} - t$. Then $a = \dfrac{\sqrt{2}\,z}{\sin z}$, where $z > 0$. Now $\sin z < z$ for $z > 0$, so $a > \sqrt{2}$.

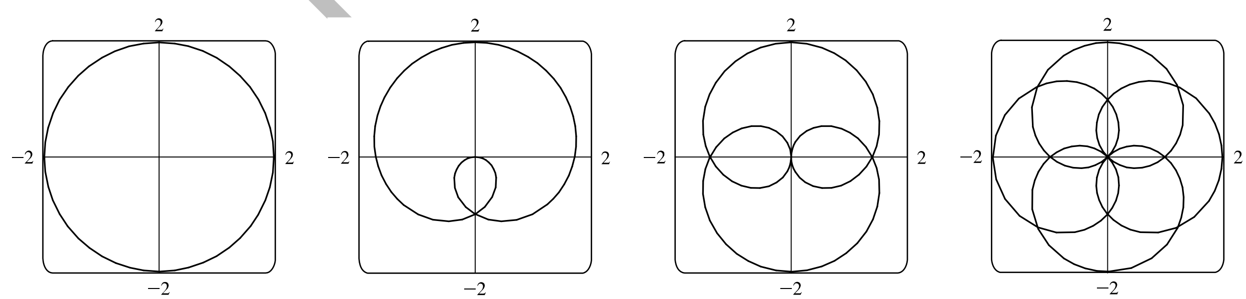
$\left[\text{As } z \to 0^+, \text{ that is, as } t \to \left(\frac{3\pi}{4}\right)^-, a \to \sqrt{2}\right]$.

50. Consider the curves $x = \sin t + \sin nt$, $y = \cos t + \cos nt$, where n is a positive integer. For $n = 1$, we get a circle of radius 2 centered at the origin. For $n > 1$, we get a curve lying on or inside that circle that traces out $n - 1$ loops as t ranges from 0 to 2π.

Note:

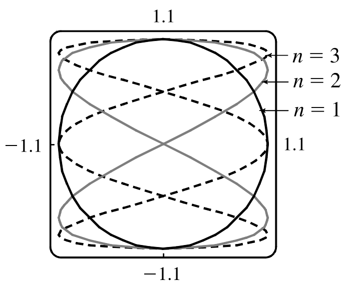
$$\begin{aligned} x^2 + y^2 &= (\sin t + \sin nt)^2 + (\cos t + \cos nt)^2 \\ &= \sin^2 t + 2\sin t\,\sin nt + \sin^2 nt + \cos^2 t + 2\cos t\,\cos nt + \cos^2 nt \\ &= (\sin^2 t + \cos^2 t) + (\sin^2 nt + \cos^2 nt) + 2(\cos t\,\cos nt + \sin t\,\sin nt) \\ &= 1 + 1 + 2\cos(t - nt) = 2 + 2\cos((1-n)t) \le 4 = 2^2, \end{aligned}$$

with equality for $n = 1$. This shows that each curve lies on or inside the curve for $n = 1$, which is a circle of radius 2 centered at the origin.

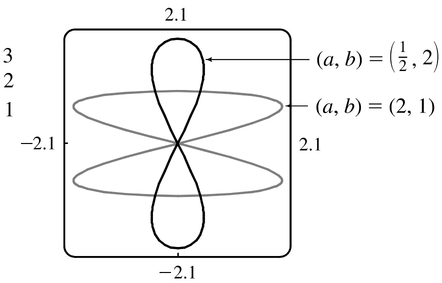


$n = 1$ $n = 2$ $n = 3$ $n = 5$

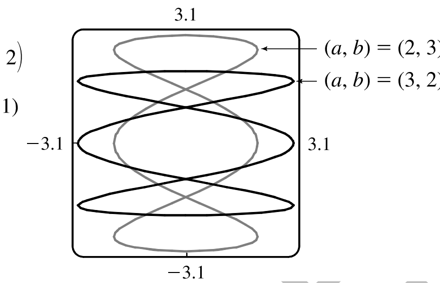
51. Note that all the Lissajous figures are symmetric about the x-axis. The parameters a and b simply stretch the graph in the x- and y-directions respectively. For $a = b = n = 1$ the graph is simply a circle with radius 1. For $n = 2$ the graph crosses itself at the origin and there are loops above and below the x-axis. In general, the figures have $n-1$ points of intersection, all of which are on the y-axis, and a total of n closed loops.



$a=b=1$

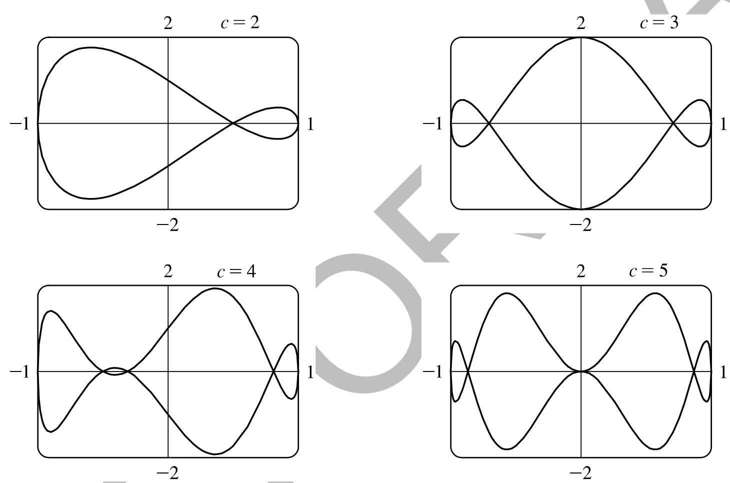


$n=2$

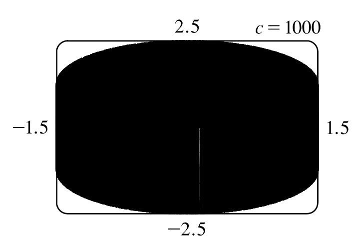


$n=3$

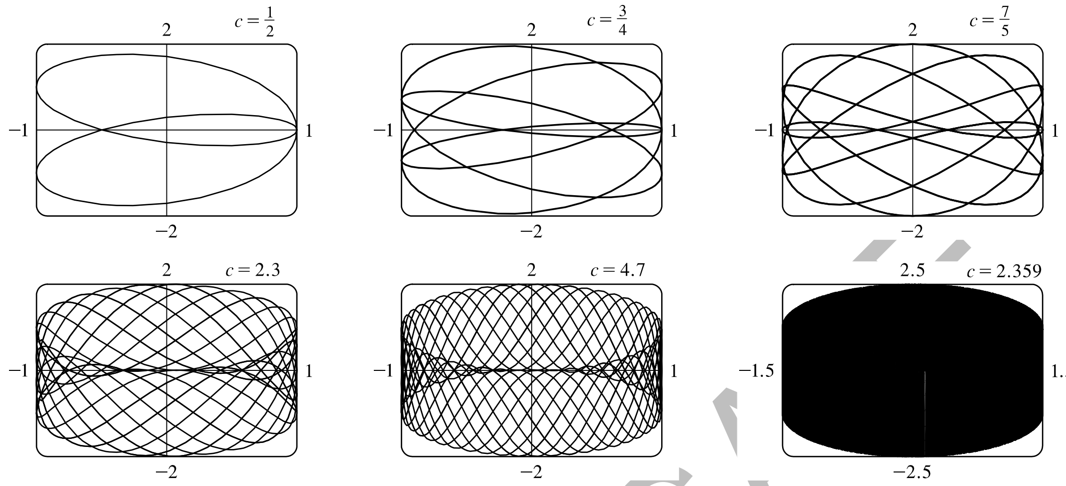
52. $x=\cos t$, $y=\sin t-\sin ct$. If $c=1$, then $y=0$, and the curve is simply the line segment from $(-1,0)$ to $(1,0)$. The graphs are shown for $c=2,3,4$ and 5.



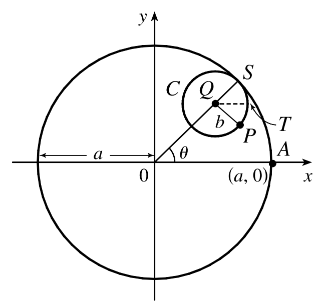
It is easy to see that all the curves lie in the rectangle $[-1,1]$ by $[-2,2]$. When c is an integer, $x(t+2\pi)=x(t)$ and $y(t+2\pi)=y(t)$, so the curve is closed. When c is a positive integer greater than 1, the curve intersects the x-axis $c+1$ times and has c loops (one of which degenerates to a tangency at the origin when c is an odd integer of the form $4k+1$). As c increases, the curve's loops become thinner, but stay in the region bounded by the semicircles $y=\pm\left(1+\sqrt{1-x^2}\,\right)$ and the line segments from $(-1,-1)$ to $(-1,1)$ and from $(1,-1)$ to $(1,1)$. This is true because $|y|=|\sin t-\sin ct|\leq|\sin t|+|\sin ct|\leq\sqrt{1-x^2}+1$. This curve appears to fill the entire region when c is very large, as shown in the figure for $c=1000$.



When c is a fraction, we get a variety of shapes with multiple loops, but always within the same region. For some fractional values, such as $c = 2.359$, the curve again appears to fill the region.



LABORATORY PROJECT Running Circles Around Circles

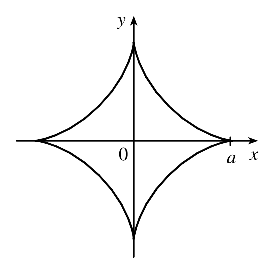


1. The center Q of the smaller circle has coordinates $((a-b)\cos\theta, (a-b)\sin\theta)$. Arc PS on circle C has length $a\theta$ since it is equal in length to arc AS (the smaller circle rolls without slipping against the larger.)

Thus, $\angle PQS = \frac{a}{b}\theta$ and $\angle PQT = \frac{a}{b}\theta - \theta$, so P has coordinates

$$x = (a-b)\cos\theta + b\cos(\angle PQT) = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$$

and $$y = (a-b)\sin\theta - b\sin(\angle PQT) = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right).$$

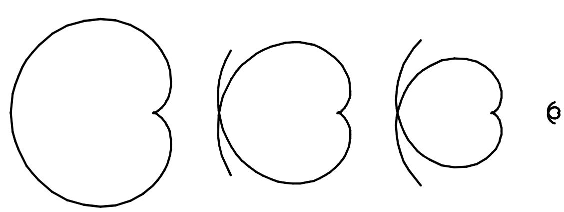


2. With $b = 1$ and a a positive integer greater than 2, we obtain a hypocycloid of a cusps. Shown in the figure is the graph for $a = 4$. Let $a = 4$ and $b = 1$. Using the sum identities to expand $\cos 3\theta$ and $\sin 3\theta$, we obtain

$$x = 3\cos\theta + \cos 3\theta = 3\cos\theta + \left(4\cos^3\theta - 3\cos\theta\right) = 4\cos^3\theta$$

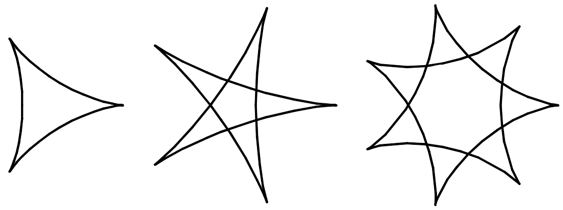
and $$y = 3\sin\theta - \sin 3\theta = 3\sin\theta - \left(3\sin\theta - 4\sin^3\theta\right) = 4\sin^3\theta.$$

3. The graphs at the right are obtained with $b = 1$ and $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, and $\frac{1}{10}$ with $-2\pi \le \theta \le 2\pi$. We conclude that as the denominator d increases, the graph gets smaller, but maintains the basic shape shown.

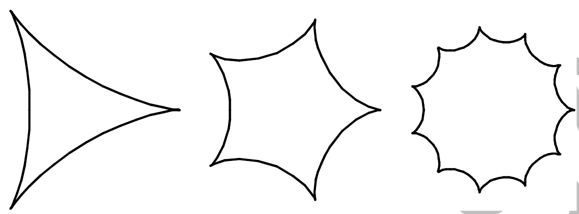


[continued]

Letting $d = 2$ and $n = 3$, 5, and 7 with $-2\pi \le \theta \le 2\pi$ gives us the following:



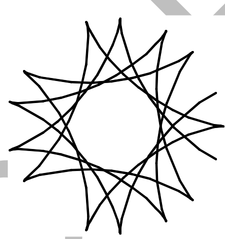
So if d is held constant and n varies, we get a graph with n cusps (assuming n/d is in lowest form). When $n = d + 1$, we obtain a hypocycloid of n cusps. As n increases, we must expand the range of θ in order to get a closed curve. The following graphs have $a = \frac{3}{2}$, $\frac{5}{4}$, and $\frac{11}{10}$.



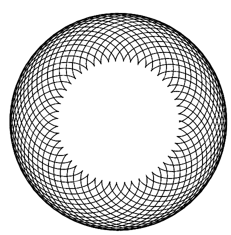
4. If $b = 1$, the equations for the hypocycloid are

$$x = (a-1)\cos\theta + \cos((a-1)\theta) \qquad y = (a-1)\sin\theta - \sin((a-1)\theta)$$

which is a hypocycloid of a cusps (from Problem 2). In general, if $a > 1$, we get a figure with cusps on the "outside ring" and if $a < 1$, the cusps are on the "inside ring". In any case, as the values of θ get larger, we get a figure that looks more and more like a washer. If we were to graph the hypocycloid for all values of θ, every point on the washer would eventually be arbitrarily close to a point on the curve.



$a = \sqrt{2}, \quad -10\pi \le \theta \le 10\pi$

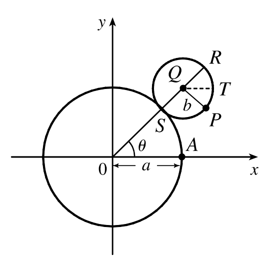


$a = e - 2, \quad 0 \le \theta \le 446$

5. The center Q of the smaller circle has coordinates $((a+b)\cos\theta, (a+b)\sin\theta)$.

Arc PS has length $a\theta$ (as in Problem 1), so that $\angle PQS = \dfrac{a\theta}{b}$, $\angle PQR = \pi - \dfrac{a\theta}{b}$,

and $\angle PQT = \pi - \dfrac{a\theta}{b} - \theta = \pi - \left(\dfrac{a+b}{b}\right)\theta$ since $\angle RQT = \theta$.



Thus, the coordinates of P are

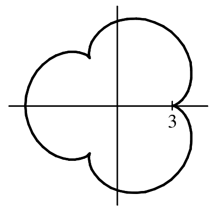
$$x = (a+b)\cos\theta + b\cos\left(\pi - \frac{a+b}{b}\theta\right) = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$$

and $$y = (a+b)\sin\theta - b\sin\left(\pi - \frac{a+b}{b}\theta\right) = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right).$$

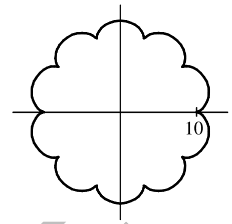
6. Let $b = 1$ and the equations become

$$x = (a + 1)\cos\theta - \cos((a + 1)\theta) \qquad y = (a + 1)\sin\theta - \sin((a + 1)\theta)$$

If $a = 1$, we have a cardioid. If a is a positive integer greater than 1, we get the graph of an "a-leafed clover", with cusps that are a units from the origin. (Some of the pairs of figures are not to scale.)

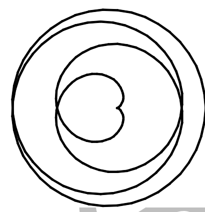


$a = 3, -2\pi \le \theta \le 2\pi$

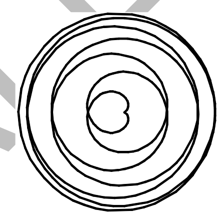


$a = 10, -2\pi \le \theta \le 2\pi$

If $a = n/d$ with $n = 1$, we obtain a figure that does not increase in size and requires $-d\pi \le \theta \le d\pi$ to be a closed curve traced exactly once.

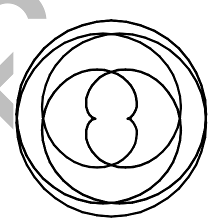


$a = \frac{1}{4}, -4\pi \le \theta \le 4\pi$

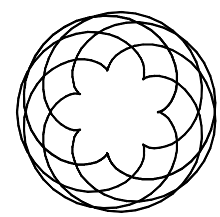


$a = \frac{1}{7}, -7\pi \le \theta \le 7\pi$

Next, we keep d constant and let n vary. As n increases, so does the size of the figure. There is an n-pointed star in the middle.

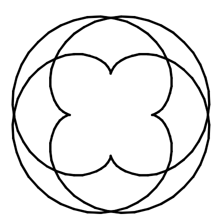


$a = \frac{2}{5}, -5\pi \le \theta \le 5\pi$

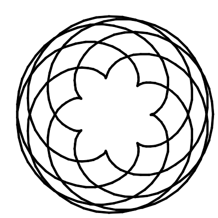


$a = \frac{7}{5}, -5\pi \le \theta \le 5\pi$

Now if $n = d + 1$ we obtain figures similar to the previous ones, but the size of the figure does not increase.

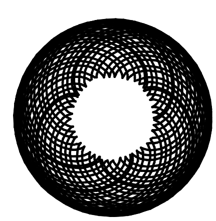


$a = \frac{4}{3}, -3\pi \le \theta \le 3\pi$

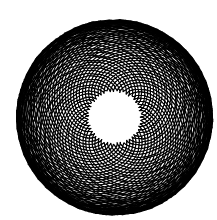


$a = \frac{7}{6}, -6\pi \le \theta \le 6\pi$

If a is irrational, we get washers that increase in size as a increases.



$a = \sqrt{2}, 0 \le \theta \le 200$



$a = e - 2, 0 \le \theta \le 446$

10.2 Calculus with Parametric Curves

1. $x = \dfrac{t}{1+t},\ y = \sqrt{1+t} \quad \Rightarrow \quad \dfrac{dy}{dt} = \dfrac{1}{2}(1+t)^{-1/2} = \dfrac{1}{2\sqrt{1+t}},\ \dfrac{dx}{dt} = \dfrac{(1+t)(1) - t(1)}{(1+t)^2} = \dfrac{1}{(1+t)^2}$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1/(2\sqrt{1+t}\,)}{1/(1+t)^2} = \dfrac{(1+t)^2}{2\sqrt{1+t}} = \dfrac{1}{2}(1+t)^{3/2}$.

2. $x = te^t,\ y = t + \sin t \quad \Rightarrow \quad \dfrac{dy}{dt} = 1 + \cos t,\ \dfrac{dx}{dt} = te^t + e^t = e^t(t+1)$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1+\cos t}{e^t(t+1)}$.

3. $x = t^3 + 1,\ y = t^4 + t;\quad t = -1.\ \dfrac{dy}{dt} = 4t^3 + 1,\ \dfrac{dx}{dt} = 3t^2$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t^3+1}{3t^2}$. When $t = -1$, $(x, y) = (0, 0)$ and $dy/dx = -3/3 = -1$, so an equation of the tangent to the curve at the point corresponding to $t = -1$ is $y - 0 = -1(x - 0)$, or $y = -x$.

4. $x = \sqrt{t},\ y = t^2 - 2t;\quad t = 4.\ \dfrac{dy}{dt} = 2t - 2,\ \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = (2t-2)2\sqrt{t} = 4(t-1)\sqrt{t}$. When $t = 4$, $(x, y) = (2, 8)$ and $dy/dx = 4(3)(2) = 24$, so an equation of the tangent to the curve at the point corresponding to $t = 4$ is $y - 8 = 24(x - 2)$, or $y = 24x - 40$.

5. $x = t\cos t,\ y = t\sin t;\ t = \pi.\quad \dfrac{dy}{dt} = t\cos t + \sin t,\ \dfrac{dx}{dt} = t(-\sin t) + \cos t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t\cos t + \sin t}{-t\sin t + \cos t}$. When $t = \pi$, $(x, y) = (-\pi, 0)$ and $dy/dx = -\pi/(-1) = \pi$, so an equation of the tangent to the curve at the point corresponding to $t = \pi$ is $y - 0 = \pi[x - (-\pi)]$, or $y = \pi x + \pi^2$.

6. $x = e^t \sin \pi t,\ y = e^{2t};\quad t = 0.\ \dfrac{dy}{dt} = 2e^{2t},\ \dfrac{dx}{dt} = e^t(\pi\cos\pi t) + (\sin\pi t)e^t = e^t(\pi\cos\pi t + \sin\pi t)$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2e^{2t}}{e^t(\pi\cos\pi t + \sin\pi t)} = \dfrac{2e^t}{\pi\cos\pi t + \sin\pi t}$. When $t = 0$, $(x, y) = (0, 1)$ and $dy/dx = 2/\pi$, so an equation of the tangent to the curve at the point corresponding to $t = 0$ is $y - 1 = \frac{2}{\pi}(x - 0)$, or $y = \frac{2}{\pi}x + 1$.

7. (a) $x = 1 + \ln t,\ y = t^2 + 2;\ (1, 3).\quad \dfrac{dy}{dt} = 2t,\ \dfrac{dx}{dt} = \dfrac{1}{t}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{1/t} = 2t^2$. At $(1, 3)$,

$x = 1 + \ln t = 1 \quad \Rightarrow \quad \ln t = 0 \quad \Rightarrow \quad t = 1$ and $\dfrac{dy}{dx} = 2$, so an equation of the tangent is $y - 3 = 2(x - 1)$, or $y = 2x + 1$.

(b) $x = 1 + \ln t \quad \Rightarrow \quad \ln t = x - 1 \quad \Rightarrow \quad t = e^{x-1}$, so $y = t^2 + 2 = (e^{x-1})^2 + 2 = e^{2x-2} + 2$, and $y' = e^{2x-2} \cdot 2$. At $(1, 3)$, $y' = e^{2(1)-2} \cdot 2 = 2$, so an equation of the tangent is $y - 3 = 2(x - 1)$, or $y = 2x + 1$.

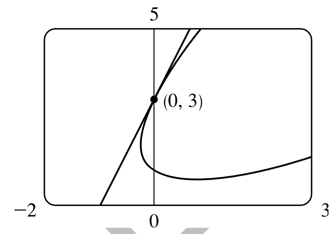
8. (a) $x = 1 + \sqrt{t},\ y = e^{t^2};\ (2, e).\quad \dfrac{dy}{dt} = e^{t^2} \cdot 2t,\ \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2te^{t^2}}{1/(2\sqrt{t}\,)} = 4t^{3/2}e^{t^2}$. At $(2, e)$,

$x = 1 + \sqrt{t} = 2 \quad \Rightarrow \quad \sqrt{t} = 1 \quad \Rightarrow \quad t = 1$ and $\dfrac{dy}{dx} = 4e$, so an equation of the tangent is $y - e = 4e(x - 2)$, or $y = 4ex - 7e$.

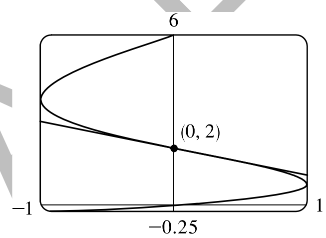
(b) $x = 1 + \sqrt{t} \;\Rightarrow\; \sqrt{t} = x - 1 \;\Rightarrow\; t = (x-1)^2$, so $y = e^{t^2} = e^{(x-1)^4}$, and $y' = e^{(x-1)^4} \cdot 4(x-1)^3$.

At $(2, e)$, $y' = e \cdot 4 = 4e$, so an equation of the tangent is $y - e = 4e(x - 2)$, or $y = 4ex - 7e$.

9. $x = t^2 - t,\; y = t^2 + t + 1;\; (0, 3)$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{2t-1}$. To find the value of t corresponding to the point $(0, 3)$, solve $x = 0 \;\Rightarrow\; t^2 - t = 0 \;\Rightarrow\; t(t-1) = 0 \;\Rightarrow\; t = 0$ or $t = 1$. Only $t = 1$ gives $y = 3$. With $t = 1$, $dy/dx = 3$, and an equation of the tangent is $y - 3 = 3(x - 0)$, or $y = 3x + 3$.



10. $x = \sin \pi t,\; y = t^2 + t;\; (0, 2)$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{\pi \cos \pi t}$. To find the value of t corresponding to the point $(0, 2)$, solve $y = 2 \;\Rightarrow\; t^2 + t - 2 = 0 \;\Rightarrow\; (t+2)(t-1) = 0 \;\Rightarrow\; t = -2$ or $t = 1$. Either value gives $dy/dx = -3/\pi$, so an equation of the tangent is $y - 2 = -\frac{3}{\pi}(x - 0)$, or $y = -\frac{3}{\pi}x + 2$.



11. $x = t^2 + 1,\; y = t^2 + t \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{2t} = 1 + \dfrac{1}{2t} \;\Rightarrow\; \dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{-1/(2t^2)}{2t} = -\dfrac{1}{4t^3}$.

The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t < 0$.

12. $x = t^3 + 1,\; y = t^2 - t \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t-1}{3t^2} = \dfrac{2}{3t} - \dfrac{1}{3t^2} \;\Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{-\frac{2}{3t^2} + \frac{2}{3t^3}}{3t^2} = \frac{\frac{2-2t}{3t^3}}{3t^2} = \frac{2(1-t)}{9t^5}.$$ The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $0 < t < 1$.

13. $x = e^t,\; y = te^{-t} \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-te^{-t} + e^{-t}}{e^t} = \dfrac{e^{-t}(1-t)}{e^t} = e^{-2t}(1-t) \;\Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{e^{-2t}(-1) + (1-t)(-2e^{-2t})}{e^t} = \frac{e^{-2t}(-1-2+2t)}{e^t} = e^{-3t}(2t-3).$$ The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t > \frac{3}{2}$.

14. $x = t^2 + 1,\; y = e^t - 1 \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{e^t}{2t} \;\Rightarrow\; \dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{\frac{2te^t - e^t \cdot 2}{(2t)^2}}{2t} = \dfrac{2e^t(t-1)}{(2t)^3} = \dfrac{e^t(t-1)}{4t^3}$.

The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t < 0$ or $t > 1$.

15. $x = t - \ln t,\ y = t + \ln t$ [note that $t > 0$] $\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 + 1/t}{1 - 1/t} = \dfrac{t+1}{t-1}$ $\Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \frac{\dfrac{(t-1)(1) - (t+1)(1)}{(t-1)^2}}{(t-1)/t} = \frac{-2t}{(t-1)^3}.$$ The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $0 < t < 1$.

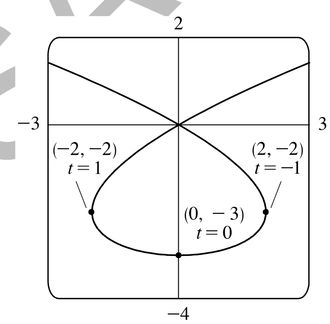
16. $x = \cos t,\ y = \sin 2t,\ 0 < t < \pi$ $\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2\cos 2t}{-\sin t}$ $\Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \frac{\dfrac{(-\sin t)(-4\sin 2t) - (2\cos 2t)(-\cos t)}{(-\sin t)^2}}{-\sin t} = \frac{(\sin t)(8\sin t\,\cos t) + [2(1 - 2\sin^2 t)](\cos t)}{(-\sin t)\sin^2 t}$$

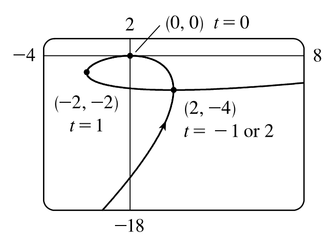
$$= \frac{(\cos t)(8\sin^2 t + 2 - 4\sin^2 t)}{(-\sin t)\sin^2 t} = -\frac{\cos t}{\sin t}\cdot\frac{4\sin^2 t + 2}{\sin^2 t} \quad [\,(-\cot t)\cdot \text{positive expression}]$$

The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $-\cot t > 0$ $\Leftrightarrow$ $\cot t < 0$ $\Leftrightarrow$ $\frac{\pi}{2} < t < \pi$.

17. $x = t^3 - 3t,\ y = t^2 - 3.$ $\dfrac{dy}{dt} = 2t$, so $\dfrac{dy}{dt} = 0$ $\Leftrightarrow$ $t = 0$ $\Leftrightarrow$ $(x, y) = (0, -3).$ $\dfrac{dx}{dt} = 3t^2 - 3 = 3(t+1)(t-1)$, so $\dfrac{dx}{dt} = 0$ $\Leftrightarrow$ $t = -1$ or 1 $\Leftrightarrow$ $(x, y) = (2, -2)$ or $(-2, -2)$. The curve has a horizontal tangent at $(0, -3)$ and vertical tangents at $(2, -2)$ and $(-2, -2)$.



18. $x = t^3 - 3t,\ y = t^3 - 3t^2.$ $\dfrac{dy}{dt} = 3t^2 - 6t = 3t(t-2)$, so $\dfrac{dy}{dt} = 0$ $\Leftrightarrow$ $t = 0$ or 2 $\Leftrightarrow$ $(x, y) = (0, 0)$ or $(2, -4)$. $\dfrac{dx}{dt} = 3t^2 - 3 = 3(t+1)(t-1)$, so $\dfrac{dx}{dt} = 0$ $\Leftrightarrow$ $t = -1$ or 1 $\Leftrightarrow$ $(x, y) = (2, -4)$ or $(-2, -2)$. The curve has horizontal tangents at $(0, 0)$ and $(2, -4)$, and vertical tangents at $(2, -4)$ and $(-2, -2)$.



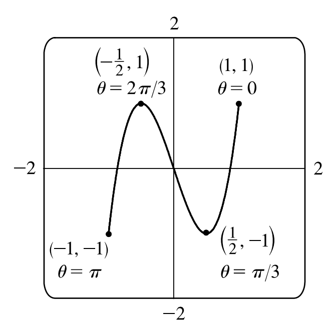
19. $x = \cos\theta,\ y = \cos 3\theta$. The whole curve is traced out for $0 \le \theta \le \pi$.

$\dfrac{dy}{d\theta} = -3\sin 3\theta$, so $\dfrac{dy}{d\theta} = 0$ $\Leftrightarrow$ $\sin 3\theta = 0$ $\Leftrightarrow$ $3\theta = 0, \pi, 2\pi$, or 3π $\Leftrightarrow$

$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, or π $\Leftrightarrow$ $(x, y) = (1, 1), \left(\frac{1}{2}, -1\right), \left(-\frac{1}{2}, 1\right)$, or $(-1, -1)$.

$\dfrac{dx}{d\theta} = -\sin\theta$, so $\dfrac{dx}{d\theta} = 0$ $\Leftrightarrow$ $\sin\theta = 0$ $\Leftrightarrow$ $\theta = 0$ or π $\Leftrightarrow$

$(x, y) = (1, 1)$ or $(-1, -1)$. Both $\dfrac{dy}{d\theta}$ and $\dfrac{dx}{d\theta}$ equal 0 when $\theta = 0$ and π.



To find the slope when $\theta = 0$, we find $\displaystyle\lim_{\theta\to 0}\frac{dy}{dx} = \lim_{\theta\to 0}\frac{-3\sin 3\theta}{-\sin\theta} \overset{\text{H}}{=} \lim_{\theta\to 0}\frac{-9\cos 3\theta}{-\cos\theta} = 9$, which is the same slope when $\theta = \pi$.

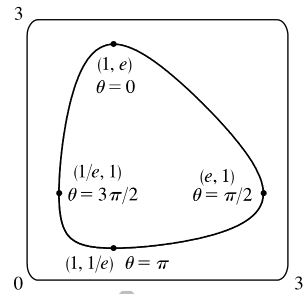
Thus, the curve has horizontal tangents at $\left(\frac{1}{2}, -1\right)$ and $\left(-\frac{1}{2}, 1\right)$, and there are no vertical tangents.

20. $x = e^{\sin\theta}$, $y = e^{\cos\theta}$. The whole curve is traced out for $0 \le \theta < 2\pi$.

$\dfrac{dy}{d\theta} = -\sin\theta\, e^{\cos\theta}$, so $\dfrac{dy}{d\theta} = \theta \quad\Leftrightarrow\quad \sin\theta = 0 \quad\Leftrightarrow\quad \theta = 0$ or $\pi \quad\Leftrightarrow$

$(x, y) = (1, e)$ or $(1, 1/e)$. $\quad \dfrac{dx}{d\theta} = \cos\theta\, e^{\sin\theta}$, so $\dfrac{dx}{d\theta} = 0 \quad\Leftrightarrow\quad \cos\theta = 0 \quad\Leftrightarrow$

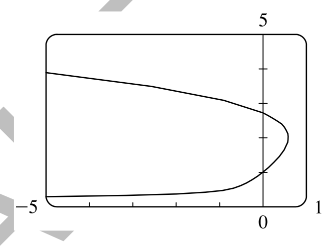
$\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \quad\Leftrightarrow\quad (x, y) = (e, 1)$ or $(1/e, 1)$. The curve has horizontal tangents at $(1, e)$ and $(1, 1/e)$, and vertical tangents at $(e, 1)$ and $(1/e, 1)$.



21. From the graph, it appears that the rightmost point on the curve $x = t - t^6$, $y = e^t$ is about $(0.6, 2)$. To find the exact coordinates, we find the value of t for which the graph has a vertical tangent, that is, $0 = dx/dt = 1 - 6t^5 \quad\Leftrightarrow\quad t = 1/\sqrt[5]{6}$.

Hence, the rightmost point is

$$\left(1/\sqrt[5]{6} - 1/\left(6\sqrt[5]{6}\right), e^{1/\sqrt[5]{6}}\right) = \left(5 \cdot 6^{-6/5}, e^{6^{-1/5}}\right) \approx (0.58, 2.01).$$



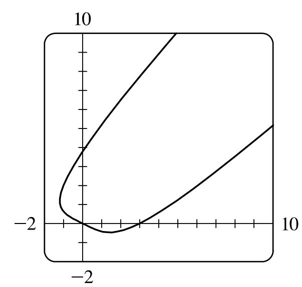
22. From the graph, it appears that the lowest point and the leftmost point on the curve $x = t^4 - 2t$, $y = t + t^4$ are $(1.5, -0.5)$ and $(-1.2, 1.2)$, respectively. To find the exact coordinates, we solve $dy/dt = 0$ (horizontal tangents) and $dx/dt = 0$ (vertical tangents).

$\dfrac{dy}{dt} = 0 \quad\Leftrightarrow\quad 1 + 4t^3 = 0 \quad\Leftrightarrow\quad t = -\dfrac{1}{\sqrt[3]{4}}$, so the lowest point is

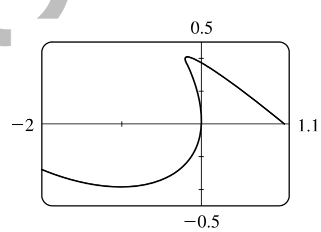
$$\left(\frac{1}{\sqrt[3]{256}} + \frac{2}{\sqrt[3]{4}}, -\frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{256}}\right) = \left(\frac{9}{\sqrt[3]{256}}, -\frac{3}{\sqrt[3]{256}}\right) \approx (1.42, -0.47).$$

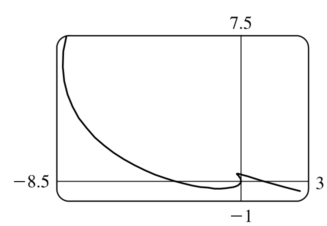
$\dfrac{dx}{dt} = 0 \quad\Leftrightarrow\quad 4t^3 - 2 = 0 \quad\Leftrightarrow\quad t = \dfrac{1}{\sqrt[3]{2}}$, so the leftmost point is

$$\left(\frac{1}{\sqrt[3]{16}} - \frac{2}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{16}}\right) = \left(-\frac{3}{\sqrt[3]{16}}, \frac{3}{\sqrt[3]{16}}\right) \approx (-1.19, 1.19).$$



23. We graph the curve $x = t^4 - 2t^3 - 2t^2$, $y = t^3 - t$ in the viewing rectangle $[-2, 1.1]$ by $[-0.5, 0.5]$. This rectangle corresponds approximately to $t \in [-1, 0.8]$.

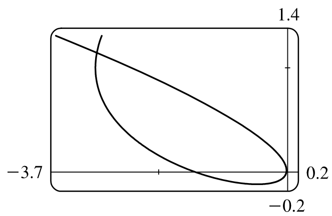


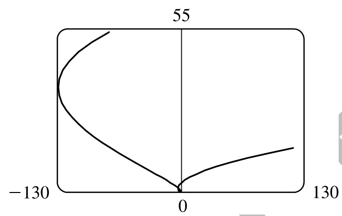


We estimate that the curve has horizontal tangents at about $(-1, -0.4)$ and $(-0.17, 0.39)$ and vertical tangents at about $(0, 0)$ and $(-0.19, 0.37)$. We calculate $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 1}{4t^3 - 6t^2 - 4t}$. The horizontal tangents occur when $dy/dt = 3t^2 - 1 = 0 \quad\Leftrightarrow\quad t = \pm\frac{1}{\sqrt{3}}$, so both horizontal tangents are shown in our graph. The vertical tangents occur when

$dx/dt = 2t(2t^2 - 3t - 2) = 0 \quad \Leftrightarrow \quad 2t(2t+1)(t-2) = 0 \quad \Leftrightarrow \quad t = 0, -\frac{1}{2}$ or 2. It seems that we have missed one vertical tangent, and indeed if we plot the curve on the t-interval $[-1.2, 2.2]$ we see that there is another vertical tangent at $(-8, 6)$.

24. We graph the curve $x = t^4 + 4t^3 - 8t^2$, $y = 2t^2 - t$ in the viewing rectangle $[-3.7, 0.2]$ by $[-0.2, 1.4]$. It appears that there is a horizontal tangent at about $(-0.4, -0.1)$, and vertical tangents at about $(-3, 1)$ and $(0, 0)$.

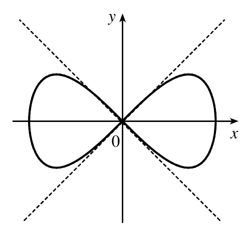




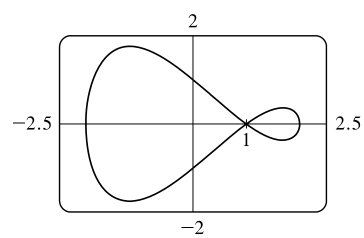
We calculate $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t-1}{4t^3 + 12t^2 - 16t}$, so there is a horizontal tangent where $dy/dt = 4t - 1 = 0 \quad \Leftrightarrow \quad t = \frac{1}{4}$.

This point (the lowest point) is shown in the first graph. There are vertical tangents where $dx/dt = 4t^3 + 12t^2 - 16t = 0 \quad \Leftrightarrow$ $4t(t^2 + 3t - 4) = 0 \quad \Leftrightarrow \quad 4t(t+4)(t-1) = 0$. We have missed one vertical tangent corresponding to $t = -4$, and if we plot the graph for $t \in [-5, 3]$, we see that the curve has another vertical tangent line at approximately $(-128, 36)$.

25. $x = \cos t,\ y = \sin t \cos t. \quad dx/dt = -\sin t$,
$dy/dt = -\sin^2 t + \cos^2 t = \cos 2t$. $(x, y) = (0, 0) \quad \Leftrightarrow \quad \cos t = 0 \quad \Leftrightarrow \quad t$ is an odd multiple of $\frac{\pi}{2}$. When $t = \frac{\pi}{2}$, $dx/dt = -1$ and $dy/dt = -1$, so $dy/dx = 1$. When $t = \frac{3\pi}{2}$, $dx/dt = 1$ and $dy/dt = -1$. So $dy/dx = -1$. Thus, $y = x$ and $y = -x$ are both tangent to the curve at $(0, 0)$.



26. $x = -2\cos t,\ y = \sin t + \sin 2t$. From the graph, it appears that the curve crosses itself at the point $(1, 0)$. If this is true, then $x = 1 \quad \Leftrightarrow$ $-2\cos t = 1 \quad \Leftrightarrow \quad \cos t = -\frac{1}{2} \quad \Leftrightarrow \quad t = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ for $0 \le t \le 2\pi$. Substituting either value of t into y gives $y = 0$, confirming that $(1, 0)$ is the point where the curve crosses itself. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos t + 2\cos 2t}{2\sin t}$.



When $t = \dfrac{2\pi}{3}$, $\dfrac{dy}{dx} = \dfrac{-1/2 + 2(-1/2)}{2(\sqrt{3}/2)} = \dfrac{-3/2}{\sqrt{3}} = -\dfrac{\sqrt{3}}{2}$, so an equation of the tangent line is $y - 0 = -\dfrac{\sqrt{3}}{2}(x - 1)$,

or $y = -\dfrac{\sqrt{3}}{2}x + \dfrac{\sqrt{3}}{2}$. Similarly, when $t = \dfrac{4\pi}{3}$, an equation of the tangent line is $y = \dfrac{\sqrt{3}}{2}x - \dfrac{\sqrt{3}}{2}$.

27. $x = r\theta - d\sin\theta,\ y = r - d\cos\theta$.

(a) $\dfrac{dx}{d\theta} = r - d\cos\theta$, $\dfrac{dy}{d\theta} = d\sin\theta$, so $\dfrac{dy}{dx} = \dfrac{d\sin\theta}{r - d\cos\theta}$.

(b) If $0 < d < r$, then $|d\cos\theta| \le d < r$, so $r - d\cos\theta \ge r - d > 0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d < r$.

28. $x = a\cos^3\theta,\ y = a\sin^3\theta$.

(a) $\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$, so $\dfrac{dy}{dx} = -\dfrac{\sin\theta}{\cos\theta} = -\tan\theta$.

(b) The tangent is horizontal $\Leftrightarrow$ $dy/dx = 0$ $\Leftrightarrow$ $\tan\theta = 0$ $\Leftrightarrow$ $\theta = n\pi$ $\Leftrightarrow$ $(x,y) = (\pm a, 0)$.

The tangent is vertical $\Leftrightarrow$ $\cos\theta = 0$ $\Leftrightarrow$ θ is an odd multiple of $\frac{\pi}{2}$ $\Leftrightarrow$ $(x,y) = (0, \pm a)$.

(c) $dy/dx = \pm 1$ $\Leftrightarrow$ $\tan\theta = \pm 1$ $\Leftrightarrow$ θ is an odd multiple of $\frac{\pi}{4}$ $\Leftrightarrow$ $(x,y) = \left(\pm\frac{\sqrt{2}}{4}a, \pm\frac{\sqrt{2}}{4}a\right)$

[All sign choices are valid.]

29. $x = 3t^2 + 1,\ y = t^3 - 1$ $\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2}{6t} = \dfrac{t}{2}$. The tangent line has slope $\dfrac{1}{2}$ when $\dfrac{t}{2} = \dfrac{1}{2}$ $\Leftrightarrow$ $t = 1$, so the point is $(4, 0)$.

30. $x = 3t^2 + 1, y = 2t^3 + 1$, $\dfrac{dx}{dt} = 6t$, $\dfrac{dy}{dt} = 6t^2$, so $\dfrac{dy}{dx} = \dfrac{6t^2}{6t} = t$ [even where $t = 0$].

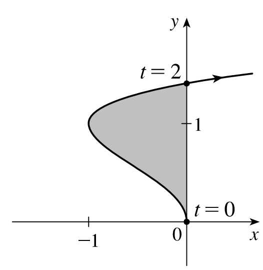
So at the point corresponding to parameter value t, an equation of the tangent line is $y - (2t^3 + 1) = t[x - (3t^2 + 1)]$.

If this line is to pass through $(4, 3)$, we must have $3 - (2t^3 + 1) = t[4 - (3t^2 + 1)]$ $\Leftrightarrow$ $2t^3 - 2 = 3t^3 - 3t$ $\Leftrightarrow$ $t^3 - 3t + 2 = 0$ $\Leftrightarrow$ $(t - 1)^2(t + 2) = 0$ $\Leftrightarrow$ $t = 1$ or -2. Hence, the desired equations are $y - 3 = x - 4$, or $y = x - 1$, tangent to the curve at $(4, 3)$, and $y - (-15) = -2(x - 13)$, or $y = -2x + 11$, tangent to the curve at $(13, -15)$.

31. By symmetry of the ellipse about the x- and y-axes,

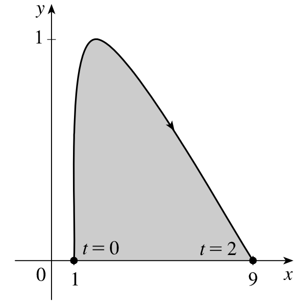
$$A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 b\sin\theta\,(-a\sin\theta)\,d\theta = 4ab\int_0^{\pi/2}\sin^2\theta\,d\theta = 4ab\int_0^{\pi/2}\tfrac{1}{2}(1 - \cos 2\theta)\,d\theta$$
$$= 2ab\big[\theta - \tfrac{1}{2}\sin 2\theta\big]_0^{\pi/2} = 2ab\big(\tfrac{\pi}{2}\big) = \pi ab$$

32. The curve $x = t^2 - 2t = t(t - 2)$, $y = \sqrt{t}$ intersects the y-axis when $x = 0$, that is, when $t = 0$ and $t = 2$. The corresponding values of y are 0 and $\sqrt{2}$. The shaded area is given by



$$\int_{y=0}^{y=\sqrt{2}} (x_R - x_L)\,dy = \int_{t=0}^{t=2} [0 - x(t)]\,y'(t)\,dt = -\int_0^2 (t^2 - 2t)\left(\frac{1}{2\sqrt{t}}\,dt\right)$$
$$= -\int_0^2 \left(\tfrac{1}{2}t^{3/2} - t^{1/2}\right)dt = -\left[\tfrac{1}{5}t^{5/2} - \tfrac{2}{3}t^{3/2}\right]_0^2$$
$$= -\left(\tfrac{1}{5}\cdot 2^{5/2} - \tfrac{2}{3}\cdot 2^{3/2}\right) = -2^{1/2}\left(\tfrac{4}{5} - \tfrac{4}{3}\right)$$
$$= -\sqrt{2}\left(-\tfrac{8}{15}\right) = \tfrac{8}{15}\sqrt{2}$$

33. The curve $x = t^3 + 1$, $y = 2t - t^2 = t(2 - t)$ intersects the x-axis when $y = 0$, that is, when $t = 0$ and $t = 2$. The corresponding values of x are 1 and 9. The shaded area is given by



$$\int_{x=1}^{x=9} (y_T - y_B)\,dx = \int_{t=0}^{t=2} [y(t) - 0]\,x'(t)\,dt = \int_0^2 (2t - t^2)(3t^2)\,dt$$
$$= 3\int_0^2 (2t^3 - t^4)\,dt = 3\left[\tfrac{1}{2}t^4 - \tfrac{1}{5}t^5\right]_0^2 = 3\left(8 - \tfrac{32}{5}\right) = \tfrac{24}{5}$$

34. By symmetry, $A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 a\sin^3\theta(-3a\cos^2\theta\,\sin\theta)\,d\theta = 12a^2\int_0^{\pi/2}\sin^4\theta\,\cos^2\theta\,d\theta$. Now

$$\begin{aligned}\int\sin^4\theta\,\cos^2\theta\,d\theta &= \int\sin^2\theta\left(\tfrac{1}{4}\sin^2 2\theta\right)d\theta = \tfrac{1}{8}\int(1-\cos 2\theta)\sin^2 2\theta\,d\theta \\ &= \tfrac{1}{8}\int\left[\tfrac{1}{2}(1-\cos 4\theta) - \sin^2 2\theta\,\cos 2\theta\right]d\theta = \tfrac{1}{16}\theta - \tfrac{1}{64}\sin 4\theta - \tfrac{1}{48}\sin^3 2\theta + C\end{aligned}$$

so $\int_0^{\pi/2}\sin^4\theta\,\cos^2\theta\,d\theta = \left[\tfrac{1}{16}\theta - \tfrac{1}{64}\sin 4\theta - \tfrac{1}{48}\sin^3 2\theta\right]_0^{\pi/2} = \tfrac{\pi}{32}$. Thus, $A = 12a^2\left(\tfrac{\pi}{32}\right) = \tfrac{3}{8}\pi a^2$.

35. $x = r\theta - d\sin\theta,\ y = r - d\cos\theta$.

$$\begin{aligned}A &= \int_0^{2\pi r} y\,dx = \int_0^{2\pi}(r - d\cos\theta)(r - d\cos\theta)\,d\theta = \int_0^{2\pi}(r^2 - 2dr\cos\theta + d^2\cos^2\theta)\,d\theta \\ &= \left[r^2\theta - 2dr\sin\theta + \tfrac{1}{2}d^2\left(\theta + \tfrac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = 2\pi r^2 + \pi d^2\end{aligned}$$

36. (a) By symmetry, the area of $\mathcal{R}$ is twice the area inside $\mathcal{R}$ above the x-axis. The top half of the loop is described by $x = t^2$, $y = t^3 - 3t$, $-\sqrt{3} \le t \le 0$, so, using the Substitution Rule with $y = t^3 - 3t$ and $dx = 2t\,dt$, we find that

$$\begin{aligned}\text{area} &= 2\int_0^3 y\,dx = 2\int_0^{-\sqrt{3}}(t^3 - 3t)2t\,dt = 2\int_0^{-\sqrt{3}}(2t^4 - 6t^2)\,dt = 2\left[\tfrac{2}{5}t^5 - 2t^3\right]_0^{-\sqrt{3}} \\ &= 2\left[\tfrac{2}{5}(-3^{1/2})^5 - 2(-3^{1/2})^3\right] = 2\left[\tfrac{2}{5}\left(-9\sqrt{3}\right) - 2\left(-3\sqrt{3}\right)\right] = \tfrac{24}{5}\sqrt{3}\end{aligned}$$

(b) Here we use the formula for disks and use the Substitution Rule as in part (a):

$$\begin{aligned}\text{volume} &= \pi\int_0^3 y^2\,dx = \pi\int_0^{-\sqrt{3}}(t^3 - 3t)^2 2t\,dt = 2\pi\int_0^{-\sqrt{3}}(t^6 - 6t^4 + 9t^2)t\,dt = 2\pi\left[\tfrac{1}{8}t^8 - t^6 + \tfrac{9}{4}t^4\right]_0^{-\sqrt{3}} \\ &= 2\pi\left[\tfrac{1}{8}(-3^{1/2})^8 - (-3^{1/2})^6 + \tfrac{9}{4}(-3^{1/2})^4\right] = 2\pi\left[\tfrac{81}{8} - 27 + \tfrac{81}{4}\right] = \tfrac{27}{4}\pi\end{aligned}$$

(c) By symmetry, the y-coordinate of the centroid is 0. To find the x-coordinate, we note that it is the same as the x-coordinate of the centroid of the top half of $\mathcal{R}$, the area of which is $\tfrac{1}{2}\cdot\tfrac{24}{5}\sqrt{3} = \tfrac{12}{5}\sqrt{3}$. So, using Formula 8.3.8 with $A = \tfrac{12}{5}\sqrt{3}$, we get

$$\begin{aligned}\overline{x} &= \tfrac{5}{12\sqrt{3}}\int_0^3 xy\,dx = \tfrac{5}{12\sqrt{3}}\int_0^{-\sqrt{3}} t^2(t^3 - 3t)2t\,dt = \tfrac{5}{6\sqrt{3}}\left[\tfrac{1}{7}t^7 - \tfrac{3}{5}t^5\right]_0^{-\sqrt{3}} \\ &= \tfrac{5}{6\sqrt{3}}\left[\tfrac{1}{7}(-3^{1/2})^7 - \tfrac{3}{5}(-3^{1/2})^5\right] = \tfrac{5}{6\sqrt{3}}\left[-\tfrac{27}{7}\sqrt{3} + \tfrac{27}{5}\sqrt{3}\right] = \tfrac{9}{7}\end{aligned}$$

So the coordinates of the centroid of $\mathcal{R}$ are $(x, y) = \left(\tfrac{9}{7}, 0\right)$.

37. $x = t + e^{-t},\ y = t - e^{-t},\ 0 \le t \le 2$. $dx/dt = 1 - e^{-t}$ and $dy/dt = 1 + e^{-t}$, so

$(dx/dt)^2 + (dy/dt)^2 = (1 - e^{-t})^2 + (1 + e^{-t})^2 = 1 - 2e^{-t} + e^{-2t} + 1 + 2e^{-t} + e^{-2t} = 2 + 2e^{-2t}$.

Thus, $L = \int_a^b\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^2\sqrt{2 + 2e^{-2t}}\,dt \approx 3.1416$.

38. $x = t^2 - t,\ y = t^4,\ 1 \le t \le 4$. $dx/dt = 2t - 1$ and $dy/dt = 4t^3$, so

$(dx/dt)^2 + (dy/dt)^2 = (2t - 1)^2 + (4t^3)^2 = 4t^2 - 4t + 1 + 16t^6$.

Thus, $L = \int_a^b\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_1^4\sqrt{16t^6 + 4t^2 - 4t + 1}\,dt \approx 255.3756$.

39. $x = t - 2\sin t,\ y = 1 - 2\cos t,\ 0 \le t \le 4\pi$. $dx/dt = 1 - 2\cos t$ and $dy/dt = 2\sin t$, so

$(dx/dt)^2 + (dy/dt)^2 = (1 - 2\cos t)^2 + (2\sin t)^2 = 1 - 4\cos t + 4\cos^2 t + 4\sin^2 t = 5 - 4\cos t$.

Thus, $L = \int_a^b\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^{4\pi}\sqrt{5 - 4\cos t}\,dt \approx 26.7298$.

40. $x = t + \sqrt{t},\ y = t - \sqrt{t},\ 0 \le t \le 1.$ $\dfrac{dx}{dt} = 1 + \dfrac{1}{2\sqrt{t}}$ and $\dfrac{dy}{dt} = 1 - \dfrac{1}{2\sqrt{t}}$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(1 + \frac{1}{2\sqrt{t}}\right)^2 + \left(1 - \frac{1}{2\sqrt{t}}\right)^2 = 1 + \frac{1}{\sqrt{t}} + \frac{1}{4t} + 1 - \frac{1}{\sqrt{t}} + \frac{1}{4t} = 2 + \frac{1}{2t}.$$

Thus, $L = \displaystyle\int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^1 \sqrt{2 + \frac{1}{2t}}\,dt = \lim_{t\to 0^+} \int_t^1 \sqrt{2 + \frac{1}{2t}}\,dt \approx 2.0915.$

41. $x = 1 + 3t^2,\ \ y = 4 + 2t^3,\ \ 0 \le t \le 1.$ $dx/dt = 6t$ and $dy/dt = 6t^2$, so $(dx/dt)^2 + (dy/dt)^2 = 36t^2 + 36t^4$.

Thus, $L = \displaystyle\int_0^1 \sqrt{36t^2 + 36t^4}\,dt = \int_0^1 6t\sqrt{1 + t^2}\,dt = 6\int_1^2 \sqrt{u}\,\left(\tfrac{1}{2}\,du\right)$ $[u = 1 + t^2,\ du = 2t\,dt]$

$$= 3\left[\tfrac{2}{3}u^{3/2}\right]_1^2 = 2(2^{3/2} - 1) = 2\left(2\sqrt{2} - 1\right)$$

42. $x = e^t - t,\ y = 4e^{t/2},\ 0 \le t \le 2.$ $dx/dt = e^t - 1$ and $dy/dt = 2e^{t/2}$, so

$(dx/dt)^2 + (dy/dt)^2 = (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} - 2e^t + 1 + 4e^t = e^{2t} + 2e^t + 1 = (e^t + 1)^2$. Thus,

$$L = \int_0^2 \sqrt{(e^t + 1)^2}\,dt = \int_0^2 \left|e^t + 1\right|\,dt = \int_0^2 (e^t + 1)\,dt = \left[e^t + t\right]_0^2 = (e^2 + 2) - (1 + 0) = e^2 + 1.$$

43. $x = t\sin t,\ y = t\cos t,\ 0 \le t \le 1.$ $\dfrac{dx}{dt} = t\cos t + \sin t$ and $\dfrac{dy}{dt} = -t\sin t + \cos t$, so

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= t^2\cos^2 t + 2t\sin t\,\cos t + \sin^2 t + t^2\sin^2 t - 2t\sin t\,\cos t + \cos^2 t\\ &= t^2(\cos^2 t + \sin^2 t) + \sin^2 t + \cos^2 t = t^2 + 1.\end{aligned}$$

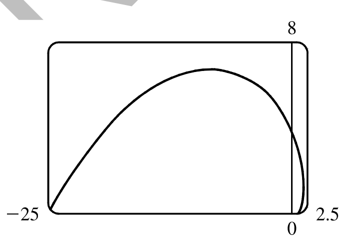
Thus, $L = \int_0^1 \sqrt{t^2 + 1}\,dt \overset{21}{=} \left[\tfrac{1}{2}t\sqrt{t^2 + 1} + \tfrac{1}{2}\ln\left(t + \sqrt{t^2 + 1}\right)\right]_0^1 = \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\ln\left(1 + \sqrt{2}\right).$

44. $x = 3\cos t - \cos 3t,\ y = 3\sin t - \sin 3t,\ 0 \le t \le \pi.$ $\dfrac{dx}{dt} = -3\sin t + 3\sin 3t$ and $\dfrac{dy}{dt} = 3\cos t - 3\cos 3t$, so

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 9\sin^2 t - 18\sin t\,\sin 3t + 9\sin^2(3t) + 9\cos^2 t - 18\cos t\,\cos 3t + 9\cos^2(3t)\\ &= 9(\cos^2 t + \sin^2 t) - 18(\cos t\,\cos 3t + \sin t\,\sin 3t) + 9[\cos^2(3t) + \sin^2(3t)]\\ &= 9(1) - 18\cos(t - 3t) + 9(1) = 18 - 18\cos(-2t) = 18(1 - \cos 2t)\\ &= 18[1 - (1 - 2\sin^2 t)] = 36\sin^2 t.\end{aligned}$$

Thus, $L = \int_0^\pi \sqrt{36\sin^2 t}\,dt = 6\int_0^\pi |\sin t|\,dt = 6\int_0^\pi \sin t\,dt = -6\left[\cos t\right]_0^\pi = -6\,(-1 - 1) = 12.$

45.



$x = e^t\cos t,\ y = e^t\sin t,\ 0 \le t \le \pi.$

$$\begin{aligned}\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 &= [e^t(\cos t - \sin t)]^2 + [e^t(\sin t + \cos t)]^2\\ &= (e^t)^2(\cos^2 t - 2\cos t\,\sin t + \sin^2 t)\\ &\qquad + (e^t)^2(\sin^2 t + 2\sin t\,\cos t + \cos^2 t\\ &= e^{2t}(2\cos^2 t + 2\sin^2 t) = 2e^{2t}\end{aligned}$$

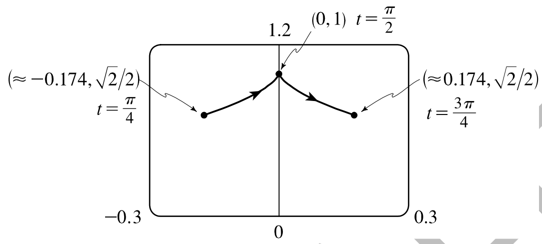
Thus, $L = \int_0^\pi \sqrt{2e^{2t}}\,dt = \int_0^\pi \sqrt{2}\,e^t\,dt = \sqrt{2}\,\left[\,e^t\,\right]_0^\pi = \sqrt{2}\,(e^\pi - 1).$

46. $x = \cos t + \ln(\tan \frac{1}{2}t), \quad y = \sin t, \quad \pi/4 \le t \le 3\pi/4.$

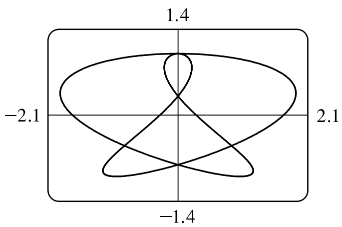
$$\frac{dx}{dt} = -\sin t + \frac{\frac{1}{2}\sec^2(t/2)}{\tan(t/2)} = -\sin t + \frac{1}{2\sin(t/2)\,\cos(t/2)} = -\sin t + \frac{1}{\sin t} \text{ and } \frac{dy}{dt} = \cos t, \text{ so}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2 t - 2 + \frac{1}{\sin^2 t} + \cos^2 t = 1 - 2 + \csc^2 t = \cot^2 t. \text{ Thus,}$$

$$L = \int_{\pi/4}^{3\pi/4} |\cot t|\, dt = 2\int_{\pi/4}^{\pi/2} \cot t\, dt$$
$$= 2\Big[\ln|\sin t|\Big]_{\pi/4}^{\pi/2} = 2\left(\ln 1 - \ln\frac{1}{\sqrt{2}}\right)$$
$$= 2\big(0 + \ln\sqrt{2}\big) = 2\big(\tfrac{1}{2}\ln 2\big) = \ln 2.$$



47.



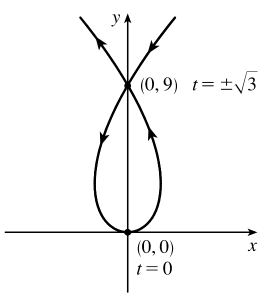
The figure shows the curve $x = \sin t + \sin 1.5t$, $y = \cos t$ for $0 \le t \le 4\pi$. $dx/dt = \cos t + 1.5\cos 1.5t$ and $dy/dt = -\sin t$, so $(dx/dt)^2 + (dy/dt)^2 = \cos^2 t + 3\cos t\,\cos 1.5t + 2.25\cos^2 1.5t + \sin^2 t$. Thus, $L = \int_0^{4\pi} \sqrt{1 + 3\cos t\,\cos 1.5t + 2.25\cos^2 1.5t}\, dt \approx 16.7102$.

48. $x = 3t - t^3$, $y = 3t^2$. $dx/dt = 3 - 3t^2$ and $dy/dt = 6t$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2 = (3 + 3t^2)^2$$

and the length of the loop is given by

$$L = \int_{-\sqrt{3}}^{\sqrt{3}} (3 + 3t^2)\, dt = 2\int_0^{\sqrt{3}} (3 + 3t^2)\, dt = 2\left[3t + t^3\right]_0^{\sqrt{3}}$$
$$= 2\big(3\sqrt{3} + 3\sqrt{3}\big) = 12\sqrt{3}$$



49. $x = t - e^t,\ y = t + e^t,\ -6 \le t \le 6.$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 - e^t)^2 + (1 + e^t)^2 = (1 - 2e^t + e^{2t}) + (1 + 2e^t + e^{2t}) = 2 + 2e^{2t}$, so $L = \int_{-6}^{6} \sqrt{2 + 2e^{2t}}\, dt$.

Set $f(t) = \sqrt{2 + 2e^{2t}}$. Then by Simpson's Rule with $n = 6$ and $\Delta t = \frac{6-(-6)}{6} = 2$, we get

$L \approx \frac{2}{3}[f(-6) + 4f(-4) + 2f(-2) + 4f(0) + 2f(2) + 4f(4) + f(6)] \approx 612.3053$.

50. $x = 2a\cot\theta \quad\Rightarrow\quad dx/dt = -2a\csc^2\theta$ and $y = 2a\sin^2\theta \quad\Rightarrow\quad dy/dt = 4a\sin\theta\,\cos\theta = 2a\sin 2\theta$.

So $L = \int_{\pi/4}^{\pi/2} \sqrt{4a^2\csc^4\theta + 4a^2\sin^2 2\theta}\, d\theta = 2a\int_{\pi/4}^{\pi/2} \sqrt{\csc^4\theta + \sin^2 2\theta}\, d\theta$. Using Simpson's Rule with

$n = 4$, $\Delta\theta = \frac{\pi/2 - \pi/4}{4} = \frac{\pi}{16}$, and $f(\theta) = \sqrt{\csc^4\theta + \sin^2 2\theta}$, we get

$L \approx 2a \cdot S_4 = (2a)\,\frac{\pi}{16\cdot 3}\left[f\left(\frac{\pi}{4}\right) + 4f\left(\frac{5\pi}{16}\right) + 2f\left(\frac{3\pi}{8}\right) + 4f\left(\frac{7\pi}{16}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 2.2605a$.

51. $x = \sin^2 t,\ y = \cos^2 t,\ 0 \le t \le 3\pi.$

$(dx/dt)^2 + (dy/dt)^2 = (2\sin t\,\cos t)^2 + (-2\cos t\sin t)^2 = 8\sin^2 t\cos^2 t = 2\sin^2 2t \quad\Rightarrow$

Distance $= \int_0^{3\pi} \sqrt{2}\,|\sin 2t|\ dt = 6\sqrt{2}\int_0^{\pi/2} \sin 2t\,dt$ [by symmetry] $= -3\sqrt{2}\Big[\cos 2t\Big]_0^{\pi/2} = -3\sqrt{2}\,(-1-1) = 6\sqrt{2}$.

The full curve is traversed as t goes from 0 to $\frac{\pi}{2}$, because the curve is the segment of $x + y = 1$ that lies in the first quadrant (since $x, y \geq 0$), and this segment is completely traversed as t goes from 0 to $\frac{\pi}{2}$. Thus, $L = \int_0^{\pi/2} \sin 2t\,dt = \sqrt{2}$, as above.

52. $x = \cos^2 t,\ y = \cos t,\ 0 \leq t \leq 4\pi.\quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-2\cos t\sin t)^2 + (-\sin t)^2 = \sin^2 t\,(4\cos^2 t + 1)$

$$\begin{aligned}\text{Distance } &= \int_0^{4\pi} |\sin t|\,\sqrt{4\cos^2 t + 1}\,dt = 4\int_0^{\pi} \sin t\,\sqrt{4\cos^2 t + 1}\,dt \\ &= -4\int_1^{-1} \sqrt{4u^2+1}\,du \quad [u = \cos t,\ du = -\sin t\,dt] \quad = 4\int_{-1}^{1} \sqrt{4u^2+1}\,du \\ &= 8\int_0^1 \sqrt{4u^2+1}\,du = 8\int_0^{\tan^{-1} 2} \sec\theta \cdot \tfrac{1}{2}\sec^2\theta\,d\theta \qquad [2u = \tan\theta,\ 2\,du = \sec^2\theta\,d\theta] \\ &= 4\int_0^{\tan^{-1} 2} \sec^3\theta\,d\theta \overset{71}{=} \Big[2\sec\theta\tan\theta + 2\ln|\sec\theta + \tan\theta|\Big]_0^{\tan^{-1} 2} = 4\sqrt{5} + 2\ln\!\left(\sqrt{5}+2\right)\end{aligned}$$

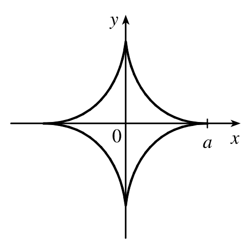
Thus, $L = \int_0^{\pi} |\sin t|\,\sqrt{4\cos^2 t + 1}\,dt = \sqrt{5} + \frac{1}{2}\ln\!\left(\sqrt{5}+2\right)$.

53. $x = a\sin\theta,\ y = b\cos\theta,\ 0 \leq \theta \leq 2\pi.$

$$\begin{aligned}\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 &= (a\cos\theta)^2 + (-b\sin\theta)^2 = a^2\cos^2\theta + b^2\sin^2\theta = a^2(1-\sin^2\theta) + b^2\sin^2\theta \\ &= a^2 - (a^2-b^2)\sin^2\theta = a^2 - c^2\sin^2\theta = a^2\left(1 - \frac{c^2}{a^2}\sin^2\theta\right) = a^2(1 - e^2\sin^2\theta)\end{aligned}$$

So $L = 4\int_0^{\pi/2} \sqrt{a^2\left(1 - e^2\sin^2\theta\right)}\,d\theta$ [by symmetry] $= 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\,d\theta$.

54. $x = a\cos^3\theta,\ y = a\sin^3\theta.$

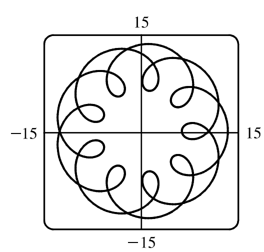


$$\begin{aligned}\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 &= (-3a\cos^2\theta\ \sin\theta)^2 + (3a\sin^2\theta\ \cos\theta)^2 \\ &= 9a^2\cos^4\theta\ \sin^2\theta + 9a^2\sin^4\theta\ \cos^2\theta \\ &= 9a^2\sin^2\theta\ \cos^2\theta(\cos^2\theta + \sin^2\theta) = 9a^2\sin^2\theta\ \cos^2\theta.\end{aligned}$$

The graph has four-fold symmetry and the curve in the first quadrant corresponds to $0 \leq \theta \leq \pi/2$. Thus,

$$\begin{aligned}L &= 4\int_0^{\pi/2} 3a\sin\theta\ \cos\theta\,d\theta \qquad [\text{since } a > 0 \text{ and } \sin\theta \text{ and } \cos\theta \text{ are positive for } 0 \leq \theta \leq \pi/2] \\ &= 12a\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} = 12a\left(\tfrac{1}{2} - 0\right) = 6a\end{aligned}$$

55. (a) $x = 11\cos t - 4\cos(11t/2),\ y = 11\sin t - 4\sin(11t/2).$

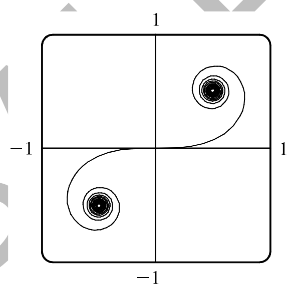


Notice that $0 \leq t \leq 2\pi$ does not give the complete curve because $x(0) \neq x(2\pi)$. In fact, we must take $t \in [0, 4\pi]$ in order to obtain the complete curve, since the first term in each of the parametric equations has period 2π and the second has period $\frac{2\pi}{11/2} = \frac{4\pi}{11}$, and the least common integer multiple of these two numbers is 4π.

(b) We use the CAS to find the derivatives dx/dt and dy/dt, and then use Theorem 5 to find the arc length. Recent versions of Maple express the integral $\int_0^{4\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$ as $88E\big(2\sqrt{2}\,i\big)$, where $E(x)$ is the elliptic integral $\displaystyle\int_0^1 \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}\,dt$ and i is the imaginary number $\sqrt{-1}$.

Some earlier versions of Maple (as well as Mathematica) cannot do the integral exactly, so we use the command `evalf(Int(sqrt(diff(x,t)^2+diff(y,t)^2),t=0..4*Pi));` to estimate the length, and find that the arc length is approximately 294.03. Derive's `Para_arc_length` function in the utility file `Int_apps` simplifies the integral to $11\int_0^{4\pi} \sqrt{-4\cos t\,\cos\left(\frac{11t}{2}\right) - 4\sin t\,\sin\left(\frac{11t}{2}\right) + 5}\,dt$.

56. (a) It appears that as $t \to \infty$, $(x,y) \to \left(\frac{1}{2}, \frac{1}{2}\right)$, and as $t \to -\infty$, $(x,y) \to \left(-\frac{1}{2}, -\frac{1}{2}\right)$.



(b) By the Fundamental Theorem of Calculus, $dx/dt = \cos\left(\frac{\pi}{2}t^2\right)$ and $dy/dt = \sin\left(\frac{\pi}{2}t^2\right)$, so by Theorem 5, the length of the curve from the origin to the point with parameter value t is

$$L = \int_0^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\,du = \int_0^t \sqrt{\cos^2\left(\frac{\pi}{2}u^2\right) + \sin^2\left(\frac{\pi}{2}u^2\right)}\,du$$
$$= \int_0^t 1\,du = t \qquad [\text{or } -t \text{ if } t < 0]$$

We have used u as the dummy variable so as not to confuse it with the upper limit of integration.

57. $x = t\sin t,\ y = t\cos t,\ 0 \le t \le \pi/2$. $dx/dt = t\cos t + \sin t$ and $dy/dt = -t\sin t + \cos t$, so

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= t^2\cos^2 t + 2t\sin t\,\cos t + \sin^2 t + t^2\sin^2 t - 2t\sin t\,\cos t + \cos^2 t \\ &= t^2(\cos^2 t + \sin^2 t) + \sin^2 t + \cos^2 t = t^2 + 1\end{aligned}$$

$S = \int 2\pi y\,ds = \int_0^{\pi/2} 2\pi t\cos t\sqrt{t^2+1}\,dt \approx 4.7394$.

58. $x = \sin t,\ y = \sin 2t,\ 0 \le t \le \pi/2$. $dx/dt = \cos t$ and $dy/dt = 2\cos 2t$, so $(dx/dt)^2 + (dy/dt)^2 = \cos^2 t + 4\cos^2 2t$.

$S = \int 2\pi y\,ds = \int_0^{\pi/2} 2\pi\sin 2t\sqrt{\cos^2 t + 4\cos^2 2t}\,dt \approx 8.0285$.

59. $x = t + e^t,\ y = e^{-t},\ 0 \le t \le 1$.

$dx/dt = 1 + e^t$ and $dy/dt = -e^{-t}$, so $(dx/dt)^2 + (dy/dt)^2 = (1+e^t)^2 + (-e^{-t})^2 = 1 + 2e^t + e^{2t} + e^{-2t}$.

$S = \int 2\pi y\,ds = \int_0^1 2\pi e^{-t}\sqrt{1 + 2e^t + e^{2t} + e^{-2t}}\,dt \approx 10.6705$.

60. $x = t^2 - t^3,\ y = t + t^4,\ 0 \le t \le 1$.

$(dx/dt)^2 + (dy/dt)^2 = (2t - 3t^2)^2 + (1 + 4t^3)^2 = 4t^2 - 12t^3 + 9t^4 + 1 + 8t^3 + 16t^6$, so

$S = \int 2\pi y\,ds = \int_0^1 2\pi(t + t^4)\sqrt{16t^6 + 9t^4 - 4t^3 + 4t^2 + 1}\,dt \approx 12.7176$.

61. $x = t^3,\ y = t^2,\ 0 \le t \le 1.\quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(3t^2\right)^2 + (2t)^2 = 9t^4 + 4t^2.$

$$S = \int_0^1 2\pi y\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt = \int_0^1 2\pi t^2\sqrt{9t^4 + 4t^2}\,dt = 2\pi\int_0^1 t^2\sqrt{t^2(9t^2+4)}\,dt$$

$$= 2\pi\int_4^{13}\left(\frac{u-4}{9}\right)\sqrt{u}\left(\tfrac{1}{18}\,du\right)\quad \begin{bmatrix} u = 9t^2+4,\ t^2 = (u-4)/9, \\ du = 18t\,dt, \text{so } t\,dt = \frac{1}{18}\,du \end{bmatrix}\quad = \frac{2\pi}{9\cdot 18}\int_4^{13}\left(u^{3/2} - 4u^{1/2}\right)du$$

$$= \tfrac{\pi}{81}\left[\tfrac{2}{5}u^{5/2} - \tfrac{8}{3}u^{3/2}\right]_4^{13} = \tfrac{\pi}{81}\cdot\tfrac{2}{15}\left[3u^{5/2} - 20u^{3/2}\right]_4^{13}$$

$$= \tfrac{2\pi}{1215}\left[\left(3\cdot 13^2\sqrt{13} - 20\cdot 13\sqrt{13}\right) - (3\cdot 32 - 20\cdot 8)\right] = \tfrac{2\pi}{1215}\left(247\sqrt{13} + 64\right)$$

62. $x = 2t^2 + 1/t,\ y = 8\sqrt{t},\ 1 \le t \le 3.$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(4t - \frac{1}{t^2}\right)^2 + \left(\frac{4}{\sqrt{t}}\right)^2 = 16t^2 - \frac{8}{t} + \frac{1}{t^4} + \frac{16}{t} = 16t^2 + \frac{8}{t} + \frac{1}{t^4} = \left(4t + \frac{1}{t^2}\right)^2.$$

$$S = \int_1^3 2\pi y\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt = \int_1^3 2\pi\left(8\sqrt{t}\right)\sqrt{\left(4t + \frac{1}{t^2}\right)^2}\,dt = 16\pi\int_1^3 t^{1/2}\left(4t + t^{-2}\right)dt$$

$$= 16\pi\int_1^3\left(4t^{3/2} + t^{-3/2}\right)dt = 16\pi\left[\tfrac{8}{5}t^{5/2} - 2t^{-1/2}\right]_1^3 = 16\pi\left[\left(\tfrac{72}{5}\sqrt{3} - \tfrac{2}{3}\sqrt{3}\right) - \left(\tfrac{8}{5} - 2\right)\right]$$

$$= 16\pi\left(\tfrac{206}{15}\sqrt{3} + \tfrac{6}{15}\right) = \tfrac{32\pi}{15}\left(103\sqrt{3} + 3\right)$$

63. $x = a\cos^3\theta,\ y = a\sin^3\theta,\ 0 \le \theta \le \frac{\pi}{2}.\quad \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(-3a\cos^2\theta\,\sin\theta\right)^2 + \left(3a\sin^2\theta\,\cos\theta\right)^2 = 9a^2\sin^2\theta\,\cos^2\theta.$

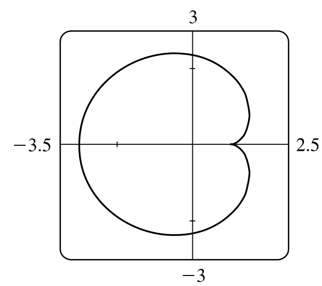
$$S = \int_0^{\pi/2} 2\pi\cdot a\sin^3\theta\cdot 3a\sin\theta\,\cos\theta\,d\theta = 6\pi a^2\int_0^{\pi/2}\sin^4\theta\,\cos\theta\,d\theta = \tfrac{6}{5}\pi a^2\left[\sin^5\theta\right]_0^{\pi/2} = \tfrac{6}{5}\pi a^2$$

64. $x = 2\cos\theta - \cos 2\theta,\ y = 2\sin\theta - \sin 2\theta \quad\Rightarrow$

$$\left(\tfrac{dx}{d\theta}\right)^2 + \left(\tfrac{dy}{d\theta}\right)^2 = (-2\sin\theta + 2\sin 2\theta)^2 + (2\cos\theta - 2\cos 2\theta)^2$$

$$= 4[(\sin^2\theta - 2\sin\theta\,\sin 2\theta + \sin^2 2\theta) + (\cos^2\theta - 2\cos\theta\,\cos 2\theta + \cos^2 2\theta)]$$

$$= 4[1 + 1 - 2(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta)] = 8[1 - \cos(2\theta - \theta)] = 8(1 - \cos\theta)$$



We plot the graph with parameter interval $[0, 2\pi]$, and see that we should only integrate between 0 and π. (If the interval $[0, 2\pi]$ were taken, the surface of revolution would be generated twice.) Also note that $y = 2\sin\theta - \sin 2\theta = 2\sin\theta(1 - \cos\theta)$. So

$$S = \int_0^{\pi} 2\pi\cdot 2\sin\theta(1 - \cos\theta)\,2\sqrt{2}\sqrt{1 - \cos\theta}\,d\theta$$

$$= 8\sqrt{2}\pi\int_0^{\pi}(1 - \cos\theta)^{3/2}\sin\theta\,d\theta = 8\sqrt{2}\pi\int_0^2\sqrt{u^3}\,du\quad \begin{bmatrix} u = 1 - \cos\theta, \\ du = \sin\theta\,d\theta \end{bmatrix}$$

$$= 8\sqrt{2}\pi\left[\left(\tfrac{2}{5}\right)u^{5/2}\right]_0^2 = \tfrac{16}{5}\sqrt{2}\pi\left(2^{5/2}\right) = \tfrac{128}{5}\pi$$

65. $x = 3t^2,\ y = 2t^3,\ 0 \le t \le 5 \quad\Rightarrow\quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (6t)^2 + (6t^2)^2 = 36t^2(1 + t^2) \quad\Rightarrow$

$$S = \int_0^5 2\pi x\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^5 2\pi(3t^2)6t\sqrt{1 + t^2}\,dt = 18\pi\int_0^5 t^2\sqrt{1 + t^2}\,2t\,dt$$

$$= 18\pi\int_1^{26}(u - 1)\sqrt{u}\,du\quad \begin{bmatrix} u = 1 + t^2, \\ du = 2t\,dt \end{bmatrix}\quad = 18\pi\int_1^{26}\left(u^{3/2} - u^{1/2}\right)du = 18\pi\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^{26}$$

$$= 18\pi\left[\left(\tfrac{2}{5}\cdot 676\sqrt{26} - \tfrac{2}{3}\cdot 26\sqrt{26}\right) - \left(\tfrac{2}{5} - \tfrac{2}{3}\right)\right] = \tfrac{24}{5}\pi\left(949\sqrt{26} + 1\right)$$

66. $x = e^t - t$, $y = 4e^{t/2}$, $0 \le t \le 1$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} + 2e^t + 1 = (e^t + 1)^2$.

$$S = \int_0^1 2\pi(e^t - t)\sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\,dt = \int_0^1 2\pi(e^t - t)(e^t + 1)d$$
$$= 2\pi\left[\tfrac{1}{2}e^{2t} + e^t - (t-1)e^t - \tfrac{1}{2}t^2\right]_0^1 = \pi(e^2 + 2e - 6)$$

67. If f' is continuous and $f'(t) \neq 0$ for $a \le t \le b$, then either $f'(t) > 0$ for all t in $[a, b]$ or $f'(t) < 0$ for all t in $[a, b]$. Thus, f is monotonic (in fact, strictly increasing or strictly decreasing) on $[a, b]$. It follows that f has an inverse. Set $F = g \circ f^{-1}$, that is, define F by $F(x) = g(f^{-1}(x))$. Then $x = f(t) \;\Rightarrow\; f^{-1}(x) = t$, so $y = g(t) = g(f^{-1}(x)) = F(x)$.

68. By Formula 8.2.5 with $y = F(x)$, $S = \int_a^b 2\pi F(x)\sqrt{1 + [F'(x)]^2}\,dx$. But by Formula 10.2.1,

$1 + [F'(x)]^2 = 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \left(\dfrac{dy/dt}{dx/dt}\right)^2 = \dfrac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}$. Using the Substitution Rule with $x = x(t)$, where $a = x(\alpha)$ and $b = x(\beta)$, we have $\left[\text{since } dx = \dfrac{dx}{dt}\,dt\right]$

$S = \displaystyle\int_\alpha^\beta 2\pi\, F(x(t))\sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}}\,\frac{dx}{dt}\,dt = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$, which is Formula 10.2.6.

69. (a) $\phi = \tan^{-1}\left(\dfrac{dy}{dx}\right) \;\Rightarrow\; \dfrac{d\phi}{dt} = \dfrac{d}{dt}\tan^{-1}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{1 + (dy/dx)^2}\left[\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\right]$. But $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\dot{y}}{\dot{x}} \;\Rightarrow$

$\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{\dot{y}}{\dot{x}}\right) = \dfrac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2} \;\Rightarrow\; \dfrac{d\phi}{dt} = \dfrac{1}{1 + (\dot{y}/\dot{x})^2}\left(\dfrac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2}\right) = \dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}$. Using the Chain Rule, and the fact that $s = \displaystyle\int_0^t \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt \;\Rightarrow\; \frac{ds}{dt} = \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2} = \left(\dot{x}^2 + \dot{y}^2\right)^{1/2}$, we have that

$\dfrac{d\phi}{ds} = \dfrac{d\phi/dt}{ds/dt} = \left(\dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}\right)\dfrac{1}{(\dot{x}^2 + \dot{y}^2)^{1/2}} = \dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$. So $\kappa = \left|\dfrac{d\phi}{ds}\right| = \left|\dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\right| = \dfrac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$.

(b) $x = x$ and $y = f(x) \;\Rightarrow\; \dot{x} = 1$, $\ddot{x} = 0$ and $\dot{y} = \dfrac{dy}{dx}$, $\ddot{y} = \dfrac{d^2y}{dx^2}$.

So $\kappa = \dfrac{\left|1 \cdot (d^2y/dx^2) - 0 \cdot (dy/dx)\right|}{[1 + (dy/dx)^2]^{3/2}} = \dfrac{\left|d^2y/dx^2\right|}{[1 + (dy/dx)^2]^{3/2}}$.

70. (a) $y = x^2 \;\Rightarrow\; \dfrac{dy}{dx} = 2x \;\Rightarrow\; \dfrac{d^2y}{dx^2} = 2$. So $\kappa = \dfrac{\left|d^2y/dx^2\right|}{[1 + (dy/dx)^2]^{3/2}} = \dfrac{2}{(1 + 4x^2)^{3/2}}$, and at $(1, 1)$,

$\kappa = \dfrac{2}{5^{3/2}} = \dfrac{2}{5\sqrt{5}}$.

(b) $\kappa' = \dfrac{d\kappa}{dx} = -3(1 + 4x^2)^{-5/2}(8x) = 0 \;\Leftrightarrow\; x = 0 \;\Rightarrow\; y = 0$. This is a maximum since $\kappa' > 0$ for $x < 0$ and $\kappa' < 0$ for $x > 0$. So the parabola $y = x^2$ has maximum curvature at the origin.

71. $x = \theta - \sin\theta \;\Rightarrow\; \dot{x} = 1 - \cos\theta \;\Rightarrow\; \ddot{x} = \sin\theta$, and $y = 1 - \cos\theta \;\Rightarrow\; \dot{y} = \sin\theta \;\Rightarrow\; \ddot{y} = \cos\theta$. Therefore,

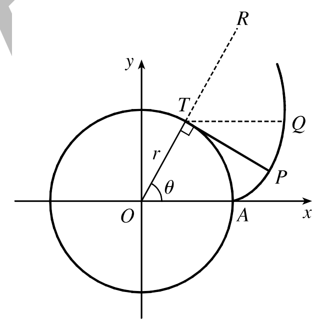
$\kappa = \dfrac{\left|\cos\theta - \cos^2\theta - \sin^2\theta\right|}{[(1 - \cos\theta)^2 + \sin^2\theta]^{3/2}} = \dfrac{\left|\cos\theta - (\cos^2\theta + \sin^2\theta)\right|}{(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)^{3/2}} = \dfrac{|\cos\theta - 1|}{(2 - 2\cos\theta)^{3/2}}$. The top of the arch is characterized by a horizontal tangent, and from Example 2(b) in Section 10.2, the tangent is horizontal when $\theta = (2n-1)\pi$, so take $n = 1$ and substitute $\theta = \pi$ into the expression for κ: $\kappa = \dfrac{|\cos\pi - 1|}{(2 - 2\cos\pi)^{3/2}} = \dfrac{|-1-1|}{[2-2(-1)]^{3/2}} = \dfrac{1}{4}$.

72. (a) Every straight line has parametrizations of the form $x = a + vt$, $y = b + wt$, where a, b are arbitrary and $v, w \neq 0$. For example, a straight line passing through distinct points (a, b) and (c, d) can be described as the parametrized curve $x = a + (c-a)t$, $y = b + (d-b)t$. Starting with $x = a + vt$, $y = b + wt$, we compute $\dot{x} = v$, $\dot{y} = w$, $\ddot{x} = \ddot{y} = 0$, and $\kappa = \dfrac{|v \cdot 0 - w \cdot 0|}{(v^2+w^2)^{3/2}} = 0$.

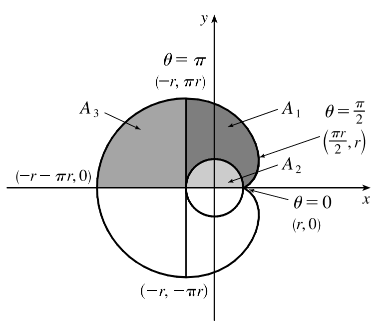
(b) Parametric equations for a circle of radius r are $x = r\cos\theta$ and $y = r\sin\theta$. We can take the center to be the origin. So $\dot{x} = -r\sin\theta \;\Rightarrow\; \ddot{x} = -r\cos\theta$ and $\dot{y} = r\cos\theta \;\Rightarrow\; \ddot{y} = -r\sin\theta$. Therefore,

$\kappa = \dfrac{\left|r^2\sin^2\theta + r^2\cos^2\theta\right|}{(r^2\sin^2\theta + r^2\cos^2\theta)^{3/2}} = \dfrac{r^2}{r^3} = \dfrac{1}{r}$. And so for any θ (and thus any point), $\kappa = \dfrac{1}{r}$.

73. The coordinates of T are $(r\cos\theta, r\sin\theta)$. Since TP was unwound from arc TA, TP has length $r\theta$. Also $\angle PTQ = \angle PTR - \angle QTR = \frac{1}{2}\pi - \theta$, so P has coordinates $x = r\cos\theta + r\theta\cos\left(\frac{1}{2}\pi - \theta\right) = r(\cos\theta + \theta\sin\theta)$, $y = r\sin\theta - r\theta\sin\left(\frac{1}{2}\pi - \theta\right) = r(\sin\theta - \theta\cos\theta)$.



74. If the cow walks with the rope taut, it traces out the portion of the involute in Exercise 73 corresponding to the range $0 \leq \theta \leq \pi$, arriving at the point $(-r, \pi r)$ when $\theta = \pi$. With the rope now fully extended, the cow walks in a semicircle of radius πr, arriving at $(-r, -\pi r)$. Finally, the cow traces out another portion of the involute, namely the reflection about the x-axis of the initial involute path. (This corresponds to the range $-\pi \leq \theta \leq 0$.) Referring to the figure, we see that the total grazing area is $2(A_1 + A_3)$. A_3 is one-quarter of the area of a circle of radius πr, so $A_3 = \frac{1}{4}\pi(\pi r)^2 = \frac{1}{4}\pi^3 r^2$. We will compute $A_1 + A_2$ and then subtract $A_2 = \frac{1}{2}\pi r^2$ to obtain A_1.



To find $A_1 + A_2$, first note that the rightmost point of the involute is $\left(\frac{\pi}{2}r, r\right)$. [To see this, note that $dx/d\theta = 0$ when $\theta = 0$ or $\frac{\pi}{2}$. $\theta = 0$ corresponds to the cusp at $(r, 0)$ and $\theta = \frac{\pi}{2}$ corresponds to $\left(\frac{\pi}{2}r, r\right)$.] The leftmost point of the involute is $(-r, \pi r)$. Thus, $A_1 + A_2 = \int_{\theta=\pi}^{\pi/2} y\,dx - \int_{\theta=0}^{\pi/2} y\,dx = \int_{\theta=\pi}^{0} y\,dx$.

Now $y\,dx = r(\sin\theta - \theta\cos\theta)\,r\theta\cos\theta\,d\theta = r^2(\theta\sin\theta\,\cos\theta - \theta^2\cos^2\theta)d\theta$. Integrate:

$(1/r^2)\int y\,dx = -\theta\cos^2\theta - \frac{1}{2}\left(\theta^2 - 1\right)\sin\theta\,\cos\theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta + C$. This enables us to compute

$$A_1 + A_2 = r^2\left[-\theta\cos^2\theta - \tfrac{1}{2}(\theta^2 - 1)\,\sin\theta\,\cos\theta - \tfrac{1}{6}\theta^3 + \tfrac{1}{2}\theta\right]_\pi^0 = r^2\left[0 - \left(-\pi - \frac{\pi^3}{6} + \frac{\pi}{2}\right)\right] = r^2\left(\frac{\pi}{2} + \frac{\pi^3}{6}\right)$$

Therefore, $A_1 = (A_1 + A_2) - A_2 = \frac{1}{6}\pi^3 r^2$, so the grazing area is $2(A_1 + A_3) = 2\left(\frac{1}{6}\pi^3 r^2 + \frac{1}{4}\pi^3 r^2\right) = \frac{5}{6}\pi^3 r^2$.

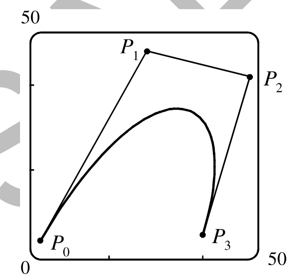
LABORATORY PROJECT Bézier Curves

1. The parametric equations for a cubic Bézier curve are

$$x = x_0(1-t)^3 + 3x_1 t(1-t)^2 + 3x_2 t^2(1-t) + x_3 t^3$$
$$y = y_0(1-t)^3 + 3y_1 t(1-t)^2 + 3y_2 t^2(1-t) + y_3 t^3$$

where $0 \le t \le 1$. We are given the points $P_0(x_0, y_0) = (4, 1)$, $P_1(x_1, y_1) = (28, 48)$, $P_2(x_2, y_2) = (50, 42)$, and $P_3(x_3, y_3) = (40, 5)$. The curve is then given by

$$x(t) = 4(1-t)^3 + 3 \cdot 28t(1-t)^2 + 3 \cdot 50t^2(1-t) + 40t^3$$
$$y(t) = 1(1-t)^3 + 3 \cdot 48t(1-t)^2 + 3 \cdot 42t^2(1-t) + 5t^3$$



where $0 \le t \le 1$. The line segments are of the form $x = x_0 + (x_1 - x_0)t$, $y = y_0 + (y_1 - y_0)t$:

P_0P_1	$x = 4 + 24t,$	$y = 1 + 47t$
P_1P_2	$x = 28 + 22t,$	$y = 48 - 6t$
P_2P_3	$x = 50 - 10t,$	$y = 42 - 37t$

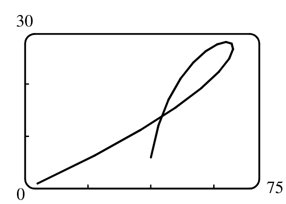
2. It suffices to show that the slope of the tangent at P_0 is the same as that of line segment P_0P_1, namely $\dfrac{y_1 - y_0}{x_1 - x_0}$.

We calculate the slope of the tangent to the Bézier curve:

$$\frac{dy/dt}{dx/dt} = \frac{-3y_0(1-t)^2 + 3y_1\left[-2t(1-t) + (1-t)^2\right] + 3y_2\left[-t^2 + (2t)(1-t)\right] + 3y_3t^2}{-3x_0^2(1-t) + 3x_1[-2t(1-t) + (1-t)^2] + 3x_2[-t^2 + (2t)(1-t)] + 3x_3t^2}$$

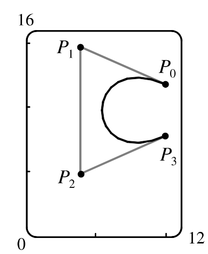
At point P_0, $t = 0$, so the slope of the tangent is $\dfrac{-3y_0 + 3y_1}{-3x_0 + 3x_1} = \dfrac{y_1 - y_0}{x_1 - x_0}$. So the tangent to the curve at P_0 passes through P_1. Similarly, the slope of the tangent at point P_3 [where $t = 1$] is $\dfrac{-3y_2 + 3y_3}{-3x_2 + 3x_3} = \dfrac{y_3 - y_2}{x_3 - x_2}$, which is also the slope of line P_2P_3.

3. It seems that if P_1 were to the right of P_2, a loop would appear.

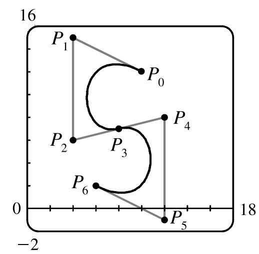


We try setting $P_1 = (110, 30)$, and the resulting curve does indeed have a loop.

4. Based on the behavior of the Bézier curve in Problems 1–3, we suspect that the four control points should be in an exaggerated C shape. We try $P_0(10, 12)$, $P_1(4, 15)$, $P_2(4, 5)$, and $P_3(10, 8)$, and these produce a decent C. If you are using a CAS, it may be necessary to instruct it to make the x- and y-scales the same so as not to distort the figure (this is called a "constrained projection" in Maple.)

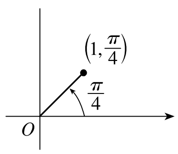


5. We use the same P_0 and P_1 as in Problem 4, and use part of our C as the top of an S. To prevent the center line from slanting up too much, we move P_2 up to $(4, 6)$ and P_3 down and to the left, to $(8, 7)$. In order to have a smooth joint between the top and bottom halves of the S (and a symmetric S), we determine points P_4, P_5, and P_6 by rotating points P_2, P_1, and P_0 about the center of the letter (point P_3). The points are therefore $P_4(12, 8)$, $P_5(12, -1)$, and $P_6(6, 2)$.



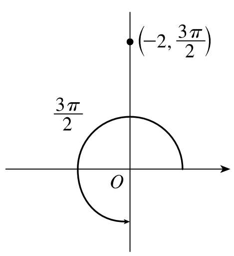
10.3 Polar Coordinates

1. (a) $\left(1, \frac{\pi}{4}\right)$



By adding 2π to $\frac{\pi}{4}$, we obtain the point $\left(1, \frac{9\pi}{4}\right)$, which satisfies the $r > 0$ requirement. The direction opposite $\frac{\pi}{4}$ is $\frac{5\pi}{4}$, so $\left(-1, \frac{5\pi}{4}\right)$ is a point that satisfies the $r < 0$ requirement.

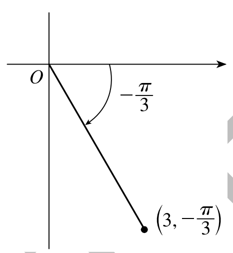
(b) $\left(-2, \frac{3\pi}{2}\right)$



$\boldsymbol{r} > \mathbf{0}$: $\left(-(-2), \frac{3\pi}{2} - \pi\right) = \left(2, \frac{\pi}{2}\right)$

$\boldsymbol{r} < \mathbf{0}$: $\left(-2, \frac{3\pi}{2} + 2\pi\right) = \left(-2, \frac{7\pi}{2}\right)$

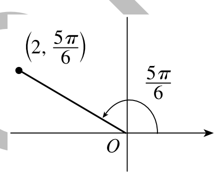
(c) $\left(3, -\frac{\pi}{3}\right)$



$\boldsymbol{r} > \mathbf{0}$: $\left(3, -\frac{\pi}{3} + 2\pi\right) = \left(3, \frac{5\pi}{3}\right)$

$\boldsymbol{r} < \mathbf{0}$: $\left(-3, -\frac{\pi}{3} + \pi\right) = \left(-3, \frac{2\pi}{3}\right)$

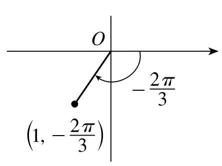
2. (a) $\left(2, \frac{5\pi}{6}\right)$



$\boldsymbol{r} > \mathbf{0}$: $\left(2, \frac{5\pi}{6} + 2\pi\right) = \left(2, \frac{17\pi}{6}\right)$

$\boldsymbol{r} < \mathbf{0}$: $\left(-2, \frac{5\pi}{6} - \pi\right) = \left(-2, -\frac{\pi}{6}\right)$

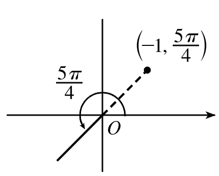
(b) $\left(1, -\frac{2\pi}{3}\right)$



$\boldsymbol{r} > \mathbf{0}$:$\left(1, -\frac{2\pi}{3} + 2\pi\right) = \left(1, \frac{4\pi}{3}\right)$

$\boldsymbol{r} < \mathbf{0}$: $\left(-1, -\frac{2\pi}{3} + \pi\right) = \left(-1, \frac{\pi}{3}\right)$

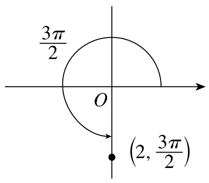
(c) $\left(-1, \frac{5\pi}{4}\right)$



$\boldsymbol{r} > \mathbf{0}$: $\left(-(-1), \frac{5\pi}{4} - \pi\right) = \left(1, \frac{\pi}{4}\right)$

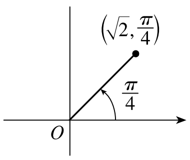
$\boldsymbol{r} < \mathbf{0}$: $\left(-1, \frac{5\pi}{4} - 2\pi\right) = \left(-1, -\frac{3\pi}{4}\right)$

3. (a)



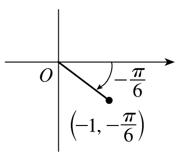
$x = 2\cos\frac{3\pi}{2} = 2(0) = 0$ and $y = 2\sin\frac{3\pi}{2} = 2(-1) = -2$ give us the Cartesian coordinates $(0, -2)$.

(b)



$x = \sqrt{2}\cos\dfrac{\pi}{4} = \sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right) = 1$ and $y = \sqrt{2}\sin\dfrac{\pi}{4} = \sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right) = 1$ give us the Cartesian coordinates $(1, 1)$.

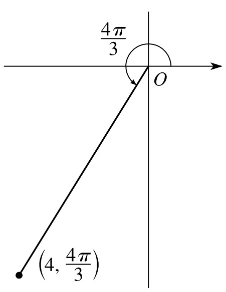
(c)



$x = -1\cos\left(-\dfrac{\pi}{6}\right) = -1\left(\dfrac{\sqrt{3}}{2}\right) = -\dfrac{\sqrt{3}}{2}$ and

$y = -1\sin\left(-\dfrac{\pi}{6}\right) = -1\left(-\dfrac{1}{2}\right) = \dfrac{1}{2}$ give us the Cartesian coordinates $\left(-\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$.

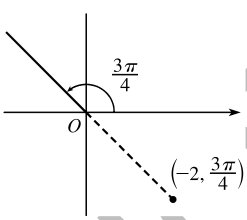
4. (a)



$x = 4\cos\dfrac{4\pi}{3} = 4\left(-\dfrac{1}{2}\right) = -2$ and

$y = 4\sin\dfrac{4\pi}{3} = 4\left(-\dfrac{\sqrt{3}}{2}\right) = -2\sqrt{3}$ give us the Cartesian coordinates $\left(-2, -2\sqrt{3}\right)$.

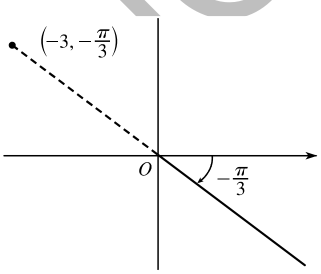
(b)



$x = -2\cos\dfrac{3\pi}{4} = -2\left(-\dfrac{\sqrt{2}}{2}\right) = \sqrt{2}$ and

$y = -2\sin\dfrac{3\pi}{4} = -2\left(\dfrac{\sqrt{2}}{2}\right) = -\sqrt{2}$ give us the Cartesian coordinates $\left(\sqrt{2}, -\sqrt{2}\right)$.

(c)



$x = -3\cos\left(-\dfrac{\pi}{3}\right) = -3\left(\dfrac{1}{2}\right) = -\dfrac{3}{2}$ and

$y = -3\sin\left(-\dfrac{\pi}{3}\right) = -3\left(-\dfrac{\sqrt{3}}{2}\right) = \dfrac{3\sqrt{3}}{2}$ give us the Cartesian coordinates $\left(-\dfrac{3}{2}, \dfrac{3\sqrt{3}}{2}\right)$.

5. (a) $x = -4$ and $y = 4 \quad\Rightarrow\quad r = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$ and $\tan\theta = \frac{4}{-4} = -1 \quad [\theta = -\frac{\pi}{4} + n\pi]$. Since $(-4, 4)$ is in the second quadrant, the polar coordinates are (i) $\left(4\sqrt{2}, \frac{3\pi}{4}\right)$ and (ii) $\left(-4\sqrt{2}, \frac{7\pi}{4}\right)$.

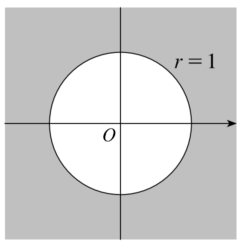
(b) $x = 3$ and $y = 3\sqrt{3} \;\;\Rightarrow\;\; r = \sqrt{3^2 + \left(3\sqrt{3}\right)^2} = \sqrt{9 + 27} = 6$ and $\tan\theta = \dfrac{3\sqrt{3}}{3} = \sqrt{3}$ $\;\left[\theta = \frac{\pi}{3} + n\pi\right]$.

Since $\left(3, 3\sqrt{3}\right)$ is in the first quadrant, the polar coordinates are (i) $\left(6, \frac{\pi}{3}\right)$ and (ii) $\left(-6, \frac{4\pi}{3}\right)$.

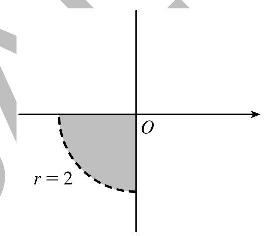
6. (a) $x = \sqrt{3}$ and $y = -1 \;\;\Rightarrow\;\; r = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$ and $\tan\theta = \dfrac{-1}{\sqrt{3}}$ $\;\left[\theta = -\frac{\pi}{6} + n\pi\right]$. Since $\left(\sqrt{3}, -1\right)$ is in the fourth quadrant, the polar coordinates are (i) $\left(2, \frac{11\pi}{6}\right)$ and (ii) $\left(-2, \frac{5\pi}{6}\right)$.

(b) $x = -6$ and $y = 0 \;\;\Rightarrow\;\; r = \sqrt{(-6)^2 + 0^2} = 6$ and $\tan\theta = \frac{0}{-6} = 0$ $\;[\theta = n\pi]$. Since $(-6, 0)$ is on the negative x-axis, the polar coordinates are (i) $(6, \pi)$ and (ii) $(-6, 0)$.

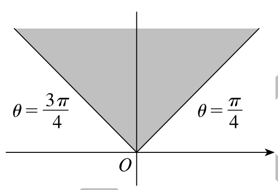
7. $r \geq 1$. The curve $r = 1$ represents a circle with center O and radius 1. So $r \geq 1$ represents the region on or outside the circle. Note that θ can take on any value.



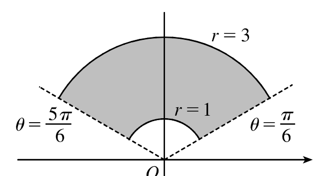
8. $0 \leq r < 2,\; \pi \leq \theta \leq 3\pi/2$. This is the region inside the circle $r = 2$ in the third quadrant.



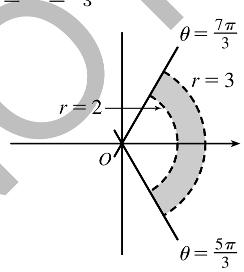
9. $r \geq 0,\; \pi/4 \leq \theta \leq 3\pi/4$.
$\theta = k$ represents a line through O.



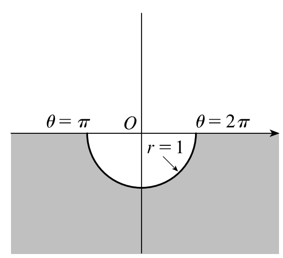
10. $1 \leq r \leq 3,\; \pi/6 < \theta < 5\pi/6$



11. $2 < r < 3,\; \frac{5\pi}{3} \leq \theta \leq \frac{7\pi}{3}$



12. $r \geq 1,\; \pi \leq \theta \leq 2\pi$



13. Converting the polar coordinates $\left(4, \frac{4\pi}{3}\right)$ and $\left(6, \frac{5\pi}{3}\right)$ to Cartesian coordinates gives us $\left(4\cos\frac{4\pi}{3}, 4\sin\frac{4\pi}{3}\right) = \left(-2, -2\sqrt{3}\right)$ and $\left(6\cos\frac{5\pi}{3}, 6\sin\frac{5\pi}{3}\right) = \left(3, -3\sqrt{3}\right)$. Now use the distance formula

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[3 - (-2)]^2 + \left[-3\sqrt{3} - \left(-2\sqrt{3}\right)\right]^2} \\ &= \sqrt{5^2 + \left(-\sqrt{3}\right)^2} = \sqrt{25 + 3} = \sqrt{28} = 2\sqrt{7} \end{aligned}$$

14. The points (r_1, θ_1) and (r_2, θ_2) in Cartesian coordinates are $(r_1 \cos\theta_1, r_1 \sin\theta_1)$ and $(r_2 \cos\theta_2, r_2 \sin\theta_2)$, respectively. The *square* of the distance between them is

$$\begin{aligned}(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2 \\ &= \left(r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\cos^2\theta_1\right) + \left(r_2^2\sin^2\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_1^2\sin^2\theta_1\right) \\ &= r_1^2\left(\sin^2\theta_1 + \cos^2\theta_1\right) + r_2^2\left(\sin^2\theta_2 + \cos^2\theta_2\right) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\ &= r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2,\end{aligned}$$

so the distance between them is $\sqrt{r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2}$.

15. $r^2 = 5 \quad\Leftrightarrow\quad x^2 + y^2 = 5$, a circle of radius $\sqrt{5}$ centered at the origin.

16. $r = 4\sec\theta \quad\Leftrightarrow\quad \dfrac{r}{\sec\theta} = 4 \quad\Leftrightarrow\quad r\cos\theta = 4 \quad\Leftrightarrow\quad x = 4$, a vertical line.

17. $r = 5\cos\theta \quad\Rightarrow\quad r^2 = 5r\cos\theta \quad\Leftrightarrow\quad x^2 + y^2 = 5x \quad\Leftrightarrow\quad x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4} \quad\Leftrightarrow\quad \left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$, a circle of radius $\frac{5}{2}$ centered at $\left(\frac{5}{2}, 0\right)$. The first two equations are actually equivalent since $r^2 = 5r\cos\theta \quad\Rightarrow$ $r(r - 5\cos\theta) = 0 \quad\Rightarrow\quad r = 0$ or $r = 5\cos\theta$. But $r = 5\cos\theta$ gives the point $r = 0$ (the pole) when $\theta = 0$. Thus, the equation $r = 5\cos\theta$ is equivalent to the compound condition ($r = 0$ or $r = 5\cos\theta$).

18. $\theta = \dfrac{\pi}{3} \quad\Rightarrow\quad \tan\theta = \tan\dfrac{\pi}{3} \quad\Rightarrow\quad \dfrac{y}{x} = \sqrt{3} \quad\Leftrightarrow\quad y = \sqrt{3}\,x$, a line through the origin.

19. $r^2\cos 2\theta = 1 \quad\Leftrightarrow\quad r^2(\cos^2\theta - \sin^2\theta) = 1 \quad\Leftrightarrow\quad (r\cos\theta)^2 - (r\sin\theta)^2 = 1 \quad\Leftrightarrow\quad x^2 - y^2 = 1$, a hyperbola centered at the origin with foci on the x-axis.

20. $r^2\sin 2\theta = 1 \quad\Leftrightarrow\quad r^2(2\sin\theta\,\cos\theta) = 1 \quad\Leftrightarrow\quad 2(r\cos\theta)(r\sin\theta) = 1 \quad\Leftrightarrow\quad 2xy = 1 \quad\Leftrightarrow\quad xy = \frac{1}{2}$, a hyperbola centered at the origin with foci on the line $y = x$.

21. $y = 2 \quad\Leftrightarrow\quad r\sin\theta = 2 \quad\Leftrightarrow\quad r = \dfrac{2}{\sin\theta} \quad\Leftrightarrow\quad r = 2\csc\theta$

22. $y = x \quad\Rightarrow\quad \dfrac{y}{x} = 1\ [x \neq 0] \quad\Rightarrow\quad \tan\theta = 1 \quad\Rightarrow\quad \theta = \tan^{-1}1 \quad\Rightarrow\quad \theta = \dfrac{\pi}{4}$ or $\theta = \dfrac{5\pi}{4}$ [either includes the pole]

23. $y = 1 + 3x \quad\Leftrightarrow\quad r\sin\theta = 1 + 3r\cos\theta \quad\Leftrightarrow\quad r\sin\theta - 3r\cos\theta = 1 \quad\Leftrightarrow\quad r(\sin\theta - 3\cos\theta) = 1 \quad\Leftrightarrow$ $r = \dfrac{1}{\sin\theta - 3\cos\theta}$

24. $4y^2 = x \quad\Leftrightarrow\quad 4(r\sin\theta)^2 = r\cos\theta \quad\Leftrightarrow\quad 4r^2\sin^2\theta - r\cos\theta = 0 \quad\Leftrightarrow\quad r(4r\sin^2\theta - \cos\theta) = 0 \quad\Leftrightarrow\quad r = 0$ or $r = \dfrac{\cos\theta}{4\sin^2\theta} \quad\Leftrightarrow\quad r = 0$ or $r = \frac{1}{4}\cot\theta\,\csc\theta$. $r = 0$ is included in $r = \frac{1}{4}\cot\theta\,\csc\theta$ when $\theta = \frac{\pi}{2}$, so the curve is represented by the single equation $r = \frac{1}{4}\cot\theta\,\csc\theta$.

25. $x^2 + y^2 = 2cx \quad\Leftrightarrow\quad r^2 = 2cr\cos\theta \quad\Leftrightarrow\quad r^2 - 2cr\cos\theta = 0 \quad\Leftrightarrow\quad r(r - 2c\cos\theta) = 0 \quad\Leftrightarrow\quad r = 0$ or $r = 2c\cos\theta$. $r = 0$ is included in $r = 2c\cos\theta$ when $\theta = \frac{\pi}{2} + n\pi$, so the curve is represented by the single equation $r = 2c\cos\theta$.

26. $x^2 - y^2 = 4 \iff (r\cos\theta)^2 - (r\sin\theta)^2 = 4 \iff r^2\cos^2\theta - r^2\sin^2\theta = 4 \iff r^2(\cos^2\theta - \sin^2\theta) = 4 \iff$ $r^2\cos 2\theta = 4$

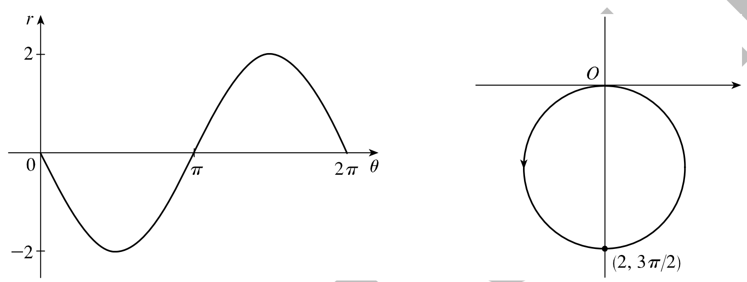
27. (a) The description leads immediately to the polar equation $\theta = \frac{\pi}{6}$, and the Cartesian equation $y = \tan\left(\frac{\pi}{6}\right)x = \frac{1}{\sqrt{3}}x$ is slightly more difficult to derive.

(b) The easier description here is the Cartesian equation $x = 3$.

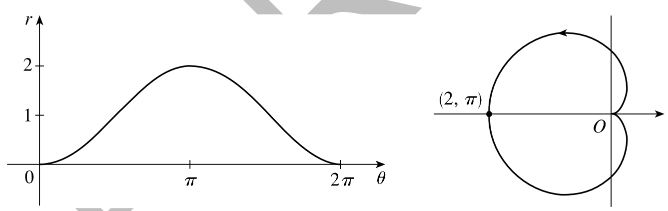
28. (a) Because its center is not at the origin, it is more easily described by its Cartesian equation, $(x-2)^2 + (y-3)^2 = 5^2$.

(b) This circle is more easily given in polar coordinates: $r = 4$. The Cartesian equation is also simple: $x^2 + y^2 = 16$.

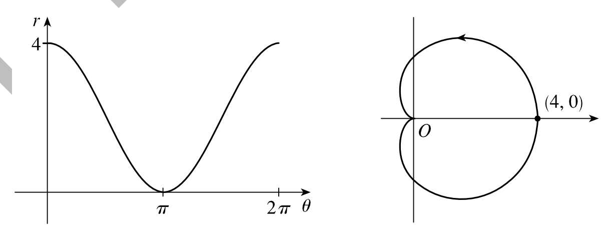
29. $r = -2\sin\theta$



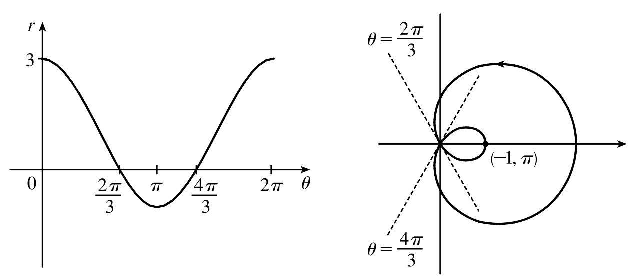
30. $r = 1 - \cos\theta$



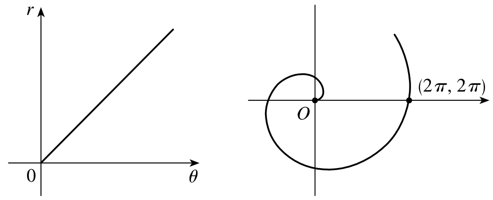
31. $r = 2(1 + \cos\theta)$



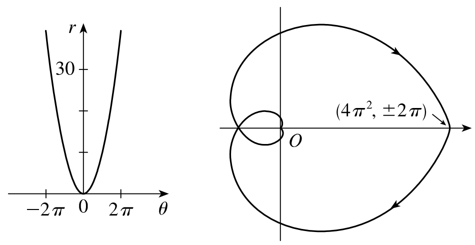
32. $r = 1 + 2\cos\theta$



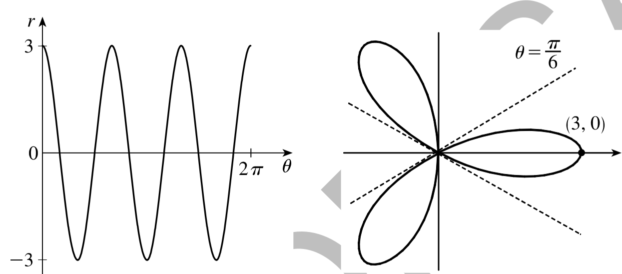
33. $r = \theta, \quad \theta \geq 0$



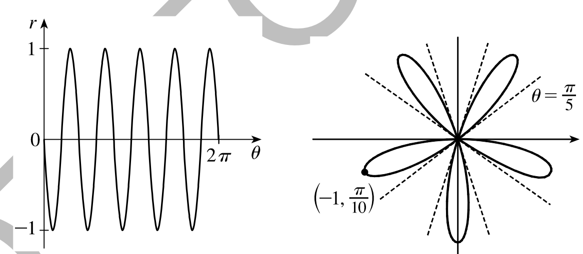
34. $r = \theta^2, \ -2\pi \leq \theta \leq 2\pi$



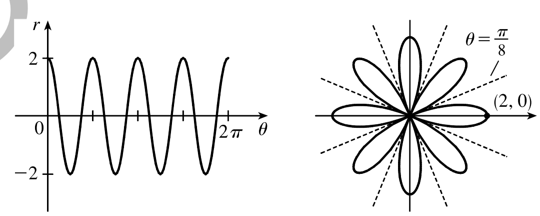
35. $r = 3\cos 3\theta$



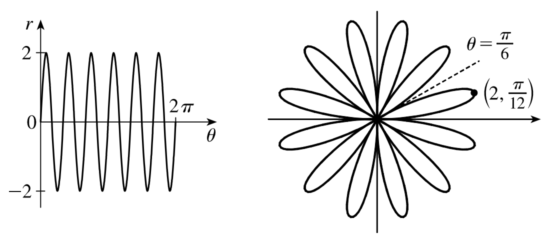
36. $r = -\sin 5\theta$



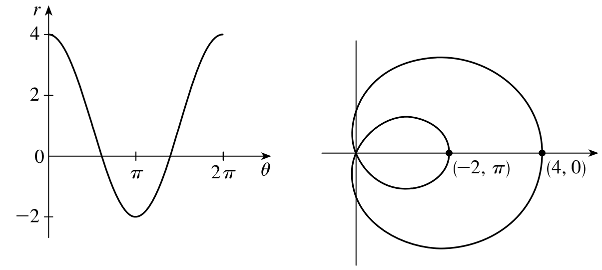
37. $r = 2\cos 4\theta$



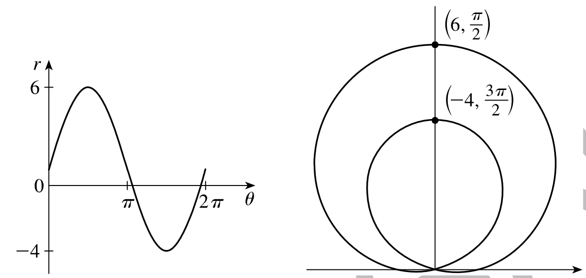
38. $r = 2\sin 6\theta$



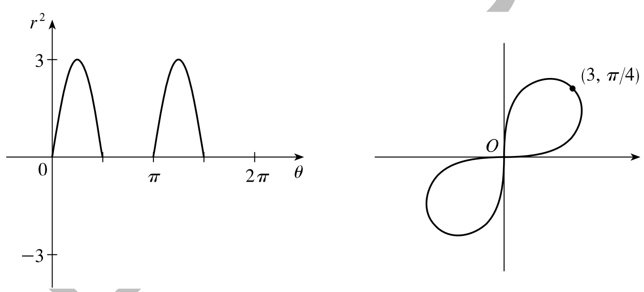
39. $r = 1 + 3\cos\theta$



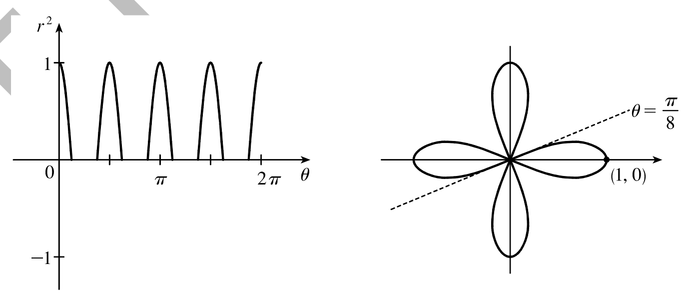
40. $r = 1 + 5\sin\theta$



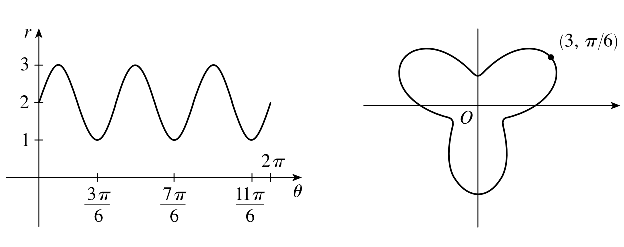
41. $r^2 = 9\sin 2\theta$



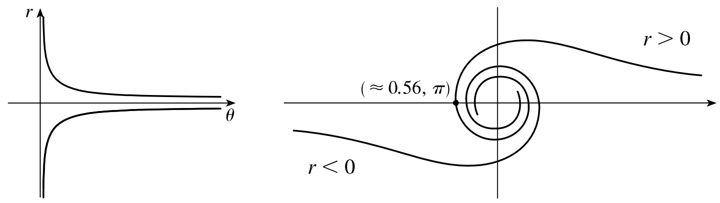
42. $r^2 = \cos 4\theta$



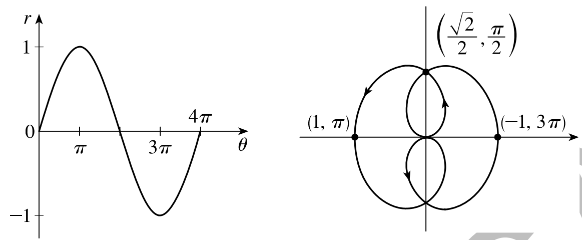
43. $r = 2 + \sin 3\theta$



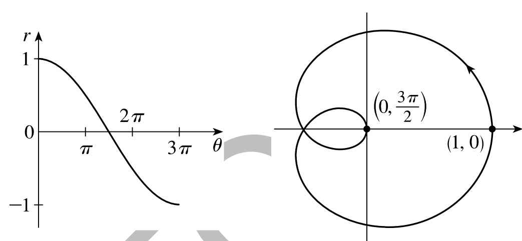
44. $r^2\theta = 1 \quad \Leftrightarrow \quad r = \pm 1/\sqrt{\theta}$ for $\theta > 0$



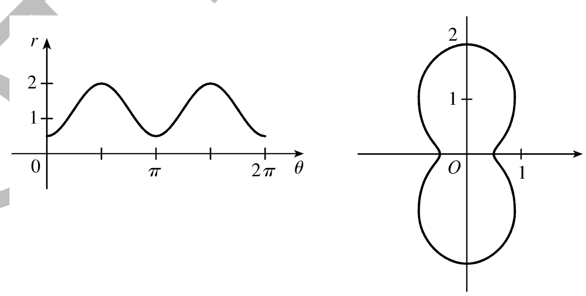
45. $r = \sin(\theta/2)$



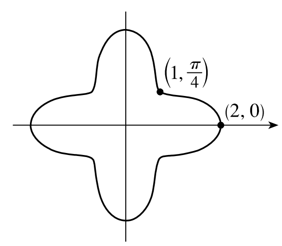
46. $r = \cos(\theta/3)$



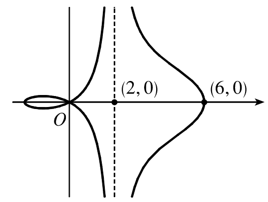
47. For $\theta = 0$, π, and 2π, r has its minimum value of about 0.5. For $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, r attains its maximum value of 2. We see that the graph has a similar shape for $0 \leq \theta \leq \pi$ and $\pi \leq \theta \leq 2\pi$.



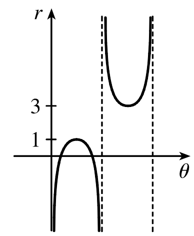
48. The given graph has a maximum of 2 for $\theta = 0$, a minimum of 1 for $\theta = \frac{\pi}{4}$, and then a maximum of 2 for $\theta = \frac{\pi}{2}$. This pattern is repeated 4 times for $0 \leq \theta \leq 2\pi$.

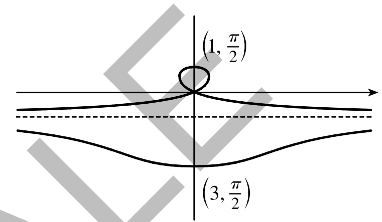


49. $x = r\cos\theta = (4 + 2\sec\theta)\cos\theta = 4\cos\theta + 2$. Now, $r \to \infty \quad \Rightarrow$ $(4 + 2\sec\theta) \to \infty \quad \Rightarrow \quad \theta \to \left(\frac{\pi}{2}\right)^-$ or $\theta \to \left(\frac{3\pi}{2}\right)^+$ [since we need only consider $0 \le \theta < 2\pi$], so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-}(4\cos\theta + 2) = 2$. Also, $r \to -\infty \quad \Rightarrow \quad (4 + 2\sec\theta) \to -\infty \quad \Rightarrow \quad \theta \to \left(\frac{\pi}{2}\right)^+$ or $\theta \to \left(\frac{3\pi}{2}\right)^-$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+}(4\cos\theta + 2) = 2$. Therefore, $\lim\limits_{r\to\pm\infty} x = 2 \quad \Rightarrow \quad x = 2$ is a vertical asymptote.



50. $y = r\sin\theta = 2\sin\theta - \csc\theta\sin\theta = 2\sin\theta - 1$.
$r \to \infty \quad \Rightarrow \quad (2 - \csc\theta) \to \infty \quad \Rightarrow$ $\csc\theta \to -\infty \quad \Rightarrow \quad \theta \to \pi^+$ [since we need only consider $0 \le \theta < 2\pi$] and so $\lim\limits_{r\to\infty} y = \lim\limits_{\theta\to\pi^+} 2\sin\theta - 1 = -1$.



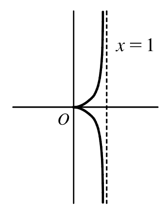


Also $r \to -\infty \quad \Rightarrow \quad (2 - \csc\theta) \to -\infty \quad \Rightarrow \quad \csc\theta \to \infty \quad \Rightarrow \quad \theta \to \pi^-$ and so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi^-} 2\sin\theta - 1 = -1$.

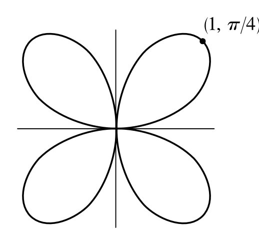
Therefore $\lim\limits_{r\to\pm\infty} y = -1 \quad \Rightarrow \quad y = -1$ is a horizontal asymptote.

51. To show that $x = 1$ is an asymptote we must prove $\lim\limits_{r\to\pm\infty} x = 1$.

$x = (r)\cos\theta = (\sin\theta\,\tan\theta)\cos\theta = \sin^2\theta$. Now, $r \to \infty \quad \Rightarrow \quad \sin\theta\,\tan\theta \to \infty \quad \Rightarrow$ $\theta \to \left(\frac{\pi}{2}\right)^-$, so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-} \sin^2\theta = 1$. Also, $r \to -\infty \quad \Rightarrow \quad \sin\theta\,\tan\theta \to -\infty \quad \Rightarrow$ $\theta \to \left(\frac{\pi}{2}\right)^+$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} \sin^2\theta = 1$. Therefore, $\lim\limits_{r\to\pm\infty} x = 1 \quad \Rightarrow \quad x = 1$ is a vertical asymptote. Also notice that $x = \sin^2\theta \ge 0$ for all θ, and $x = \sin^2\theta \le 1$ for all θ. And $x \ne 1$, since the curve is not defined at odd multiples of $\frac{\pi}{2}$. Therefore, the curve lies entirely within the vertical strip $0 \le x < 1$.



52. The equation is $(x^2 + y^2)^3 = 4x^2y^2$, but using polar coordinates we know that $x^2 + y^2 = r^2$ and $x = r\cos\theta$ and $y = r\sin\theta$. Substituting into the given equation: $r^6 = 4r^2\cos^2\theta\, r^2\sin^2\theta \quad \Rightarrow \quad r^2 = 4\cos^2\theta\,\sin^2\theta \quad \Rightarrow$ $r = \pm 2\cos\theta\,\sin\theta = \pm\sin 2\theta$. $r = \pm\sin 2\theta$ is sketched at right.



53. (a) We see that the curve $r = 1 + c\sin\theta$ crosses itself at the origin, where $r = 0$ (in fact the inner loop corresponds to negative r-values,) so we solve the equation of the limaçon for $r = 0 \quad \Leftrightarrow \quad c\sin\theta = -1 \quad \Leftrightarrow \quad \sin\theta = -1/c$. Now if $|c| < 1$, then this equation has no solution and hence there is no inner loop. But if $c < -1$, then on the interval $(0, 2\pi)$ the equation has the two solutions $\theta = \sin^{-1}(-1/c)$ and $\theta = \pi - \sin^{-1}(-1/c)$, and if $c > 1$, the solutions are $\theta = \pi + \sin^{-1}(1/c)$ and $\theta = 2\pi - \sin^{-1}(1/c)$. In each case, $r < 0$ for θ between the two solutions, indicating a loop.

(b) For $0 < c < 1$, the dimple (if it exists) is characterized by the fact that y has a local maximum at $\theta = \frac{3\pi}{2}$. So we determine for what c-values $\dfrac{d^2y}{d\theta^2}$ is negative at $\theta = \frac{3\pi}{2}$, since by the Second Derivative Test this indicates a maximum:

$y = r\sin\theta = \sin\theta + c\sin^2\theta \quad\Rightarrow\quad \dfrac{dy}{d\theta} = \cos\theta + 2c\sin\theta\,\cos\theta = \cos\theta + c\sin 2\theta \quad\Rightarrow\quad \dfrac{d^2y}{d\theta^2} = -\sin\theta + 2c\cos 2\theta.$

At $\theta = \frac{3\pi}{2}$, this is equal to $-(-1) + 2c(-1) = 1 - 2c$, which is negative only for $c > \frac{1}{2}$. A similar argument shows that for $-1 < c < 0$, y only has a local minimum at $\theta = \frac{\pi}{2}$ (indicating a dimple) for $c < -\frac{1}{2}$.

54. (a) $r = \ln\theta,\ 1 \le \theta \le 6\pi$. r increases as θ increases and there are almost three full revolutions. The graph must be either III or VI. As θ increases, r grows slowly in VI and quickly in III. Since $r = \ln\theta$ grows slowly, its graph must be VI.

(b) $r = \theta^2,\ 0 \le \theta \le 8\pi$. See part (a). This is graph III.

(c) The graph of $r = \cos 3\theta$ is a three-leaved rose, which is graph II.

(d) Since $-1 \le \cos 3\theta \le 1$, $1 \le 2 + \cos 3\theta \le 3$, so $r = 2 + \cos 3\theta$ is never 0; that is, the curve never intersects the pole. The graph must be I or IV. For $0 \le \theta \le 2\pi$, the graph assumes its minimum r-value of 1 three times, at $\theta = \frac{\pi}{3}$, π, and $\frac{5\pi}{3}$, so it must be graph IV.

(e) $r = \cos(\theta/2)$. For $\theta = 0$, $r = 1$, and as θ increases to π, r decreases to 0. Only graph V satisfies those values.

(f) $r = 2 + \cos(3\theta/2)$. As in part (d), this graph never intersects the pole, so it must be graph I.

55. $r = 2\cos\theta \quad\Rightarrow\quad x = r\cos\theta = 2\cos^2\theta,\ y = r\sin\theta = 2\sin\theta\,\cos\theta = \sin 2\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos 2\theta}{2\cdot 2\cos\theta(-\sin\theta)} = \frac{\cos 2\theta}{-\sin 2\theta} = -\cot 2\theta$$

When $\theta = \dfrac{\pi}{3}$, $\dfrac{dy}{dx} = -\cot\left(2\cdot\dfrac{\pi}{3}\right) = \cot\dfrac{\pi}{3} = \dfrac{1}{\sqrt{3}}$. [*Another method:* Use Equation 3.]

56. $r = 2 + \sin 3\theta \quad\Rightarrow\quad x = r\cos\theta = (2 + \sin 3\theta)\cos\theta,\ y = r\sin\theta = (2 + \sin 3\theta)\sin\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(2 + \sin 3\theta)\cos\theta + \sin\theta(3\cos 3\theta)}{(2 + \sin 3\theta)(-\sin\theta) + \cos\theta(3\cos 3\theta)}$$

When $\theta = \dfrac{\pi}{4}$, $\dfrac{dy}{dx} = \dfrac{\left(2 + \sin\frac{3\pi}{4}\right)\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\left(3\cos\frac{3\pi}{4}\right)}{\left(2 + \sin\frac{3\pi}{4}\right)\left(-\sin\frac{\pi}{4}\right) + \cos\frac{\pi}{4}\left(3\cos\frac{3\pi}{4}\right)} = \dfrac{\left(2 + \frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot 3\left(-\frac{\sqrt{2}}{2}\right)}{\left(2 + \frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}\cdot 3\left(-\frac{\sqrt{2}}{2}\right)}$

$$= \frac{\sqrt{2} + \frac{1}{2} - \frac{3}{2}}{-\sqrt{2} - \frac{1}{2} - \frac{3}{2}} = \frac{\sqrt{2} - 1}{-\sqrt{2} - 2}, \text{ or, equivalently, } 2 - \frac{3}{2}\sqrt{2}.$$

57. $r = 1/\theta \quad\Rightarrow\quad x = r\cos\theta = (\cos\theta)/\theta,\ y = r\sin\theta = (\sin\theta)/\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta(-1/\theta^2) + (1/\theta)\cos\theta}{\cos\theta(-1/\theta^2) - (1/\theta)\sin\theta}\cdot\frac{\theta^2}{\theta^2} = \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta}$$

When $\theta = \pi$, $\dfrac{dy}{dx} = \dfrac{-0 + \pi(-1)}{-(-1) - \pi(0)} = \dfrac{-\pi}{1} = -\pi$.

58. $r = \cos(\theta/3) \quad\Rightarrow\quad x = r\cos\theta = \cos(\theta/3)\cos\theta,\ y = r\sin\theta = \cos(\theta/3)\sin\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos(\theta/3)\cos\theta + \sin\theta\left(-\frac{1}{3}\sin(\theta/3)\right)}{\cos(\theta/3)(-\sin\theta) + \cos\theta\left(-\frac{1}{3}\sin(\theta/3)\right)}$$

When $\theta = \pi$, $\dfrac{dy}{dx} = \dfrac{\frac{1}{2}(-1) + (0)\left(-\sqrt{3}/6\right)}{\frac{1}{2}(0) + (-1)\left(-\sqrt{3}/6\right)} = \dfrac{-1/2}{\sqrt{3}/6} = -\dfrac{3}{\sqrt{3}} = -\sqrt{3}$.

59. $r = \cos 2\theta \quad \Rightarrow \quad x = r\cos\theta = \cos 2\theta\,\cos\theta,\, y = r\sin\theta = \cos 2\theta\,\sin\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos 2\theta\,\cos\theta + \sin\theta\,(-2\sin 2\theta)}{\cos 2\theta\,(-\sin\theta) + \cos\theta\,(-2\sin 2\theta)}$$

When $\theta = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{0\left(\sqrt{2}/2\right) + \left(\sqrt{2}/2\right)(-2)}{0\left(-\sqrt{2}/2\right) + \left(\sqrt{2}/2\right)(-2)} = \frac{-\sqrt{2}}{-\sqrt{2}} = 1.$

60. $r = 1 + 2\cos\theta \quad \Rightarrow \quad x = r\cos\theta = (1 + 2\cos\theta)\cos\theta,\, y = r\sin\theta = (1 + 2\cos\theta)\sin\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(1 + 2\cos\theta)\cos\theta + \sin\theta\,(-2\sin\theta)}{(1 + 2\cos\theta)(-\sin\theta) + \cos\theta\,(-2\sin\theta)}$$

When $\theta = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{2\left(\frac{1}{2}\right) + \left(\sqrt{3}/2\right)\left(-\sqrt{3}\right)}{2\left(-\sqrt{3}/2\right) + \frac{1}{2}\left(-\sqrt{3}\right)} \cdot \frac{2}{2} = \frac{2 - 3}{-2\sqrt{3} - \sqrt{3}} = \frac{-1}{-3\sqrt{3}} = \frac{\sqrt{3}}{9}.$

61. $r = 3\cos\theta \quad \Rightarrow \quad x = r\cos\theta = 3\cos\theta\,\cos\theta,\ y = r\sin\theta = 3\cos\theta\,\sin\theta \quad \Rightarrow$

$\frac{dy}{d\theta} = -3\sin^2\theta + 3\cos^2\theta = 3\cos 2\theta = 0 \quad \Rightarrow \quad 2\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \quad \Leftrightarrow \quad \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

So the tangent is horizontal at $\left(\frac{3}{\sqrt{2}}, \frac{\pi}{4}\right)$ and $\left(-\frac{3}{\sqrt{2}}, \frac{3\pi}{4}\right)$ $\left[\text{same as } \left(\frac{3}{\sqrt{2}}, -\frac{\pi}{4}\right)\right]$.

$\frac{dx}{d\theta} = -6\sin\theta\cos\theta = -3\sin 2\theta = 0 \quad \Rightarrow \quad 2\theta = 0$ or $\pi \quad \Leftrightarrow \quad \theta = 0$ or $\frac{\pi}{2}$. So the tangent is vertical at $(3, 0)$ and $\left(0, \frac{\pi}{2}\right)$.

62. $r = 1 - \sin\theta \quad \Rightarrow \quad x = r\cos\theta = \cos\theta\,(1 - \sin\theta),\ y = r\sin\theta = \sin\theta\,(1 - \sin\theta) \quad \Rightarrow$

$\frac{dy}{d\theta} = \sin\theta\,(-\cos\theta) + (1 - \sin\theta)\,\cos\theta = \cos\theta\,(1 - 2\sin\theta) = 0 \quad \Rightarrow \quad \cos\theta = 0$ or $\sin\theta = \frac{1}{2} \quad \Rightarrow$

$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, or $\frac{3\pi}{2} \quad \Rightarrow \quad$ horizontal tangent at $\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$, and $\left(2, \frac{3\pi}{2}\right)$.

$$\begin{aligned}\frac{dx}{d\theta} &= \cos\theta\,(-\cos\theta) + (1 - \sin\theta)(-\sin\theta) = -\cos^2\theta - \sin\theta + \sin^2\theta = 2\sin^2\theta - \sin\theta - 1\\ &= (2\sin\theta + 1)(\sin\theta - 1) = 0 \quad \Rightarrow\end{aligned}$$

$\sin\theta = -\frac{1}{2}$ or $1 \quad \Rightarrow \quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$, or $\frac{\pi}{2} \quad \Rightarrow \quad$ vertical tangent at $\left(\frac{3}{2}, \frac{7\pi}{6}\right)$, $\left(\frac{3}{2}, \frac{11\pi}{6}\right)$, and $\left(0, \frac{\pi}{2}\right)$.

Note that the tangent is vertical, not horizontal, when $\theta = \frac{\pi}{2}$, since

$$\lim_{\theta \to (\pi/2)^-} \frac{dy/d\theta}{dx/d\theta} = \lim_{\theta \to (\pi/2)^-} \frac{\cos\theta\,(1 - 2\sin\theta)}{(2\sin\theta + 1)(\sin\theta - 1)} = \infty \text{ and } \lim_{\theta \to (\pi/2)^+} \frac{dy/d\theta}{dx/d\theta} = -\infty.$$

63. $r = 1 + \cos\theta \quad \Rightarrow \quad x = r\cos\theta = \cos\theta\,(1 + \cos\theta),\ y = r\sin\theta = \sin\theta\,(1 + \cos\theta) \quad \Rightarrow$

$\frac{dy}{d\theta} = (1 + \cos\theta)\,\cos\theta - \sin^2\theta = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0 \quad \Rightarrow \quad \cos\theta = \frac{1}{2}$ or $-1 \quad \Rightarrow$

$\theta = \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3} \quad \Rightarrow \quad$ horizontal tangent at $\left(\frac{3}{2}, \frac{\pi}{3}\right)$, $(0, \pi)$, and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$.

$\frac{dx}{d\theta} = -(1 + \cos\theta)\sin\theta - \cos\theta\sin\theta = -\sin\theta\,(1 + 2\cos\theta) = 0 \quad \Rightarrow \quad \sin\theta = 0$ or $\cos\theta = -\frac{1}{2} \quad \Rightarrow$

$\theta = 0, \pi, \frac{2\pi}{3}$, or $\frac{4\pi}{3} \quad \Rightarrow \quad$ vertical tangent at $(2, 0)$, $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$, and $\left(\frac{1}{2}, \frac{4\pi}{3}\right)$.

Note that the tangent is horizontal, not vertical when $\theta = \pi$, since $\lim\limits_{\theta \to \pi} \frac{dy/d\theta}{dx/d\theta} = 0$.

64. $r = e^\theta \quad \Rightarrow \quad x = r\cos\theta = e^\theta\cos\theta,\ y = r\sin\theta = e^\theta\sin\theta \quad \Rightarrow$

$\frac{dy}{d\theta} = e^\theta\sin\theta + e^\theta\cos\theta = e^\theta(\sin\theta + \cos\theta) = 0 \quad \Rightarrow \quad \sin\theta = -\cos\theta \quad \Rightarrow \quad \tan\theta = -1 \quad \Rightarrow$

$\theta = -\frac{1}{4}\pi + n\pi$ [n any integer] $\Rightarrow$ horizontal tangents at $\left(e^{\pi(n-1/4)}, \pi\left(n-\frac{1}{4}\right)\right)$.

$\frac{dx}{d\theta} = e^{\theta}\cos\theta - e^{\theta}\sin\theta = e^{\theta}(\cos\theta - \sin\theta) = 0 \quad\Rightarrow\quad \sin\theta = \cos\theta \quad\Rightarrow\quad \tan\theta = 1 \quad\Rightarrow$

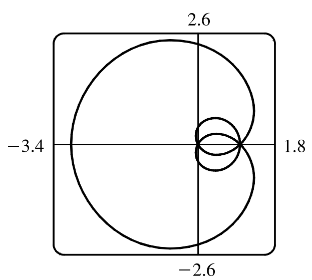
$\theta = \frac{1}{4}\pi + n\pi$ [n any integer] $\Rightarrow$ vertical tangents at $\left(e^{\pi(n+1/4)}, \pi\left(n+\frac{1}{4}\right)\right)$.

65. $r = a\sin\theta + b\cos\theta \quad\Rightarrow\quad r^2 = ar\sin\theta + br\cos\theta \quad\Rightarrow\quad x^2 + y^2 = ay + bx \quad\Rightarrow$

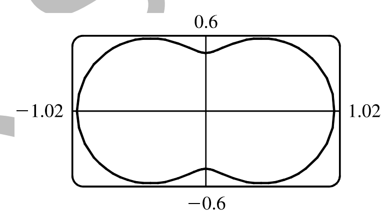
$x^2 - bx + \left(\frac{1}{2}b\right)^2 + y^2 - ay + \left(\frac{1}{2}a\right)^2 = \left(\frac{1}{2}b\right)^2 + \left(\frac{1}{2}a\right)^2 \quad\Rightarrow\quad \left(x - \frac{1}{2}b\right)^2 + \left(y - \frac{1}{2}a\right)^2 = \frac{1}{4}(a^2 + b^2)$, and this is a circle with center $\left(\frac{1}{2}b, \frac{1}{2}a\right)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2}$.

66. These curves are circles which intersect at the origin and at $\left(\frac{1}{\sqrt{2}}\,a, \frac{\pi}{4}\right)$. At the origin, the first circle has a horizontal tangent and the second a vertical one, so the tangents are perpendicular here. For the first circle [$r = a\sin\theta$], $dy/d\theta = a\cos\theta\,\sin\theta + a\sin\theta\,\cos\theta = a\sin 2\theta = a$ at $\theta = \frac{\pi}{4}$ and $dx/d\theta = a\cos^2\theta - a\sin^2\theta = a\cos 2\theta = 0$ at $\theta = \frac{\pi}{4}$, so the tangent here is vertical. Similarly, for the second circle [$r = a\cos\theta$], $dy/d\theta = a\cos 2\theta = 0$ and $dx/d\theta = -a\sin 2\theta = -a$ at $\theta = \frac{\pi}{4}$, so the tangent is horizontal, and again the tangents are perpendicular.

67. $r = 1 + 2\sin(\theta/2)$. The parameter interval is $[0, 4\pi]$.

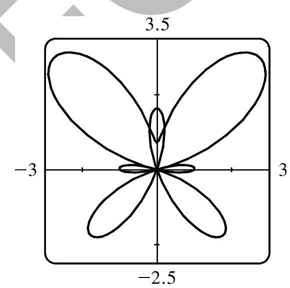


68. $r = \sqrt{1 - 0.8\sin^2\theta}$. The parameter interval is $[0, 2\pi]$.



69. $r = e^{\sin\theta} - 2\cos(4\theta)$.

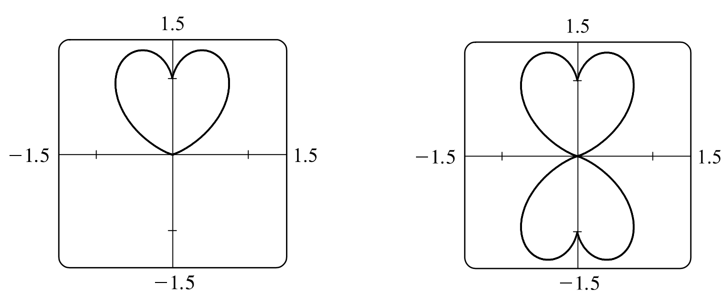
The parameter interval is $[0, 2\pi]$.



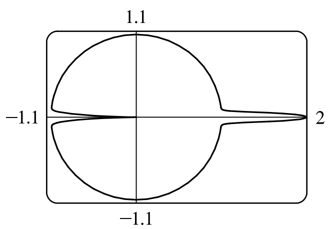
70. $r = |\tan\theta|^{|\cot\theta|}$.

The parameter interval $[0, \pi]$ produces the heart-shaped valentine curve shown in the first window.

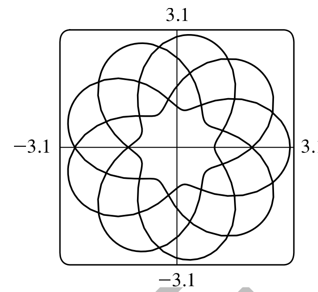
The complete curve, including the reflected heart, is produced by the parameter interval $[0, 2\pi]$, but perhaps you'll agree that the first curve is more appropriate.



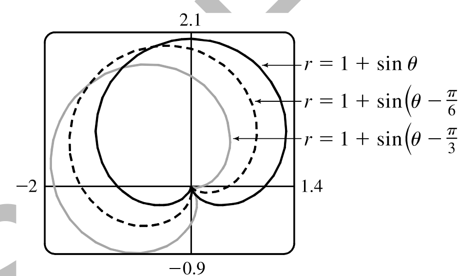
71. $r = 1 + \cos^{999}\theta$. The parameter interval is $[0, 2\pi]$.



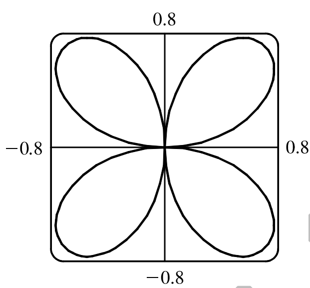
72. $r = 2 + \cos(9\theta/4)$. The parameter interval is $[0, 8\pi]$.



73. It appears that the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{6}\right)$ is the same shape as the graph of $r = 1 + \sin\theta$, but rotated counterclockwise about the origin by $\frac{\pi}{6}$. Similarly, the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{3}\right)$ is rotated by $\frac{\pi}{3}$. In general, the graph of $r = f(\theta - \alpha)$ is the same shape as that of $r = f(\theta)$, but rotated counterclockwise through α about the origin. That is, for any point (r_0, θ_0) on the curve $r = f(\theta)$, the point $(r_0, \theta_0 + \alpha)$ is on the curve $r = f(\theta - \alpha)$, since $r_0 = f(\theta_0) = f((\theta_0 + \alpha) - \alpha)$.



74.



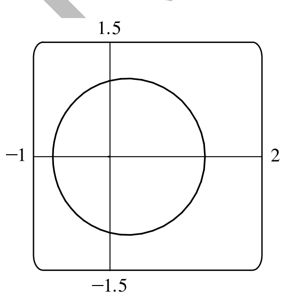
From the graph, the highest points seem to have $y \approx 0.77$. To find the exact value, we solve $dy/d\theta = 0$. $y = r\sin\theta = \sin\theta\,\sin 2\theta \quad\Rightarrow$

$$\begin{aligned} dy/d\theta &= 2\sin\theta\,\cos 2\theta + \cos\theta\,\sin 2\theta \\ &= 2\sin\theta\,(2\cos^2\theta - 1) + \cos\theta\,(2\sin\theta\,\cos\theta) \\ &= 2\sin\theta\,(3\cos^2\theta - 1) \end{aligned}$$

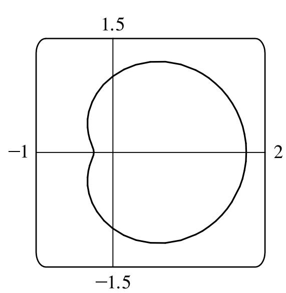
In the first quadrant, this is 0 when $\cos\theta = \frac{1}{\sqrt{3}} \quad\Leftrightarrow\quad \sin\theta = \sqrt{\frac{2}{3}} \quad\Leftrightarrow$

$y = 2\sin^2\theta\cos\theta = 2\cdot\frac{2}{3}\cdot\frac{1}{\sqrt{3}} = \frac{4}{9}\sqrt{3} \approx 0.77$.

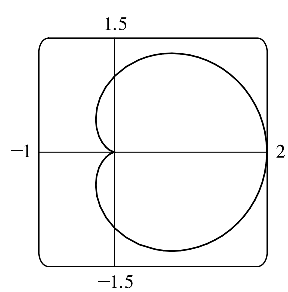
75. Consider curves with polar equation $r = 1 + c\cos\theta$, where c is a real number. If $c = 0$, we get a circle of radius 1 centered at the pole. For $0 < c \leq 0.5$, the curve gets slightly larger, moves right, and flattens out a bit on the left side. For $0.5 < c < 1$, the left side has a dimple shape. For $c = 1$, the dimple becomes a cusp. For $c > 1$, there is an internal loop. For $c \geq 0$, the rightmost point on the curve is $(1 + c, 0)$. For $c < 0$, the curves are reflections through the vertical axis of the curves with $c > 0$.



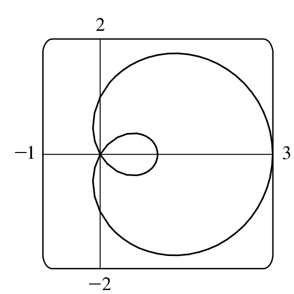
$c = 0.25$



$c = 0.75$

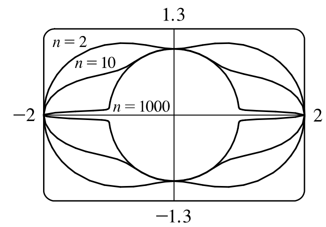


$c = 1$

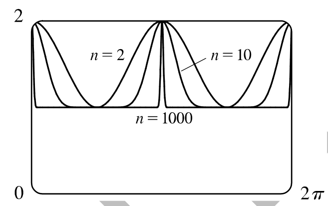


$c = 2$

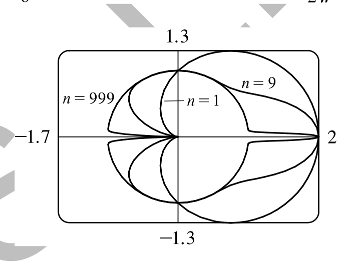
76. Consider the polar curves $r = 1 + \cos^n \theta$, where n is a positive integer. First, let $\boldsymbol{n}$ **be an even positive integer**. The first figure shows that the curve has a peanut shape for $n = 2$, but as n increases, the ends are squeezed. As n becomes large, the curves look more and more like the unit circle, but with spikes to the points $(2, 0)$ and $(2, \pi)$.



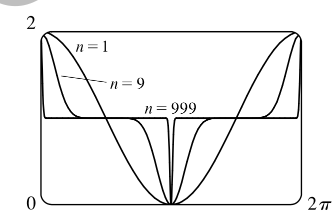
The second figure shows r as a function of θ in Cartesian coordinates for the same values of n. We can see that for large n, the graph is similar to the graph of $y = 1$, but with spikes to $y = 2$ for $x = 0$, π, and 2π. (Note that when $0 < \cos\theta < 1$, $\cos^{1000}\theta$ is very small.)



Next, let $\boldsymbol{n}$ **be an odd positive integer**. The third figure shows that the curve is a cardioid for $n = 1$, but as n increases, the heart shape becomes more pronounced. As n becomes large, the curves again look more like the unit circle, but with an outward spike to $(2, 0)$ and an inward spike to $(0, \pi)$.



The fourth figure shows r as a function of θ in Cartesian coordinates for the same values of n. We can see that for large n, the graph is similar to the graph of $y = 1$, but spikes to $y = 2$ for $x = 0$ and π, and to $y = 0$ for $x = \pi$.

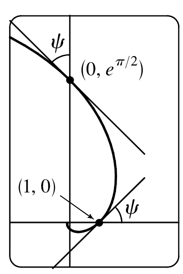


77. $$\tan\psi = \tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\,\tan\theta} = \frac{\dfrac{dy}{dx} - \tan\theta}{1 + \dfrac{dy}{dx}\tan\theta} = \frac{\dfrac{dy/d\theta}{dx/d\theta} - \tan\theta}{1 + \dfrac{dy/d\theta}{dx/d\theta}\tan\theta}$$

$$= \frac{\dfrac{dy}{d\theta} - \dfrac{dx}{d\theta}\tan\theta}{\dfrac{dx}{d\theta} + \dfrac{dy}{d\theta}\tan\theta} = \frac{\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right) - \tan\theta\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right)}{\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right) + \tan\theta\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right)} = \frac{r\cos\theta + r\cdot\dfrac{\sin^2\theta}{\cos\theta}}{\dfrac{dr}{d\theta}\cos\theta + \dfrac{dr}{d\theta}\cdot\dfrac{\sin^2\theta}{\cos\theta}}$$

$$= \frac{r\cos^2\theta + r\sin^2\theta}{\dfrac{dr}{d\theta}\cos^2\theta + \dfrac{dr}{d\theta}\sin^2\theta} = \frac{r}{dr/d\theta}$$

78. (a) $r = e^\theta \;\Rightarrow\; dr/d\theta = e^\theta$, so by Exercise 77, $\tan\psi = r/e^\theta = 1 \;\Rightarrow\; \psi = \arctan 1 = \frac{\pi}{4}$.

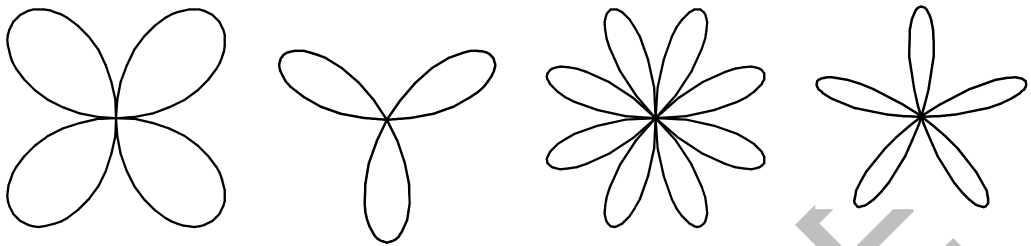


(b) The Cartesian equation of the tangent line at $(1, 0)$ is $y = x - 1$, and that of the tangent line at $(0, e^{\pi/2})$ is $y = e^{\pi/2} - x$.

(c) Let a be the tangent of the angle between the tangent and radial lines, that is, $a = \tan\psi$. Then, by Exercise 77, $a = \dfrac{r}{dr/d\theta} \;\Rightarrow\; \dfrac{dr}{d\theta} = \dfrac{1}{a}r \;\Rightarrow\; r = Ce^{\theta/a}$ (by Theorem 9.4.2).

LABORATORY PROJECT Families of Polar Curves

1. (a) $r = \sin n\theta$.

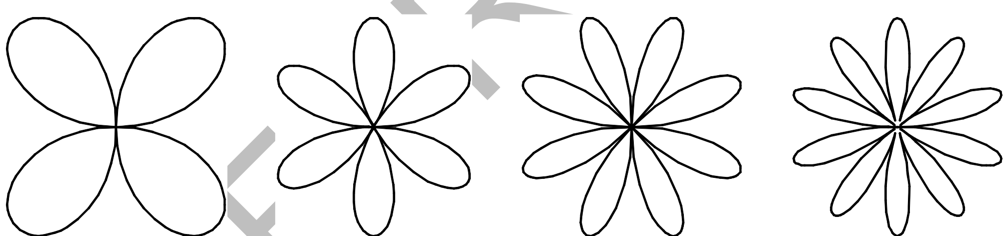


$n = 2$ $n = 3$ $n = 4$ $n = 5$

From the graphs, it seems that when n is even, the number of loops in the curve (called a rose) is $2n$, and when n is odd, the number of loops is simply n. This is because in the case of n odd, every point on the graph is traversed twice, due to the fact that

$$r(\theta + \pi) = \sin[n(\theta + \pi)] = \sin n\theta \, \cos n\pi + \cos n\theta \, \sin n\pi = \begin{cases} \sin n\theta & \text{if } n \text{ is even} \\ -\sin n\theta & \text{if } n \text{ is odd} \end{cases}$$

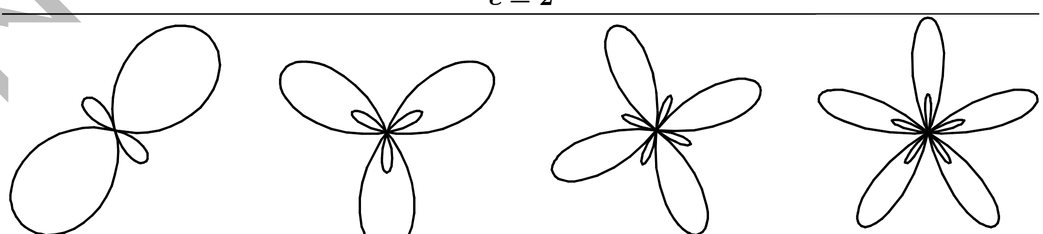
(b) The graph of $r = |\sin n\theta|$ has $2n$ loops whether n is odd or even, since $r(\theta + \pi) = r(\theta)$.



$n = 2$ $n = 3$ $n = 4$ $n = 5$

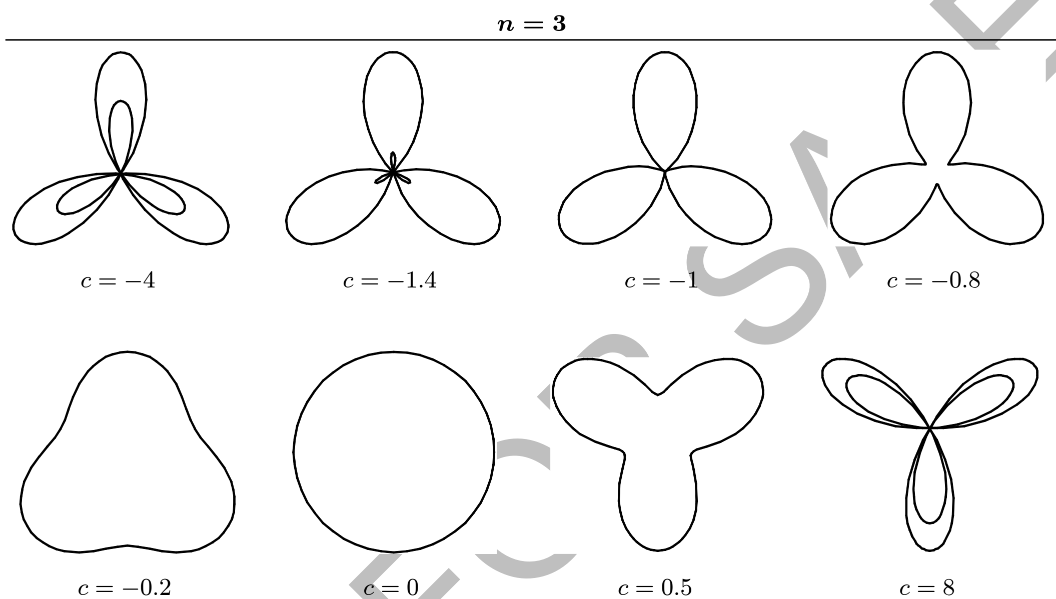
2. $r = 1 + c \sin n\theta$. We vary n while keeping c constant at 2. As n changes, the curves change in the same way as those in Exercise 1: the number of loops increases. Note that if n is even, the smaller loops are outside the larger ones; if n is odd, they are inside.

$c = 2$



$n = 2$ $n = 3$ $n = 4$ $n = 5$

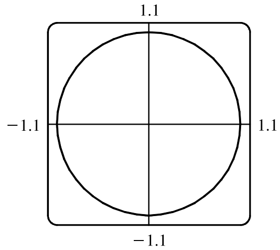
Now we vary c while keeping $n = 3$. As c increases toward 0, the entire graph gets smaller (the graphs below are not to scale) and the smaller loops shrink in relation to the large ones. At $c = -1$, the small loops disappear entirely, and for $-1 < c < 1$, the graph is a simple, closed curve (at $c = 0$ it is a circle). As c continues to increase, the same changes are seen, but in reverse order, since $1 + (-c)\sin n\theta = 1 + c\sin n(\theta + \pi)$, so the graph for $c = c_0$ is the same as that for $c = -c_0$, with a rotation through π. As $c \to \infty$, the smaller loops get relatively closer in size to the large ones. Note that the distance between the outermost points of corresponding inner and outer loops is always 2. Maple's `animate` command (or Mathematica's `Animate`) is very useful for seeing the changes that occur as c varies.



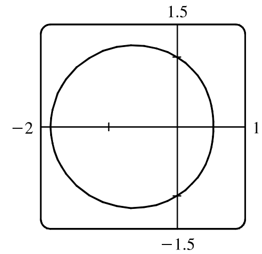
3. $r = \dfrac{1 - a\cos\theta}{1 + a\cos\theta}$. We start with $a = 0$, since in this case the curve is simply the circle $r = 1$.

As a increases, the graph moves to the left, and its right side becomes flattened. As a increases through about 0.4, the right side seems to grow a dimple, which upon closer investigation (with narrower θ-ranges) seems to appear at $a \approx 0.42$ [the actual value is $\sqrt{2} - 1$]. As $a \to 1$, this dimple becomes more pronounced, and the curve begins to stretch out horizontally, until at $a = 1$ the denominator vanishes at $\theta = \pi$, and the dimple becomes an actual cusp. For $a > 1$ we must choose our parameter interval carefully, since $r \to \infty$ as $1 + a\cos\theta \to 0 \quad \Leftrightarrow \quad \theta \to \pm\cos^{-1}(-1/a)$. As a increases from 1, the curve splits into two parts. The left part has a loop, which grows larger as a increases, and the right part grows broader vertically, and its left tip develops a dimple when $a \approx 2.42$ [actually, $\sqrt{2} + 1$]. As a increases, the dimple grows more and more pronounced. If $a < 0$, we get the same graph as we do for the corresponding positive a-value, but with a rotation through π about the pole, as happened when c was replaced with $-c$ in Exercise 2.

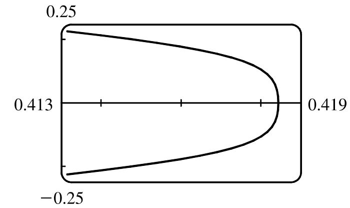
[continued]



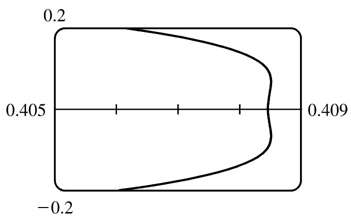
$a = 0$



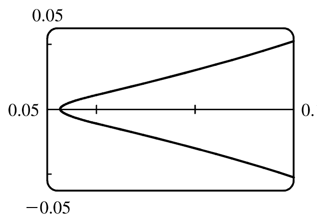
$a = 0.3$



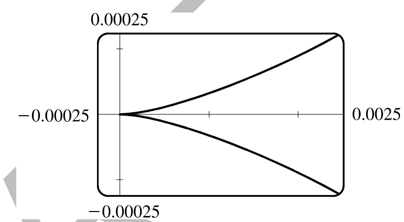
$a = 0.41, |\theta| \leq 0.5$



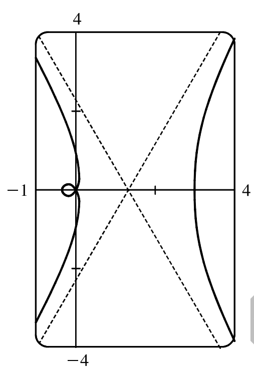
$a = 0.42, |\theta| \leq 0.5$



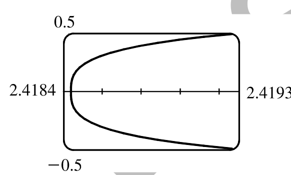
$a = 0.9, |\theta| \leq 0.5$



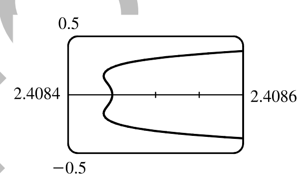
$a = 1, |\theta| \leq 0.1$



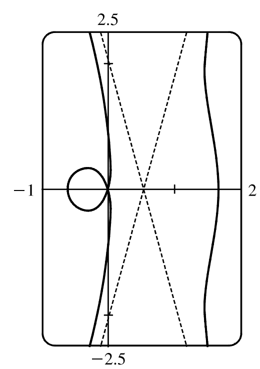
$a = 2$



$a = 2.41, |\theta - \pi| \leq 0.2$



$a = 2.42, |\theta - \pi| \leq 0.2$



$a = 4$

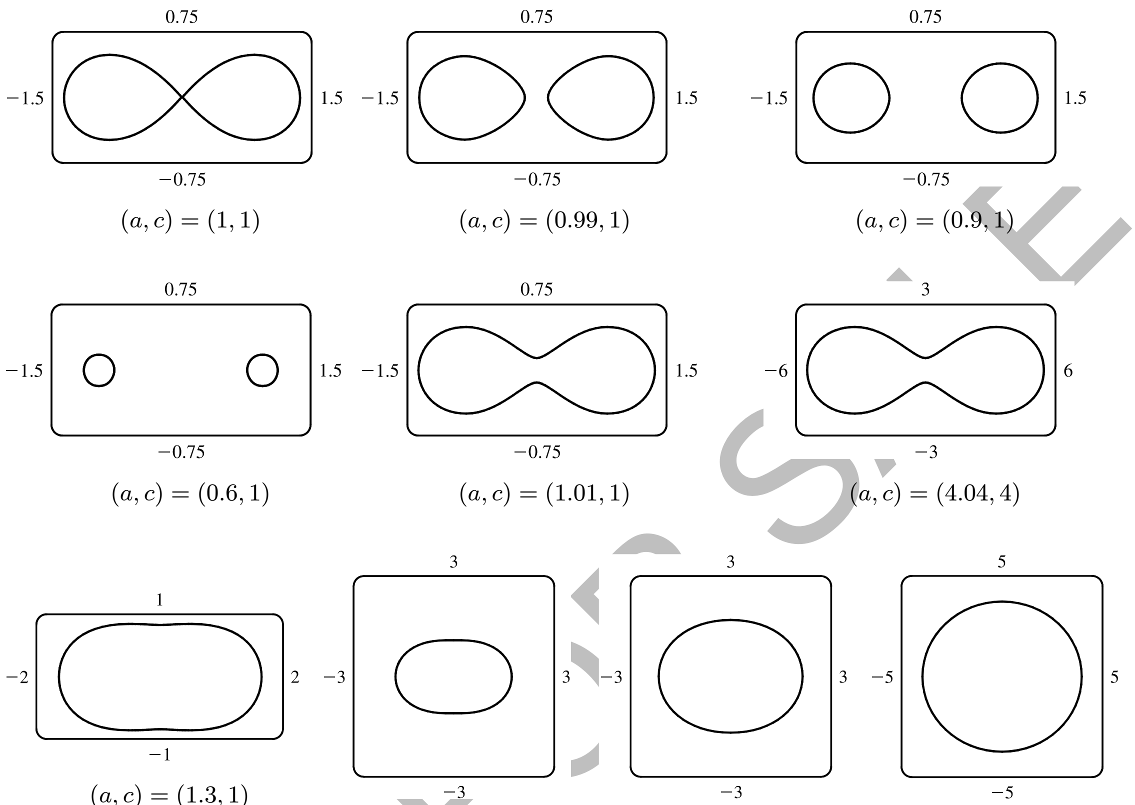
4. Most graphing devices cannot plot implicit polar equations, so we must first find an explicit expression (or expressions) for r in terms of θ, a, and c. We note that the given equation, $r^4 - 2c^2r^2\cos 2\theta + c^4 - a^4 = 0$, is a quadratic in r^2, so we use the quadratic formula and find that

$$r^2 = \frac{2c^2\cos 2\theta \pm \sqrt{4c^4\cos^2 2\theta - 4(c^4 - a^4)}}{2} = c^2\cos 2\theta \pm \sqrt{a^4 - c^4\sin^2 2\theta}$$

so $r = \pm\sqrt{c^2\cos 2\theta \pm \sqrt{a^4 - c^4\sin^2 2\theta}}$. So for each graph, we must plot four curves to be sure of plotting all the points which satisfy the given equation. Note that all four functions have period π.

We start with the case $a = c = 1$, and the resulting curve resembles the symbol for infinity. If we let a decrease, the curve splits into two symmetric parts, and as a decreases further, the parts become smaller, further apart, and rounder. If instead we let a increase from 1, the two lobes of the curve join together, and as a increases further they continue to merge, until at

$a \approx 1.4$, the graph no longer has dimples, and has an oval shape. As $a \to \infty$, the oval becomes larger and rounder, since the c^2 and c^4 terms lose their significance. Note that the shape of the graph seems to depend only on the ratio c/a, while the size of the graph varies as c and a jointly increase.



$(a, c) = (1, 1)$ $(a, c) = (0.99, 1)$ $(a, c) = (0.9, 1)$

$(a, c) = (0.6, 1)$ $(a, c) = (1.01, 1)$ $(a, c) = (4.04, 4)$

$(a, c) = (1.3, 1)$ $(a, c) = (1.5, 1)$ $(a, c) = (2, 1)$ $(a, c) = (4, 1)$

10.4 Areas and Lengths in Polar Coordinates

1. $r = e^{-\theta/4},\ \pi/2 \le \theta \le \pi$.

$$A = \int_{\pi/2}^{\pi} \tfrac{1}{2} r^2\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2}(e^{-\theta/4})^2\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2} e^{-\theta/2}\, d\theta = \tfrac{1}{2}\Big[-2e^{-\theta/2}\Big]_{\pi/2}^{\pi} = -1(e^{-\pi/2} - e^{-\pi/4}) = e^{-\pi/4} - e^{-\pi/2}$$

2. $r = \cos\theta,\ 0 \le \theta \le \pi/6$.

$$A = \int_0^{\pi/6} \tfrac{1}{2} r^2\, d\theta = \int_0^{\pi/6} \tfrac{1}{2}\cos^2\theta\, d\theta = \tfrac{1}{2}\int_0^{\pi/6} \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta = \tfrac{1}{4}\big[\theta + \tfrac{1}{2}\sin 2\theta\big]_0^{\pi/6}$$
$$= \tfrac{1}{4}\big(\tfrac{\pi}{6} + \tfrac{1}{2}\cdot\tfrac{1}{2}\sqrt{3}\big) = \tfrac{\pi}{24} + \tfrac{1}{16}\sqrt{3}$$

3. $r = \sin\theta + \cos\theta,\ 0 \le \theta \le \pi$.

$$A = \int_0^{\pi} \frac{1}{2} r^2\, d\theta = \int_0^{\pi} \frac{1}{2}(\sin\theta + \cos\theta)^2\, d\theta = \int_0^{\pi} \frac{1}{2}(\sin^2\theta + 2\sin\theta\,\cos\theta + \cos^2\theta)\, d\theta = \int_0^{\pi} \frac{1}{2}(1 + \sin 2\theta)\, d\theta$$
$$= \tfrac{1}{2}\Big[\theta - \tfrac{1}{2}\cos 2\theta\Big]_0^{\pi} = \tfrac{1}{2}\big[\big(\pi - \tfrac{1}{2}\big) - \big(0 - \tfrac{1}{2}\big)\big] = \tfrac{\pi}{2}$$

4. $r = 1/\theta,\ \pi/2 \le \theta \le 2\pi$.

$$A = \int_{\pi/2}^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_{\pi/2}^{2\pi} \frac{1}{2}\left(\frac{1}{\theta}\right)^2 d\theta = \int_{\pi/2}^{2\pi} \frac{1}{2}\theta^{-2}\, d\theta = \frac{1}{2}\left[-\frac{1}{\theta}\right]_{\pi/2}^{2\pi}$$
$$= \tfrac{1}{2}\left(-\tfrac{1}{2\pi} + \tfrac{2}{\pi}\right) = \tfrac{1}{2}\left(-\tfrac{1}{2\pi} + \tfrac{4}{2\pi}\right) = \tfrac{3}{4\pi}$$

5. $r^2 = \sin 2\theta,\ 0 \le \theta \le \pi/2$.

$$A = \int_0^{\pi/2} \tfrac{1}{2} r^2\, d\theta = \int_0^{\pi/2} \tfrac{1}{2} \sin 2\theta\, d\theta = \left[-\tfrac{1}{4}\cos 2\theta\right]_0^{\pi/2} = -\tfrac{1}{4}(\cos\pi - \cos 0) = -\tfrac{1}{4}(-1-1) = \tfrac{1}{2}$$

6. $r = 2 + \cos\theta,\ \pi/2 \le \theta \le \pi$.

$$A = \int_{\pi/2}^{\pi} \tfrac{1}{2} r^2\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2}(2+\cos\theta)^2\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2}(4 + 4\cos\theta + \cos^2\theta)\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2}\left[4 + 4\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right] d\theta$$
$$= \int_{\pi/2}^{\pi} \left(\tfrac{9}{4} + 2\cos\theta + \tfrac{1}{4}\cos 2\theta\right) d\theta = \left[\tfrac{9}{4}\theta + 2\sin\theta + \tfrac{1}{8}\sin 2\theta\right]_{\pi/2}^{\pi} = \left(\tfrac{9\pi}{4} + 0 + 0\right) - \left(\tfrac{9\pi}{8} + 2 + 0\right) = \tfrac{9\pi}{8} - 2$$

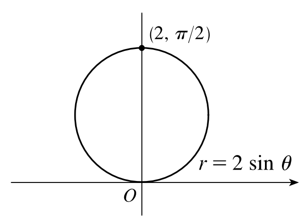
7. $r = 4 + 3\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$A = \int_{-\pi/2}^{\pi/2} \tfrac{1}{2}((4 + 3\sin\theta)^2\, d\theta = \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} (16 + 24\sin\theta + 9\sin^2\theta)\, d\theta$$
$$= \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} (16 + 9\sin^2\theta)\, d\theta \qquad \text{[by Theorem 5.5.7(b)]}$$
$$= \tfrac{1}{2}\cdot 2\int_0^{\pi/2} \left[16 + 9\cdot\tfrac{1}{2}(1 - \cos 2\theta)\right] d\theta \qquad \text{[by Theorem 5.5.7(a)]}$$
$$= \int_0^{\pi/2} \left(\tfrac{41}{2} - \tfrac{9}{2}\cos 2\theta\right) d\theta = \left[\tfrac{41}{2}\theta - \tfrac{9}{4}\sin 2\theta\right]_0^{\pi/2} = \left(\tfrac{41\pi}{4} - 0\right) - (0 - 0) = \tfrac{41\pi}{4}$$

8. $r = \sqrt{\ln\theta},\ 1 \le \theta \le 2\pi$.

$$A = \int_1^{2\pi} \tfrac{1}{2}\left(\sqrt{\ln\theta}\right)^2 d\theta = \int_1^{2\pi} \tfrac{1}{2}\ln\theta\, d\theta = \left[\tfrac{1}{2}\theta\ln\theta\right]_1^{2\pi} - \int_1^{2\pi} \tfrac{1}{2}\, d\theta \qquad \begin{bmatrix} u = \ln\theta, & dv = \frac{1}{2}\, d\theta \\ du = (1/\theta)\, d\theta, & v = \frac{1}{2}\theta \end{bmatrix}$$
$$= [\pi\ln(2\pi) - 0] - \left[\tfrac{1}{2}\theta\right]_1^{2\pi} = \pi\ln(2\pi) - \pi + \tfrac{1}{2}$$

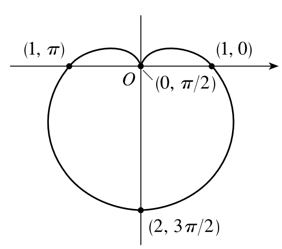
9. The area is bounded by $r = 2\sin\theta$ for $\theta = 0$ to $\theta = \pi$.



$$A = \int_0^{\pi} \tfrac{1}{2} r^2\, d\theta = \tfrac{1}{2}\int_0^{\pi} (2\sin\theta)^2\, d\theta = \tfrac{1}{2}\int_0^{\pi} 4\sin^2\theta\, d\theta$$
$$= 2\int_0^{\pi} \tfrac{1}{2}(1 - \cos 2\theta) d\theta = \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi} = \pi$$

Also, note that this is a circle with radius 1, so its area is $\pi(1)^2 = \pi$.

10. $A = \int_0^{2\pi} \tfrac{1}{2} r^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2}(1 - \sin\theta)^2\, d\theta$



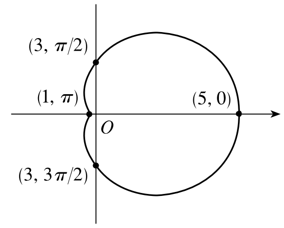
$$= \tfrac{1}{2}\int_0^{2\pi} (1 - 2\sin\theta + \sin^2\theta)\, d\theta = \tfrac{1}{2}\int_0^{2\pi} \left[1 - 2\sin\theta + \tfrac{1}{2}(1 - \cos 2\theta)\right] d\theta$$
$$= \tfrac{1}{2}\int_0^{2\pi} \left(\tfrac{3}{2} - 2\sin\theta - \tfrac{1}{2}\cos 2\theta\right) d\theta = \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}$$
$$= \tfrac{1}{2}[(3\pi + 2) - (2)] = \tfrac{3\pi}{2}$$

11. $A = \int_0^{2\pi} \frac{1}{2} r^2\, d\theta = \int_0^{2\pi} \frac{1}{2}(3 + 2\cos\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} (9 + 12\cos\theta + 4\cos^2\theta)\, d\theta$

$= \frac{1}{2}\int_0^{2\pi} \left[9 + 12\cos\theta + 4 \cdot \frac{1}{2}(1 + \cos 2\theta)\right] d\theta$

$= \frac{1}{2}\int_0^{2\pi} (11 + 12\cos\theta + 2\cos 2\theta)\, d\theta = \frac{1}{2}\left[11\theta + 12\sin\theta + \sin 2\theta\right]_0^{2\pi}$

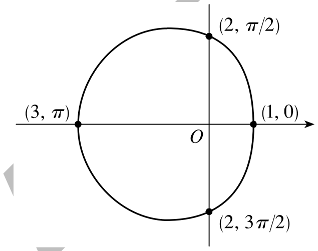
$= \frac{1}{2}(22\pi) = 11\pi$



12. $A = \int_0^{2\pi} \frac{1}{2} r^2\, d\theta = \int_0^{2\pi} \frac{1}{2}(2 - \cos\theta)^2\, d\theta = \int_0^{2\pi} \frac{1}{2}(4 - 4\cos\theta + \cos^2\theta)\, d\theta$

$= \int_0^{2\pi} \frac{1}{2}\left[4 - 4\cos\theta + \frac{1}{2}(1 + \cos 2\theta)\right] d\theta = \int_0^{2\pi} \left(\frac{9}{4} - 2\cos\theta + \frac{1}{4}\cos 2\theta\right) d\theta$

$= \left[\frac{9}{4}\theta - 2\sin\theta + \frac{1}{8}\sin 2\theta\right]_0^{2\pi} = \left(\frac{9\pi}{2} - 0 + 0\right) - (0 - 0 + 0) = \frac{9\pi}{2}$

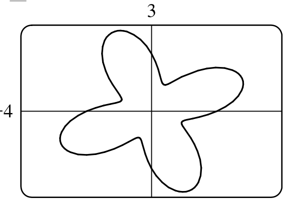


13. $A = \int_0^{2\pi} \frac{1}{2} r^2\, d\theta = \int_0^{2\pi} \frac{1}{2}(2 + \sin 4\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} (4 + 4\sin 4\theta + \sin^2 4\theta)\, d\theta$

$= \frac{1}{2}\int_0^{2\pi} \left[4 + 4\sin 4\theta + \frac{1}{2}(1 - \cos 8\theta)\right] d\theta$

$= \frac{1}{2}\int_0^{2\pi} \left(\frac{9}{2} + 4\sin 4\theta - \frac{1}{2}\cos 8\theta\right) d\theta = \frac{1}{2}\left[\frac{9}{2}\theta - \cos 4\theta - \frac{1}{16}\sin 8\theta\right]_0^{2\pi}$

$= \frac{1}{2}[(9\pi - 1) - (-1)] = \frac{9}{2}\pi$

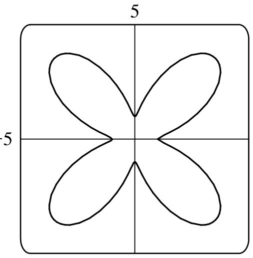


14. $A = \int_0^{2\pi} \frac{1}{2} r^2\, d\theta = \int_0^{2\pi} \frac{1}{2}(3 - 2\cos 4\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} (9 - 12\cos 4\theta + 4\cos^2 4\theta)\, d\theta$

$= \frac{1}{2}\int_0^{2\pi} \left[9 - 12\cos 4\theta + 4 \cdot \frac{1}{2}(1 + \cos 8\theta)\right] d\theta$

$= \frac{1}{2}\int_0^{2\pi} (11 - 12\cos 4\theta + 2\cos 8\theta)\, d\theta = \frac{1}{2}\left[11\theta - 3\sin 4\theta + \frac{1}{4}\sin 8\theta\right]_0^{2\pi}$

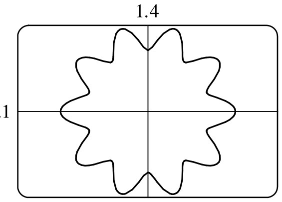
$= \frac{1}{2}(22\pi) = 11\pi$



15. $A = \int_0^{2\pi} \frac{1}{2} r^2\, d\theta = \int_0^{2\pi} \frac{1}{2}\left(\sqrt{1 + \cos^2 5\theta}\right)^2 d\theta$

$= \frac{1}{2}\int_0^{2\pi} (1 + \cos^2 5\theta)\, d\theta = \frac{1}{2}\int_0^{2\pi} \left[1 + \frac{1}{2}(1 + \cos 10\theta)\right] d\theta$

$= \frac{1}{2}\left[\frac{3}{2}\theta + \frac{1}{20}\sin 10\theta\right]_0^{2\pi} = \frac{1}{2}(3\pi) = \frac{3}{2}\pi$



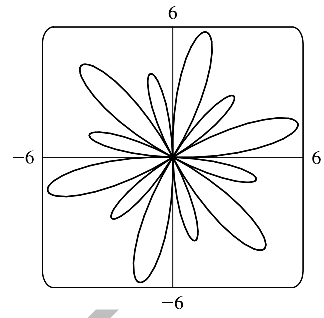
16. $A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}(1 + 5\sin 6\theta)^2 \, d\theta$

$= \frac{1}{2} \int_0^{2\pi} (1 + 10\sin 6\theta + 25\sin^2 6\theta) \, d\theta$

$= \frac{1}{2} \int_0^{2\pi} \left[1 + 10\sin 6\theta + 25 \cdot \frac{1}{2}(1 - \cos 12\theta)\right] d\theta$

$= \frac{1}{2} \int_0^{2\pi} \left[\frac{27}{2} + 10\sin 6\theta - \frac{25}{2}\cos 12\theta\right] d\theta = \frac{1}{2}\left[\frac{27}{2}\theta - \frac{5}{3}\cos 6\theta - \frac{25}{24}\sin 12\theta\right]_0^{2\pi}$

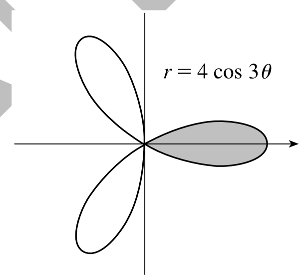
$= \frac{1}{2}\left[\left(27\pi - \frac{5}{3}\right) - \left(-\frac{5}{3}\right)\right] = \frac{27}{2}\pi$



17. The curve passes through the pole when $r = 0 \;\Rightarrow\; 4\cos 3\theta = 0 \;\Rightarrow\; \cos 3\theta = 0 \;\Rightarrow\; 3\theta = \frac{\pi}{2} + \pi n \;\Rightarrow$ $\theta = \frac{\pi}{6} + \frac{\pi}{3} n$. The part of the shaded loop above the polar axis is traced out for $\theta = 0$ to $\theta = \pi/6$, so we'll use $-\pi/6$ and $\pi/6$ as our limits of integration.

$A = \int_{-\pi/6}^{\pi/6} \frac{1}{2}(4\cos 3\theta)^2 \, d\theta = 2\int_0^{\pi/6} \frac{1}{2}(16\cos^2 3\theta) \, d\theta$

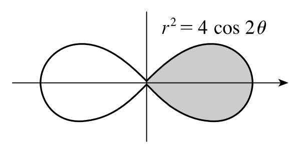
$= 16 \int_0^{\pi/6} \frac{1}{2}(1 + \cos 6\theta) \, d\theta = 8\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = 8\left(\frac{\pi}{6}\right) = \frac{4}{3}\pi$



18. The curve given by $r^2 = 4\cos 2\theta$ passes through the pole when $r = 0 \;\Rightarrow\; 4\cos 2\theta = 0 \;\Rightarrow\; \cos 2\theta = 0 \;\Rightarrow$ $2\theta = \frac{\pi}{2} + \pi n \;\Rightarrow\; \theta = \frac{\pi}{4} + \frac{\pi}{2} n$. The part of the shaded loop above the polar axis is traced out for $\theta = 0$ to $\theta = \pi/4$, so we'll use $-\pi/4$ to $\pi/4$ as our limits of integration.

$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2}(4\cos 2\theta) \, d\theta = 2\int_0^{\pi/4} 2\cos 2\theta \, d\theta = 2\Big[\sin 2\theta\Big]_0^{\pi/4}$

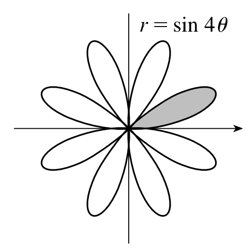
$= 2\sin\frac{\pi}{2} = 2(1) = 2$



19. $r = 0 \;\Rightarrow\; \sin 4\theta = 0 \;\Rightarrow\; 4\theta = \pi n \;\Rightarrow\; \theta = \frac{\pi}{4} n$.

$A = \int_0^{\pi/4} \frac{1}{2}(\sin 4\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\pi/4} \sin^2 4\theta \, d\theta = \frac{1}{2}\int_0^{\pi/4} \frac{1}{2}(1 - \cos 8\theta) \, d\theta$

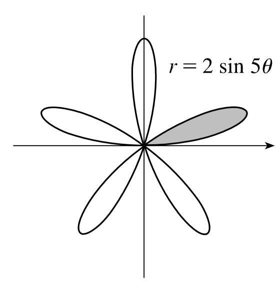
$= \frac{1}{4}\left[\theta - \frac{1}{8}\sin 8\theta\right]_0^{\pi/4} = \frac{1}{4}\left(\frac{\pi}{4}\right) = \frac{1}{16}\pi$

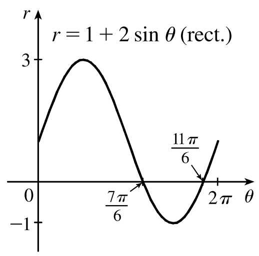


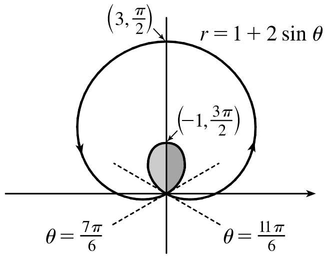
20. $r = 0 \;\Rightarrow\; 2\sin 5\theta = 0 \;\Rightarrow\; \sin 5\theta = 0 \;\Rightarrow\; 5\theta = \pi n \;\Rightarrow\; \theta = \frac{\pi}{5} n$.

$A = \int_0^{\pi/5} \frac{1}{2}(2\sin 5\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\pi/5} 4\sin^2 5\theta \, d\theta$

$= 2\int_0^{\pi/5} \frac{1}{2}(1 - \cos 10\theta) \, d\theta = \left[\theta - \frac{1}{10}\sin 10\theta\right]_0^{\pi/5} = \frac{\pi}{5}$



21. 



This is a limaçon, with inner loop traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ [found by solving $r = 0$].

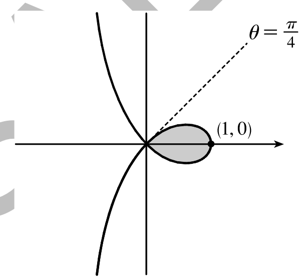
$$A = 2\int_{7\pi/6}^{3\pi/2} \tfrac{1}{2}(1 + 2\sin\theta)^2\,d\theta = \int_{7\pi/6}^{3\pi/2}\left(1 + 4\sin\theta + 4\sin^2\theta\right)d\theta = \int_{7\pi/6}^{3\pi/2}\left[1 + 4\sin\theta + 4\cdot\tfrac{1}{2}(1 - \cos 2\theta)\right]d\theta$$

$$= \Big[\theta - 4\cos\theta + 2\theta - \sin 2\theta\Big]_{7\pi/6}^{3\pi/2} = \left(\tfrac{9\pi}{2}\right) - \left(\tfrac{7\pi}{2} + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\right) = \pi - \tfrac{3\sqrt{3}}{2}$$

22. To determine when the strophoid $r = 2\cos\theta - \sec\theta$ passes through the pole, we solve

$$r = 0 \;\Rightarrow\; 2\cos\theta - \frac{1}{\cos\theta} = 0 \;\Rightarrow\; 2\cos^2\theta - 1 = 0 \;\Rightarrow\; \cos^2\theta = \frac{1}{2} \;\Rightarrow$$

$$\cos\theta = \pm\frac{1}{\sqrt{2}} \;\Rightarrow\; \theta = \tfrac{\pi}{4} \text{ or } \theta = \tfrac{3\pi}{4} \text{ for } 0 \le \theta \le \pi \text{ with } \theta \ne \tfrac{\pi}{2}.$$



$$A = 2\int_0^{\pi/4} \tfrac{1}{2}(2\cos\theta - \sec\theta)^2\,d\theta = \int_0^{\pi/4}(4\cos^2\theta - 4 + \sec^2\theta)\,d\theta$$

$$= \int_0^{\pi/4}\left[4\cdot\tfrac{1}{2}(1 + \cos 2\theta) - 4 + \sec^2\theta\right]d\theta = \int_0^{\pi/4}(-2 + 2\cos 2\theta + \sec^2\theta)\,d\theta$$

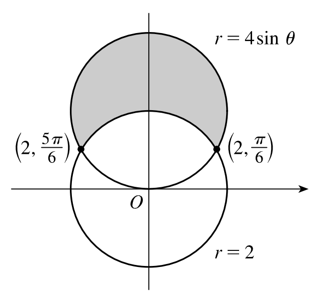
$$= \big[-2\theta + \sin 2\theta + \tan\theta\big]_0^{\pi/4} = \left(-\tfrac{\pi}{2} + 1 + 1\right) - 0 = 2 - \tfrac{\pi}{2}$$

23. $4\sin\theta = 2 \;\Rightarrow\; \sin\theta = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6} \;\Rightarrow$

$$A = \int_{\pi/6}^{5\pi/6} \tfrac{1}{2}[(4\sin\theta)^2 - 2^2]\,d\theta = 2\int_{\pi/6}^{\pi/2}\tfrac{1}{2}(16\sin^2\theta - 4)\,d\theta$$

$$= \int_{\pi/6}^{\pi/2}\left[16\cdot\tfrac{1}{2}(1 - \cos 2\theta) - 4\right]d\theta = \int_{\pi/6}^{\pi/2}(4 - 8\cos 2\theta)\,d\theta$$

$$= \Big[4\theta - 4\sin 2\theta\Big]_{\pi/6}^{\pi/2} = (2\pi - 0) - \left(\tfrac{2\pi}{3} - 2\sqrt{3}\right) = \tfrac{4\pi}{3} + 2\sqrt{3}$$

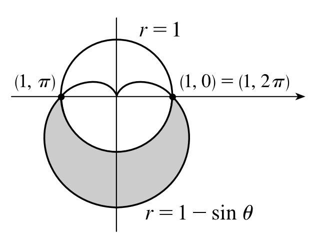


24. $1 - \sin\theta = 1 \;\Rightarrow\; \sin\theta = 0 \;\Rightarrow\; \theta = 0$ or $\pi \;\Rightarrow$

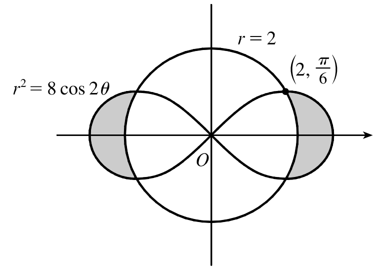
$$A = \int_{\pi}^{2\pi} \tfrac{1}{2}\left[(1 - \sin\theta)^2 - 1\right]d\theta = \tfrac{1}{2}\int_{\pi}^{2\pi}(\sin^2\theta - 2\sin\theta)\,d\theta$$

$$= \tfrac{1}{4}\int_{\pi}^{2\pi}(1 - \cos 2\theta - 4\sin\theta)\,d\theta = \tfrac{1}{4}\Big[\theta - \tfrac{1}{2}\sin 2\theta + 4\cos\theta\Big]_{\pi}^{2\pi}$$

$$= \tfrac{1}{4}\pi + 2$$



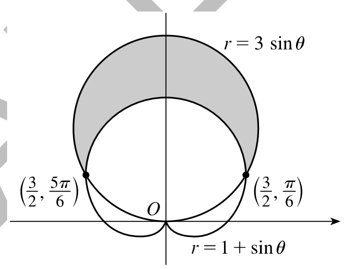
25. To find the area inside the leminiscate $r^2 = 8\cos 2\theta$ and outside the circle $r = 2$, we first note that the two curves intersect when $r^2 = 8\cos 2\theta$ and $r = 2$, that is, when $\cos 2\theta = \frac{1}{2}$. For $-\pi < \theta \le \pi$, $\cos 2\theta = \frac{1}{2} \quad \Leftrightarrow \quad 2\theta = \pm\pi/3$ or $\pm 5\pi/3 \quad \Leftrightarrow \quad \theta = \pm\pi/6$ or $\pm 5\pi/6$. The figure shows that the desired area is 4 times the area between the curves from 0 to $\pi/6$. Thus,



$$A = 4\int_0^{\pi/6}\left[\tfrac{1}{2}(8\cos 2\theta) - \tfrac{1}{2}(2)^2\right]d\theta = 8\int_0^{\pi/6}(2\cos 2\theta - 1)\,d\theta$$

$$= 8\left[\sin 2\theta - \theta\right]_0^{\pi/6} = 8\left(\sqrt{3}/2 - \pi/6\right) = 4\sqrt{3} - 4\pi/3$$

26. $3\sin\theta = 1 + \sin\theta \quad \Rightarrow \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6} \quad \Rightarrow$



$$A = \int_{\pi/6}^{5\pi/6} \tfrac{1}{2}\left[(3\sin\theta)^2 - (1 + \sin\theta)^2\right]d\theta$$

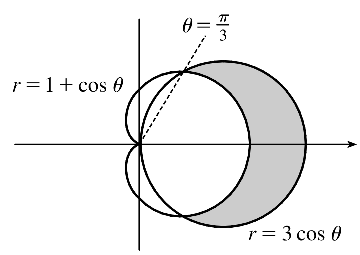
$$= 2\int_{\pi/6}^{\pi/2} \tfrac{1}{2}(9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta)\,d\theta$$

$$= \int_{\pi/6}^{\pi/2}(8\sin^2\theta - 1 - 2\sin\theta)\,d\theta$$

$$= \int_{\pi/6}^{\pi/2}\left[8\cdot\tfrac{1}{2}(1 - \cos 2\theta) - 1 - 2\sin\theta\right]d\theta = \int_{\pi/6}^{\pi/2}(3 - 4\cos 2\theta - 2\sin\theta)\,d\theta$$

$$= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\pi/6}^{\pi/2} = \left(\tfrac{3\pi}{2} - 0 + 0\right) - \left(\tfrac{\pi}{2} - \sqrt{3} + \sqrt{3}\right) = \pi$$

27. $3\cos\theta = 1 + \cos\theta \quad \Leftrightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$.



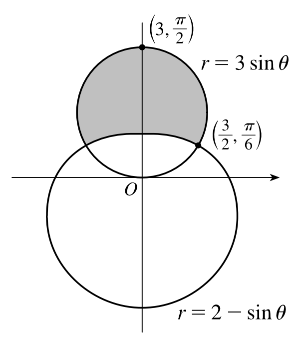
$$A = 2\int_0^{\pi/3}\tfrac{1}{2}\left[(3\cos\theta)^2 - (1 + \cos\theta)^2\right]d\theta$$

$$= \int_0^{\pi/3}(8\cos^2\theta - 2\cos\theta - 1)\,d\theta = \int_0^{\pi/3}\left[4(1 + \cos 2\theta) - 2\cos\theta - 1\right]d\theta$$

$$= \int_0^{\pi/3}(3 + 4\cos 2\theta - 2\cos\theta)\,d\theta = \left[3\theta + 2\sin 2\theta - 2\sin\theta\right]_0^{\pi/3}$$

$$= \pi + \sqrt{3} - \sqrt{3} = \pi$$

28. $3\sin\theta = 2 - \sin\theta \quad \Rightarrow \quad 4\sin\theta = 2 \quad \Rightarrow \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.



$$A = 2\int_{\pi/6}^{\pi/2}\tfrac{1}{2}\left[(3\sin\theta)^2 - (2 - \sin\theta)^2\right]d\theta$$

$$= \int_{\pi/6}^{\pi/2}(9\sin^2\theta - 4 + 4\sin\theta - \sin^2\theta]\,d\theta$$

$$= \int_{\pi/6}^{\pi/2}(8\sin^2\theta + 4\sin\theta - 4)\,d\theta$$

$$= 4\int_{\pi/6}^{\pi/2}\left[2\cdot\tfrac{1}{2}(1 - \cos 2\theta) + \sin\theta - 1\right]d\theta$$

$$= 4\int_{\pi/6}^{\pi/2}(\sin\theta - \cos 2\theta)\,d\theta = 4\left[-\cos\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/2}$$

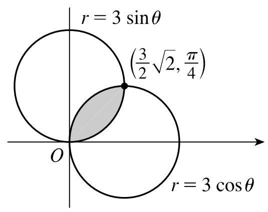
$$= 4\left[(0 - 0) - \left(-\tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{4}\right)\right] = 4\left(\tfrac{3\sqrt{3}}{4}\right) = 3\sqrt{3}$$

29. $3\sin\theta = 3\cos\theta \Rightarrow \dfrac{3\sin\theta}{3\cos\theta} = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4} \Rightarrow$

$$A = 2\int_0^{\pi/4} \tfrac{1}{2}(3\sin\theta)^2\,d\theta = \int_0^{\pi/4} 9\sin^2\theta\,d\theta = \int_0^{\pi/4} 9\cdot\tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= \int_0^{\pi/4}\left(\tfrac{9}{2} - \tfrac{9}{2}\cos 2\theta\right)d\theta = \left[\tfrac{9}{2}\theta - \tfrac{9}{4}\sin 2\theta\right]_0^{\pi/4} = \left(\tfrac{9\pi}{8} - \tfrac{9}{4}\right) - (0-0)$$

$$= \tfrac{9\pi}{8} - \tfrac{9}{4}$$

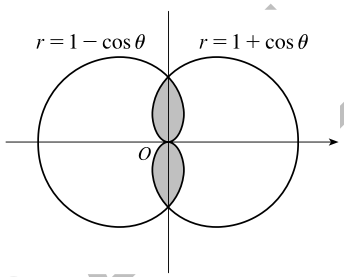


30. $A = 4\int_0^{\pi/2}\frac{1}{2}(1-\cos\theta)^2\,d\theta = 2\int_0^{\pi/2}(1 - 2\cos\theta + \cos^2\theta)\,d\theta$

$$= 2\int_0^{\pi/2}\left[1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$= 2\int_0^{\pi/2}\left(\tfrac{3}{2} - 2\cos\theta + \tfrac{1}{2}\cos 2\theta\right)d\theta = \int_0^{\pi/2}(3 - 4\cos\theta + \cos 2\theta)\,d\theta$$

$$= \left[3\theta - 4\sin\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{3\pi}{2} - 4$$

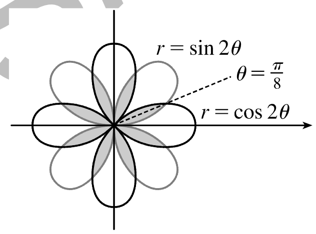


31. $\sin 2\theta = \cos 2\theta \Rightarrow \dfrac{\sin 2\theta}{\cos 2\theta} = 1 \Rightarrow \tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4} \Rightarrow$

$\theta = \frac{\pi}{8} \Rightarrow$

$$A = 8\cdot 2\int_0^{\pi/8}\tfrac{1}{2}\sin^2 2\theta\,d\theta = 8\int_0^{\pi/8}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= 4\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/8} = 4\left(\tfrac{\pi}{8} - \tfrac{1}{4}\cdot 1\right) = \tfrac{\pi}{2} - 1$$



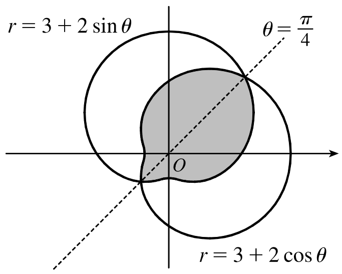
32. $3 + 2\cos\theta = 3 + 2\sin\theta \Rightarrow \cos\theta = \sin\theta \Rightarrow \theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

$$A = 2\int_{\pi/4}^{5\pi/4}\tfrac{1}{2}(3+2\cos\theta)^2\,d\theta = \int_{\pi/4}^{5\pi/4}(9 + 12\cos\theta + 4\cos^2\theta)\,d\theta$$

$$= \int_{\pi/4}^{5\pi/4}\left[9 + 12\cos\theta + 4\cdot\tfrac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$= \int_{\pi/4}^{5\pi/4}(11 + 12\cos\theta + 2\cos 2\theta)\,d\theta = \left[11\theta + 12\sin\theta + \sin 2\theta\right]_{\pi/4}^{5\pi/4}$$

$$= \left(\tfrac{55\pi}{4} - 6\sqrt{2} + 1\right) - \left(\tfrac{11\pi}{4} + 6\sqrt{2} + 1\right) = 11\pi - 12\sqrt{2}$$

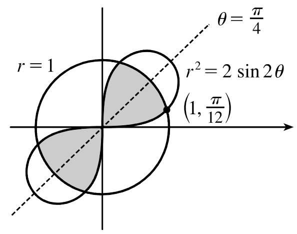


33. From the figure, we see that the shaded region is 4 times the shaded region from $\theta = 0$ to $\theta = \pi/4$. $r^2 = 2\sin 2\theta$ and $r = 1 \Rightarrow$ $2\sin 2\theta = 1^2 \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{12}$.

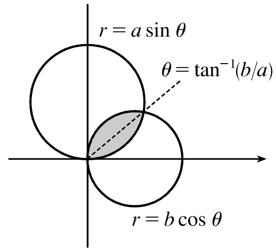
$$A = 4\int_0^{\pi/12}\tfrac{1}{2}(2\sin 2\theta)\,d\theta + 4\int_{\pi/12}^{\pi/4}\tfrac{1}{2}(1)^2\,d\theta$$

$$= \int_0^{\pi/12} 4\sin 2\theta\,d\theta + \int_{\pi/12}^{\pi/4} 2\,d\theta = \left[-2\cos 2\theta\right]_0^{\pi/12} + \left[2\theta\right]_{\pi/12}^{\pi/4}$$

$$= \left(-\sqrt{3} + 2\right) + \left(\tfrac{\pi}{2} - \tfrac{\pi}{6}\right) = -\sqrt{3} + 2 + \tfrac{\pi}{3}$$

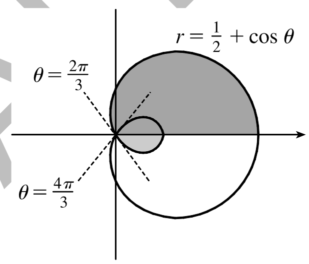


34. Let $\alpha = \tan^{-1}(b/a)$. Then



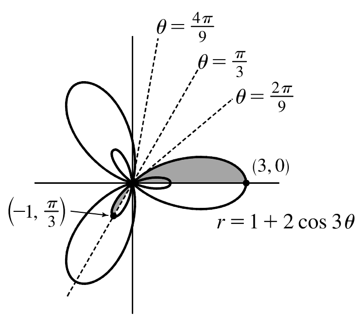
$$A = \int_0^\alpha \tfrac{1}{2}(a\sin\theta)^2\,d\theta + \int_\alpha^{\pi/2}\tfrac{1}{2}(b\cos\theta)^2\,d\theta$$
$$= \tfrac{1}{4}a^2\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^\alpha + \tfrac{1}{4}b^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_\alpha^{\pi/2}$$
$$= \tfrac{1}{4}\alpha(a^2 - b^2) + \tfrac{1}{8}\pi b^2 - \tfrac{1}{4}(a^2+b^2)(\sin\alpha\,\cos\alpha)$$
$$= \tfrac{1}{4}(a^2 - b^2)\,\tan^{-1}(b/a) + \tfrac{1}{8}\pi b^2 - \tfrac{1}{4}ab$$

35. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/3$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/3$ to $\theta = \pi$), and then double that difference to obtain the desired area.



$$A = 2\left[\int_0^{2\pi/3}\tfrac{1}{2}\left(\tfrac{1}{2}+\cos\theta\right)^2 d\theta - \int_{2\pi/3}^{\pi}\tfrac{1}{2}\left(\tfrac{1}{2}+\cos\theta\right)^2 d\theta\right]$$
$$= \int_0^{2\pi/3}\left(\tfrac{1}{4}+\cos\theta+\cos^2\theta\right)d\theta - \int_{2\pi/3}^{\pi}\left(\tfrac{1}{4}+\cos\theta+\cos^2\theta\right)d\theta$$
$$= \int_0^{2\pi/3}\left[\tfrac{1}{4}+\cos\theta+\tfrac{1}{2}(1+\cos 2\theta)\right]d\theta - \int_{2\pi/3}^{\pi}\left[\tfrac{1}{4}+\cos\theta+\tfrac{1}{2}(1+\cos 2\theta)\right]d\theta$$
$$= \left[\frac{\theta}{4}+\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{2\pi/3} - \left[\frac{\theta}{4}+\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_{2\pi/3}^{\pi}$$
$$= \left(\tfrac{\pi}{6}+\tfrac{\sqrt{3}}{2}+\tfrac{\pi}{3}-\tfrac{\sqrt{3}}{8}\right) - \left(\tfrac{\pi}{4}+\tfrac{\pi}{2}\right) + \left(\tfrac{\pi}{6}+\tfrac{\sqrt{3}}{2}+\tfrac{\pi}{3}-\tfrac{\sqrt{3}}{8}\right)$$
$$= \tfrac{\pi}{4}+\tfrac{3}{4}\sqrt{3} = \tfrac{1}{4}\left(\pi+3\sqrt{3}\right)$$

36. $r = 0 \;\Rightarrow\; 1 + 2\cos 3\theta = 0 \;\Rightarrow\; \cos 3\theta = -\frac{1}{2} \;\Rightarrow\; 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ [for $0 \le 3\theta \le 2\pi$] $\Rightarrow\; \theta = \frac{2\pi}{9}, \frac{4\pi}{9}$. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/9$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/9$ to $\theta = \pi/3$), and then double that difference to obtain the desired area.



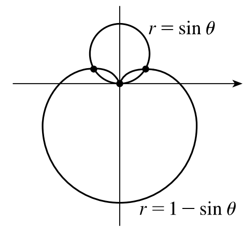
$$A = 2\left[\int_0^{2\pi/9}\tfrac{1}{2}(1+2\cos 3\theta)^2\,d\theta - \int_{2\pi/9}^{\pi/3}\tfrac{1}{2}(1+2\cos 3\theta)^2\,d\theta\right]$$

Now
$$r^2 = (1+2\cos 3\theta)^2 = 1 + 4\cos 3\theta + 4\cos^2 3\theta = 1 + 4\cos 3\theta + 4\cdot\tfrac{1}{2}(1+\cos 6\theta)$$
$$= 1 + 4\cos 3\theta + 2 + 2\cos 6\theta = 3 + 4\cos 3\theta + 2\cos 6\theta$$

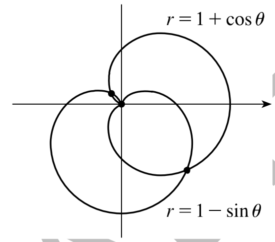
and $\int r^2 d\theta = 3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta + C$, so

$$A = \left[3\theta + \tfrac{4}{3}\sin 3\theta + \tfrac{1}{3}\sin 6\theta\right]_0^{2\pi/9} - \left[3\theta + \tfrac{4}{3}\sin 3\theta + \tfrac{1}{3}\sin 6\theta\right]_{2\pi/9}^{\pi/3}$$
$$= \left[\left(\tfrac{2\pi}{3}+\tfrac{4}{3}\cdot\tfrac{\sqrt{3}}{2}+\tfrac{1}{3}\cdot\tfrac{-\sqrt{3}}{2}\right)-0\right] - \left[(\pi+0+0)-\left(\tfrac{2\pi}{3}+\tfrac{4}{3}\cdot\tfrac{\sqrt{3}}{2}+\tfrac{1}{3}\cdot\tfrac{-\sqrt{3}}{2}\right)\right]$$
$$= \tfrac{4\pi}{3}+\tfrac{4}{3}\sqrt{3}-\tfrac{1}{3}\sqrt{3}-\pi = \tfrac{\pi}{3}+\sqrt{3}$$

37. The pole is a point of intersection. $\sin\theta = 1 - \sin\theta \Rightarrow 2\sin\theta = 1 \Rightarrow$ $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. So the other points of intersection are $\left(\frac{1}{2}, \frac{\pi}{6}\right)$ and $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$.



38. The pole is a point of intersection. $1 + \cos\theta = 1 - \sin\theta \Rightarrow$ $\cos\theta = -\sin\theta \Rightarrow \dfrac{\cos\theta}{\sin\theta} = -1 \Rightarrow \cot\theta = -1 \Rightarrow \theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. So the other points of intersection are $\left(1 - \frac{1}{2}\sqrt{2}, \frac{3\pi}{4}\right)$ and $\left(1 + \frac{1}{2}\sqrt{2}, \frac{7\pi}{4}\right)$.



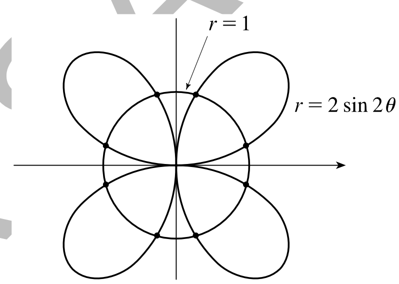
39. $2\sin 2\theta = 1 \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$.

By symmetry, the eight points of intersection are given by

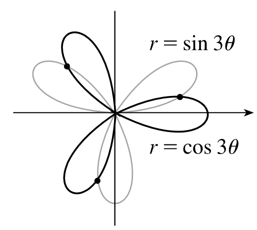
$(1, \theta)$, where $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}$, and $\frac{17\pi}{12}$, and

$(-1, \theta)$, where $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}$, and $\frac{23\pi}{12}$.

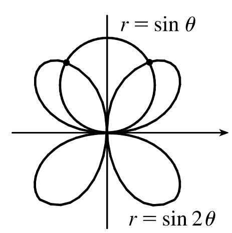
[There are many ways to describe these points.]



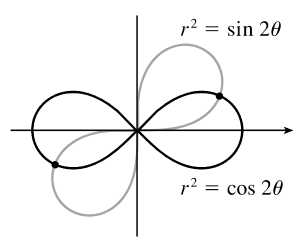
40. Clearly the pole lies on both curves. $\sin 3\theta = \cos 3\theta \Rightarrow \tan 3\theta = 1 \Rightarrow$ $3\theta = \frac{\pi}{4} + n\pi$ [n any integer] $\Rightarrow \theta = \frac{\pi}{12} + \frac{\pi}{3}n \Rightarrow$ $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$, or $\frac{3\pi}{4}$, so the three remaining intersection points are $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{12}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12}\right)$, and $\left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right)$.



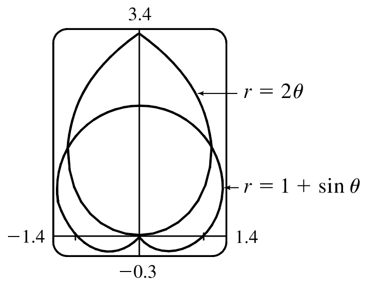
41. The pole is a point of intersection. $\sin\theta = \sin 2\theta = 2\sin\theta\,\cos\theta \Leftrightarrow$ $\sin\theta\,(1 - 2\cos\theta) = 0 \Leftrightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2} \Rightarrow$ $\theta = 0, \pi, \frac{\pi}{3}$, or $-\frac{\pi}{3} \Rightarrow$ the other intersection points are $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$ [by symmetry].

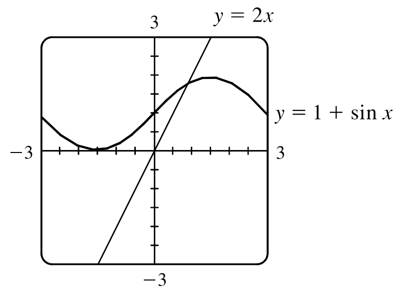


42. Clearly the pole is a point of intersection. $\sin 2\theta = \cos 2\theta \Rightarrow$ $\tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4} + 2n\pi$ [since $\sin 2\theta$ and $\cos 2\theta$ must be positive in the equations] $\Rightarrow \theta = \frac{\pi}{8} + n\pi \Rightarrow \theta = \frac{\pi}{8}$ or $\frac{9\pi}{8}$. So the curves also intersect at $\left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8}\right)$ and $\left(\frac{1}{\sqrt[4]{2}}, \frac{9\pi}{8}\right)$.



43.



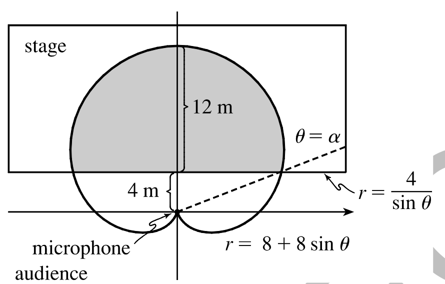


From the first graph, we see that the pole is one point of intersection. By zooming in or using the cursor, we find the θ-values of the intersection points to be $\alpha \approx 0.88786 \approx 0.89$ and $\pi - \alpha \approx 2.25$. (The first of these values may be more easily estimated by plotting $y = 1 + \sin x$ and $y = 2x$ in rectangular coordinates; see the second graph.) By symmetry, the total area contained is twice the area contained in the first quadrant, that is,

$$A = 2\int_0^\alpha \tfrac{1}{2}(2\theta)^2\,d\theta + 2\int_\alpha^{\pi/2} \tfrac{1}{2}(1+\sin\theta)^2\,d\theta = \int_0^\alpha 4\theta^2\,d\theta + \int_\alpha^{\pi/2}\left[1 + 2\sin\theta + \tfrac{1}{2}(1-\cos 2\theta)\right]d\theta$$

$$= \left[\tfrac{4}{3}\theta^3\right]_0^\alpha + \left[\theta - 2\cos\theta + \left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right]_\alpha^{\pi/2} = \tfrac{4}{3}\alpha^3 + \left[\left(\tfrac{\pi}{2} + \tfrac{\pi}{4}\right) - \left(\alpha - 2\cos\alpha + \tfrac{1}{2}\alpha - \tfrac{1}{4}\sin 2\alpha\right)\right] \approx 3.4645$$

44.



We need to find the shaded area A in the figure. The horizontal line representing the front of the stage has equation $y = 4 \;\Leftrightarrow\; r\sin\theta = 4 \;\Rightarrow\; r = 4/\sin\theta$. This line intersects the curve $r = 8 + 8\sin\theta$ when $8 + 8\sin\theta = \dfrac{4}{\sin\theta} \;\Rightarrow$

$8\sin\theta + 8\sin^2\theta = 4 \;\Rightarrow\; 2\sin^2\theta + 2\sin\theta - 1 = 0 \;\Rightarrow$

$$\sin\theta = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} = \frac{-1+\sqrt{3}}{2} \quad \text{[the other value is less than } -1] \;\Rightarrow\; \theta = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right).$$

This angle is about 21.5° and is denoted by α in the figure.

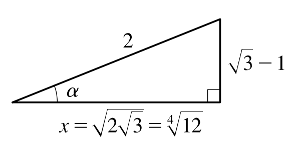
$$A = 2\int_\alpha^{\pi/2} \tfrac{1}{2}(8+8\sin\theta)^2\,d\theta - 2\int_\alpha^{\pi/2} \tfrac{1}{2}(4\csc\theta)^2\,d\theta = 64\int_\alpha^{\pi/2}(1 + 2\sin\theta + \sin^2\theta)\,d\theta - 16\int_\alpha^{\pi/2}\csc^2\theta\,d\theta$$

$$= 64\int_\alpha^{\pi/2}\left(1 + 2\sin\theta + \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right)d\theta + 16\int_\alpha^{\pi/2}(-\csc^2\theta)\,d\theta = 64\left[\tfrac{3}{2}\theta - 2\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_\alpha^{\pi/2} + 16\left[\cot\theta\right]_\alpha^{\pi/2}$$

$$= 16\left[6\theta - 8\cos\theta - \sin 2\theta + \cot\theta\right]_\alpha^{\pi/a} = 16[(3\pi - 0 - 0 + 0) - (6\alpha - 8\cos\alpha - \sin 2\alpha + \cot\alpha)]$$

$$= 48\pi - 96\alpha + 128\cos\alpha + 16\sin 2\alpha - 16\cot\alpha$$

From the figure, $x^2 + \left(\sqrt{3}-1\right)^2 = 2^2 \;\Rightarrow\; x^2 = 4 - \left(3 - 2\sqrt{3} + 1\right) \;\Rightarrow$ $x^2 = 2\sqrt{3} = \sqrt{12}$, so $x = \sqrt{2\sqrt{3}} = \sqrt[4]{12}$. Using the trigonometric relationships for a right triangle and the identity $\sin 2\alpha = 2\sin\alpha\,\cos\alpha$, we continue:



$$A = 48\pi - 96\alpha + 128\cdot\frac{\sqrt[4]{12}}{2} + 16\cdot 2\cdot\frac{\sqrt{3}-1}{2}\cdot\frac{\sqrt[4]{12}}{2} - 16\cdot\frac{\sqrt[4]{12}}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= 48\pi - 96\alpha + 64\sqrt[4]{12} + 8\sqrt[4]{12}\left(\sqrt{3}-1\right) - 8\sqrt[4]{12}\left(\sqrt{3}+1\right) = 48\pi + 48\sqrt[4]{12} - 96\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

$$\approx 204.16\ \text{m}^2$$

45. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^\pi \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$

$= \int_0^\pi \sqrt{4(\cos^2\theta + \sin^2\theta)}\, d\theta = \int_0^\pi \sqrt{4}\, d\theta = \big[2\theta\big]_0^\pi = 2\pi$

As a check, note that the curve is a circle of radius 1, so its circumference is $2\pi(1) = 2\pi$.

46. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{(5^\theta)^2 + (5^\theta \ln 5)^2}\, d\theta = \int_0^{2\pi} \sqrt{5^{2\theta}[1 + (\ln 5)^2]}\, d\theta$

$= \sqrt{1 + (\ln 5)^2} \int_0^{2\pi} \sqrt{5^{2\theta}}\, d\theta = \sqrt{1 + (\ln 5)^2} \int_0^{2\pi} 5^\theta\, d\theta = \sqrt{1 + (\ln 5)^2} \left[\dfrac{5^\theta}{\ln 5}\right]_0^{2\pi}$

$= \sqrt{1 + (\ln 5)^2} \left(\dfrac{5^{2\pi}}{\ln 5} - \dfrac{1}{\ln 5}\right) = \dfrac{\sqrt{1 + (\ln 5)^2}}{\ln 5}(5^{2\pi} - 1)$

47. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{\theta^2(\theta^2 + 4)}\, d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\, d\theta$

Now let $u = \theta^2 + 4$, so that $du = 2\theta\, d\theta$ $\left[\theta\, d\theta = \frac{1}{2}\, du\right]$ and

$$\int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\, d\theta = \int_4^{4\pi^2+4} \tfrac{1}{2}\sqrt{u}\, du = \tfrac{1}{2} \cdot \tfrac{2}{3}\left[u^{3/2}\right]_4^{4(\pi^2+1)} = \tfrac{1}{3}[4^{3/2}(\pi^2+1)^{3/2} - 4^{3/2}] = \tfrac{8}{3}[(\pi^2+1)^{3/2} - 1]$$

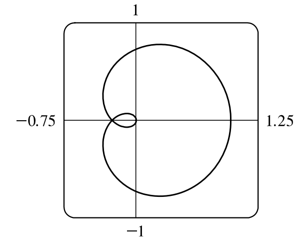
48. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{[2(1+\cos\theta)]^2 + (-2\sin\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta$

$= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta = \sqrt{8}\int_0^{2\pi} \sqrt{1 + \cos\theta}\, d\theta = \sqrt{8}\int_0^{2\pi} \sqrt{2 \cdot \tfrac{1}{2}(1 + \cos\theta)}\, d\theta$

$= \sqrt{8}\int_0^{2\pi} \sqrt{2\cos^2\dfrac{\theta}{2}}\, d\theta = \sqrt{8}\sqrt{2}\int_0^{2\pi} \left|\cos\dfrac{\theta}{2}\right| d\theta = 4 \cdot 2\int_0^{\pi} \cos\dfrac{\theta}{2}\, d\theta$ [by symmetry]

$= 8\left[2\sin\dfrac{\theta}{2}\right]_0^\pi = 8(2) = 16$

49. The curve $r = \cos^4(\theta/4)$ is completely traced with $0 \le \theta \le 4\pi$.



$r^2 + (dr/d\theta)^2 = [\cos^4(\theta/4)]^2 + \left[4\cos^3(\theta/4) \cdot (-\sin(\theta/4)) \cdot \tfrac{1}{4}\right]^2$

$= \cos^8(\theta/4) + \cos^6(\theta/4)\sin^2(\theta/4)$

$= \cos^6(\theta/4)[\cos^2(\theta/4) + \sin^2(\theta/4)] = \cos^6(\theta/4)$

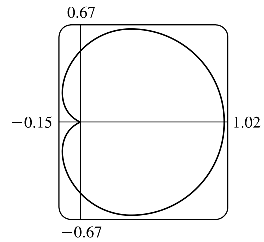
$L = \int_0^{4\pi} \sqrt{\cos^6(\theta/4)}\, d\theta = \int_0^{4\pi} \left|\cos^3(\theta/4)\right|\, d\theta$

$= 2\int_0^{2\pi} \cos^3(\theta/4)\, d\theta$ [since $\cos^3(\theta/4) \ge 0$ for $0 \le \theta \le 2\pi$] $= 8\int_0^{\pi/2} \cos^3 u\, du$ $\left[u = \tfrac{1}{4}\theta\right]$

$= 8\int_0^{\pi/2} (1 - \sin^2 u)\cos u\, du = 8\int_0^1 (1 - x^2)\, dx$ $\left[\begin{array}{l} x = \sin u, \\ dx = \cos u\, du \end{array}\right]$

$= 8\left[x - \tfrac{1}{3}x^3\right]_0^1 = 8\left(1 - \tfrac{1}{3}\right) = \tfrac{16}{3}$

50. The curve $r = \cos^2(\theta/2)$ is completely traced with $0 \le \theta \le 2\pi$.



$$\begin{aligned} r^2 + (dr/d\theta)^2 &= [\cos^2(\theta/2)]^2 + \left[2\cos(\theta/2)\cdot(-\sin(\theta/2))\cdot\tfrac{1}{2}\right]^2 \\ &= \cos^4(\theta/2) + \cos^2(\theta/2)\sin^2(\theta/2) \\ &= \cos^2(\theta/2)[\cos^2(\theta/2) + \sin^2(\theta/2)] \\ &= \cos^2(\theta/2) \end{aligned}$$

$L = \int_0^{2\pi} \sqrt{\cos^2(\theta/2)}\,d\theta = \int_0^{2\pi} |\cos(\theta/2)|\,d\theta = 2\int_0^{\pi} \cos(\theta/2)\,d\theta$ [since $\cos(\theta/2) \ge 0$ for $0 \le \theta \le \pi$]

$= 4\int_0^{\pi/2} \cos u\,du \quad \left[u = \tfrac{1}{2}\theta\right] \quad = 4\big[\sin u\big]_0^{\pi/2} = 4(1-0) = 4$

51. One loop of the curve $r = \cos 2\theta$ is traced with $-\pi/4 \le \theta \le \pi/4$.

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \cos^2 2\theta + (-2\sin 2\theta)^2 = \cos^2 2\theta + 4\sin^2 2\theta = 1 + 3\sin^2 2\theta \quad \Rightarrow$$

$$L = \int_{-\pi/4}^{\pi/4} \sqrt{1 + 3\sin^2 2\theta}\,d\theta \approx 2.4221.$$

52. $r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = \tan^2\theta + (\sec^2\theta)^2 \quad \Rightarrow \quad L = \displaystyle\int_{\pi/6}^{\pi/3} \sqrt{\tan^2\theta + \sec^4\theta}\,d\theta \approx 1.2789$

53. The curve $r = \sin(6\sin\theta)$ is completely traced with $0 \le \theta \le \pi$. $r = \sin(6\sin\theta) \quad \Rightarrow$

$\dfrac{dr}{d\theta} = \cos(6\sin\theta)\cdot 6\cos\theta$, so $r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = \sin^2(6\sin\theta) + 36\cos^2\theta\cos^2(6\sin\theta) \quad \Rightarrow$

$$L = \int_0^{\pi} \sqrt{\sin^2(6\sin\theta) + 36\cos^2\theta\,\cos^2(6\sin\theta)}\,d\theta \approx 8.0091.$$

54. The curve $r = \sin(\theta/4)$ is completely traced with $0 \le \theta \le 8\pi$. $r = \sin(\theta/4) \quad \Rightarrow \quad \dfrac{dr}{d\theta} = \frac{1}{4}\cos(\theta/4)$, so

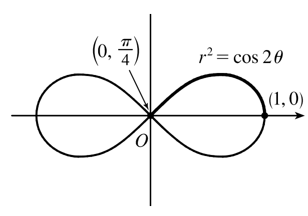
$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \sin^2(\theta/4) + \tfrac{1}{16}\cos^2(\theta/4) \quad \Rightarrow \quad L = \int_0^{8\pi} \sqrt{\sin^2(\theta/4) + \tfrac{1}{16}\cos^2(\theta/4)}\,d\theta \approx 17.1568.$$

55. (a) From (10.2.6),

$$\begin{aligned} S &= \int_a^b 2\pi y\sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta \\ &= \int_a^b 2\pi y\sqrt{r^2 + (dr/d\theta)^2}\,d\theta \qquad \text{[from the derivation of Equation 10.4.5]} \\ &= \int_a^b 2\pi r\sin\theta\sqrt{r^2 + (dr/d\theta)^2}\,d\theta \end{aligned}$$

(b) The curve $r^2 = \cos 2\theta$ goes through the pole when $\cos 2\theta = 0 \quad \Rightarrow$

$2\theta = \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{4}$. We'll rotate the curve from $\theta = 0$ to $\theta = \frac{\pi}{4}$ and double this value to obtain the total surface area generated.



$r^2 = \cos 2\theta \quad \Rightarrow \quad 2r\,\dfrac{dr}{d\theta} = -2\sin 2\theta \quad \Rightarrow \quad \left(\dfrac{dr}{d\theta}\right)^2 = \dfrac{\sin^2 2\theta}{r^2} = \dfrac{\sin^2 2\theta}{\cos 2\theta}.$

$$S = 2\int_0^{\pi/4} 2\pi\sqrt{\cos 2\theta}\sin\theta\sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta}\,d\theta = 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\sin\theta\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}\,d\theta$$

$$= 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\sin\theta\,\frac{1}{\sqrt{\cos 2\theta}}\,d\theta = 4\pi\int_0^{\pi/4}\sin\theta\,d\theta = 4\pi\big[-\cos\theta\big]_0^{\pi/4} = -4\pi\left(\tfrac{\sqrt{2}}{2} - 1\right) = 2\pi\left(2 - \sqrt{2}\right)$$

56. (a) Rotation around $\theta = \frac{\pi}{2}$ is the same as rotation around the y-axis, that is, $S = \int_a^b 2\pi x\, ds$ where $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$ for a parametric equation, and for the special case of a polar equation, $x = r\cos\theta$ and $ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta = \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$ [see the derivation of Equation 10.4.5]. Therefore, for a polar equation rotated around $\theta = \frac{\pi}{2}$, $S = \int_a^b 2\pi r\cos\theta\sqrt{r^2 + (dr/d\theta)^2}\, d\theta$.

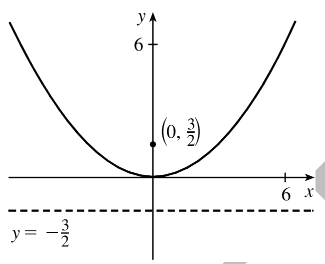
(b) As in the solution for Exercise 55(b), we can double the surface area generated by rotating the curve from $\theta = 0$ to $\theta = \frac{\pi}{4}$ to obtain the total surface area.

$$S = 2\int_0^{\pi/4} 2\pi\sqrt{\cos 2\theta}\cos\theta\sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta}\, d\theta = 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\cos\theta\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}\, d\theta$$

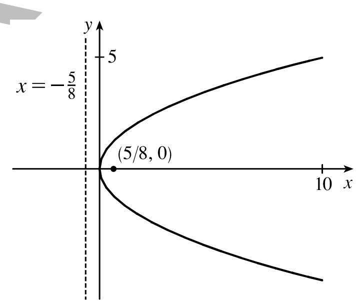
$$= 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\cos\theta\,\frac{1}{\sqrt{\cos 2\theta}}\, d\theta = 4\pi\int_0^{\pi/4}\cos\theta\, d\theta = 4\pi\big[\sin\theta\big]_0^{\pi/4} = 4\pi\left(\frac{\sqrt{2}}{2} - 0\right) = 2\sqrt{2}\,\pi$$

10.5 Conic Sections

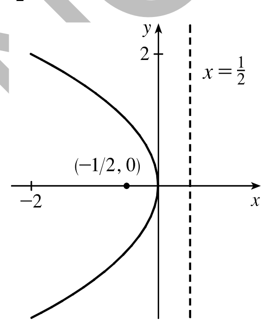
1. $x^2 = 6y$ and $x^2 = 4py \Rightarrow 4p = 6 \Rightarrow p = \frac{3}{2}$. The vertex is $(0, 0)$, the focus is $\left(0, \frac{3}{2}\right)$, and the directrix is $y = -\frac{3}{2}$.



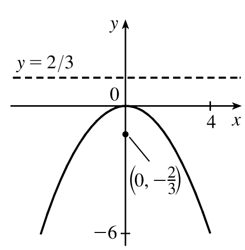
2. $2y^2 = 5x \Rightarrow y^2 = \frac{5}{2}x$. $4p = \frac{5}{2} \Rightarrow p = \frac{5}{8}$. The vertex is $(0, 0)$, the focus is $\left(\frac{5}{8}, 0\right)$, and the directrix is $x = -\frac{5}{8}$.



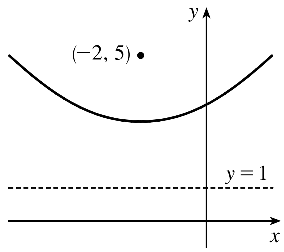
3. $2x = -y^2 \Rightarrow y^2 = -2x$. $4p = -2 \Rightarrow p = -\frac{1}{2}$. The vertex is $(0, 0)$, the focus is $\left(-\frac{1}{2}, 0\right)$, and the directrix is $x = \frac{1}{2}$.



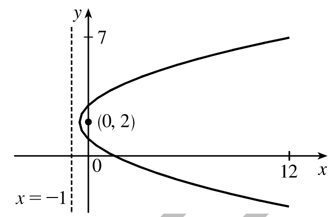
4. $3x^2 + 8y = 0 \Rightarrow 3x^2 = -8y \Rightarrow x^2 = -\frac{8}{3}y$. $4p = -\frac{8}{3} \Rightarrow p = -\frac{2}{3}$. The vertex is $(0, 0)$, the focus is $\left(0, -\frac{2}{3}\right)$, and the directrix is $y = \frac{2}{3}$.



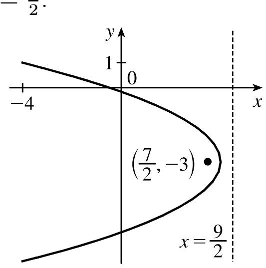
5. $(x+2)^2 = 8(y-3)$. $4p = 8$, so $p = 2$. The vertex is $(-2, 3)$, the focus is $(-2, 5)$, and the directrix is $y = 1$.



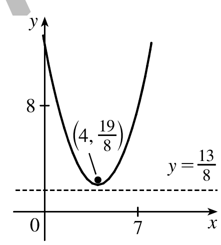
6. $(y-2)^2 = 2x + 1 = 2\left(x + \frac{1}{2}\right)$. $4p = 2$, so $p = \frac{1}{2}$. The vertex is $\left(-\frac{1}{2}, 2\right)$, the focus is $(0, 2)$, and the directrix is $x = -1$.



7. $y^2 + 6y + 2x + 1 = 0 \quad \Leftrightarrow \quad y^2 + 6y = -2x - 1$
$\Leftrightarrow \quad y^2 + 6y + 9 = -2x + 8 \quad \Leftrightarrow$
$(y+3)^2 = -2(x-4)$. $4p = -2$, so $p = -\frac{1}{2}$.
The vertex is $(4, -3)$, the focus is $\left(\frac{7}{2}, -3\right)$, and the directrix is $x = \frac{9}{2}$.



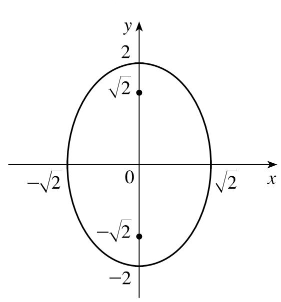
8. $2x^2 - 16x - 3y + 38 = 0 \quad \Leftrightarrow \quad 2x^2 - 16x = 3y - 38$
$\Leftrightarrow \quad 2(x^2 - 8x + 16) = 3y - 38 + 32 \quad \Leftrightarrow$
$2(x-4)^2 = 3y - 6 \quad \Leftrightarrow \quad (x-4)^2 = \frac{3}{2}(y-2)$.
$4p = \frac{3}{2}$, so $p = \frac{3}{8}$. The vertex is $(4, 2)$, the focus is $\left(4, \frac{19}{8}\right)$, and the directrix is $y = \frac{13}{8}$.



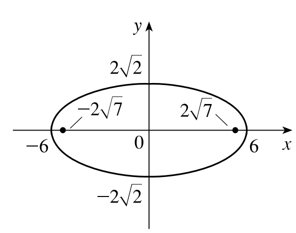
9. The equation has the form $y^2 = 4px$, where $p < 0$. Since the parabola passes through $(-1, 1)$, we have $1^2 = 4p(-1)$, so $4p = -1$ and an equation is $y^2 = -x$ or $x = -y^2$. $4p = -1$, so $p = -\frac{1}{4}$ and the focus is $\left(-\frac{1}{4}, 0\right)$ while the directrix is $x = \frac{1}{4}$.

10. The vertex is $(2, -2)$, so the equation is of the form $(x-2)^2 = 4p(y+2)$, where $p > 0$. The point $(0, 0)$ is on the parabola, so $4 = 4p(2)$ and $4p = 2$. Thus, an equation is $(x-2)^2 = 2(y+2)$. $4p = 2$, so $p = \frac{1}{2}$ and the focus is $\left(2, -\frac{3}{2}\right)$ while the directrix is $y = -\frac{5}{2}$.

11. $\dfrac{x^2}{2} + \dfrac{y^2}{4} = 1 \quad \Rightarrow \quad a = \sqrt{4} = 2,\ b = \sqrt{2}$,
$c = \sqrt{a^2 - b^2} = \sqrt{4-2} = \sqrt{2}$. The ellipse is centered at $(0, 0)$, with vertices at $(0, \pm 2)$. The foci are $\left(0, \pm\sqrt{2}\right)$.



12. $\dfrac{x^2}{36} + \dfrac{y^2}{8} = 1 \quad \Rightarrow \quad a = \sqrt{36} = 6,\ b = \sqrt{8}$,
$c = \sqrt{a^2 - b^2} = \sqrt{36 - 8} = \sqrt{28} = 2\sqrt{7}$. The ellipse is centered at $(0, 0)$, with vertices at $(\pm 6, 0)$. The foci are $(\pm 2\sqrt{7}, 0)$.

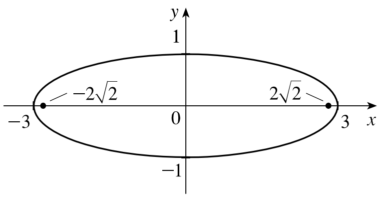


13. $x^2 + 9y^2 = 9 \quad \Leftrightarrow \quad \dfrac{x^2}{9} + \dfrac{y^2}{1} = 1 \quad \Rightarrow \quad a = \sqrt{9} = 3,$

$b = \sqrt{1} = 1, c = \sqrt{a^2 - b^2} = \sqrt{9-1} = \sqrt{8} = 2\sqrt{2}.$

The ellipse is centered at $(0,0)$, with vertices $(\pm 3, 0)$.

The foci are $(\pm 2\sqrt{2}, 0)$.

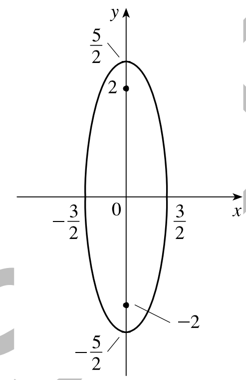


14. $100x^2 + 36y^2 = 225 \quad \Leftrightarrow \quad \dfrac{x^2}{\frac{225}{100}} + \dfrac{y^2}{\frac{225}{36}} = 1 \quad \Leftrightarrow$

$\dfrac{x^2}{\frac{9}{4}} + \dfrac{y^2}{\frac{25}{4}} = 1 \quad \Rightarrow \quad a = \sqrt{\frac{25}{4}} = \frac{5}{2}, b = \sqrt{\frac{9}{4}} = \frac{3}{2},$

$c = \sqrt{a^2 - b^2} = \sqrt{\frac{25}{4} - \frac{9}{4}} = 2.$ The ellipse is centered

at $(0,0)$, with vertices $\left(0, \pm\frac{5}{2}\right)$. The foci are $(0, \pm 2)$.



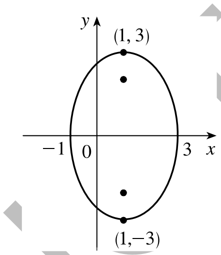
15. $9x^2 - 18x + 4y^2 = 27 \quad \Leftrightarrow$

$9(x^2 - 2x + 1) + 4y^2 = 27 + 9 \quad \Leftrightarrow$

$9(x-1)^2 + 4y^2 = 36 \quad \Leftrightarrow \quad \dfrac{(x-1)^2}{4} + \dfrac{y^2}{9} = 1 \quad \Rightarrow$

$a = 3, b = 2, c = \sqrt{5} \quad \Rightarrow$ center $(1, 0)$,

vertices $(1, \pm 3)$, foci $\left(1, \pm\sqrt{5}\right)$



16. $x^2 + 3y^2 + 2x - 12y + 10 = 0 \quad \Leftrightarrow$

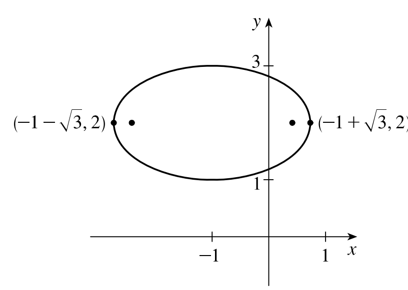
$x^2 + 2x + 1 + 3(y^2 - 4y + 4) = -10 + 1 + 12 \quad \Leftrightarrow$

$(x+1)^2 + 3(y-2)^2 = 3 \quad \Leftrightarrow$

$\dfrac{(x+1)^2}{3} + \dfrac{(y-2)^2}{1} = 1 \quad \Rightarrow \quad a = \sqrt{3}, b = 1,$

$c = \sqrt{2} \quad \Rightarrow$ center $(-1, 2)$, vertices $\left(-1 \pm \sqrt{3}, 2\right)$,

foci $\left(-1 \pm \sqrt{2}, 2\right)$



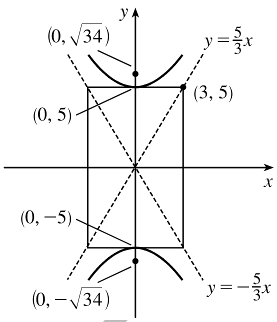
17. The center is $(0,0)$, $a = 3$, and $b = 2$, so an equation is $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. $c = \sqrt{a^2 - b^2} = \sqrt{5}$, so the foci are $\left(0, \pm\sqrt{5}\right)$.

18. The ellipse is centered at $(2,1)$, with $a = 3$ and $b = 2$. An equation is $\dfrac{(x-2)^2}{9} + \dfrac{(y-1)^2}{4} = 1$. $c = \sqrt{a^2 - b^2} = \sqrt{5}$, so the foci are $\left(2 \pm \sqrt{5}, 1\right)$.

19. $\dfrac{y^2}{25} - \dfrac{x^2}{9} = 1 \quad \Rightarrow \quad a = 5, b = 3, c = \sqrt{25 + 9} = \sqrt{34} \quad \Rightarrow$

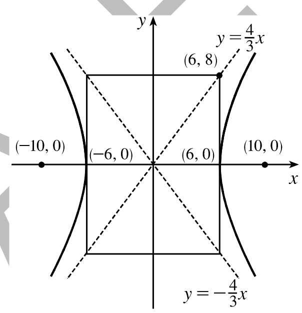
center $(0, 0)$, vertices $(0, \pm 5)$, foci $\left(0, \pm\sqrt{34}\right)$, asymptotes $y = \pm\frac{5}{3}x$.

Note: It is helpful to draw a $2a$-by-$2b$ rectangle whose center is the center of the hyperbola. The asymptotes are the extended diagonals of the rectangle.



20. $\dfrac{x^2}{36} - \dfrac{y^2}{64} = 1 \quad \Rightarrow \quad a = 6, b = 8, c = \sqrt{36 + 64} = 10 \quad \Rightarrow$

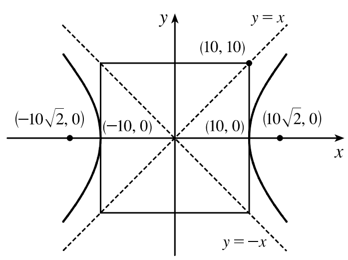
center $(0, 0)$, vertices $(\pm 6, 0)$, foci $(\pm 10, 0)$, asymptotes $y = \pm\frac{8}{6}x = \pm\frac{4}{3}x$



21. $x^2 - y^2 = 100 \quad \Leftrightarrow \quad \dfrac{x^2}{100} - \dfrac{y^2}{100} = 1 \quad \Rightarrow \quad a = b = 10,$

$c = \sqrt{100 + 100} = 10\sqrt{2} \quad \Rightarrow \quad$ center $(0, 0)$, vertices $(\pm 10, 0)$,

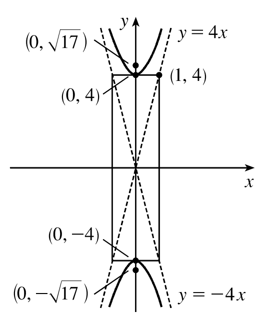
foci $\left(\pm 10\sqrt{2}, 0\right)$, asymptotes $y = \pm\frac{10}{10}x = \pm x$



22. $y^2 - 16x^2 = 16 \quad \Leftrightarrow \quad \dfrac{y^2}{16} - \dfrac{x^2}{1} = 1 \quad \Rightarrow \quad a = 4, b = 1,$

$c = \sqrt{16 + 1} = \sqrt{17} \quad \Rightarrow \quad$ center $(0, 0)$, vertices $(0, \pm 4)$,

foci $\left(0, \pm\sqrt{17}\right)$, asymptotes $y = \pm\frac{4}{1}x = \pm 4x$

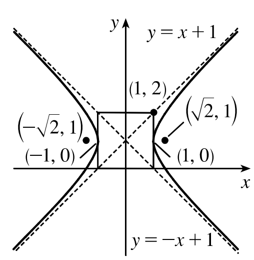


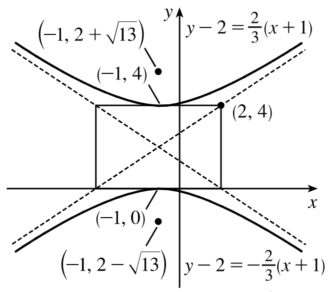
23. $x^2 - y^2 + 2y = 2 \quad \Leftrightarrow \quad x^2 - (y^2 - 2y + 1) = 2 - 1 \quad \Leftrightarrow$

$\dfrac{x^2}{1} - \dfrac{(y-1)^2}{1} = 1 \quad \Rightarrow \quad a = b = 1, c = \sqrt{1 + 1} = \sqrt{2} \quad \Rightarrow$

center $(0, 1)$, vertices $(\pm 1, 1)$, foci $\left(\pm\sqrt{2}, 1\right)$,

asymptotes $y - 1 = \pm\frac{1}{1}x = \pm x$.





24. $9y^2 - 4x^2 - 36y - 8x = 4 \quad\Leftrightarrow$

$9(y^2 - 4y + 4) - 4(x^2 + 2x + 1) = 4 + 36 - 4 \quad\Leftrightarrow$

$9(y-2)^2 - 4(x+1)^2 = 36 \quad\Leftrightarrow\quad \dfrac{(y-2)^2}{4} - \dfrac{(x+1)^2}{9} = 1 \quad\Rightarrow$

$a = 2$, $b = 3$, $c = \sqrt{4+9} = \sqrt{13} \quad\Rightarrow\quad$ center $(-1, 2)$, vertices $(-1, 2 \pm 2)$, foci $\left(-1, 2 \pm \sqrt{13}\right)$, asymptotes $y - 2 = \pm\frac{2}{3}(x+1)$.

25. $4x^2 = y^2 + 4 \quad\Leftrightarrow\quad 4x^2 - y^2 = 4 \quad\Leftrightarrow\quad \dfrac{x^2}{1} - \dfrac{y^2}{4} = 1$. This is an equation of a *hyperbola* with vertices $(\pm 1, 0)$. The foci are at $\left(\pm\sqrt{1+4}, 0\right) = \left(\pm\sqrt{5}, 0\right)$.

26. $4x^2 = y + 4 \quad\Leftrightarrow\quad x^2 = \frac{1}{4}(y+4)$. This is an equation of a *parabola* with $4p = \frac{1}{4}$, so $p = \frac{1}{16}$. The vertex is $(0, -4)$ and the focus is $\left(0, -4 + \frac{1}{16}\right) = \left(0, -\frac{63}{16}\right)$.

27. $x^2 = 4y - 2y^2 \quad\Leftrightarrow\quad x^2 + 2y^2 - 4y = 0 \quad\Leftrightarrow\quad x^2 + 2(y^2 - 2y + 1) = 2 \quad\Leftrightarrow\quad x^2 + 2(y-1)^2 = 2 \quad\Leftrightarrow$

$\dfrac{x^2}{2} + \dfrac{(y-1)^2}{1} = 1$. This is an equation of an *ellipse* with vertices at $\left(\pm\sqrt{2}, 1\right)$. The foci are at $\left(\pm\sqrt{2-1}, 1\right) = (\pm 1, 1)$.

28. $y^2 - 2 = x^2 - 2x \quad\Leftrightarrow\quad y^2 - x^2 + 2x = 2 \quad\Leftrightarrow\quad y^2 - (x^2 - 2x + 1) = 2 - 1 \quad\Leftrightarrow\quad \dfrac{y^2}{1} - \dfrac{(x-1)^2}{1} = 1$. This is an equation of a *hyperbola* with vertices $(1, \pm 1)$. The foci are at $\left(1, \pm\sqrt{1+1}\right) = \left(1, \pm\sqrt{2}\right)$.

29. $3x^2 - 6x - 2y = 1 \quad\Leftrightarrow\quad 3x^2 - 6x = 2y + 1 \quad\Leftrightarrow\quad 3(x^2 - 2x + 1) = 2y + 1 + 3 \quad\Leftrightarrow\quad 3(x-1)^2 = 2y + 4 \quad\Leftrightarrow$

$(x-1)^2 = \frac{2}{3}(y+2)$. This is an equation of a *parabola* with $4p = \frac{2}{3}$, so $p = \frac{1}{6}$. The vertex is $(1, -2)$ and the focus is $\left(1, -2 + \frac{1}{6}\right) = \left(1, -\frac{11}{6}\right)$.

30. $x^2 - 2x + 2y^2 - 8y + 7 = 0 \quad\Leftrightarrow\quad (x^2 - 2x + 1) + 2(y^2 - 4y + 4) = -7 + 1 + 8 \quad\Leftrightarrow\quad (x-1)^2 + 2(y-2)^2 = 2 \quad\Leftrightarrow$

$\dfrac{(x-1)^2}{2} + \dfrac{(y-2)^2}{1} = 1$. This is an equation of an *ellipse* with vertices at $\left(1 \pm \sqrt{2}, 2\right)$. The foci are at $\left(1 \pm \sqrt{2-1}, 2\right) = (1 \pm 1, 2)$.

31. The parabola with vertex $(0, 0)$ and focus $(1, 0)$ opens to the right and has $p = 1$, so its equation is $y^2 = 4px$, or $y^2 = 4x$.

32. The parabola with focus $(0, 0)$ and directrix $y = 6$ has vertex $(0, 3)$ and opens downward, so $p = -3$ and its equation is $(x-0)^2 = 4p(y-3)$, or $x^2 = -12(y-3)$.

33. The distance from the focus $(-4, 0)$ to the directrix $x = 2$ is $2 - (-4) = 6$, so the distance from the focus to the vertex is $\frac{1}{2}(6) = 3$ and the vertex is $(-1, 0)$. Since the focus is to the left of the vertex, $p = -3$. An equation is $y^2 = 4p(x+1) \quad\Rightarrow$ $y^2 = -12(x+1)$.

34. The parabola with vertex $(2, 3)$ and focus $(2, -1)$ opens downward and has $p = -1 - 3 = -4$, so its equation is $(x-2)^2 = 4p(y-3)$, or $(x-2)^2 = -16(y-3)$.

35. The parabola with vertex $(3,-1)$ having a horizontal axis has equation $[y-(-1)]^2 = 4p(x-3)$. Since it passes through $(-15,2)$, $(2+1)^2 = 4p(-15-3) \quad\Rightarrow\quad 9 = 4p(-18) \quad\Rightarrow\quad 4p = -\frac{1}{2}$. An equation is $(y+1)^2 = -\frac{1}{2}(x-3)$.

36. The parabola with vertical axis and passing through $(0,4)$ has equation $y = ax^2 + bx + 4$. It also passes through $(1,3)$ and $(-2,-6)$, so

$$\begin{cases} 3 = a+b+4 \\ -6 = 4a-2b+4 \end{cases} \quad\Rightarrow\quad \begin{cases} -1 = a+b \\ -10 = 4a-2b \end{cases} \quad\Rightarrow\quad \begin{cases} -1 = a+b \\ -5 = 2a-b \end{cases}$$

Adding the last two equations gives us $3a = -6$, or $a = -2$. Since $a+b=-1$, we have $b=1$, and an equation is $y = -2x^2 + x + 4$.

37. The ellipse with foci $(\pm 2, 0)$ and vertices $(\pm 5, 0)$ has center $(0,0)$ and a horizontal major axis, with $a=5$ and $c=2$, so $b^2 = a^2 - c^2 = 25 - 4 = 21$. An equation is $\dfrac{x^2}{25} + \dfrac{y^2}{21} = 1$.

38. The ellipse with foci $\left(0, \pm\sqrt{2}\right)$ and vertices $(0,\pm 2)$ has center $(0,0)$ and a vertical major axis, with $a=2$ and $c=\sqrt{2}$, so $b^2 = a^2 - c^2 = 4 - 2 = 2$. An equation is $\dfrac{x^2}{2} + \dfrac{y^2}{4} = 1$.

39. Since the vertices are $(0,0)$ and $(0,8)$, the ellipse has center $(0,4)$ with a vertical axis and $a=4$. The foci at $(0,2)$ and $(0,6)$ are 2 units from the center, so $c=2$ and $b = \sqrt{a^2-c^2} = \sqrt{4^2-2^2} = \sqrt{12}$. An equation is $\dfrac{(x-0)^2}{b^2} + \dfrac{(y-4)^2}{a^2} = 1 \quad\Rightarrow$ $\dfrac{x^2}{12} + \dfrac{(y-4)^2}{16} = 1$.

40. Since the foci are $(0,-1)$ and $(8,-1)$, the ellipse has center $(4,-1)$ with a horizontal axis and $c=4$. The vertex $(9,-1)$ is 5 units from the center, so $a=5$ and $b = \sqrt{a^2-c^2} = \sqrt{5^2-4^2} = \sqrt{9}$. An equation is $\dfrac{(x-4)^2}{a^2} + \dfrac{(y+1)^2}{b^2} = 1 \quad\Rightarrow\quad \dfrac{(x-4)^2}{25} + \dfrac{(y+1)^2}{9} = 1$.

41. An equation of an ellipse with center $(-1,4)$ and vertex $(-1,0)$ is $\dfrac{(x+1)^2}{b^2} + \dfrac{(y-4)^2}{4^2} = 1$. The focus $(-1,6)$ is 2 units from the center, so $c=2$. Thus, $b^2 + 2^2 = 4^2 \quad\Rightarrow\quad b^2 = 12$, and the equation is $\dfrac{(x+1)^2}{12} + \dfrac{(y-4)^2}{16} = 1$.

42. Foci $F_1(-4,0)$ and $F_2(4,0) \quad\Rightarrow\quad c=4$ and an equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The ellipse passes through $P(-4, 1.8)$, so $2a = |PF_1| + |PF_2| \quad\Rightarrow\quad 2a = 1.8 + \sqrt{8^2 + (1.8)^2} \quad\Rightarrow\quad 2a = 1.8 + 8.2 \quad\Rightarrow\quad a = 5$.
$b^2 = a^2 - c^2 = 25 - 16 = 9$ and the equation is $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$.

43. An equation of a hyperbola with vertices $(\pm 3, 0)$ is $\dfrac{x^2}{3^2} - \dfrac{y^2}{b^2} = 1$. Foci $(\pm 5, 0) \quad\Rightarrow\quad c=5$ and $3^2 + b^2 = 5^2 \quad\Rightarrow$ $b^2 = 25 - 9 = 16$, so the equation is $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$.

44. An equation of a hyperbola with vertices $(0, \pm 2)$ is $\dfrac{y^2}{2^2} - \dfrac{x^2}{b^2} = 1$. Foci $(0, \pm 5) \Rightarrow c = 5$ and $2^2 + b^2 = 5^2 \Rightarrow$ $b^2 = 25 - 4 = 21$, so the equation is $\dfrac{y^2}{4} - \dfrac{x^2}{21} = 1$.

45. The center of a hyperbola with vertices $(-3, -4)$ and $(-3, 6)$ is $(-3, 1)$, so $a = 5$ and an equation is $\dfrac{(y-1)^2}{5^2} - \dfrac{(x+3)^2}{b^2} = 1$. Foci $(-3, -7)$ and $(-3, 9) \Rightarrow c = 8$, so $5^2 + b^2 = 8^2 \Rightarrow b^2 = 64 - 25 = 39$ and the equation is $\dfrac{(y-1)^2}{25} - \dfrac{(x+3)^2}{39} = 1$.

46. The center of a hyperbola with vertices $(-1, 2)$ and $(7, 2)$ is $(3, 2)$, so $a = 4$ and an equation is $\dfrac{(x-3)^2}{4^2} - \dfrac{(y-2)^2}{b^2} = 1$. Foci $(-2, 2)$ and $(8, 2) \Rightarrow c = 5$, so $4^2 + b^2 = 5^2 \Rightarrow b^2 = 25 - 16 = 9$ and the equation is $\dfrac{(x-3)^2}{16} - \dfrac{(y-2)^2}{9} = 1$.

47. The center of a hyperbola with vertices $(\pm 3, 0)$ is $(0, 0)$, so $a = 3$ and an equation is $\dfrac{x^2}{3^2} - \dfrac{y^2}{b^2} = 1$. Asymptotes $y = \pm 2x \Rightarrow \dfrac{b}{a} = 2 \Rightarrow b = 2(3) = 6$ and the equation is $\dfrac{x^2}{9} - \dfrac{y^2}{36} = 1$.

48. The center of a hyperbola with foci $(2, 0)$ and $(2, 8)$ is $(2, 4)$, so $c = 4$ and an equation is $\dfrac{(y-4)^2}{a^2} - \dfrac{(x-2)^2}{b^2} = 1$. The asymptote $y = 3 + \frac{1}{2}x$ has slope $\frac{1}{2}$, so $\dfrac{a}{b} = \dfrac{1}{2} \Rightarrow b = 2a$ and $a^2 + b^2 = c^2 \Rightarrow a^2 + (2a)^2 = 4^2 \Rightarrow$ $5a^2 = 16 \Rightarrow a^2 = \frac{16}{5}$ and so $b^2 = 16 - \frac{16}{5} = \frac{64}{5}$. Thus, an equation is $\dfrac{(y-4)^2}{16/5} - \dfrac{(x-2)^2}{64/5} = 1$.

49. In Figure 8, we see that the point on the ellipse closest to a focus is the closer vertex (which is a distance $a - c$ from it) while the farthest point is the other vertex (at a distance of $a + c$). So for this lunar orbit, $(a - c) + (a + c) = 2a = (1728 + 110) + (1728 + 314)$, or $a = 1940$; and $(a + c) - (a - c) = 2c = 314 - 110$, or $c = 102$. Thus, $b^2 = a^2 - c^2 = 3{,}753{,}196$, and the equation is $\dfrac{x^2}{3{,}763{,}600} + \dfrac{y^2}{3{,}753{,}196} = 1$.

50. (a) Choose V to be the origin, with x-axis through V and F. Then F is $(p, 0)$, A is $(p, 5)$, so substituting A into the equation $y^2 = 4px$ gives $25 = 4p^2$ so $p = \frac{5}{2}$ and $y^2 = 10x$.

(b) $x = 11 \Rightarrow y = \sqrt{110} \Rightarrow |CD| = 2\sqrt{110}$

51. (a) Set up the coordinate system so that A is $(-200, 0)$ and B is $(200, 0)$.

$|PA| - |PB| = (1200)(980) = 1{,}176{,}000 \text{ ft} = \frac{2450}{11} \text{ mi} = 2a \Rightarrow a = \frac{1225}{11}$, and $c = 200$ so

$b^2 = c^2 - a^2 = \dfrac{3{,}339{,}375}{121} \Rightarrow \dfrac{121x^2}{1{,}500{,}625} - \dfrac{121y^2}{3{,}339{,}375} = 1$.

(b) Due north of $B \Rightarrow x = 200 \Rightarrow \dfrac{(121)(200)^2}{1{,}500{,}625} - \dfrac{121y^2}{3{,}339{,}375} = 1 \Rightarrow y = \dfrac{133{,}575}{539} \approx 248$ mi

52. $|PF_1| - |PF_2| = \pm 2a \Leftrightarrow \sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = \pm 2a \Leftrightarrow$

$\sqrt{(x+c)^2+y^2} = \sqrt{(x-c)^2+y^2} \pm 2a \Leftrightarrow (x+c)^2+y^2 = (x-c)^2+y^2+4a^2 \pm 4a\sqrt{(x-c)^2+y^2} \Leftrightarrow$

$4cx - 4a^2 = \pm 4a\sqrt{(x-c)^2+y^2} \Leftrightarrow c^2x^2 - 2a^2cx + a^4 = a^2(x^2 - 2cx + c^2 + y^2) \Leftrightarrow$

$(c^2-a^2)x^2 - a^2y^2 = a^2(c^2-a^2) \Leftrightarrow b^2x^2 - a^2y^2 = a^2b^2$ [where $b^2 = c^2 - a^2$] $\Leftrightarrow \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

53. The function whose graph is the upper branch of this hyperbola is concave upward. The function is

$y = f(x) = a\sqrt{1 + \dfrac{x^2}{b^2}} = \dfrac{a}{b}\sqrt{b^2+x^2}$, so $y' = \dfrac{a}{b}x(b^2+x^2)^{-1/2}$ and

$y'' = \dfrac{a}{b}\left[(b^2+x^2)^{-1/2} - x^2(b^2+x^2)^{-3/2}\right] = ab(b^2+x^2)^{-3/2} > 0$ for all x, and so f is concave upward.

54. We can follow exactly the same sequence of steps as in the derivation of Formula 4, except we use the points $(1,1)$ and $(-1,-1)$ in the distance formula (first equation of that derivation) so $\sqrt{(x-1)^2+(y-1)^2} + \sqrt{(x+1)^2+(y+1)^2} = 4$ will lead (after moving the second term to the right, squaring, and simplifying) to $2\sqrt{(x+1)^2+(y+1)^2} = x + y + 4$, which, after squaring and simplifying again, leads to $3x^2 - 2xy + 3y^2 = 8$.

55. (a) If $k > 16$, then $k - 16 > 0$, and $\dfrac{x^2}{k} + \dfrac{y^2}{k-16} = 1$ is an *ellipse* since it is the sum of two squares on the left side.

(b) If $0 < k < 16$, then $k - 16 < 0$, and $\dfrac{x^2}{k} + \dfrac{y^2}{k-16} = 1$ is a *hyperbola* since it is the difference of two squares on the left side.

(c) If $k < 0$, then $k - 16 < 0$, and there is *no curve* since the left side is the sum of two negative terms, which cannot equal 1.

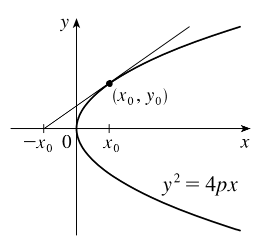
(d) In case (a), $a^2 = k$, $b^2 = k - 16$, and $c^2 = a^2 - b^2 = 16$, so the foci are at $(\pm 4, 0)$. In case (b), $k - 16 < 0$, so $a^2 = k$, $b^2 = 16 - k$, and $c^2 = a^2 + b^2 = 16$, and so again the foci are at $(\pm 4, 0)$.

56. (a) $y^2 = 4px \Rightarrow 2yy' = 4p \Rightarrow y' = \dfrac{2p}{y}$, so the tangent line is

$y - y_0 = \dfrac{2p}{y_0}(x - x_0) \Rightarrow yy_0 - y_0^2 = 2p(x - x_0) \Leftrightarrow$

$yy_0 - 4px_0 = 2px - 2px_0 \Rightarrow yy_0 = 2p(x + x_0)$.

(b) The x-intercept is $-x_0$.



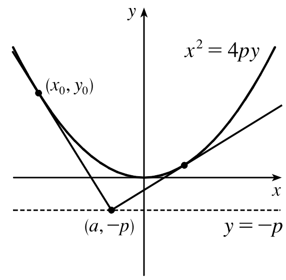
57. $x^2 = 4py \Rightarrow 2x = 4py' \Rightarrow y' = \dfrac{x}{2p}$, so the tangent line at (x_0, y_0) is $y - \dfrac{x_0^2}{4p} = \dfrac{x_0}{2p}(x - x_0)$. This line passes through the point $(a, -p)$ on the directrix, so $-p - \dfrac{x_0^2}{4p} = \dfrac{x_0}{2p}(a - x_0) \Rightarrow -4p^2 - x_0^2 = 2ax_0 - 2x_0^2 \Leftrightarrow$

$x_0^2 - 2ax_0 - 4p^2 = 0 \;\Leftrightarrow\; x_0^2 - 2ax_0 + a^2 = a^2 + 4p^2 \;\Leftrightarrow$

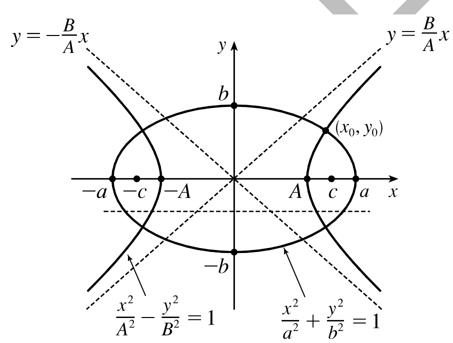
$(x_0 - a)^2 = a^2 + 4p^2 \;\Leftrightarrow\; x_0 = a \pm \sqrt{a^2 + 4p^2}$. The slopes of the tangent lines at $x = a \pm \sqrt{a^2+4p^2}$ are $\dfrac{a \pm \sqrt{a^2+4p^2}}{2p}$, so the product of the two slopes is

$$\frac{a+\sqrt{a^2+4p^2}}{2p}\cdot\frac{a-\sqrt{a^2+4p^2}}{2p} = \frac{a^2-(a^2+4p^2)}{4p^2} = \frac{-4p^2}{4p^2} = -1,$$

showing that the tangent lines are perpendicular.



58. Without a loss of generality, let the ellipse, hyperbola, and foci be as shown in the figure.



The curves intersect (eliminate y^2) $\Rightarrow$

$B^2\left(\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2}\right) + b^2\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right) = B^2 + b^2 \;\Rightarrow$

$\dfrac{B^2x^2}{A^2} + \dfrac{b^2x^2}{a^2} = B^2 + b^2 \;\Rightarrow\; x^2\left(\dfrac{B^2}{A^2} + \dfrac{b^2}{a^2}\right) = B^2 + b^2 \;\Rightarrow$

$x^2 = \dfrac{B^2 + b^2}{\dfrac{a^2B^2 + b^2A^2}{A^2a^2}} = \dfrac{A^2a^2(B^2+b^2)}{a^2B^2 + b^2A^2}$.

Similarly, $y^2 = \dfrac{B^2b^2(a^2 - A^2)}{b^2A^2 + a^2B^2}$.

Next we find the slopes of the tangent lines of the curves: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \;\Rightarrow\; \dfrac{yy'}{b^2} = -\dfrac{x}{a^2} \;\Rightarrow$ $y_E' = -\dfrac{b^2}{a^2}\dfrac{x}{y}$ and $\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1 \;\Rightarrow\; \dfrac{2x}{A^2} - \dfrac{2yy'}{B^2} = 0 \;\Rightarrow\; \dfrac{yy'}{B^2} = \dfrac{x}{A^2} \;\Rightarrow\; y_H' = \dfrac{B^2}{A^2}\dfrac{x}{y}$. The product of the slopes at (x_0, y_0) is $y_E'\,y_H' = -\dfrac{b^2B^2x_0^2}{a^2A^2y_0^2} = -\dfrac{b^2B^2\left[\dfrac{A^2a^2(B^2+b^2)}{a^2B^2+b^2A^2}\right]}{a^2A^2\left[\dfrac{B^2b^2(a^2-A^2)}{b^2A^2+a^2B^2}\right]} = -\dfrac{B^2+b^2}{a^2-A^2}$. Since $a^2 - b^2 = c^2$ and $A^2 + B^2 = c^2$, we have $a^2 - b^2 = A^2 + B^2 \;\Rightarrow\; a^2 - A^2 = b^2 + B^2$, so the product of the slopes is -1, and hence, the tangent lines at each point of intersection are perpendicular.

59. $9x^2 + 4y^2 = 36 \;\Leftrightarrow\; \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. We use the parametrization $x = 2\cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$. The circumference is given by

$$\begin{aligned} L &= \int_0^{2\pi}\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^{2\pi}\sqrt{(-2\sin t)^2 + (3\cos t)^2}\,dt \\ &= \int_0^{2\pi}\sqrt{4\sin^2 t + 9\cos^2 t}\,dt = \int_0^{2\pi}\sqrt{4 + 5\cos^2 t}\,dt \end{aligned}$$

Now use Simpson's Rule with $n = 8$, $\Delta t = \dfrac{2\pi - 0}{8} = \dfrac{\pi}{4}$, and $f(t) = \sqrt{4 + 5\cos^2 t}$ to get

$$L \approx S_8 = \tfrac{\pi/4}{3}\left[f(0) + 4f\left(\tfrac{\pi}{4}\right) + 2f\left(\tfrac{\pi}{2}\right) + 4f\left(\tfrac{3\pi}{4}\right) + 2f(\pi) + 4f\left(\tfrac{5\pi}{4}\right) + 2f\left(\tfrac{3\pi}{2}\right) + 4f\left(\tfrac{7\pi}{4}\right) + f(2\pi)\right] \approx 15.9.$$

60. The length of the major axis is $2a$, so $a = \frac{1}{2}(1.18 \times 10^{10}) = 5.9 \times 10^9$. The length of the minor axis is $2b$, so $b = \frac{1}{2}(1.14 \times 10^{10}) = 5.7 \times 10^9$. An equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, or converting into parametric equations,

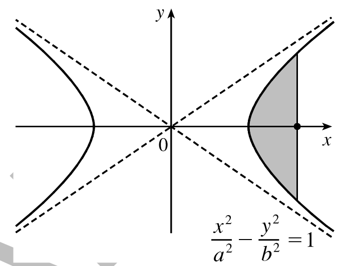
$x = a\cos\theta$ and $y = b\sin\theta$. So

$$L = 4\int_0^{\pi/2}\sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta = 4\int_0^{\pi/2}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta$$

Using Simpson's Rule with $n = 10$, $\Delta\theta = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$, and $f(\theta) = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$, we get

$$L \approx 4 \cdot S_{10} = 4 \cdot \tfrac{\pi}{20\cdot 3}\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{2\pi}{20}\right) + \cdots + 2f\left(\tfrac{8\pi}{20}\right) + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right] \approx 3.64 \times 10^{10}\text{ km}$$

61. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{y^2}{b^2} = \dfrac{x^2 - a^2}{a^2} \;\Rightarrow\; y = \pm\dfrac{b}{a}\sqrt{x^2 - a^2}$.



$$A = 2\int_a^c \frac{b}{a}\sqrt{x^2 - a^2}\,dx \overset{39}{=} \frac{2b}{a}\left[\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right|\right]_a^c$$

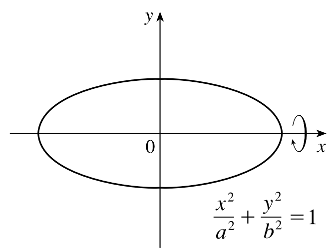
$$= \frac{b}{a}\left[c\sqrt{c^2 - a^2} - a^2\ln\left|c + \sqrt{c^2 - a^2}\right| + a^2\ln|a|\right]$$

Since $a^2 + b^2 = c^2$, $c^2 - a^2 = b^2$, and $\sqrt{c^2 - a^2} = b$.

$$= \frac{b}{a}\left[cb - a^2\ln(c + b) + a^2\ln a\right] = \frac{b}{a}\left[cb + a^2(\ln a - \ln(b + c))\right]$$

$$= b^2c/a + ab\ln[a/(b + c)],\ \text{where } c^2 = a^2 + b^2.$$

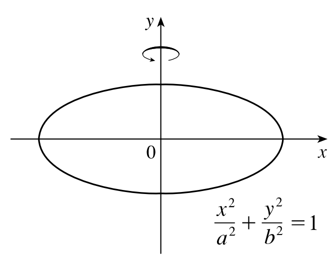
62. (a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{y^2}{b^2} = \dfrac{a^2 - x^2}{a^2} \;\Rightarrow\; y = \pm\dfrac{b}{a}\sqrt{a^2 - x^2}$.



$$V = \int_{-a}^{a}\pi\left(\frac{b}{a}\sqrt{a^2 - x^2}\right)^2 dx = 2\pi\frac{b^2}{a^2}\int_0^a (a^2 - x^2)\,dx$$

$$= \frac{2\pi b^2}{a^2}\left[a^2x - \tfrac{1}{3}x^3\right]_0^a = \frac{2\pi b^2}{a^2}\left(\frac{2a^3}{3}\right) = \frac{4}{3}\pi b^2 a$$

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{x^2}{a^2} = \dfrac{b^2 - y^2}{b^2} \;\Rightarrow\; x = \pm\dfrac{a}{b}\sqrt{b^2 - y^2}$.



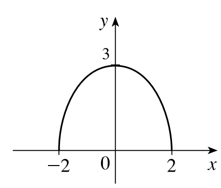
$$V = \int_{-b}^{b}\pi\left(\frac{a}{b}\sqrt{b^2 - y^2}\right)^2 dy = 2\pi\frac{a^2}{b^2}\int_0^b (b^2 - y^2)\,dy$$

$$= \frac{2\pi a^2}{b^2}\left[b^2y - \tfrac{1}{3}y^3\right]_0^b = \frac{2\pi a^2}{b^2}\left(\frac{2b^3}{3}\right) = \frac{4}{3}\pi a^2 b$$

63. $9x^2 + 4y^2 = 36 \;\Leftrightarrow\; \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1 \;\Rightarrow\; a = 3, b = 2$. By symmetry, $\overline{x} = 0$. By Example 2 in Section 7.3, the area of the top half of the ellipse is $\frac{1}{2}(\pi ab) = 3\pi$. Solve $9x^2 + 4y^2 = 36$ for y to get an equation for the top half of the ellipse: $9x^2 + 4y^2 = 36 \;\Leftrightarrow\; 4y^2 = 36 - 9x^2 \;\Leftrightarrow\; y^2 = \frac{9}{4}(4 - x^2) \;\Rightarrow\; y = \frac{3}{2}\sqrt{4 - x^2}$. Now

$$\overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx = \frac{1}{3\pi}\int_{-2}^{2}\frac{1}{2}\left(\frac{3}{2}\sqrt{4 - x^2}\right)^2 dx = \frac{3}{8\pi}\int_{-2}^{2}(4 - x^2)\,dx$$

$$= \frac{3}{8\pi}\cdot 2\int_0^2 (4 - x^2)\,dx = \frac{3}{4\pi}\left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{3}{4\pi}\left(\frac{16}{3}\right) = \frac{4}{\pi}$$



so the centroid is $(0, 4/\pi)$.

64. (a) Consider the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with $a>b$, so that the major axis is the x-axis. Let the ellipse be parametrized by $x=a\cos t,\ y=b\sin t,\ 0\le t\le 2\pi$. Then

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=a^2\sin^2 t+b^2\cos^2 t=a^2(1-\cos^2 t)+b^2\cos^2 t=a^2+(b^2-a^2)\cos^2 t=a^2-c^2\cos^2 t,$$

where $c^2=a^2-b^2$. Using symmetry and rotating the ellipse about the major axis gives us surface area

$$S=\int 2\pi y\,ds=2\int_0^{\pi/2}2\pi(b\sin t)\sqrt{a^2-c^2\cos^2 t}\,dt=4\pi b\int_c^0\sqrt{a^2-u^2}\left(-\frac{1}{c}\,du\right)\qquad\begin{bmatrix}u=c\cos t\\ du=-c\sin t\,dt\end{bmatrix}$$

$$=\frac{4\pi b}{c}\int_0^c\sqrt{a^2-u^2}\,du\overset{30}{=}\frac{4\pi b}{c}\left[\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)\right]_0^c=\frac{2\pi b}{c}\left[c\sqrt{a^2-c^2}+a^2\sin^{-1}\left(\frac{c}{a}\right)\right]$$

$$=\frac{2\pi b}{c}\left[bc+a^2\sin^{-1}\left(\frac{c}{a}\right)\right]$$

(b) As in part (a),

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=a^2\sin^2 t+b^2\cos^2 t=a^2\sin^2 t+b^2(1-\sin^2 t)=b^2+(a^2-b^2)\sin^2 t=b^2+c^2\sin^2 t.$$

Rotating about the minor axis gives us

$$S=\int 2\pi x\,ds=2\int_0^{\pi/2}2\pi(a\cos t)\sqrt{b^2+c^2\sin^2 t}\,dt=4\pi a\int_0^c\sqrt{b^2+u^2}\left(\frac{1}{c}\,du\right)\qquad\begin{bmatrix}u=c\sin t\\ du=c\cos t\,dt\end{bmatrix}$$

$$\overset{21}{=}\frac{4\pi a}{c}\left[\frac{u}{2}\sqrt{b^2+u^2}+\frac{b^2}{2}\ln\left(u+\sqrt{b^2+u^2}\right)\right]_0^c=\frac{2\pi a}{c}\left[c\sqrt{b^2+c^2}+b^2\ln\left(c+\sqrt{b^2+c^2}\right)-b^2\ln b\right]$$

$$=\frac{2\pi a}{c}\left[ac+b^2\ln\left(\frac{a+c}{b}\right)\right]$$

65. Differentiating implicitly, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\ \Rightarrow\ \dfrac{2x}{a^2}+\dfrac{2yy'}{b^2}=0\ \Rightarrow\ y'=-\dfrac{b^2x}{a^2y}$ $[y\ne 0]$. Thus, the slope of the tangent line at P is $-\dfrac{b^2x_1}{a^2y_1}$. The slope of F_1P is $\dfrac{y_1}{x_1+c}$ and of F_2P is $\dfrac{y_1}{x_1-c}$. By the formula in Problem 21 on text page 273, we have

$$\tan\alpha=\frac{\dfrac{y_1}{x_1+c}+\dfrac{b^2x_1}{a^2y_1}}{1-\dfrac{b^2x_1y_1}{a^2y_1(x_1+c)}}=\frac{a^2y_1^2+b^2x_1(x_1+c)}{a^2y_1(x_1+c)-b^2x_1y_1}=\frac{a^2b^2+b^2cx_1}{c^2x_1y_1+a^2cy_1}\qquad\begin{bmatrix}\text{using } b^2x_1^2+a^2y_1^2=a^2b^2,\\ \text{and } a^2-b^2=c^2\end{bmatrix}$$

$$=\frac{b^2(cx_1+a^2)}{cy_1(cx_1+a^2)}=\frac{b^2}{cy_1}$$

and $$\tan\beta=\frac{-\dfrac{b^2x_1}{a^2y_1}-\dfrac{y_1}{x_1-c}}{1-\dfrac{b^2x_1y_1}{a^2y_1(x_1-c)}}=\frac{-a^2y_1^2-b^2x_1(x_1-c)}{a^2y_1(x_1-c)-b^2x_1y_1}=\frac{-a^2b^2+b^2cx_1}{c^2x_1y_1-a^2cy_1}=\frac{b^2(cx_1-a^2)}{cy_1(cx_1-a^2)}=\frac{b^2}{cy_1}$$

Thus, $\alpha=\beta$.

66. The slopes of the line segments F_1P and F_2P are $\dfrac{y_1}{x_1+c}$ and $\dfrac{y_1}{x_1-c}$, where P is (x_1,y_1). Differentiating implicitly, $\dfrac{2x}{a^2}-\dfrac{2yy'}{b^2}=0\ \Rightarrow\ y'=\dfrac{b^2x}{a^2y}\ \Rightarrow$ the slope of the tangent at P is $\dfrac{b^2x_1}{a^2y_1}$, so by the formula in Problem 21 on text page 273,

$$\tan\alpha = \frac{\dfrac{b^2x_1}{a^2y_1} - \dfrac{y_1}{x_1+c}}{1 + \dfrac{b^2x_1y_1}{a^2y_1(x_1+c)}} = \frac{b^2x_1(x_1+c) - a^2y_1^2}{a^2y_1(x_1+c) + b^2x_1y_1} = \frac{b^2(cx_1+a^2)}{cy_1(cx_1+a^2)} \quad \left[\begin{array}{c}\text{using } x_1^2/a^2 - y_1^2/b^2 = 1, \\ \text{and } a^2+b^2=c^2\end{array}\right] = \frac{b^2}{cy_1}$$

and
$$\tan\beta = \frac{-\dfrac{b^2x_1}{a^2y_1} + \dfrac{y_1}{x_1-c}}{1 + \dfrac{b^2x_1y_1}{a^2y_1(x_1-c)}} = \frac{-b^2x_1(x_1-c) + a^2y_1^2}{a^2y_1(x_1-c) + b^2x_1y_1} = \frac{b^2(cx_1-a^2)}{cy_1(cx_1-a^2)} = \frac{b^2}{cy_1}$$

So $\alpha = \beta$.

10.6 Conic Sections in Polar Coordinates

1. The directrix $x = 4$ is to the right of the focus at the origin, so we use the form with "$+\ e\cos\theta$" in the denominator. (See Theorem 6 and Figure 2.) An equation of the ellipse is $r = \dfrac{ed}{1+e\cos\theta} = \dfrac{\frac{1}{2}\cdot 4}{1+\frac{1}{2}\cos\theta} = \dfrac{4}{2+\cos\theta}$.

2. The directrix $x = -3$ is to the left of the focus at the origin, so we use the form with "$-\ e\cos\theta$" in the denominator. $e = 1$ for a parabola, so an equation is $r = \dfrac{ed}{1-e\cos\theta} = \dfrac{1\cdot 3}{1-1\cos\theta} = \dfrac{3}{1-\cos\theta}$.

3. The directrix $y = 2$ is above the focus at the origin, so we use the form with "$+\ e\sin\theta$" in the denominator. An equation of the hyperbola is $r = \dfrac{ed}{1+e\sin\theta} = \dfrac{1.5(2)}{1+1.5\sin\theta} = \dfrac{6}{2+3\sin\theta}$.

4. The directrix $x = 3$ is to the right of the focus at the origin, so we use the form with "$+\ e\cos\theta$" in the denominator. An equation of the hyperbola is $r = \dfrac{ed}{1+e\cos\theta} = \dfrac{3\cdot 3}{1+3\cos\theta} = \dfrac{9}{1+3\cos\theta}$.

5. The vertex $(2,\pi)$ is to the left of the focus at the origin, so we use the form with "$-e\cos\theta$" in the denominator. An equation of the ellipse is $r = \dfrac{ed}{1-e\cos\theta}$. Using eccentricity $e = \dfrac{2}{3}$ with $\theta = \pi$ and $r = 2$, we get $2 = \dfrac{\frac{2}{3}d}{1-\frac{2}{3}(-1)} \Rightarrow$ $2 = \dfrac{2d}{5} \Rightarrow d = 5$, so we have $r = \dfrac{\frac{2}{3}(5)}{1-\frac{2}{3}\cos\theta} = \dfrac{10}{3-2\cos\theta}$.

6. The directrix $r = 4\csc\theta$ (equivalent to $r\sin\theta = 4$ or $y = 4$) is above the focus at the origin, so we will use the form with "$+e\sin\theta$" in the denominator. The distance from the focus to the directrix is $d = 4$, so an equation of the ellipse is $r = \dfrac{ed}{1+e\sin\theta} = \dfrac{(0.6)(4)}{1+0.6\sin\theta}\cdot\dfrac{5}{5} = \dfrac{12}{5+3\sin\theta}$.

7. The vertex $\left(3,\frac{\pi}{2}\right)$ is 3 units above the focus at the origin, so the directrix is 6 units above the focus ($d = 6$), and we use the form "$+e\sin\theta$" in the denominator. $e = 1$ for a parabola, so an equation is $r = \dfrac{ed}{1+e\sin\theta} = \dfrac{1(6)}{1+1\sin\theta} = \dfrac{6}{1+\sin\theta}$.

8. The directrix $r = -2\sec\theta$ (equivalent to $r\cos\theta = -2$ or $x = -2$) is left of the focus at the origin, so we will use the form with "$-e\cos\theta$" in the denominator. The distance from the focus to the directrix is $d = 2$, so an equation of the hyperbola is $r = \dfrac{ed}{1 - e\cos\theta} = \dfrac{2(2)}{1 - 2\cos\theta} = \dfrac{4}{1 - 2\cos\theta}$.

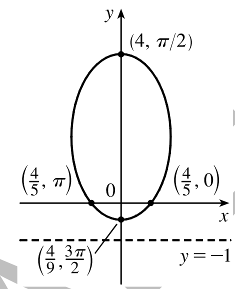
9. $r = \dfrac{4}{5 - 4\sin\theta} \cdot \dfrac{1/5}{1/5} = \dfrac{4/5}{1 - \frac{4}{5}\sin\theta}$, where $e = \frac{4}{5}$ and $ed = \frac{4}{5} \quad\Rightarrow\quad d = 1$.

(a) Eccentricity $= e = \frac{4}{5}$

(b) Since $e = \frac{4}{5} < 1$, the conic is an ellipse.

(c) Since "$-\ e\sin\theta$" appears in the denominator, the directrix is below the focus at the origin, $d = |Fl| = 1$, so an equation of the directrix is $y = -1$.

(d) The vertices are $\left(4, \frac{\pi}{2}\right)$ and $\left(\frac{4}{9}, \frac{3\pi}{2}\right)$.



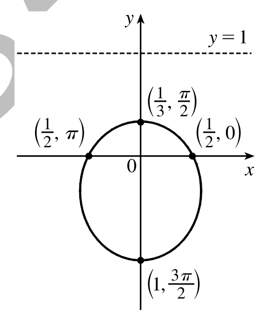
10. $r = \dfrac{1}{2 + \sin\theta} \cdot \dfrac{1/2}{1/2} = \dfrac{1/2}{1 + \frac{1}{2}\sin\theta}$, where $e = \dfrac{1}{2}$ and $ed = \dfrac{1}{2} \quad\Rightarrow\quad d = 1$.

(a) Eccentricity $= e = \dfrac{1}{2}$

(b) Since $e = \dfrac{1}{2} < 1$, the conic is an ellipse.

(c) Since "$+e\sin\theta$" appears in the denominator, the directrix is above the focus at the origin, $d = |Fl| = 1$, so an equation of the directrix is $y = 1$.

(d) The vertices are $\left(\frac{1}{3}, \frac{\pi}{2}\right)$ and $\left(1, \frac{3\pi}{2}\right)$.



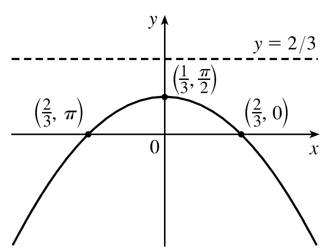
11. $r = \dfrac{2}{3 + 3\sin\theta} \cdot \dfrac{1/3}{1/3} = \dfrac{2/3}{1 + 1\sin\theta}$, where $e = 1$ and $ed = \frac{2}{3} \quad\Rightarrow\quad d = \frac{2}{3}$.

(a) Eccentricity $= e = 1$

(b) Since $e = 1$, the conic is a parabola.

(c) Since "$+\ e\sin\theta$" appears in the denominator, the directrix is above the focus at the origin. $d = |Fl| = \frac{2}{3}$, so an equation of the directrix is $y = \frac{2}{3}$.

(d) The vertex is at $\left(\frac{1}{3}, \frac{\pi}{2}\right)$, midway between the focus and directrix.



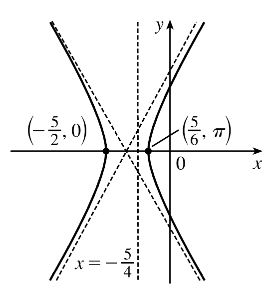
12. $r = \dfrac{5}{2 - 4\cos\theta} \cdot \dfrac{1/2}{1/2} = \dfrac{5/2}{1 - 2\cos\theta}$, where $e = 2$ and $ed = \dfrac{5}{2} \quad\Rightarrow\quad d = \dfrac{5}{4}$.

(a) Eccentricity $= e = 2$

(b) Since $e = 2 > 1$, the conic is a hyperbola.

(c) Since "$-e\cos\theta$" appears in the denominator, the directrix is to the left the focus at the origin. $d = |Fl| = \frac{5}{4}$, so an equation of the directrix is $x = -\frac{5}{4}$.

(d) The vertices are $\left(-\frac{5}{2}, 0\right)$ and $\left(\frac{5}{6}, \pi\right)$, so the center is midway between them, that is, $\left(\frac{5}{3}, \pi\right)$.



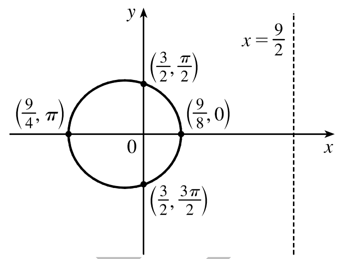
13. $r = \dfrac{9}{6+2\cos\theta} \cdot \dfrac{1/6}{1/6} = \dfrac{3/2}{1+\frac{1}{3}\cos\theta}$, where $e = \frac{1}{3}$ and $ed = \frac{3}{2} \Rightarrow d = \frac{9}{2}$.

(a) Eccentricity $= e = \frac{1}{3}$

(b) Since $e = \frac{1}{3} < 1$, the conic is an ellipse.

(c) Since "$+e\cos\theta$" appears in the denominator, the directrix is to the right of the focus at the origin. $d = |Fl| = \frac{9}{2}$, so an equation of the directrix is $x = \frac{9}{2}$.

(d) The vertices are $\left(\frac{9}{8}, 0\right)$ and $\left(\frac{9}{4}, \pi\right)$, so the center is midway between them, that is, $\left(\frac{9}{16}, \pi\right)$.



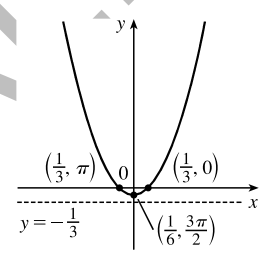
14. $r = \dfrac{1}{3-3\sin\theta} \cdot \dfrac{1/3}{1/3} = \dfrac{1/3}{1-1\sin\theta}$, where $e = 1$ and $ed = \dfrac{1}{3} \Rightarrow d = \dfrac{1}{3}$

(a) Eccentricity $= e = 1$

(b) Since $e = 1$, the conic is a parabola.

(c) Since "$-e\sin\theta$" appears in the denominator, the directrix is below the focus at the origin, $d = |Fl| = \frac{1}{3}$, so an equation of the directrix is $y = -\frac{1}{3}$.

(d) The vertex is at $\left(\frac{1}{6}, \frac{3\pi}{2}\right)$, midway between the focus and the directrix.



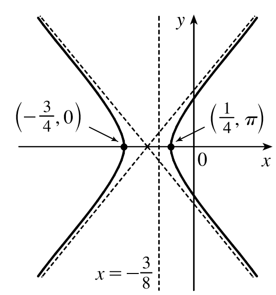
15. $r = \dfrac{3}{4-8\cos\theta} \cdot \dfrac{1/4}{1/4} = \dfrac{3/4}{1-2\cos\theta}$, where $e = 2$ and $ed = \frac{3}{4} \Rightarrow d = \frac{3}{8}$.

(a) Eccentricity $= e = 2$

(b) Since $e = 2 > 1$, the conic is a hyperbola.

(c) Since "$-e\cos\theta$" appears in the denominator, the directrix is to the left of the focus at the origin. $d = |Fl| = \frac{3}{8}$, so an equation of the directrix is $x = -\frac{3}{8}$.

(d) The vertices are $\left(-\frac{3}{4}, 0\right)$ and $\left(\frac{1}{4}, \pi\right)$, so the center is midway between them, that is, $\left(\frac{1}{2}, \pi\right)$.



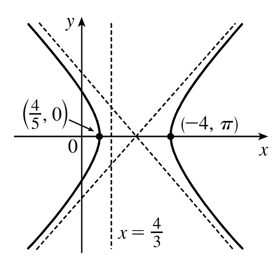
16. $r = \dfrac{4}{2+3\cos\theta} \cdot \dfrac{1/2}{1/2} = \dfrac{2}{1+\frac{3}{2}\cos\theta}$, where $e = \dfrac{3}{2}$ and $ed = 2 \Rightarrow d = \dfrac{4}{3}$.

(a) Eccentricity $= e = \dfrac{3}{2}$

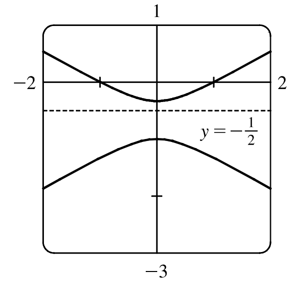
(b) Since $e = \dfrac{3}{2} > 1$, the conic is a hyperbola.

(c) Since "$+e\cos\theta$" appears in the denominator, the directrix is to the right of the focus at the origin. $d = |Fl| = \frac{4}{3}$, so an equation of the directrix is $x = \frac{4}{3}$.

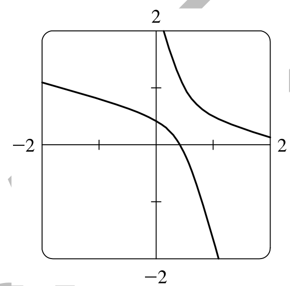
(d) The vertices are $\left(\frac{4}{5}, 0\right)$ and $(-4, \pi)$, so the center is midway between them, that is, $\left(\frac{8}{5}, 0\right)$.



17. (a) $r = \dfrac{1}{1 - 2\sin\theta}$, where $e = 2$ and $ed = 1 \;\Rightarrow\; d = \frac{1}{2}$. The eccentricity $e = 2 > 1$, so the conic is a hyperbola. Since "$-e\sin\theta$" appears in the denominator, the directrix is below the focus at the origin. $d = |Fl| = \frac{1}{2}$, so an equation of the directrix is $y = -\frac{1}{2}$. The vertices are $\left(-1, \frac{\pi}{2}\right)$ and $\left(\frac{1}{3}, \frac{3\pi}{2}\right)$, so the center is midway between them, that is, $\left(\frac{2}{3}, \frac{3\pi}{2}\right)$.



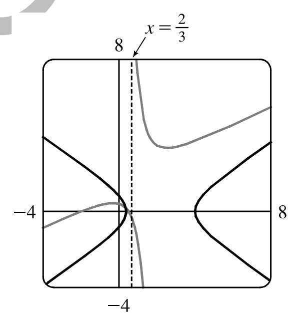
(b) By the discussion that precedes Example 4, the equation is $r = \dfrac{1}{1 - 2\sin\left(\theta - \frac{3\pi}{4}\right)}$.



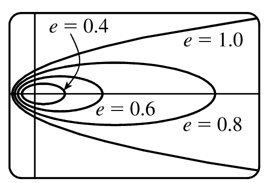
18. $r = \dfrac{4}{5 + 6\cos\theta} = \dfrac{4/5}{1 + \frac{6}{5}\cos\theta}$, so $e = \frac{6}{5}$ and $ed = \frac{4}{5} \;\Rightarrow\; d = \frac{2}{3}$.

An equation of the directrix is $x = \frac{2}{3} \;\Rightarrow\; r\cos\theta = \frac{2}{3} \;\Rightarrow\; r = \dfrac{2}{3\cos\theta}$.

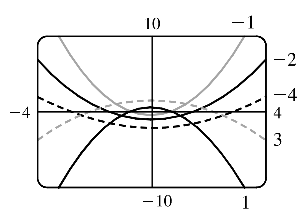
If the hyperbola is rotated about its focus (the origin) through an angle $\pi/3$, its equation is the same as that of the original, with θ replaced by $\theta - \frac{\pi}{3}$ (see Example 4), so $r = \dfrac{4}{5 + 6\cos\left(\theta - \frac{\pi}{3}\right)}$.



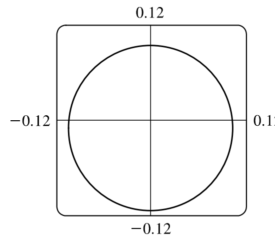
19. For $e < 1$ the curve is an ellipse. It is nearly circular when e is close to 0. As e increases, the graph is stretched out to the right, and grows larger (that is, its right-hand focus moves to the right while its left-hand focus remains at the origin.) At $e = 1$, the curve becomes a parabola with focus at the origin.



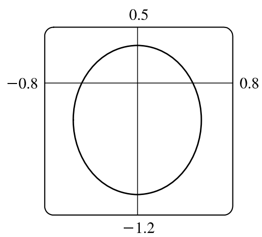
20. (a) The value of d does not seem to affect the shape of the conic (a parabola) at all, just its size, position, and orientation (for $d < 0$ it opens upward, for $d > 0$ it opens downward).



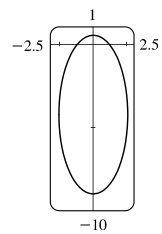
(b) We consider only positive values of e. When $0 < e < 1$, the conic is an ellipse. As $e \to 0^+$, the graph approaches perfect roundness and zero size. As e increases, the ellipse becomes more elongated, until at $e = 1$ it turns into a parabola. For $e > 1$, the conic is a hyperbola, which moves downward and gets broader as e continues to increase.



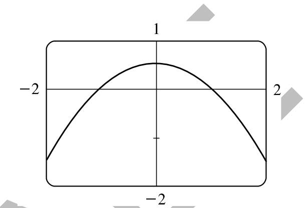
$e = 0.1$



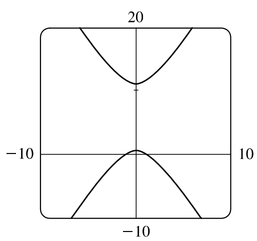
$e = 0.5$



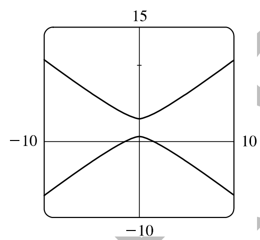
$e = 0.9$



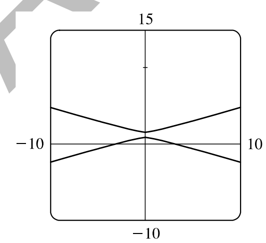
$e = 1$



$e = 1.1$

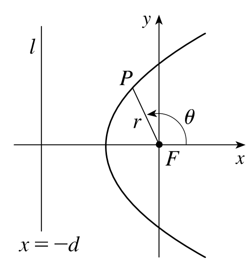


$e = 1.5$

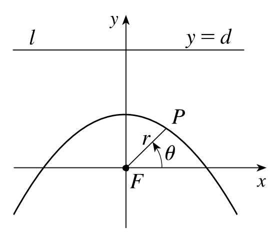


$e = 10$

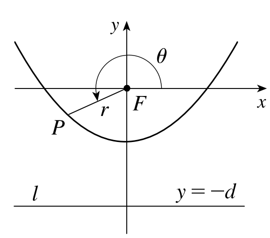
21. $|PF| = e\,|Pl| \;\Rightarrow\; r = e[d - r\cos(\pi - \theta)] = e(d + r\cos\theta) \;\Rightarrow$
$r(1 - e\cos\theta) = ed \;\Rightarrow\; r = \dfrac{ed}{1 - e\cos\theta}$



22. $|PF| = e\,|Pl| \;\Rightarrow\; r = e[d - r\sin\theta] \;\Rightarrow\; r(1 + e\sin\theta) = ed \;\Rightarrow$
$r = \dfrac{ed}{1 + e\sin\theta}$



23. $|PF| = e\,|Pl| \;\Rightarrow\; r = e[d - r\sin(\theta - \pi)] = e(d + r\sin\theta) \;\Rightarrow$
$r(1 - e\sin\theta) = ed \;\Rightarrow\; r = \dfrac{ed}{1 - e\sin\theta}$



24. The parabolas intersect at the two points where $\dfrac{c}{1+\cos\theta} = \dfrac{d}{1-\cos\theta} \quad\Rightarrow\quad \cos\theta = \dfrac{c-d}{c+d} \quad\Rightarrow\quad r = \dfrac{c+d}{2}$.

For the first parabola, $\dfrac{dr}{d\theta} = \dfrac{c\sin\theta}{(1+\cos\theta)^2}$, so

$$\frac{dy}{dx} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \frac{c\sin^2\theta + c\cos\theta(1+\cos\theta)}{c\sin\theta\cos\theta - c\sin\theta(1+\cos\theta)} = \frac{1+\cos\theta}{-\sin\theta}$$

and similarly for the second, $\dfrac{dy}{dx} = \dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta}$. Since the product of these slopes is -1, the parabolas intersect at right angles.

25. We are given $e = 0.093$ and $a = 2.28\times 10^8$. By (7), we have

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{2.28\times 10^8[1-(0.093)^2]}{1+0.093\cos\theta} \approx \frac{2.26\times 10^8}{1+0.093\cos\theta}$$

26. We are given $e = 0.048$ and $2a = 1.56\times 10^9 \quad\Rightarrow\quad a = 7.8\times 10^8$. By (7), we have

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{7.8\times 10^8[1-(0.048)^2]}{1+0.048\cos\theta} \approx \frac{7.78\times 10^8}{1+0.048\cos\theta}$$

27. Here $2a =$ length of major axis $= 36.18$ AU $\quad\Rightarrow\quad a = 18.09$ AU and $e = 0.97$. By (7), the equation of the orbit is $r = \dfrac{18.09[1-(0.97)^2]}{1+0.97\cos\theta} \approx \dfrac{1.07}{1+0.97\cos\theta}$. By (8), the maximum distance from the comet to the sun is $18.09(1+0.97) \approx 35.64$ AU or about 3.314 billion miles.

28. Here $2a =$ length of major axis $= 356.5$ AU $\quad\Rightarrow\quad a = 178.25$ AU and $e = 0.9951$. By (7), the equation of the orbit is $r = \dfrac{178.25[1-(0.9951)^2]}{1+0.9951\cos\theta} \approx \dfrac{1.7426}{1+0.9951\cos\theta}$. By (8), the minimum distance from the comet to the sun is $178.25(1-0.9951) \approx 0.8734$ AU or about 81 million miles.

29. The minimum distance is at perihelion, where $4.6\times 10^7 = r = a(1-e) = a(1-0.206) = a(0.794) \quad\Rightarrow$ $a = 4.6\times 10^7/0.794$. So the maximum distance, which is at aphelion, is $r = a(1+e) = \left(4.6\times 10^7/0.794\right)(1.206) \approx 7.0\times 10^7$ km.

30. At perihelion, $r = a(1-e) = 4.43\times 10^9$, and at aphelion, $r = a(1+e) = 7.37\times 10^9$. Adding, we get $2a = 11.80\times 10^9$, so $a = 5.90\times 10^9$ km. Therefore $1+e = a(1+e)/a = \frac{7.37}{5.90} \approx 1.249$ and $e \approx 0.249$.

31. From Exercise 29, we have $e = 0.206$ and $a(1-e) = 4.6\times 10^7$ km. Thus, $a = 4.6\times 10^7/0.794$. From (7), we can write the equation of Mercury's orbit as $r = a\dfrac{1-e^2}{1+e\cos\theta}$. So since

$$\frac{dr}{d\theta} = \frac{a(1-e^2)e\sin\theta}{(1+e\cos\theta)^2} \quad\Rightarrow$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{a^2(1-e^2)^2}{(1+e\cos\theta)^2} + \frac{a^2(1-e^2)^2\,e^2\sin^2\theta}{(1+e\cos\theta)^4} = \frac{a^2(1-e^2)^2}{(1+e\cos\theta)^4}\,(1+2e\cos\theta+e^2)$$

the length of the orbit is

$$L = \int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = a(1-e^2) \int_0^{2\pi} \frac{\sqrt{1+e^2+2e\cos\theta}}{(1+e\cos\theta)^2}\, d\theta \approx 3.6 \times 10^8 \text{ km}$$

This seems reasonable, since Mercury's orbit is nearly circular, and the circumference of a circle of radius a is $2\pi a \approx 3.6 \times 10^8$ km.

TRUE-FALSE QUIZ

1. False. Consider the curve defined by $x = f(t) = (t-1)^3$ and $y = g(t) = (t-1)^2$. Then $g'(t) = 2(t-1)$, so $g'(1) = 0$, but its graph has a *vertical* tangent when $t = 1$. *Note:* The statement is true if $f'(1) \neq 0$ when $g'(1) = 0$.

2. False. If $x = f(t)$ and $y = g(t)$ are twice differentiable, then $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$.

3. False. For example, if $f(t) = \cos t$ and $g(t) = \sin t$ for $0 \le t \le 4\pi$, then the curve is a circle of radius 1, hence its length is 2π, but $\int_0^{4\pi} \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_0^{4\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_0^{4\pi} 1\, dt = 4\pi$, since as t increases from 0 to 4π, the circle is traversed twice.

4. False. If $(r, \theta) = (1, \pi)$, then $(x, y) = (-1, 0)$, so $\tan^{-1}(y/x) = \tan^{-1} 0 = 0 \neq \theta$. The statement is true for points in quadrants I and IV.

5. True. The curve $r = 1 - \sin 2\theta$ is unchanged if we rotate it through $180°$ about O because $1 - \sin 2(\theta + \pi) = 1 - \sin(2\theta + 2\pi) = 1 - \sin 2\theta$. So it's unchanged if we replace r by $-r$. (See the discussion after Example 8 in Section 10.3.) In other words, it's the same curve as $r = -(1 - \sin 2\theta) = \sin 2\theta - 1$.

6. True. The polar equation $r = 2$, the Cartesian equation $x^2 + y^2 = 4$, and the parametric equations $x = 2\sin 3t$, $y = 2\cos 3t$ $[0 \le t \le 2\pi]$ all describe the circle of radius 2 centered at the origin.

7. False. The first pair of equations gives the portion of the parabola $y = x^2$ with $x \ge 0$, whereas the second pair of equations traces out the whole parabola $y = x^2$.

8. True. $y^2 = 2y + 3x \quad \Leftrightarrow \quad (y-1)^2 = 3x + 1 = 3\left(x + \frac{1}{3}\right) = 4\left(\frac{3}{4}\right)\left(x + \frac{1}{3}\right)$, which is the equation of a parabola with vertex $\left(-\frac{1}{3}, 1\right)$ and focus $\left(-\frac{1}{3} + \frac{3}{4}, 1\right)$, opening to the right.

9. True. By rotating and translating the parabola, we can assume it has an equation of the form $y = cx^2$, where $c > 0$. The tangent at the point $\left(a, ca^2\right)$ is the line $y - ca^2 = 2ca(x - a)$; i.e., $y = 2cax - ca^2$. This tangent meets the parabola at the points $\left(x, cx^2\right)$ where $cx^2 = 2cax - ca^2$. This equation is equivalent to $x^2 = 2ax - a^2$ [since $c > 0$]. But $x^2 = 2ax - a^2 \quad \Leftrightarrow \quad x^2 - 2ax + a^2 = 0 \quad \Leftrightarrow \quad (x-a)^2 = 0 \quad \Leftrightarrow \quad x = a \quad \Leftrightarrow$ $\left(x, cx^2\right) = \left(a, ca^2\right)$. This shows that each tangent meets the parabola at exactly one point.

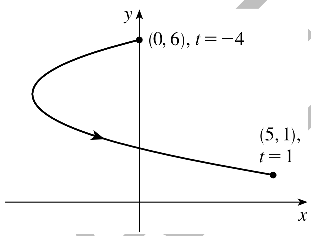
10. True. Consider a hyperbola with focus at the origin, oriented so that its polar equation is $r = \dfrac{ed}{1 + e\cos\theta}$, where $e > 1$.

The directrix is $x = d$, but along the hyperbola we have $x = r\cos\theta = \dfrac{ed\cos\theta}{1 + e\cos\theta} = d\left(\dfrac{e\cos\theta}{1 + e\cos\theta}\right) \neq d$.

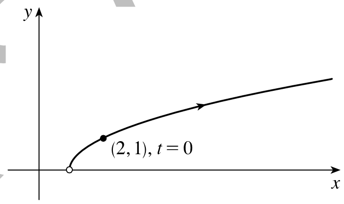
10 Review

EXERCISES

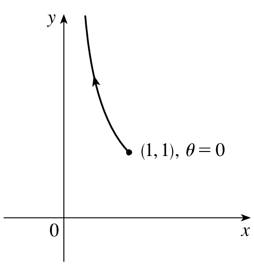
1. $x = t^2 + 4t$, $y = 2 - t$, $-4 \leq t \leq 1$. $t = 2 - y$, so
$x = (2 - y)^2 + 4(2 - y) = 4 - 4y + y^2 + 8 - 4y = y^2 - 8y + 12 \quad \Leftrightarrow$
$x + 4 = y^2 - 8y + 16 = (y - 4)^2$. This is part of a parabola with vertex $(-4, 4)$, opening to the right.



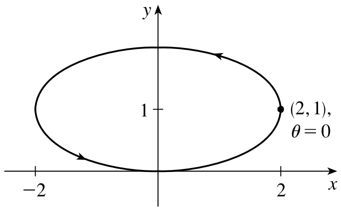
2. $x = 1 + e^{2t}$, $y = e^t$.
$x = 1 + e^{2t} = 1 + (e^t)^2 = 1 + y^2$, $y > 0$.



3. $y = \sec\theta = \dfrac{1}{\cos\theta} = \dfrac{1}{x}$. Since $0 \leq \theta \leq \pi/2$, $0 < x \leq 1$ and $y \geq 1$.
This is part of the hyperbola $y = 1/x$.



4. $x = 2\cos\theta$, $y = 1 + \sin\theta$, $\cos^2\theta + \sin^2\theta = 1 \quad \Rightarrow$
$\left(\dfrac{x}{2}\right)^2 + (y - 1)^2 = 1 \quad \Rightarrow \quad \dfrac{x^2}{4} + (y - 1)^2 = 1$. This is an ellipse, centered at $(0, 1)$, with semimajor axis of length 2 and semiminor axis of length 1.

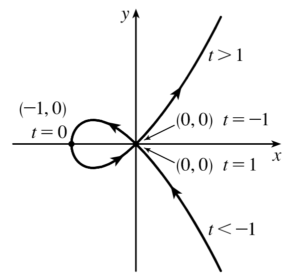


5. Three different sets of parametric equations for the curve $y = \sqrt{x}$ are

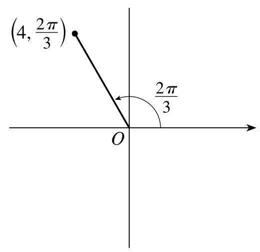
(i) $x = t, \quad y = \sqrt{t}$
(ii) $x = t^4, \quad y = t^2$
(iii) $x = \tan^2 t, \quad y = \tan t, \quad 0 \leq t < \pi/2$

There are many other sets of equations that also give this curve.

6. For $t < -1$, $x > 0$ and $y < 0$ with x decreasing and y increasing. When $t = -1$, $(x, y) = (0, 0)$. When $-1 < t < 0$, we have $-1 < x < 0$ and $0 < y < 1/2$. When $t = 0$, $(x, y) = (-1, 0)$. When $0 < t < 1$, $-1 < x < 0$ and $-\frac{1}{2} < y < 0$. When $t = 1$, $(x, y) = (0, 0)$ again. When $t > 1$, both x and y are positive and increasing.



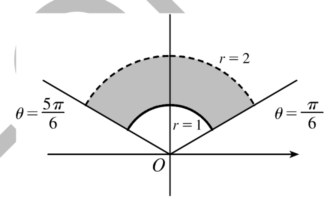
7. (a)



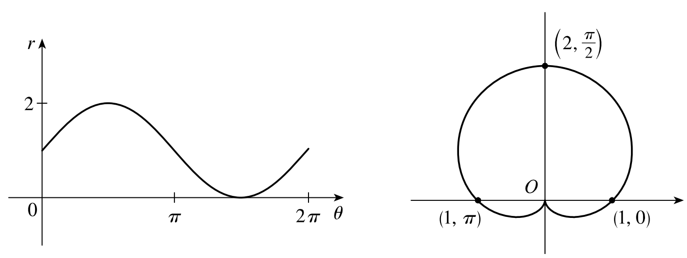
The Cartesian coordinates are $x = 4\cos\frac{2\pi}{3} = 4\left(-\frac{1}{2}\right) = -2$ and $y = 4\sin\frac{2\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$, that is, the point $\left(-2, 2\sqrt{3}\right)$.

(b) Given $x = -3$ and $y = 3$, we have $r = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$. Also, $\tan\theta = \dfrac{y}{x} \;\Rightarrow\; \tan\theta = \dfrac{3}{-3}$, and since $(-3, 3)$ is in the second quadrant, $\theta = \frac{3\pi}{4}$. Thus, one set of polar coordinates for $(-3, 3)$ is $\left(3\sqrt{2}, \frac{3\pi}{4}\right)$, and two others are $\left(3\sqrt{2}, \frac{11\pi}{4}\right)$ and $\left(-3\sqrt{2}, \frac{7\pi}{4}\right)$.

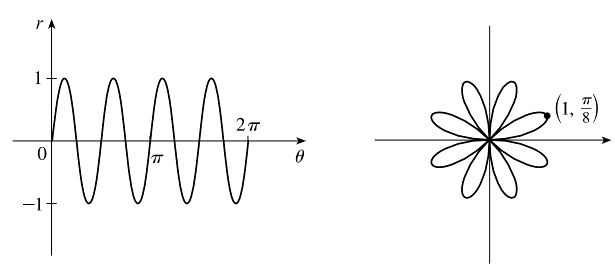
8. $1 \le r < 2$, $\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$



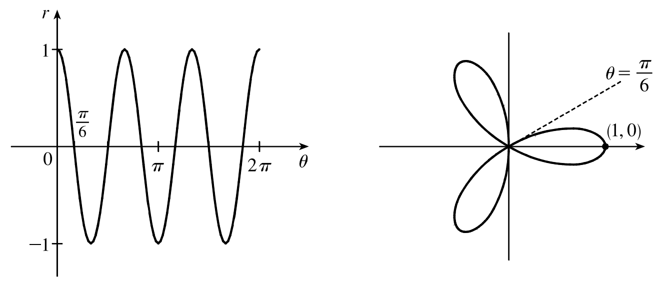
9. $r = 1 + \sin\theta$. This cardioid is symmetric about the $\theta = \pi/2$ axis.



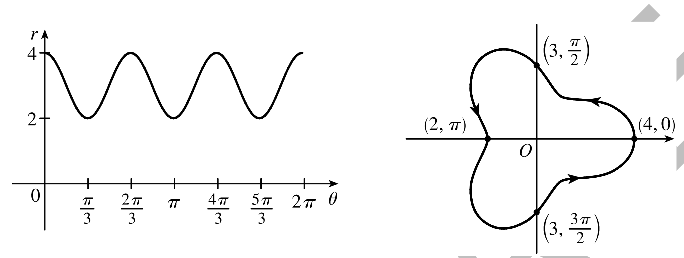
10. $r = \sin 4\theta$. This is an eight-leaved rose.



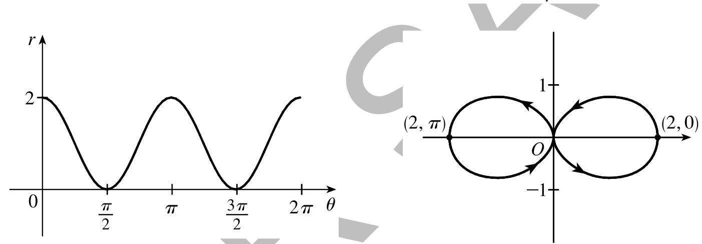
11. $r = \cos 3\theta$. This is a three-leaved rose. The curve is traced twice.



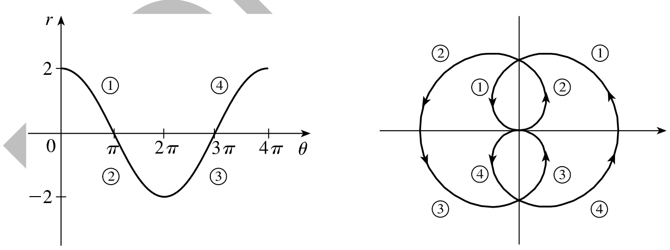
12. $r = 3 + \cos 3\theta$. The curve is symmetric about the horizontal axis.



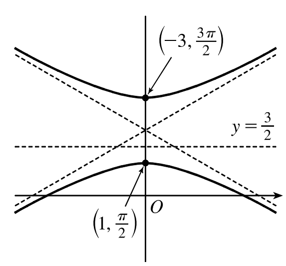
13. $r = 1 + \cos 2\theta$. The curve is symmetric about the pole and both the horizontal and vertical axes.



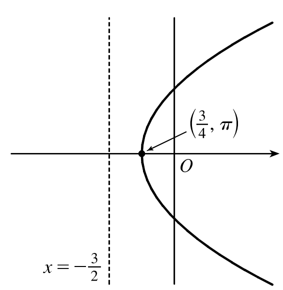
14. $r = 2\cos(\theta/2)$. The curve is symmetric about the pole and both the horizontal and vertical axes.



15. $r = \dfrac{3}{1 + 2\sin\theta} \quad\Rightarrow\quad e = 2 > 1$, so the conic is a hyperbola. $de = 3 \quad\Rightarrow$ $d = \frac{3}{2}$ and the form "$+2\sin\theta$" imply that the directrix is above the focus at the origin and has equation $y = \frac{3}{2}$. The vertices are $\left(1, \frac{\pi}{2}\right)$ and $\left(-3, \frac{3\pi}{2}\right)$.



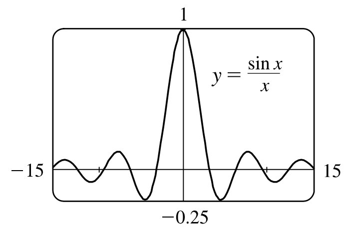
16. $r = \dfrac{3}{2 - 2\cos\theta} \cdot \dfrac{1/2}{1/2} = \dfrac{3/2}{1 - 1\cos\theta} \quad\Rightarrow\quad e = 1$, so the conic is a parabola. $de = \frac{3}{2} \quad\Rightarrow\quad d = \frac{3}{2}$ and the form "$-2\cos\theta$" imply that the directrix is to the left of the focus at the origin and has equation $x = -\frac{3}{2}$. The vertex is $\left(\frac{3}{4}, \pi\right)$.

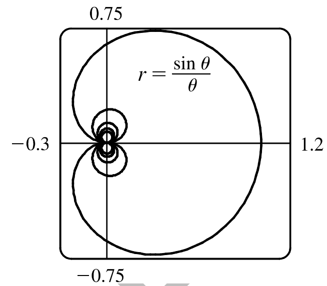


17. $x+y=2 \quad \Leftrightarrow \quad r\cos\theta + r\sin\theta = 2 \quad \Leftrightarrow \quad r(\cos\theta+\sin\theta)=2 \quad \Leftrightarrow \quad r = \dfrac{2}{\cos\theta+\sin\theta}$

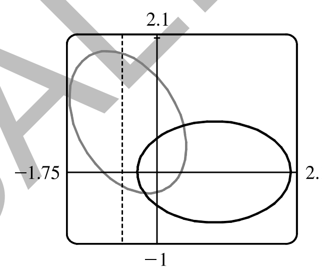
18. $x^2+y^2=2 \quad \Rightarrow \quad r^2=2 \quad \Rightarrow \quad r=\sqrt{2}$. [$r=-\sqrt{2}$ gives the same curve.]

19. $r=(\sin\theta)/\theta$. As $\theta\to\pm\infty$, $r\to 0$. As $\theta\to 0$, $r\to 1$. In the first figure, there are an infinite number of x-intercepts at $x=\pi n$, n a nonzero integer. These correspond to pole points in the second figure.





20. $r=\dfrac{2}{4-3\cos\theta}=\dfrac{1/2}{1-\frac{3}{4}\cos\theta} \quad \Rightarrow \quad e=\frac{3}{4}$ and $d=\frac{2}{3}$. The equation of the directrix is $x=-\frac{2}{3} \quad \Rightarrow \quad r=-2/(3\cos\theta)$. To obtain the equation of the rotated ellipse, we replace θ in the original equation with $\theta-\frac{2\pi}{3}$, and get $r=\dfrac{2}{4-3\cos\left(\theta-\frac{2\pi}{3}\right)}$.



21. $x=\ln t$, $y=1+t^2$; $t=1$. $\dfrac{dy}{dt}=2t$ and $\dfrac{dx}{dt}=\dfrac{1}{t}$, so $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{2t}{1/t}=2t^2$.

When $t=1$, $(x,y)=(0,2)$ and $dy/dx=2$.

22. $x=t^3+6t+1,\ y=2t-t^2;\quad t=-1$. $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{2-2t}{3t^2+6}$. When $t=-1$, $(x,y)=(-6,-3)$ and $\dfrac{dy}{dx}=\dfrac{4}{9}$.

23. $r=e^{-\theta} \quad \Rightarrow \quad y=r\sin\theta=e^{-\theta}\sin\theta$ and $x=r\cos\theta=e^{-\theta}\cos\theta \quad \Rightarrow$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{\frac{dr}{d\theta}\sin\theta+r\cos\theta}{\frac{dr}{d\theta}\cos\theta-r\sin\theta}=\frac{-e^{-\theta}\sin\theta+e^{-\theta}\cos\theta}{-e^{-\theta}\cos\theta-e^{-\theta}\sin\theta}\cdot\frac{-e^{\theta}}{-e^{\theta}}=\frac{\sin\theta-\cos\theta}{\cos\theta+\sin\theta}.$$

When $\theta=\pi$, $\dfrac{dy}{dx}=\dfrac{0-(-1)}{-1+0}=\dfrac{1}{-1}=-1$.

24. $r=3+\cos 3\theta \quad \Rightarrow \quad \dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{\frac{dr}{d\theta}\sin\theta+r\cos\theta}{\frac{dr}{d\theta}\cos\theta-r\sin\theta}=\dfrac{-3\sin 3\theta\sin\theta+(3+\cos 3\theta)\cos\theta}{-3\sin 3\theta\cos\theta-(3+\cos 3\theta)\sin\theta}$.

When $\theta=\pi/2$, $\dfrac{dy}{dx}=\dfrac{(-3)(-1)(1)+(3+0)\cdot 0}{(-3)(-1)(0)-(3+0)\cdot 1}=\dfrac{3}{-3}=-1$.

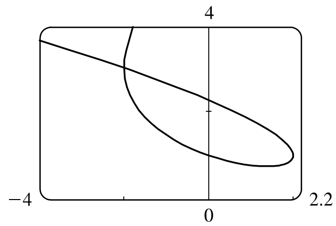
25. $x=t+\sin t,\ y=t-\cos t \quad \Rightarrow \quad \dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{1+\sin t}{1+\cos t} \quad \Rightarrow$

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}=\frac{\dfrac{(1+\cos t)\,\cos t-(1+\sin t)(-\sin t)}{(1+\cos t)^2}}{1+\cos t}=\frac{\cos t+\cos^2 t+\sin t+\sin^2 t}{(1+\cos t)^3}=\frac{1+\cos t+\sin t}{(1+\cos t)^3}$$

26. $x=1+t^2$, $y=t-t^3$. $\dfrac{dy}{dt}=1-3t^2$ and $\dfrac{dx}{dt}=2t$, so $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{1-3t^2}{2t}=\frac{1}{2}t^{-1}-\frac{3}{2}t$.

$$\frac{d^2y}{dx^2}=\frac{d(dy/dx)/dt}{dx/dt}=\frac{-\frac{1}{2}t^{-2}-\frac{3}{2}}{2t}=-\tfrac{1}{4}t^{-3}-\tfrac{3}{4}t^{-1}=-\frac{1}{4t^3}\left(1+3t^2\right)=-\frac{3t^2+1}{4t^3}.$$

27. We graph the curve $x = t^3 - 3t$, $y = t^2 + t + 1$ for $-2.2 \le t \le 1.2$. By zooming in or using a cursor, we find that the lowest point is about $(1.4, 0.75)$. To find the exact values, we find the t-value at which $dy/dt = 2t + 1 = 0 \;\Leftrightarrow\; t = -\frac{1}{2} \;\Leftrightarrow\; (x, y) = \left(\frac{11}{8}, \frac{3}{4}\right)$.



28. We estimate the coordinates of the point of intersection to be $(-2, 3)$. In fact this is exact, since both $t = -2$ and $t = 1$ give the point $(-2, 3)$. So the area enclosed by the loop is

$$\int_{t=-2}^{t=1} y\,dx = \int_{-2}^{1} (t^2 + t + 1)(3t^2 - 3)\,dt = \int_{-2}^{1} (3t^4 + 3t^3 - 3t - 3)\,dt$$
$$= \left[\tfrac{3}{5}t^5 + \tfrac{3}{4}t^4 - \tfrac{3}{2}t^2 - 3t\right]_{-2}^{1} = \left(\tfrac{3}{5} + \tfrac{3}{4} - \tfrac{3}{2} - 3\right) - \left[-\tfrac{96}{5} + 12 - 6 - (-6)\right] = \tfrac{81}{20}$$

29. $x = 2a\cos t - a\cos 2t \;\Rightarrow\; \dfrac{dx}{dt} = -2a\sin t + 2a\sin 2t = 2a\sin t(2\cos t - 1) = 0 \;\Leftrightarrow$

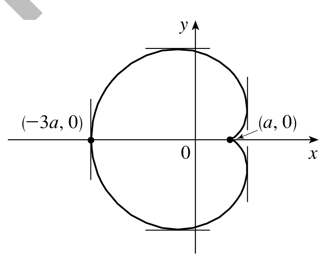
$\sin t = 0$ or $\cos t = \frac{1}{2} \;\Rightarrow\; t = 0, \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3}$.

$y = 2a\sin t - a\sin 2t \;\Rightarrow\; \dfrac{dy}{dt} = 2a\cos t - 2a\cos 2t = 2a\left(1 + \cos t - 2\cos^2 t\right) = 2a(1 - \cos t)(1 + 2\cos t) = 0 \;\Rightarrow$

$t = 0, \frac{2\pi}{3}$, or $\frac{4\pi}{3}$.

Thus the graph has vertical tangents where $t = \frac{\pi}{3}$, π and $\frac{5\pi}{3}$, and horizontal tangents where $t = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$. To determine what the slope is where $t = 0$, we use l'Hospital's Rule to evaluate $\displaystyle\lim_{t\to 0} \frac{dy/dt}{dx/dt} = 0$, so there is a horizontal tangent there.

t	x	y
0	a	0
$\frac{\pi}{3}$	$\frac{3}{2}a$	$\frac{\sqrt{3}}{2}a$
$\frac{2\pi}{3}$	$-\frac{1}{2}a$	$\frac{3\sqrt{3}}{2}a$
π	$-3a$	0
$\frac{4\pi}{3}$	$-\frac{1}{2}a$	$-\frac{3\sqrt{3}}{2}a$
$\frac{5\pi}{3}$	$\frac{3}{2}a$	$-\frac{\sqrt{3}}{2}a$



30. From Exercise 29, $x = 2a\cos t - a\cos 2t$, $y = 2a\sin t - a\sin 2t \;\Rightarrow$

$$A = 2\int_{\pi}^{0} (2a\sin t - a\sin 2t)(-2a\sin t + 2a\sin 2t)\,dt = 4a^2 \int_0^{\pi} (2\sin^2 t + \sin^2 2t - 3\sin t\sin 2t)\,dt$$
$$= 4a^2 \int_0^{\pi} \left[(1 - \cos 2t) + \tfrac{1}{2}(1 - \cos 4t) - 6\sin^2 t\,\cos t\right] dt = 4a^2\left[t - \tfrac{1}{2}\sin 2t + \tfrac{1}{2}t - \tfrac{1}{8}\sin 4t - 2\sin^3 t\right]_0^{\pi}$$
$$= 4a^2\left(\tfrac{3}{2}\right)\pi = 6\pi a^2$$

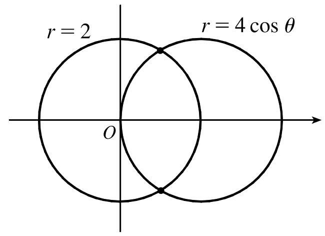
31. The curve $r^2 = 9\cos 5\theta$ has 10 "petals." For instance, for $-\frac{\pi}{10} \le \theta \le \frac{\pi}{10}$, there are two petals, one with $r > 0$ and one with $r < 0$.

$$A = 10\int_{-\pi/10}^{\pi/10} \tfrac{1}{2}r^2\,d\theta = 5\int_{-\pi/10}^{\pi/10} 9\cos 5\theta\,d\theta = 5 \cdot 9 \cdot 2\int_0^{\pi/10} \cos 5\theta\,d\theta = 18\big[\sin 5\theta\big]_0^{\pi/10} = 18$$

32. $r = 1 - 3\sin\theta$. The inner loop is traced out as θ goes from $\alpha = \sin^{-1}\left(\frac{1}{3}\right)$ to $\pi - \alpha$, so

$$A = \int_{\alpha}^{\pi-\alpha} \tfrac{1}{2}r^2\,d\theta = \int_{\alpha}^{\pi/2} (1 - 3\sin\theta)^2\,d\theta = \int_{\alpha}^{\pi/2} \left[1 - 6\sin\theta + \tfrac{9}{2}(1 - \cos 2\theta)\right] d\theta$$
$$= \left[\tfrac{11}{2}\theta + 6\cos\theta - \tfrac{9}{4}\sin 2\theta\right]_{\alpha}^{\pi/2} = \tfrac{11}{4}\pi - \tfrac{11}{2}\sin^{-1}\left(\tfrac{1}{3}\right) - 3\sqrt{2}$$

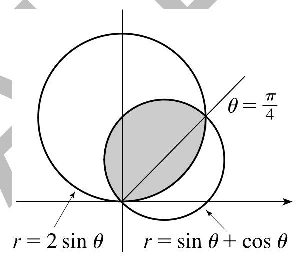
33. The curves intersect when $4\cos\theta = 2 \;\Rightarrow\; \cos\theta = \frac{1}{2} \;\Rightarrow\; \theta = \pm\frac{\pi}{3}$ for $-\pi \le \theta \le \pi$. The points of intersection are $\left(2, \frac{\pi}{3}\right)$ and $\left(2, -\frac{\pi}{3}\right)$.



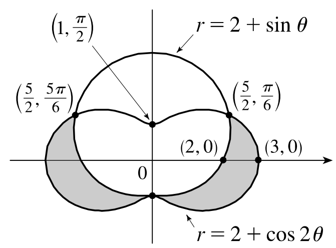
34. The two curves clearly both contain the pole. For other points of intersection, $\cot\theta = 2\cos(\theta + 2n\pi)$ or $-2\cos(\theta + \pi + 2n\pi)$, both of which reduce to $\cot\theta = 2\cos\theta \;\Leftrightarrow\; \cos\theta = 2\sin\theta\cos\theta \;\Leftrightarrow\; \cos\theta(1 - 2\sin\theta) = 0 \;\Rightarrow\; \cos\theta = 0$ or $\sin\theta = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ or $\frac{3\pi}{2} \;\Rightarrow$ intersection points are $\left(0, \frac{\pi}{2}\right)$, $\left(\sqrt{3}, \frac{\pi}{6}\right)$, and $\left(\sqrt{3}, \frac{11\pi}{6}\right)$.

35. The curves intersect where $2\sin\theta = \sin\theta + \cos\theta \;\Rightarrow\; \sin\theta = \cos\theta \;\Rightarrow \theta = \frac{\pi}{4}$, and also at the origin (at which $\theta = \frac{3\pi}{4}$ on the second curve).

$$\begin{aligned} A &= \int_0^{\pi/4} \tfrac{1}{2}(2\sin\theta)^2\,d\theta + \int_{\pi/4}^{3\pi/4} \tfrac{1}{2}(\sin\theta + \cos\theta)^2\,d\theta \\ &= \int_0^{\pi/4} (1 - \cos 2\theta)\,d\theta + \tfrac{1}{2}\int_{\pi/4}^{3\pi/4} (1 + \sin 2\theta)\,d\theta \\ &= \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} + \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\cos 2\theta\right]_{\pi/4}^{3\pi/4} = \tfrac{1}{2}(\pi - 1) \end{aligned}$$



36.
$$\begin{aligned} A &= 2\int_{-\pi/2}^{\pi/6} \tfrac{1}{2}\left[(2 + \cos 2\theta)^2 - (2 + \sin\theta)^2\right] d\theta \\ &= \int_{-\pi/2}^{\pi/6} \left[4\cos 2\theta + \cos^2 2\theta - 4\sin\theta - \sin^2\theta\right] d\theta \\ &= \left[2\sin 2\theta + \tfrac{1}{2}\theta + \tfrac{1}{8}\sin 4\theta + 4\cos\theta - \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/6} \\ &= \tfrac{51}{16}\sqrt{3} \end{aligned}$$



37. $x = 3t^2,\; y = 2t^3$.

$$\begin{aligned} L &= \int_0^2 \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^2 \sqrt{(6t)^2 + (6t^2)^2}\,dt = \int_0^2 \sqrt{36t^2 + 36t^4}\,dt = \int_0^2 \sqrt{36t^2}\sqrt{1 + t^2}\,dt \\ &= \int_0^2 6\,|t|\sqrt{1 + t^2}\,dt = 6\int_0^2 t\sqrt{1 + t^2}\,dt = 6\int_1^5 u^{1/2}\left(\tfrac{1}{2}du\right) \qquad \left[u = 1 + t^2,\, du = 2t\,dt\right] \\ &= 6 \cdot \tfrac{1}{2} \cdot \tfrac{2}{3}\left[u^{3/2}\right]_1^5 = 2(5^{3/2} - 1) = 2\left(5\sqrt{5} - 1\right) \end{aligned}$$

38. $x = 2 + 3t,\quad y = \cosh 3t \;\Rightarrow\; (dx/dt)^2 + (dy/dt)^2 = 3^2 + (3\sinh 3t)^2 = 9(1 + \sinh^2 3t) = 9\cosh^2 3t$, so $L = \int_0^1 \sqrt{9\cosh^2 3t}\,dt = \int_0^1 |3\cosh 3t|\,dt = \int_0^1 3\cosh 3t\,dt = \left[\sinh 3t\right]_0^1 = \sinh 3 - \sinh 0 = \sinh 3$.

39.
$$\begin{aligned} L &= \int_\pi^{2\pi} \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_\pi^{2\pi} \sqrt{(1/\theta)^2 + (-1/\theta^2)^2}\,d\theta = \int_\pi^{2\pi} \frac{\sqrt{\theta^2 + 1}}{\theta^2}\,d\theta \\ &\overset{24}{=} \left[-\frac{\sqrt{\theta^2 + 1}}{\theta} + \ln\left(\theta + \sqrt{\theta^2 + 1}\right)\right]_\pi^{2\pi} = \frac{\sqrt{\pi^2 + 1}}{\pi} - \frac{\sqrt{4\pi^2 + 1}}{2\pi} + \ln\left(\frac{2\pi + \sqrt{4\pi^2 + 1}}{\pi + \sqrt{\pi^2 + 1}}\right) \\ &= \frac{2\sqrt{\pi^2 + 1} - \sqrt{4\pi^2 + 1}}{2\pi} + \ln\left(\frac{2\pi + \sqrt{4\pi^2 + 1}}{\pi + \sqrt{\pi^2 + 1}}\right) \end{aligned}$$

40. $L = \int_0^\pi \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^\pi \sqrt{\sin^6\left(\frac{1}{3}\theta\right) + \sin^4\left(\frac{1}{3}\theta\right)\cos^2\left(\frac{1}{3}\theta\right)}\,d\theta$

$= \int_0^\pi \sin^2\left(\frac{1}{3}\theta\right)d\theta = \left[\frac{1}{2}\left(\theta - \frac{3}{2}\sin\left(\frac{2}{3}\theta\right)\right)\right]_0^\pi = \frac{1}{2}\pi - \frac{3}{8}\sqrt{3}$

41. $x = 4\sqrt{t},\ y = \dfrac{t^3}{3} + \dfrac{1}{2t^2},\ 1 \le t \le 4 \quad\Rightarrow$

$S = \int_1^4 2\pi y\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_1^4 2\pi\left(\frac{1}{3}t^3 + \frac{1}{2}t^{-2}\right)\sqrt{\left(2/\sqrt{t}\right)^2 + (t^2 - t^{-3})^2}\,dt$

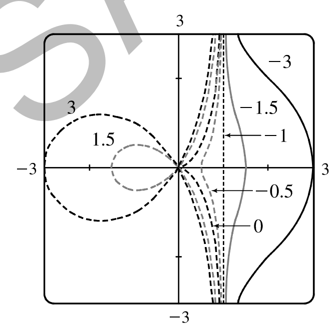
$= 2\pi\int_1^4 \left(\frac{1}{3}t^3 + \frac{1}{2}t^{-2}\right)\sqrt{(t^2 + t^{-3})^2}\,dt = 2\pi\int_1^4 \left(\frac{1}{3}t^5 + \frac{5}{6} + \frac{1}{2}t^{-5}\right)dt = 2\pi\left[\frac{1}{18}t^6 + \frac{5}{6}t - \frac{1}{8}t^{-4}\right]_1^4 = \frac{471{,}295}{1024}\pi$

42. $x = 2 + 3t, \quad y = \cosh 3t \quad\Rightarrow\quad (dx/dt)^2 + (dy/dt)^2 = 3^2 + (3\sinh 3t)^2 = 9(1 + \sinh^2 3t) = 9\cosh^2 3t$, so

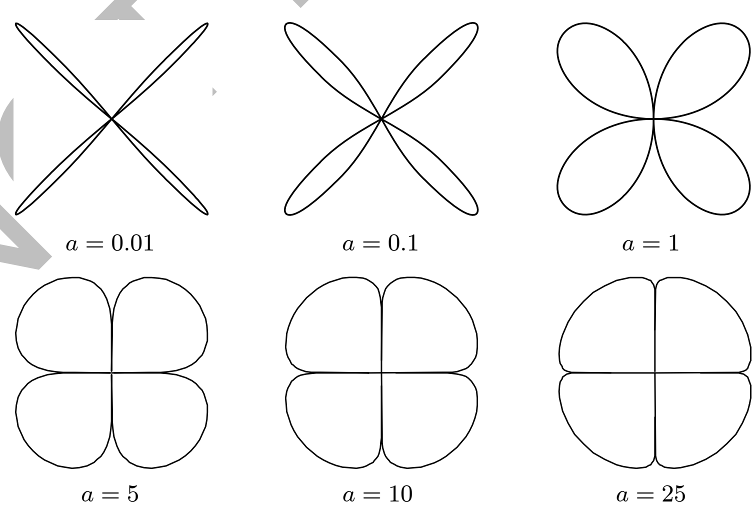
$S = \int_0^1 2\pi y\,ds = \int_0^1 2\pi\cosh 3t\sqrt{9\cosh^2 3t}\,dt = \int_0^1 2\pi\cosh 3t\,|3\cosh 3t|\,dt = \int_0^1 2\pi\cosh 3t\cdot 3\cosh 3t\,dt$

$= 6\pi\int_0^1 \cosh^2 3t\,dt = 6\pi\int_0^1 \frac{1}{2}(1 + \cosh 6t)\,dt = 3\pi\left[t + \frac{1}{6}\sinh 6t\right]_0^1 = 3\pi\left(1 + \frac{1}{6}\sinh 6\right) = 3\pi + \frac{\pi}{2}\sinh 6$

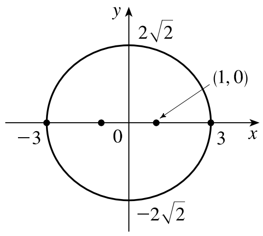
43. For all c except -1, the curve is asymptotic to the line $x = 1$. For $c < -1$, the curve bulges to the right near $y = 0$. As c increases, the bulge becomes smaller, until at $c = -1$ the curve is the straight line $x = 1$. As c continues to increase, the curve bulges to the left, until at $c = 0$ there is a cusp at the origin. For $c > 0$, there is a loop to the left of the origin, whose size and roundness increase as c increases. Note that the x-intercept of the curve is always $-c$.



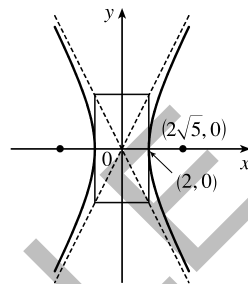
44. For a close to 0, the graph of $r^a = |\sin 2\theta|$ consists of four thin petals. As a increases, the petals get wider, until as $a \to \infty$, each petal occupies almost its entire quarter-circle.



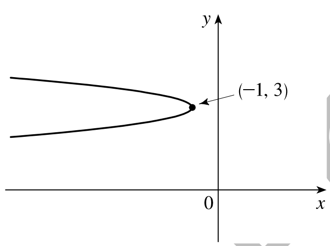
45. $\dfrac{x^2}{9}+\dfrac{y^2}{8}=1$ is an ellipse with center $(0,0)$.
$a=3$, $b=2\sqrt{2}$, $c=1 \quad\Rightarrow$
foci $(\pm1,0)$, vertices $(\pm3,0)$.



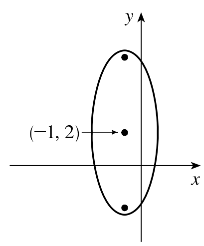
46. $4x^2-y^2=16 \quad\Leftrightarrow\quad \dfrac{x^2}{4}-\dfrac{y^2}{16}=1$ is a hyperbola with center $(0,0)$, vertices $(\pm2,0)$, $a=2$, $b=4$, $c=\sqrt{16+4}=2\sqrt{5}$, foci $\left(\pm2\sqrt{5},0\right)$ and asymptotes $y=\pm2x$.



47. $6y^2+x-36y+55=0 \quad\Leftrightarrow$
$6(y^2-6y+9)=-(x+1) \quad\Leftrightarrow$
$(y-3)^2=-\frac{1}{6}(x+1)$, a parabola with vertex $(-1,3)$, opening to the left, $p=-\frac{1}{24} \quad\Rightarrow\quad$ focus $\left(-\frac{25}{24},3\right)$ and directrix $x=-\frac{23}{24}$.



48. $25x^2+4y^2+50x-16y=59 \quad\Leftrightarrow$
$25(x+1)^2+4(y-2)^2=100 \quad\Leftrightarrow$
$\frac{1}{4}(x+1)^2+\frac{1}{25}(y-2)^2=1$ is an ellipse centered at $(-1,2)$ with foci on the line $x=-1$, vertices $(-1,7)$ and $(-1,-3)$; $a=5$, $b=2 \quad\Rightarrow\quad c=\sqrt{21} \quad\Rightarrow$
foci $\left(-1,2\pm\sqrt{21}\right)$.



49. The ellipse with foci $(\pm4,0)$ and vertices $(\pm5,0)$ has center $(0,0)$ and a horizontal major axis, with $a=5$ and $c=4$, so $b^2=a^2-c^2=5^2-4^2=9$. An equation is $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$.

50. The distance from the focus $(2,1)$ to the directrix $x=-4$ is $2-(-4)=6$, so the distance from the focus to the vertex is $\frac{1}{2}(6)=3$ and the vertex is $(-1,1)$. Since the focus is to the right of the vertex, $p=3$. An equation is $(y-1)^2=4\cdot3[x-(-1)]$, or $(y-1)^2=12(x+1)$.

51. The center of a hyperbola with foci $(0,\pm4)$ is $(0,0)$, so $c=4$ and an equation is $\dfrac{y^2}{a^2}-\dfrac{x^2}{b^2}=1$.
The asymptote $y=3x$ has slope 3, so $\dfrac{a}{b}=\dfrac{3}{1} \quad\Rightarrow\quad a=3b$ and $a^2+b^2=c^2 \quad\Rightarrow\quad (3b)^2+b^2=4^2 \quad\Rightarrow$
$10b^2=16 \quad\Rightarrow\quad b^2=\frac{8}{5}$ and so $a^2=16-\frac{8}{5}=\frac{72}{5}$. Thus, an equation is $\dfrac{y^2}{72/5}-\dfrac{x^2}{8/5}=1$, or $\dfrac{5y^2}{72}-\dfrac{5x^2}{8}=1$.

52. Center is $(3,0)$, and $a=\frac{8}{2}=4$, $c=2 \quad\Leftrightarrow\quad b=\sqrt{4^2-2^2}=\sqrt{12} \quad\Rightarrow$
an equation of the ellipse is $\dfrac{(x-3)^2}{12}+\dfrac{y^2}{16}=1$.

53. $x^2 + y = 100 \quad \Leftrightarrow \quad x^2 = -(y - 100)$ has its vertex at $(0, 100)$, so one of the vertices of the ellipse is $(0, 100)$. Another form of the equation of a parabola is $x^2 = 4p(y - 100)$ so $4p(y - 100) = -(y - 100) \quad \Rightarrow \quad 4p = -1 \quad \Rightarrow \quad p = -\frac{1}{4}$. Therefore the shared focus is found at $\left(0, \frac{399}{4}\right)$ so $2c = \frac{399}{4} - 0 \quad \Rightarrow \quad c = \frac{399}{8}$ and the center of the ellipse is $\left(0, \frac{399}{8}\right)$. So $a = 100 - \frac{399}{8} = \frac{401}{8}$ and $b^2 = a^2 - c^2 = \dfrac{401^2 - 399^2}{8^2} = 25$. So the equation of the ellipse is $\dfrac{x^2}{b^2} + \dfrac{\left(y - \frac{399}{8}\right)^2}{a^2} = 1 \quad \Rightarrow$

$\dfrac{x^2}{25} + \dfrac{\left(y - \frac{399}{8}\right)^2}{\left(\frac{401}{8}\right)^2} = 1$, or $\dfrac{x^2}{25} + \dfrac{(8y - 399)^2}{160{,}801} = 1$.

54. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{b^2}{a^2}\dfrac{x}{y}$. Therefore $\dfrac{dy}{dx} = m \quad \Leftrightarrow \quad y = -\dfrac{b^2}{a^2}\dfrac{x}{m}$. Combining this condition with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we find that $x = \pm\dfrac{a^2 m}{\sqrt{a^2 m^2 + b^2}}$. In other words, the two points on the ellipse where the tangent has slope m are $\left(\pm\dfrac{a^2 m}{\sqrt{a^2 m^2 + b^2}}, \mp\dfrac{b^2}{\sqrt{a^2 m^2 + b^2}}\right)$. The tangent lines at these points have the equations

$y \pm \dfrac{b^2}{\sqrt{a^2 m^2 + b^2}} = m\left(x \mp \dfrac{a^2 m}{\sqrt{a^2 m^2 + b^2}}\right)$ or $y = mx \mp \dfrac{a^2 m^2}{\sqrt{a^2 m^2 + b^2}} \mp \dfrac{b^2}{\sqrt{a^2 m^2 + b^2}} = mx \mp \sqrt{a^2 m^2 + b^2}$.

55. Directrix $x = 4 \quad \Rightarrow \quad d = 4$, so $e = \frac{1}{3} \quad \Rightarrow \quad r = \dfrac{ed}{1 + e\cos\theta} = \dfrac{4}{3 + \cos\theta}$.

56. See the end of the proof of Theorem 10.6.1. If $e > 1$, then $1 - e^2 < 0$ and Equations 10.6.4 become $a^2 = \dfrac{e^2 d^2}{(e^2 - 1)^2}$ and $b^2 = \dfrac{e^2 d^2}{e^2 - 1}$, so $\dfrac{b^2}{a^2} = e^2 - 1$. The asymptotes $y = \pm\dfrac{b}{a}x$ have slopes $\pm\dfrac{b}{a} = \pm\sqrt{e^2 - 1}$, so the angles they make with the polar axis are $\pm\tan^{-1}\left[\sqrt{e^2 - 1}\right] = \cos^{-1}(\pm 1/e)$.

57. In polar coordinates, an equation for the circle is $r = 2a\sin\theta$. Thus, the coordinates of Q are $x = r\cos\theta = 2a\sin\theta\,\cos\theta$ and $y = r\sin\theta = 2a\sin^2\theta$. The coordinates of R are $x = 2a\cot\theta$ and $y = 2a$. Since P is the midpoint of QR, we use the midpoint formula to get $x = a(\sin\theta\,\cos\theta + \cot\theta)$ and $y = a(1 + \sin^2\theta)$.

58. (a) If (a, b) lies on the curve, then there is some parameter value t_1 such that $\dfrac{3t_1}{1 + t_1^3} = a$ and $\dfrac{3t_1^2}{1 + t_1^3} = b$. If $t_1 = 0$, the point is $(0, 0)$, which lies on the line $y = x$. If $t_1 \neq 0$, then the point corresponding to $t = \dfrac{1}{t_1}$ is given by $x = \dfrac{3(1/t_1)}{1 + (1/t_1)^3} = \dfrac{3t_1^2}{t_1^3 + 1} = b$, $y = \dfrac{3(1/t_1)^2}{1 + (1/t_1)^3} = \dfrac{3t_1}{t_1^3 + 1} = a$. So (b, a) also lies on the curve. [Another way to see this is to do part (e) first; the result is immediate.] The curve intersects the line $y = x$ when $\dfrac{3t}{1 + t^3} = \dfrac{3t^2}{1 + t^3} \quad \Rightarrow$ $t = t^2 \quad \Rightarrow \quad t = 0$ or 1, so the points are $(0, 0)$ and $\left(\frac{3}{2}, \frac{3}{2}\right)$.

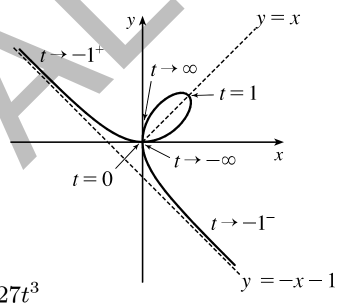
(b) $\dfrac{dy}{dt} = \dfrac{(1+t^3)(6t) - 3t^2(3t^2)}{(1+t^3)^2} = \dfrac{6t - 3t^4}{(1+t^3)^2} = 0$ when $6t - 3t^4 = 3t(2-t^3) = 0 \quad \Rightarrow \quad t = 0$ or $t = \sqrt[3]{2}$, so there are horizontal tangents at $(0,0)$ and $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using the symmetry from part (a), we see that there are vertical tangents at $(0,0)$ and $\left(\sqrt[3]{4}, \sqrt[3]{2}\right)$.

(c) Notice that as $t \to -1^+$, we have $x \to -\infty$ and $y \to \infty$. As $t \to -1^-$, we have $x \to \infty$ and $y \to -\infty$. Also $y - (-x-1) = y + x + 1 = \dfrac{3t + 3t^2 + (1+t^3)}{1+t^3} = \dfrac{(t+1)^3}{1+t^3} = \dfrac{(t+1)^2}{t^2 - t + 1} \to 0$ as $t \to -1$. So $y = -x - 1$ is a slant asymptote.

(d) $\dfrac{dx}{dt} = \dfrac{(1+t^3)(3) - 3t(3t^2)}{(1+t^3)^2} = \dfrac{3 - 6t^3}{(1+t^3)^2}$ and from part (b) we have $\dfrac{dy}{dt} = \dfrac{6t - 3t^4}{(1+t^3)^2}$. So $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t(2-t^3)}{1-2t^3}$.

Also $\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \dfrac{2(1+t^3)^4}{3(1-2t^3)^3} > 0 \quad \Leftrightarrow \quad t < \dfrac{1}{\sqrt[3]{2}}$.

So the curve is concave upward there and has a minimum point at $(0,0)$ and a maximum point at $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using this together with the information from parts (a), (b), and (c), we sketch the curve.



(e) $x^3 + y^3 = \left(\dfrac{3t}{1+t^3}\right)^3 + \left(\dfrac{3t^2}{1+t^3}\right)^3 = \dfrac{27t^3 + 27t^6}{(1+t^3)^3} = \dfrac{27t^3(1+t^3)}{(1+t^3)^3} = \dfrac{27t^3}{(1+t^3)^2}$

and $3xy = 3\left(\dfrac{3t}{1+t^3}\right)\left(\dfrac{3t^2}{1+t^3}\right) = \dfrac{27t^3}{(1+t^3)^2}$, so $x^3 + y^3 = 3xy$.

(f) We start with the equation from part (e) and substitute $x = r\cos\theta$, $y = r\sin\theta$. Then $x^3 + y^3 = 3xy \quad \Rightarrow$ $r^3\cos^3\theta + r^3\sin^3\theta = 3r^2\cos\theta\,\sin\theta$. For $r \neq 0$, this gives $r = \dfrac{3\cos\theta\,\sin\theta}{\cos^3\theta + \sin^3\theta}$. Dividing numerator and denominator by $\cos^3\theta$, we obtain $r = \dfrac{3\left(\dfrac{1}{\cos\theta}\right)\dfrac{\sin\theta}{\cos\theta}}{1 + \dfrac{\sin^3\theta}{\cos^3\theta}} = \dfrac{3\sec\theta\,\tan\theta}{1+\tan^3\theta}$.

(g) The loop corresponds to $\theta \in \left(0, \frac{\pi}{2}\right)$, so its area is

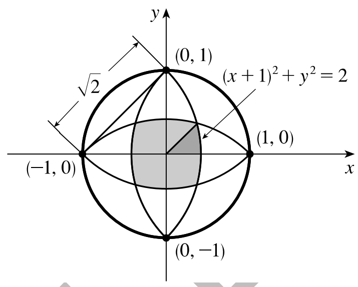
$$A = \int_0^{\pi/2} \frac{r^2}{2}\,d\theta = \frac{1}{2}\int_0^{\pi/2} \left(\frac{3\sec\theta\,\tan\theta}{1+\tan^3\theta}\right)^2 d\theta = \frac{9}{2}\int_0^{\pi/2} \frac{\sec^2\theta\,\tan^2\theta}{(1+\tan^3\theta)^2}\,d\theta = \frac{9}{2}\int_0^{\infty} \frac{u^2\,du}{(1+u^3)^2} \quad [\text{let } u = \tan\theta]$$

$$= \lim_{b\to\infty} \tfrac{9}{2}\left[-\tfrac{1}{3}(1+u^3)^{-1}\right]_0^b = \tfrac{3}{2}$$

(h) By symmetry, the area between the folium and the line $y = -x - 1$ is equal to the enclosed area in the third quadrant, plus twice the enclosed area in the fourth quadrant. The area in the third quadrant is $\frac{1}{2}$, and since $y = -x - 1 \quad \Rightarrow$ $r\sin\theta = -r\cos\theta - 1 \quad \Rightarrow \quad r = -\dfrac{1}{\sin\theta + \cos\theta}$, the area in the fourth quadrant is

$$\frac{1}{2}\int_{-\pi/2}^{-\pi/4} \left[\left(-\frac{1}{\sin\theta + \cos\theta}\right)^2 - \left(\frac{3\sec\theta\,\tan\theta}{1+\tan^3\theta}\right)^2\right] d\theta \overset{\text{CAS}}{=} \frac{1}{2}.$$ Therefore, the total area is $\frac{1}{2} + 2\left(\frac{1}{2}\right) = \frac{3}{2}$.

PROBLEMS PLUS



1. See the figure. The circle with center $(-1, 0)$ and radius $\sqrt{2}$ has equation $(x+1)^2 + y^2 = 2$ and describes the circular arc from $(0, -1)$ to $(0, 1)$. Converting the equation to polar coordinates gives us

$(r\cos\theta + 1)^2 + (r\sin\theta)^2 = 2 \quad \Rightarrow$

$r^2\cos^2\theta + 2r\cos\theta + 1 + r^2\sin^2\theta = 2 \quad \Rightarrow$

$r^2(\cos^2\theta + \sin^2\theta) + 2r\cos\theta = 1 \quad \Rightarrow \quad r^2 + 2r\cos\theta = 1$. Using the quadratic formula to solve for r gives us

$$r = \frac{-2\cos\theta \pm \sqrt{4\cos^2\theta + 4}}{2} = -\cos\theta + \sqrt{\cos^2\theta + 1} \text{ for } r > 0.$$

The darkest shaded region is $\frac{1}{8}$ of the entire shaded region A, so $\frac{1}{8}A = \int_0^{\pi/4} \frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_0^{\pi/4}(1 - 2r\cos\theta)\,d\theta \quad \Rightarrow$

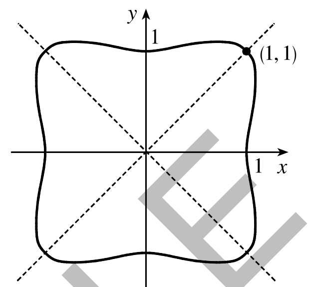
$$\begin{aligned}
\tfrac{1}{4}A &= \int_0^{\pi/4}\left[1 - 2\cos\theta\left(-\cos\theta + \sqrt{\cos^2\theta + 1}\right)\right]d\theta = \int_0^{\pi/4}\left(1 + 2\cos^2\theta - 2\cos\theta\sqrt{\cos^2\theta + 1}\right)d\theta \\
&= \int_0^{\pi/4}\left[1 + 2\cdot\tfrac{1}{2}(1 + \cos 2\theta) - 2\cos\theta\sqrt{(1 - \sin^2\theta) + 1}\right]d\theta \\
&= \int_0^{\pi/4}(2 + \cos 2\theta)\,d\theta - 2\int_0^{\pi/4}\cos\theta\sqrt{2 - \sin^2\theta}\,d\theta \\
&= \left[2\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} - 2\int_0^{1/\sqrt{2}}\sqrt{2 - u^2}\,du \qquad \left[\begin{matrix} u = \sin\theta, \\ du = \cos\theta\,d\theta \end{matrix}\right] \\
&= \left(\frac{\pi}{2} + \frac{1}{2}\right) - (0 + 0) - 2\left[\frac{u}{2}\sqrt{2 - u^2} + \sin^{-1}\frac{u}{\sqrt{2}}\right]_0^{1/\sqrt{2}} \qquad \left[\begin{matrix}\text{Formula 30,} \\ a = \sqrt{2}\end{matrix}\right] \\
&= \frac{\pi}{2} + \frac{1}{2} - 2\left(\frac{1}{2\sqrt{2}}\cdot\frac{\sqrt{3}}{\sqrt{2}} + \frac{\pi}{6}\right) = \frac{\pi}{2} + \frac{1}{2} - \frac{1}{2}\sqrt{3} - \frac{\pi}{3} = \frac{\pi}{6} + \frac{1}{2} - \frac{1}{2}\sqrt{3}.
\end{aligned}$$

Thus, $A = 4\left(\frac{\pi}{6} + \frac{1}{2} - \frac{1}{2}\sqrt{3}\right) = \frac{2\pi}{3} + 2 - 2\sqrt{3}$.

2. (a) The curve $x^4 + y^4 = x^2 + y^2$ is symmetric about both axes and about the line $y = x$ (since interchanging x and y does not change the equation) so we need only consider $y \geq x \geq 0$ to begin with. Implicit differentiation gives $4x^3 + 4y^3y' = 2x + 2yy' \quad \Rightarrow \quad y' = \dfrac{x(1 - 2x^2)}{y(2y^2 - 1)} \quad \Rightarrow \quad y' = 0$ when $x = 0$ and when $x = \pm\frac{1}{\sqrt{2}}$. If $x = 0$, then $y^4 = y^2 \quad \Rightarrow \quad y^2(y^2 - 1) = 0 \quad \Rightarrow \quad y = 0$ or ± 1. The point $(0, 0)$ can't be a highest or lowest point because it is isolated. [If $-1 < x < 1$ and $-1 < y < 1$, then $x^4 < x^2$ and $y^4 < y^2 \quad \Rightarrow \quad x^4 + y^4 < x^2 + y^2$, except for $(0, 0)$.] If $x = \frac{1}{\sqrt{2}}$, then $x^2 = \frac{1}{2}$, $x^4 = \frac{1}{4}$, so $\frac{1}{4} + y^4 = \frac{1}{2} + y^2 \quad \Rightarrow \quad 4y^4 - 4y^2 - 1 = 0 \quad \Rightarrow \quad y^2 = \frac{4 \pm \sqrt{16 + 16}}{8} = \frac{1 \pm \sqrt{2}}{2}$. But $y^2 > 0$, so $y^2 = \frac{1 + \sqrt{2}}{2} \quad \Rightarrow \quad y = \pm\sqrt{\frac{1}{2}\left(1 + \sqrt{2}\right)}$. Near the point $(0, 1)$, the denominator of y' is positive and the numerator changes from negative to positive as x increases through 0, so $(0, 1)$ is a local minimum point. At

$\left(\frac{1}{\sqrt{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right)$, y' changes from positive to negative, so that point gives a maximum. By symmetry, the highest points on the curve are $\left(\pm\frac{1}{\sqrt{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right)$ and the lowest points are $\left(\pm\frac{1}{\sqrt{2}}, -\sqrt{\frac{1+\sqrt{2}}{2}}\right)$.

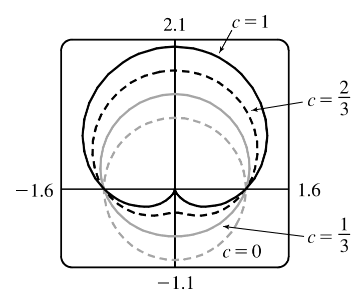
(b) We use the information from part (a), together with symmetry with respect to the axes and the lines $y = \pm x$, to sketch the curve.



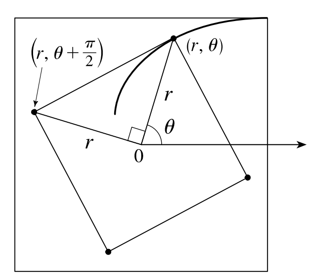
(c) In polar coordinates, $x^4 + y^4 = x^2 + y^2$ becomes $r^4\cos^4\theta + r^4\sin^4\theta = r^2$ or $r^2 = \dfrac{1}{\cos^4\theta + \sin^4\theta}$. By the symmetry shown in part (b), the area enclosed by the curve is $A = 8\displaystyle\int_0^{\pi/4} \frac{1}{2}r^2\,d\theta = 4\int_0^{\pi/4} \frac{d\theta}{\cos^4\theta + \sin^4\theta} \overset{\text{CAS}}{=} \sqrt{2}\,\pi$.

3. In terms of x and y, we have $x = r\cos\theta = (1 + c\sin\theta)\cos\theta = \cos\theta + c\sin\theta\cos\theta = \cos\theta + \frac{1}{2}c\sin 2\theta$ and $y = r\sin\theta = (1 + c\sin\theta)\sin\theta = \sin\theta + c\sin^2\theta$. Now $-1 \le \sin\theta \le 1 \;\Rightarrow\; -1 \le \sin\theta + c\sin^2\theta \le 1 + c \le 2$, so $-1 \le y \le 2$. Furthermore, $y = 2$ when $c = 1$ and $\theta = \frac{\pi}{2}$, while $y = -1$ for $c = 0$ and $\theta = \frac{3\pi}{2}$. Therefore, we need a viewing rectangle with $-1 \le y \le 2$.

To find the x-values, look at the equation $x = \cos\theta + \frac{1}{2}c\sin 2\theta$ and use the fact that $\sin 2\theta \ge 0$ for $0 \le \theta \le \frac{\pi}{2}$ and $\sin 2\theta \le 0$ for $-\frac{\pi}{2} \le \theta \le 0$. [Because $r = 1 + c\sin\theta$ is symmetric about the y-axis, we only need to consider $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.] So for $-\frac{\pi}{2} \le \theta \le 0$, x has a maximum value when $c = 0$ and then $x = \cos\theta$ has a maximum value of 1 at $\theta = 0$. Thus, the maximum value of x must occur on $\left[0, \frac{\pi}{2}\right]$ with $c = 1$. Then $x = \cos\theta + \frac{1}{2}\sin 2\theta \;\Rightarrow\; \frac{dx}{d\theta} = -\sin\theta + \cos 2\theta = -\sin\theta + 1 - 2\sin^2\theta \;\Rightarrow\; \frac{dx}{d\theta} = -(2\sin\theta - 1)(\sin\theta + 1) = 0$ when $\sin\theta = -1$ or $\frac{1}{2}$ [but $\sin\theta \ne -1$ for $0 \le \theta \le \frac{\pi}{2}$]. If $\sin\theta = \frac{1}{2}$, then $\theta = \frac{\pi}{6}$ and $x = \cos\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3} = \frac{3}{4}\sqrt{3}$. Thus, the maximum value of x is $\frac{3}{4}\sqrt{3}$, and, by symmetry, the minimum value is $-\frac{3}{4}\sqrt{3}$. Therefore, the smallest viewing rectangle that contains every member of the family of polar curves $r = 1 + c\sin\theta$, where $0 \le c \le 1$, is $\left[-\frac{3}{4}\sqrt{3}, \frac{3}{4}\sqrt{3}\right] \times [-1, 2]$.



4. (a) Let us find the polar equation of the path of the bug that starts in the upper right corner of the square. If the polar coordinates of this bug, at a particular moment, are (r, θ), then the polar coordinates of the bug that it is crawling toward must be $\left(r, \theta + \frac{\pi}{2}\right)$. (The next bug must be the same distance from the origin and the angle between the lines joining the bugs to the pole must be $\frac{\pi}{2}$.) The Cartesian coordinates of the first bug are $(r\cos\theta, r\sin\theta)$ and for the second bug we have $x = r\cos\left(\theta + \frac{\pi}{2}\right) = -r\sin\theta$, $y = r\sin\left(\theta + \frac{\pi}{2}\right) = r\cos\theta$. So the slope of the line joining the bugs is



$\dfrac{r\cos\theta - r\sin\theta}{-r\sin\theta - r\cos\theta} = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$. This must be equal to the slope of the tangent line at (r, θ), so by Equation 10.3.3 we have $\dfrac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$. Solving for $\dfrac{dr}{d\theta}$, we get

$\dfrac{dr}{d\theta}\sin^2\theta + \dfrac{dr}{d\theta}\sin\theta\cos\theta + r\sin\theta\cos\theta + r\cos^2\theta = \dfrac{dr}{d\theta}\sin\theta\cos\theta - \dfrac{dr}{d\theta}\cos^2\theta - r\sin^2\theta + r\sin\theta\cos\theta \Rightarrow$

$\dfrac{dr}{d\theta}\left(\sin^2\theta + \cos^2\theta\right) + r\left(\cos^2\theta + \sin^2\theta\right) = 0 \quad\Rightarrow\quad \dfrac{dr}{d\theta} = -r$. Solving this differential equation as a separable equation (as in Section 9.3), or using Theorem 9.4.2 with $k = -1$, we get $r = Ce^{-\theta}$. To determine C we use the fact that, at its starting position, $\theta = \frac{\pi}{4}$ and $r = \frac{1}{\sqrt{2}}a$, so $\frac{1}{\sqrt{2}}a = Ce^{-\pi/4} \quad\Rightarrow\quad C = \frac{1}{\sqrt{2}}ae^{\pi/4}$. Therefore, a polar equation of the bug's path is $r = \frac{1}{\sqrt{2}}ae^{\pi/4}e^{-\theta}$ or $r = \frac{1}{\sqrt{2}}ae^{(\pi/4)-\theta}$.

(b) The distance traveled by this bug is $L = \int_{\pi/4}^{\infty}\sqrt{r^2 + (dr/d\theta)^2}\,d\theta$, where $\dfrac{dr}{d\theta} = \dfrac{a}{\sqrt{2}}\,e^{\pi/4}\left(-e^{-\theta}\right)$ and so $r^2 + (dr/d\theta)^2 = \frac{1}{2}a^2e^{\pi/2}e^{-2\theta} + \frac{1}{2}a^2e^{\pi/2}e^{-2\theta} = a^2e^{\pi/2}e^{-2\theta}$. Thus

$$\begin{aligned} L &= \int_{\pi/4}^{\infty} ae^{\pi/4}e^{-\theta}\,d\theta = ae^{\pi/4}\lim_{t\to\infty}\int_{\pi/4}^{t} e^{-\theta}\,d\theta = ae^{\pi/4}\lim_{t\to\infty}\left[-e^{-\theta}\right]_{\pi/4}^{t} \\ &= ae^{\pi/4}\lim_{t\to\infty}\left[e^{-\pi/4} - e^{-t}\right] = ae^{\pi/4}e^{-\pi/4} = a \end{aligned}$$

5. Without loss of generality, assume the hyperbola has equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Use implicit differentiation to get $\dfrac{2x}{a^2} - \dfrac{2y\,y'}{b^2} = 0$, so $y' = \dfrac{b^2x}{a^2y}$. The tangent line at the point (c, d) on the hyperbola has equation $y - d = \dfrac{b^2c}{a^2d}(x - c)$.

The tangent line intersects the asymptote $y = \dfrac{b}{a}x$ when $\dfrac{b}{a}x - d = \dfrac{b^2c}{a^2d}(x - c) \quad\Rightarrow\quad abdx - a^2d^2 = b^2cx - b^2c^2 \quad\Rightarrow$

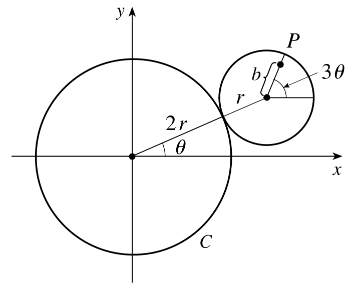
$abdx - b^2cx = a^2d^2 - b^2c^2 \quad\Rightarrow\quad x = \dfrac{a^2d^2 - b^2c^2}{b(ad - bc)} = \dfrac{ad + bc}{b}$ and the y-value is $\dfrac{b}{a}\dfrac{ad + bc}{b} = \dfrac{ad + bc}{a}$.

Similarly, the tangent line intersects $y = -\dfrac{b}{a}x$ at $\left(\dfrac{bc - ad}{b}, \dfrac{ad - bc}{a}\right)$. The midpoint of these intersection points is

$\left(\dfrac{1}{2}\left(\dfrac{ad + bc}{b} + \dfrac{bc - ad}{b}\right), \dfrac{1}{2}\left(\dfrac{ad + bc}{a} + \dfrac{ad - bc}{a}\right)\right) = \left(\dfrac{1}{2}\dfrac{2bc}{b}, \dfrac{1}{2}\dfrac{2ad}{a}\right) = (c, d)$, the point of tangency.

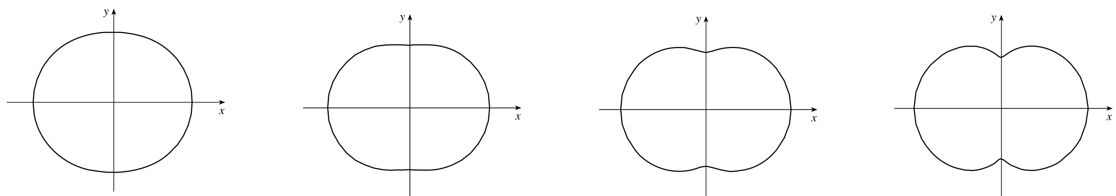
Note: If $y = 0$, then at $(\pm a, 0)$, the tangent line is $x = \pm a$, and the points of intersection are clearly equidistant from the point of tangency.

6. (a) Since the smaller circle rolls without slipping around C, the amount of arc traversed on C ($2r\theta$ in the figure) must equal the amount of arc of the smaller circle that has been in contact with C. Since the smaller circle has radius r, it must have turned through an angle of $2r\theta/r = 2\theta$. In addition to turning through an angle 2θ, the little circle has rolled through an angle θ against C. Thus, P has turned through an angle of 3θ as shown in the figure. (If the little circle had turned through an angle of 2θ with its center pinned to the x-axis,



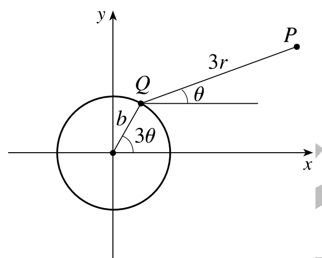
then P would have turned only 2θ instead of 3θ. The movement of the little circle around C adds θ to the angle.) From the figure, we see that the center of the small circle has coordinates $(3r\cos\theta, 3r\sin\theta)$. Thus, P has coordinates (x, y), where $x = b\cos 3\theta + 3r\cos\theta$ and $y = b\sin 3\theta + 3r\sin\theta$.

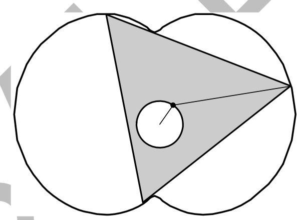
(b)



$b = \frac{1}{5}r$ $\quad$ $b = \frac{2}{5}r$ $\quad$ $b = \frac{3}{5}r$ $\quad$ $b = \frac{4}{5}r$

(c) The diagram gives an alternate description of point P on the epitrochoid. Q moves around a circle of radius b, and P rotates one-third as fast with respect to Q at a distance of $3r$. Place an equilateral triangle with sides of length $3\sqrt{3}r$ so that its centroid is at Q and one vertex is at P. (The distance from the centroid to a vertex is $\frac{1}{\sqrt{3}}$ times the length of a side of the equilateral triangle.)





As θ increases by $\frac{2\pi}{3}$, the point Q travels once around the circle of radius b, returning to its original position. At the same time, P (and the rest of the triangle) rotate through an angle of $\frac{2\pi}{3}$ about Q, so P's position is occupied by another vertex. In this way, we see that the epitrochoid traced out by P is simultaneously traced out by the other two vertices as well. The whole equilateral triangle sits inside the epitrochoid (touching it only with its vertices) and each vertex traces out the curve once while the centroid moves around the circle three times.

(d) We view the epitrochoid as being traced out in the same way as in part (c), by a rotor for which the distance from its center to each vertex is $3r$, so it has radius $6r$. To show that the rotor fits inside the epitrochoid, it suffices to show that for any position of the tracing point P, there are no points on the opposite side of the rotor which are outside the epitrochoid. But the most likely case of intersection is when P is on the y-axis, so as long as the diameter of the rotor $\left(\text{which is } 3\sqrt{3}\,r\right)$ is less than the distance between the y-intercepts, the rotor will fit. The y-intercepts occur when $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ $\Rightarrow$ $y = -b + 3r$ or $y = b - 3r$, so the distance between the intercepts is $(-b + 3r) - (b - 3r) = 6r - 2b$, and the rotor will fit if $3\sqrt{3}\,r \leq 6r - 2b \quad \Leftrightarrow \quad 2b \leq 6r - 3\sqrt{3}\,r \quad \Leftrightarrow \quad b \leq \frac{3}{2}\left(2 - \sqrt{3}\right)r$.

2 Derivatives

2.1 Derivatives and Rates of Change

SUGGESTED TIME AND EMPHASIS

1–2 classes Essential material

POINTS TO STRESS

1. The slope of the tangent line as the limit of the slopes of secant lines (visually, numerically, algebraically).
2. Physical examples of instantaneous rates of change (velocity, reaction rate, marginal cost, and so on) and their units.
3. The derivative notations $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ and $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.
4. Using f' to write an equation of the tangent line to a curve at a given point.
5. Using f' as an approximate rate of change when working with discrete data.

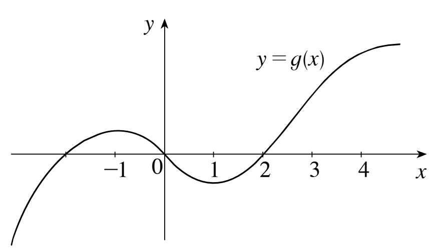
QUIZ QUESTIONS

- **TEXT QUESTION** Why is it necessary to take a limit when computing the slope of the tangent line?
  ANSWER There are several possible answers here. Examples include the following:
  - By definition, the slope of the tangent line is the limit of the slopes of secant lines.
  - You don't know where to draw the tangent line unless you pick two points very close together.

  The idea is to get them thinking about this question.

- **DRILL QUESTION** For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

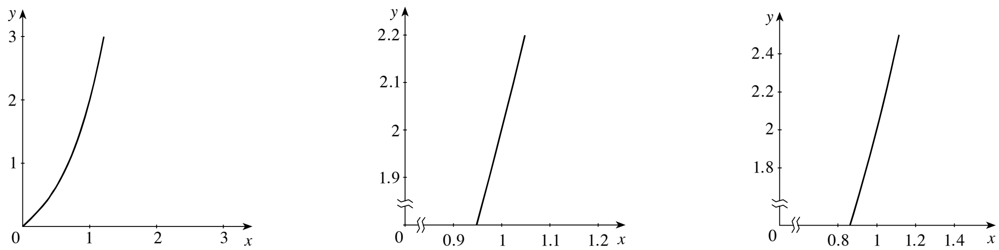
  $0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$



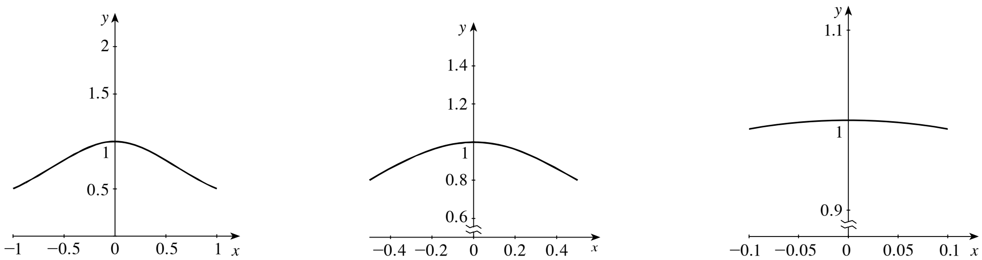
ANSWER $g'(0) < 0 < g'(4) < g'(-2) < g'(2)$

MATERIALS FOR LECTURE

- Review the geometry of the tangent line, and the concept of "locally linear". Estimate the slope of the line tangent to $y = x^3 + x$ at $(1, 2)$ by looking at the slopes of the lines between $x = 0.9$ and $x = 1.1$, $x = 0.99$ and $x = 1.01$, and so forth. Illustrate these secant lines on a graph of the function, redrawing the figure when necessary to illustrate the "zooming in" process.



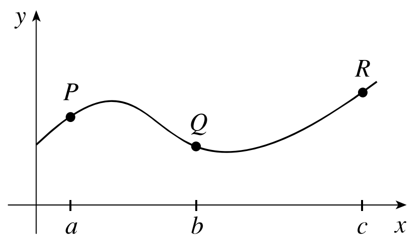
  Similarly examine $y = \frac{1}{x^2+1}$ at $(0, 1)$.



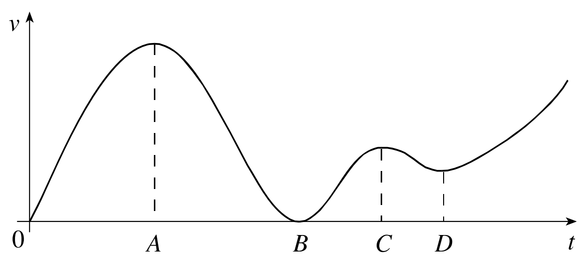
- If "A Jittery Function" was covered in Section 1.7, look at $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$ Poll the class: Is there a tangent line at $x = 0$? Then examine what happens if you look at the limits of the secant lines.
- Have students estimate the slope of the tangent line to $y = \sin x$ at various points. Foreshadow the concept of concavity by asking them some open-ended questions such as the following: What happens to the function when the slope of the tangent is increasing? Decreasing? Zero? Slowly changing?
- Discuss how physical situations can be translated into statements about derivatives. For example, the budget deficit can be viewed as the derivative of the national debt. Describe the units of derivatives in real world situations. The budget deficit, for example, is measured in billions of dollars per year. Another example: if $s(d)$ represents the sales figures for a magazine given d dollars of advertising, where s is the number of magazines sold, then $s'(d)$ is in magazines per dollar spent. Describe enough examples to make the pattern evident.
- Note that the text shows that if $f(x) = x^2 - 8x + 9$, then $f'(a) = 2a - 8$. Thus, $f'(55) = 102$ and $f'(100) = 192$. Demonstrate that these quantities cannot be easily estimated from a graph of the function. Foreshadow the treatment of a as a variable in Section 2.2.
- If a function models discrete data and the quantities involved are orders of magnitude larger than 1, we can use the approximation $f'(x) \approx f(x+1) - f(x)$. (That is, we can use $h = 1$ in the limit definition of the derivative.) For example, let $f(t)$ be the total population of the world, where t is measured in years since 1800. Then $f(211)$ is the world population in 2011, $f(212)$ is the total population in 2012, and $f'(211)$ is approximately the change in population from 2011 to 2012. In business, if $f(n)$ is the total cost of producing n objects, $f'(n)$ approximates the cost of producing the $(n+1)$th object.

WORKSHOP/DISCUSSION

- "Thumbnail" derivative estimates: graph a function on the board and have the class call out rough values of the derivative. Is it larger than 1? About 1? Between 0 and 1? About 0? Between -1 and 0? About -1? Smaller than -1? This is good preparation for Group Work 2 ("Oiling Up Your Calculators").
- Draw a function like the following, and first estimate slopes of secant lines between $x = a$ and $x = b$, and between $x = b$ and $x = c$. Then order the five quantities $f'(a)$, $f'(b)$, $f'(c)$, m_{PQ}, and m_{QR} in decreasing order. [Answer: $f'(b) < m_{PQ} < m_{QR} < f'(c) < f'(a)$.]



- Start the following problem with the students: A car is travelling down a highway away from its starting location with distance function $d(t) = 8\left(t^3 - 6t^2 + 12t\right)$, where t is in hours, and d is in miles.
  1. How far has the car travelled after 1, 2, and 3 hours?
  2. What is the average velocity over the intervals [0, 1], [1, 2], and [2, 3]?
- Consider a car's velocity function described by the graph below.



  1. Ask the students to determine when the car was stopped.
  2. Ask the students when the car was accelerating (that is, when the velocity was increasing). When was the car decelerating?
  3. Ask the students to describe what is happening at times A, C, and D in terms of both velocity and acceleration. What is happening at time B?
- Estimate the slope of the tangent line to $y = \sin x$ at $x = 1$ by looking at the following table of values.

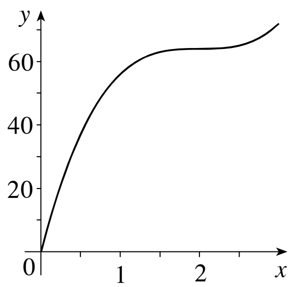
x	$\sin x$	$\dfrac{\sin x - \sin 1}{x - 1}$
0	0	0.841471
0.5	0.4794	0.724091
0.9	0.7833	0.581441
0.99	0.8360	0.544501
0.999	0.8409	0.540723
1.0001	0.8415	0.540260
1.001	0.8420	0.539881

- Demonstrate some sample computations similar to Example 4, such as finding the derivative of $f(t) = \sqrt{1+t}$ at $t = 3$, or of $g(x) = x - x^2$ at $x = 1$.

GROUP WORK 1: FOLLOW THAT CAR

Start this problem by giving the students the function $d(t) = 8\left(t^3 - 6t^2 + 12t\right)$ and having them sketch its graph. Ask them how far the car has traveled after 1, 2, and 3 hours, and then show them how to compute the average velocity for [0, 1], [1, 2], and [2, 3].

ANSWERS

1. 

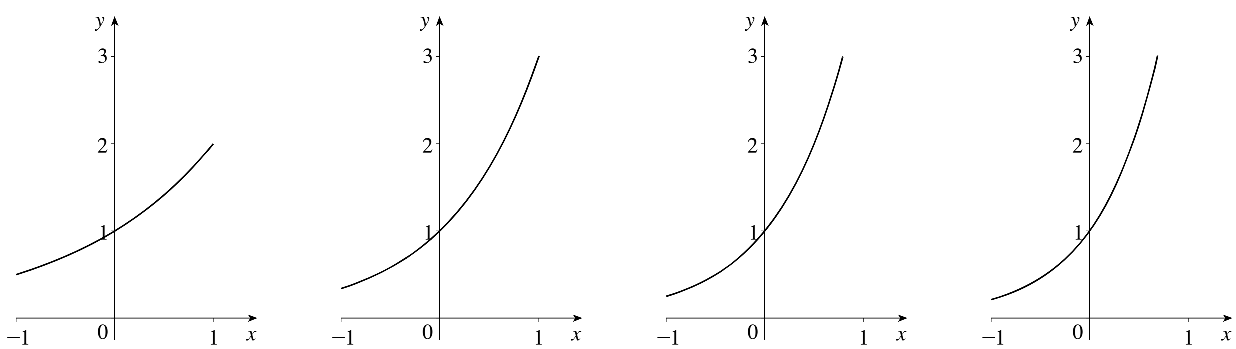
2. It appears to stop at $t = 2$.
3. 8 mi/h, 2 mi/h, 0.08 mi/h
4. 0 mi/h. This is where the car stops.

GROUP WORK 2: OILING UP YOUR CALCULATORS

As long as the students have the ability to graph a function on their calculators and to estimate the slope of a curve at a point, they don't need to have been exposed to the exponential function to do this activity. The exponential function and the number e will be covered in Chapter 6, and this exercise is a good initial introduction to the concept.

ANSWERS

1. If the students do this numerically, they should be able to get some pretty good estimates of $\ln 3 \approx 1.098612$. If they use graphs, they should be able to get 1.1 as an estimate.
2. 0.7 is a good estimate from a graph.
3. As a increases, the slope of the curve at $x = 0$ is increasing, as can be seen below.



4. The slope is less than 1 at $a = 2$ and greater than 1 at $a = 3$. Now apply the Intermediate Value Theorem.
5. The students are estimating e and should get 2.72 at a minimum level of accuracy.

GROUP WORK 3: CONNECT THE DOTS

Closure is particularly important on this activity. At this point in the course, many students will have the impression that all reasonable estimates are equally valid, so it is crucial that students discuss Problem 4. If there is student interest, this table can generate a rich discussion. Can A' ever be negative? What would that mean in real terms? What would $(A')'$ mean in real terms in this instance?

ANSWERS

1. $A'(3500) \approx 0.06$ %/\$ It is likely to be an overestimate, because the function lies below its tangent line near $p = 3500$.
2. After spending \$3500, consumer approval is increasing at the rate of about 0.06 % for every additional dollar spent.
3. Percent per dollar
4. $A'(\$3550) \approx 0.06$ %/\$. This is a better estimate because the same figures now give a two-sided approximation of the limit of the difference quotient.

HOMEWORK PROBLEMS

CORE EXERCISES 3, 5, 9, 11, 14, 22, 23, 33, 40, 48

SAMPLE ASSIGNMENT 3, 5, 9, 11, 14, 17, 22, 23, 33, 40, 48, 53, 59

EXERCISE	D	A	N	G
3		×		×
5		×		
9				×
11				×
14	×	×		
17		×		
22		×		
23		×		
33		×	×	
40				×
48				×
53	×	×		
59		×		

GROUP WORK 1, SECTION 2.1

Follow that Car

Here, we continue with the analysis of the distance $d(t) = 8\left(t^3 - 6t^2 + 12t\right)$ of a car, where d is in miles and t is in hours.

1. Draw a graph of $d(t)$ from $t = 0$ to $t = 3$.

2. Does the car ever stop?

3. What is the average velocity over [1, 3]? over [1.5, 2.5]? over [1.9, 2.1]?

4. Estimate the instantaneous velocity at $t = 2$. Give a physical interpretation of your answer.

GROUP WORK 2, SECTION 2.1

Oiling Up Your Calculators

1. Use your calculator to graph $y = 3^x$. Estimate the slope of the line tangent to this curve at $x = 0$ using a method of your choosing.

2. Use your calculator to graph $y = 2^x$. Estimate the slope of the line tangent to this curve at $x = 0$ using a method of your choosing.

3. It is a fact that, as a increases, the slope of the line tangent to $y = a^x$ at $x = 0$ also increases in a continuous way. Geometrically, why should this be the case?

4. Prove that there is a special value of a for which the slope of the line tangent to $y = a^x$ at $x = 0$ is 1.

5. By trial and error, find an estimate of this special value of a, accurate to two decimal places.

GROUP WORK 3, SECTION 2.1

Connect the Dots

A company does a study on the effect of production value p of an advertisement on its consumer approval rating A. After interviewing eight focus groups, they come up with the following data:

Production Value	Consumer Approval
\$1000	32%
\$2000	33%
\$3000	46%
\$3500	55%
\$3600	61%
\$3800	65%
\$4000	69%
\$5000	70%

Assume that $A(p)$ gives the consumer approval percentage as a function of p.

1. Estimate $A'(\$3500)$. Is this likely to be an overestimate or an underestimate?

2. Interpret your answer to Problem 1 in real terms. What does your estimate of $A'(\$3500)$ tell you?

3. What are the units of $A'(p)$?

4. Estimate $A'(\$3550)$. Is your estimate better or worse than your estimate of $A'(\$3500)$? Why?

WRITING PROJECT Early Methods for Finding Tangents

The history of calculus is a fascinating and too-often neglected subject. Most people who study history never see calculus, and vice versa. We recommend assigning this section as extra credit to any motivated class, and possibly as a required group project, especially for a class consisting of students who are not science or math majors.

The students will need clear instructions detailing what their final result should look like. For example, recommend a page or two about Fermat's or Barrow's life and career, followed by two or three technical pages describing the alternate method of finding tangent lines as in the project's directions, and completed by a final half page of meaningful conclusion.

2.2 The Derivative as a Function

SUGGESTED TIME AND EMPHASIS

2 classes Essential material

POINTS TO STRESS

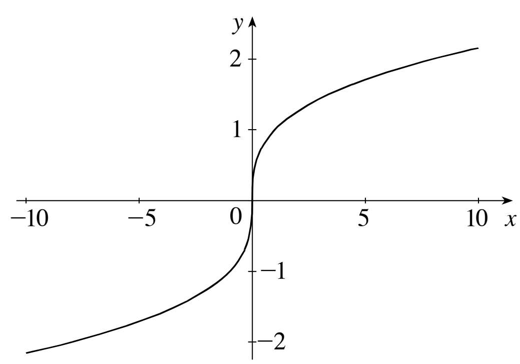
1. The concept of a differentiable function interpreted visually, algebraically, and descriptively.
2. Obtaining the derivative function f' by first considering the derivative at a point x, and then treating x as a variable.
3. How a function can fail to be differentiable.
4. Sketching the derivative function given a graph of the original function.
5. Second and higher derivatives

QUIZ QUESTIONS

TEXT QUESTION The previous section discussed the derivative $f'(a)$ for some function f. This section discusses the derivative $f'(x)$ for some function f. What is the difference, and why is it significant enough to merit separate sections?

ANSWER a is considered a constant, x is considered a variable. So $f'(a)$ is a number (the slope of the tangent line) and $f'(x)$ is a function.

DRILL QUESTION Consider the graph of $f(x) = \sqrt[3]{x}$. Is this function defined at $x = 0$? Continuous at $x = 0$? Differentiable at $x = 0$? Why?

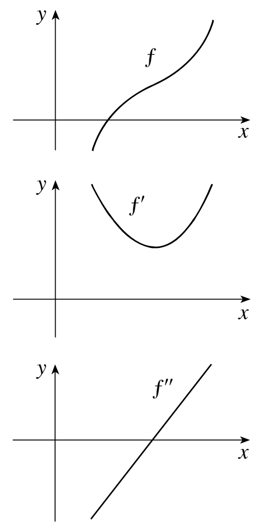


ANSWER It is defined and continuous, but not differentiable because it has a vertical tangent.

MATERIALS FOR LECTURE

- Ask the class this question: "If you were in a car, blindfolded, ears plugged, all five senses neutralized, what quantities would you still be able to perceive?" (Answers: They could feel the second derivative of motion, acceleration. They could also feel the third derivative of motion, "jerk".) Many students incorrectly add velocity to this list. Stress that acceleration is perceived as a force (hence $F = ma$) and that "jerk" causes the uncomfortable sensation when the car stops suddenly.
- Review definitions of differentiability, continuity, and the existence of a limit.
- Sketch f' from a graphical representation of $f(x) = \left|x^2 - 4\right|$, noting where f' does not exist. Then sketch $\left(f'\right)'$ from the graph of f'. Point out that differentiability implies continuity, and not vice versa.

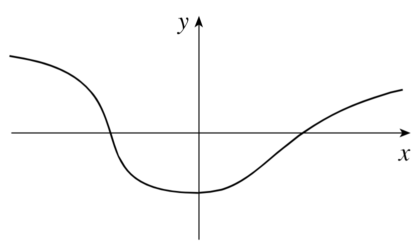
- Examine graphs of f and f' aligned vertically as shown. If you wish to foreshadow f'', add its graph below. Discuss what it means for f' to be positive, negative or zero. Then discuss what it means for f' to be increasing, decreasing or constant.



- If the group work "A Jittery Function" was covered in Section 1.7, then examine the differentiability of $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$ at $x = 0$ and elsewhere, if you have not already done so.
- Show that if $f(x) = x^4 - x^2 + x + 1$, then $f^{(5)}(x) \equiv 0$. Conclude that if $f(x)$ is a polynomial of degree m, then $f^{(m+1)}(x) \equiv 0$.

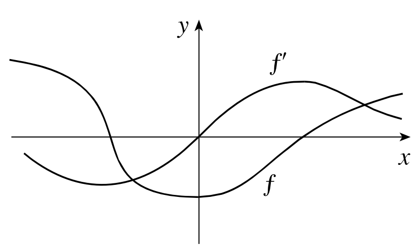
WORKSHOP/DISCUSSION

- Estimate derivatives from the graph of $f(x) = \sin x$. Do this at various points, and plot the results on the blackboard. See if the class can recognize the graph as a graph of the cosine curve.
- Given the graph of f below, have students determine where f has a horizontal tangent, where f' is positive, where f' is negative, where f' is increasing (this may require some additional discussion), and where f' is decreasing. Then have them sketch the graph of f'.



TEC has more exercises of this type using a wide variety of functions.

ANSWER There is a horizontal tangent near $x = 0$. f' is positive to the right of 0, negative to the left. f' is increasing between the x-intercepts, and decreasing outside of them.



- Compute $f'(x)$ and $g'(x)$ if $f(x) = x^2 + x + 2$ and $g(x) = x^2 + x + 4$. Point out that $f'(x) = g'(x)$ and discuss why the constant term is not important. Next, compute $h'(x)$ if $h(x) = x^2 + 2x + 2$. Point out that the graph of $h'(x)$ is just the graph of $f'(x)$ shifted up one unit, so the linear term just shifts derivatives. TEC contains more explorations on how the coefficients in polynomials and other functions affect first and second derivatives.

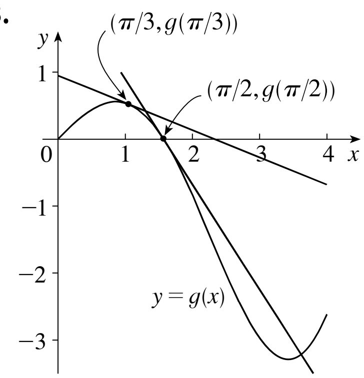
- Consider the function $f(x) = \sqrt{|x|}$. Show that it is not differentiable at 0 in two ways: by inspection (it has a cusp); and by computing the left- and right-hand limits of $f'(x)$ at $x = 0$ ($\lim_{x\to 0^+} f'(x) = \infty$, $\lim_{x\to 0^-} f'(x) = -\infty$).
- TEC can be used to develop students' ability to look at the graph of a function and visualize the graph of that function's derivative. The key feature of this module is that it allows the students to mark various features of the derivative *directly on the graph of the function* (for example, where the derivative is positive or negative). Then, after using this information and sketching a graph of the derivative, they can view the actual graph of the derivative and check their work.

GROUP WORK 1: TANGENT LINES AND THE DERIVATIVE FUNCTION

This simple activity reinforces that although we are moving to thinking of the derivative as a function of x, it is still the slope of the line tangent to the graph of f.

ANSWERS

1, 3.



2. $y = -\frac{\pi}{2}\left(x - \frac{\pi}{2}\right)$

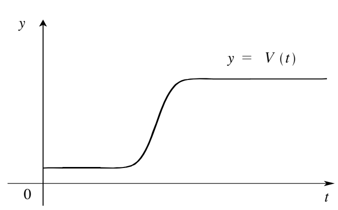
4. $y = \left(\frac{1}{2} - \frac{\pi\sqrt{3}}{6}\right)\left(x - \frac{\pi}{3}\right) + \frac{\pi}{6}$

GROUP WORK 2: THE REVENGE OF ORVILLE REDENBACHER

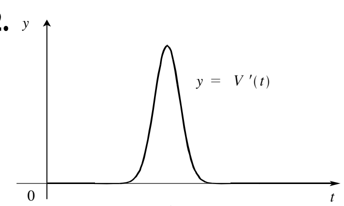
In an advanced class, or a class in which one group has finished far ahead of the others, ask the students to repeat the activity substituting "$D(t)$, the density function" for $V(t)$.

ANSWERS

1.

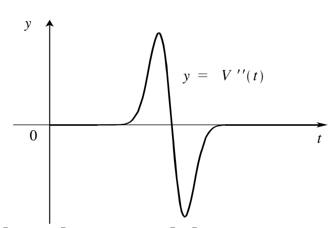


2.



Units are cm^3/s.

3.



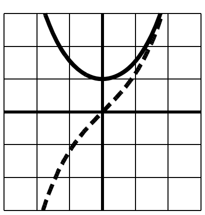
When the second derivative crosses the x-axis, the first derivative has a maximum, meaning the popcorn is expanding the fastest.

GROUP WORK 3: THE DERIVATIVE FUNCTION

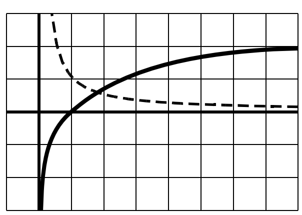
Give each group of between three and five students the picture of all eight graphs. They are to sketch the derivative functions by first estimating the slopes at points, and plotting the values of $f'(x)$. Each group should also be given a large copy of one of the graphs, perhaps on acetate. When they are ready, with this information they can draw the derivative graph on the same axes. For closure, project their solutions on the wall and point out salient features. Perhaps the students will notice that the derivatives turn out to be positive when their corresponding functions are increasing. Concavity can even be introduced at this time. Large copies of the answers are provided, in case the instructor wishes to overlay them on top of students' answers for reinforcement. Note that the derivative of graph 6 ($y = e^x$) is itself. Also note that the derivative of graph 1 ($y = \cosh x$) is *not* a straight line. Leave at least 15 minutes for closure. The whole activity should take about 45–60 minutes, but it is really, truly worth the time.

If a group finishes early, have them discuss where f' is increasing and where it is decreasing. Also show that where f is increasing, f' is positive, and where f is decreasing, f' is negative.

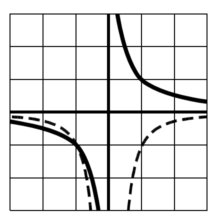
ANSWER (larger answer graphs are included after the group work)



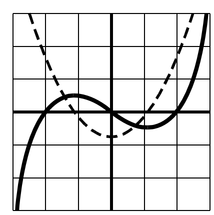
Graph 1



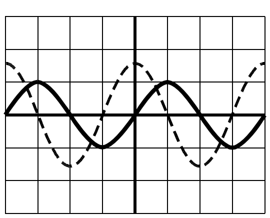
Graph 2



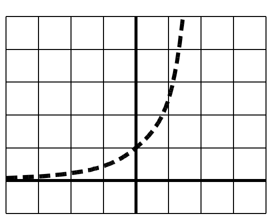
Graph 3



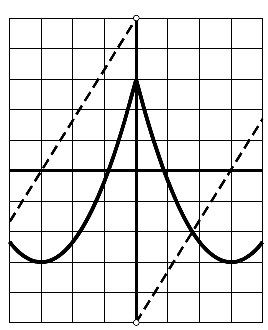
Graph 4



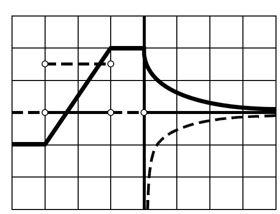
Graph 5



Graph 6



Graph 7



Graph 8

HOMEWORK PROBLEMS

CORE EXERCISES 1, 3, 5, 8, 11, 19, 33, 50

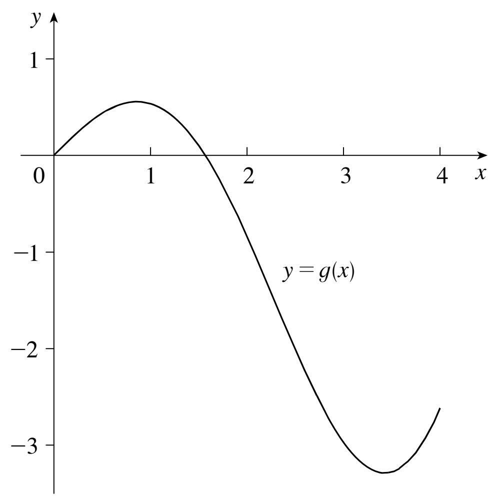
SAMPLE ASSIGNMENT 1, 3, 5, 7, 8, 11, 16, 17, 19, 33, 42, 50, 53

EXERCISE	D	A	N	G
1				×
3				×
5				×
7	×		×	
8				×
11				×
16		×		
17		×		
19		×		
33		×		
42		×		
50		×		
53		×		×

GROUP WORK 1, SECTION 2.2

Tangent Lines and the Derivative Function

The following is a graph of $g(x) = x \cos x$.



It is a fact that the derivative of this function is $g'(x) = \cos x - x \sin x$.

1. Sketch the line tangent to $g(x)$ at $x = \frac{\pi}{2} \approx 1.57$ on the graph above.

2. Find an equation of the tangent line at $x = \frac{\pi}{2}$.

3. Now sketch the line tangent to $g(x)$ at $x = \frac{\pi}{3} \approx 1.05$.

4. Find an equation of the tangent line at $x = \frac{\pi}{3}$.

GROUP WORK 2, SECTION 2.2

The Revenge of Orville Redenbacher

1. Consider a single kernel of popcorn in a microwave oven. Let $V(t)$ be the volume in cm^3 of the kernel at time t seconds. Draw a graph of $V(t)$, including as much detail as you can, up to the time that the kernel is taken from the oven.

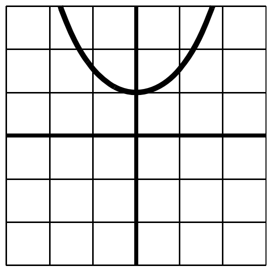
2. Now sketch a graph of the derivative function $V'(t)$. What are the units of $V'(t)$?

3. Finally, sketch a graph of $V''(t)$. What does it mean when this graph crosses the x-axis?

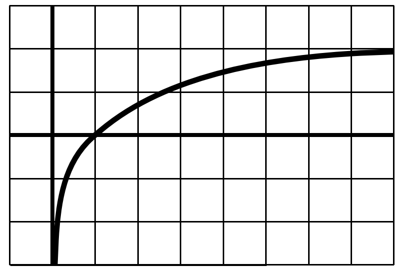
GROUP WORK 3, SECTION 2.2

The Derivative Function

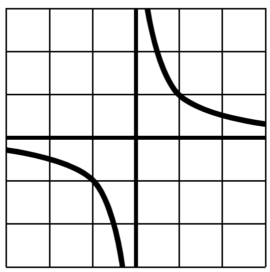
The graphs of several functions f are shown below. For each function, estimate the slope of the graph of f at various points. From your estimates, sketch graphs of f'.



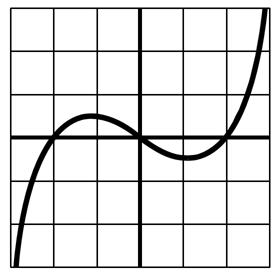
Graph 1



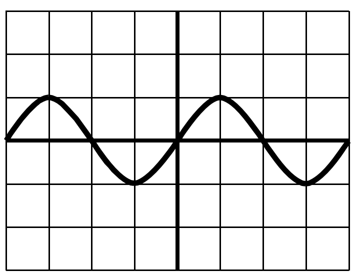
Graph 2



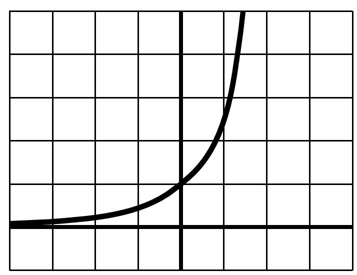
Graph 3



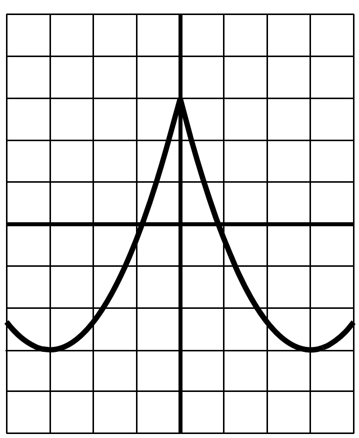
Graph 4



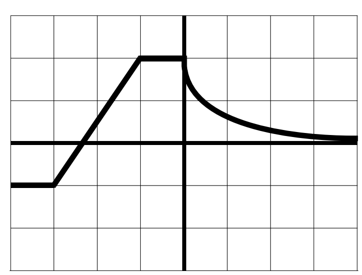
Graph 5



Graph 6

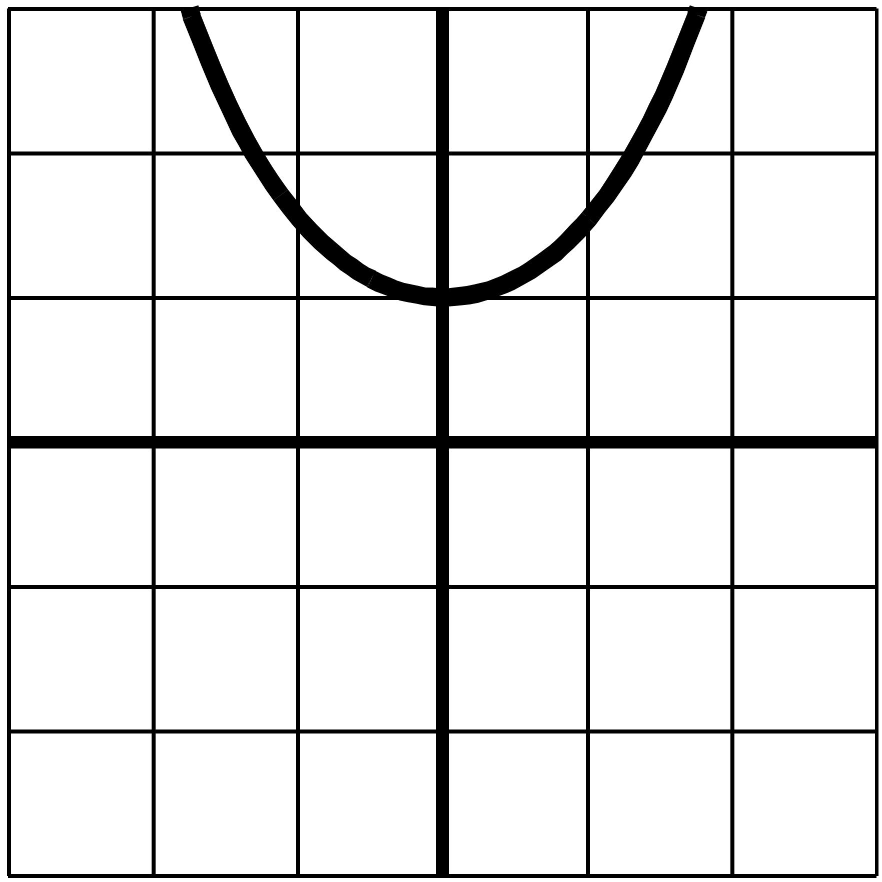


Graph 7

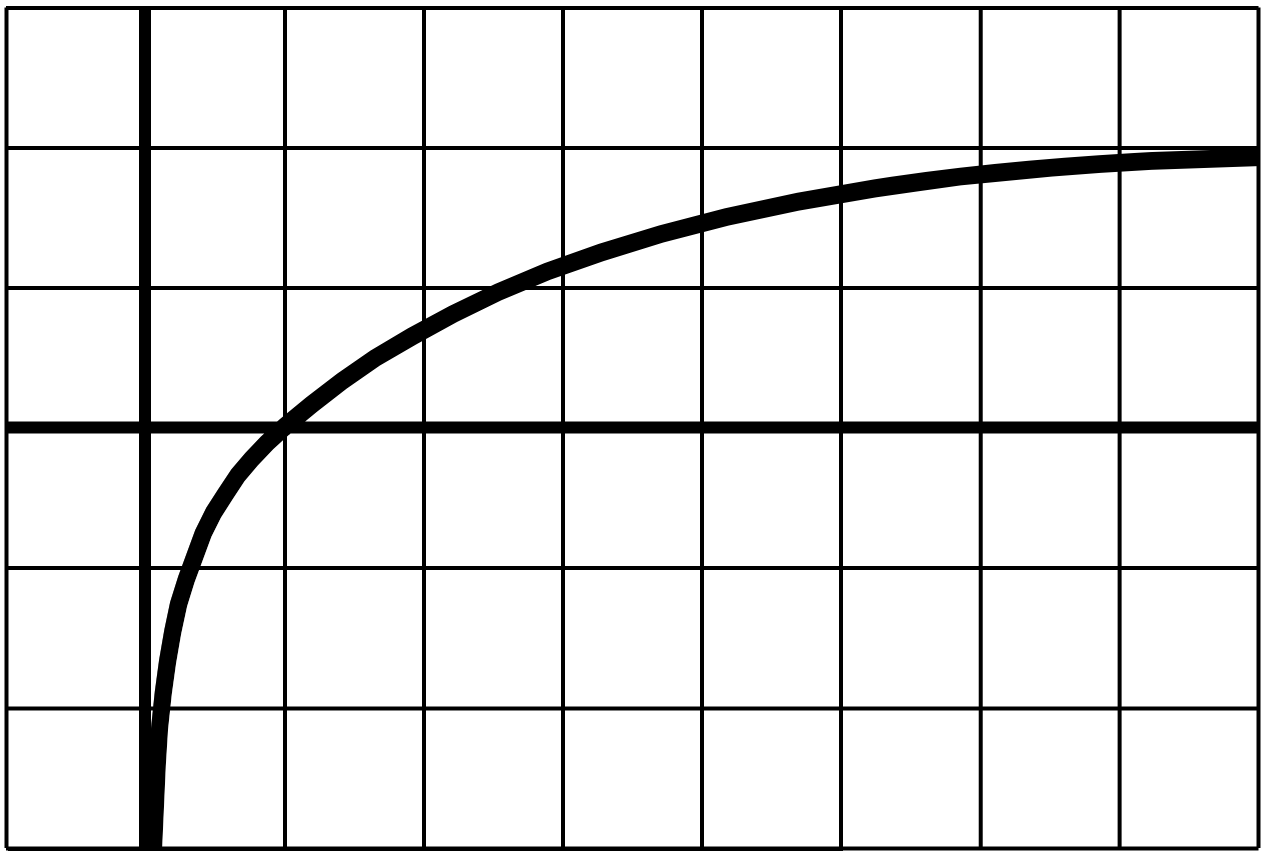


Graph 8

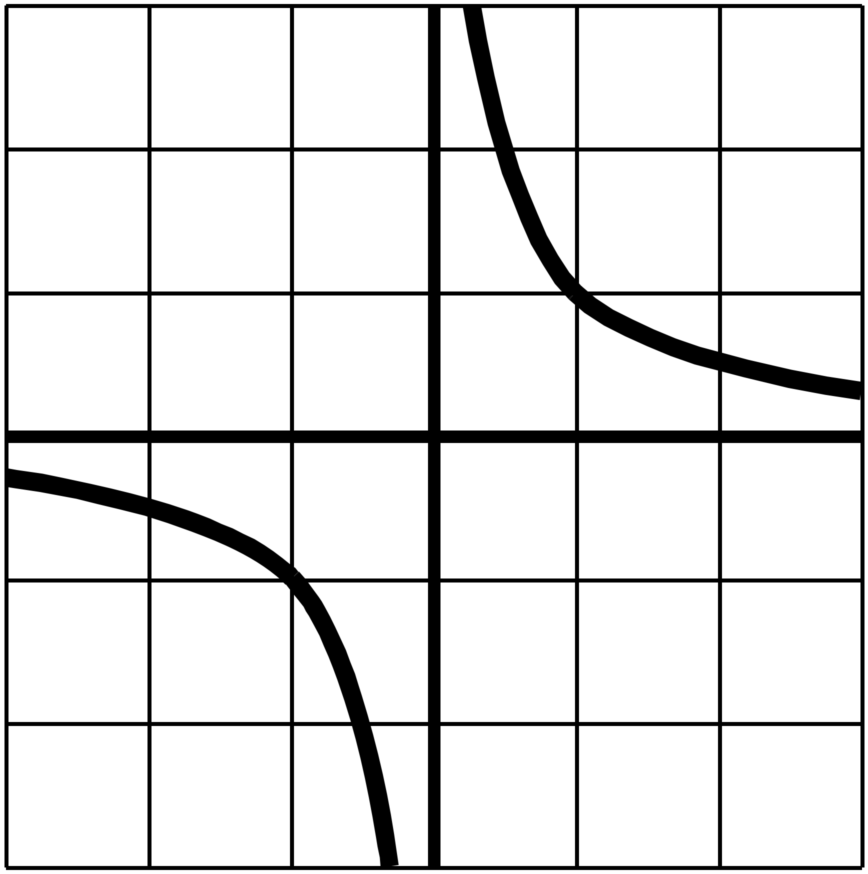
Graph 1



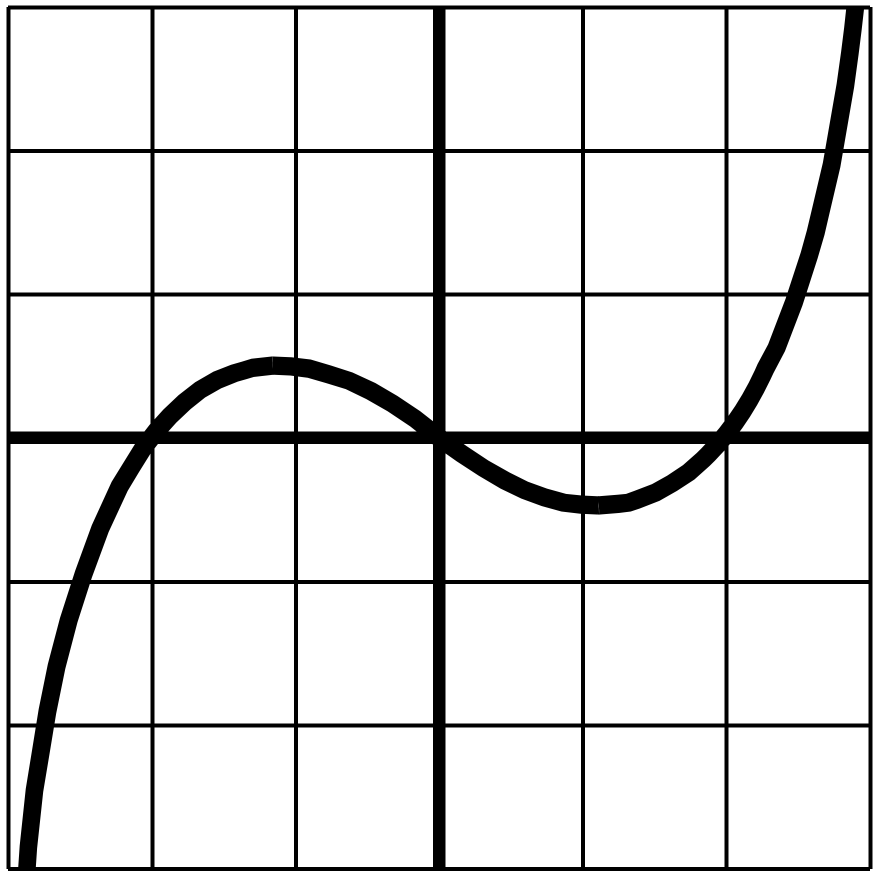
Graph 2



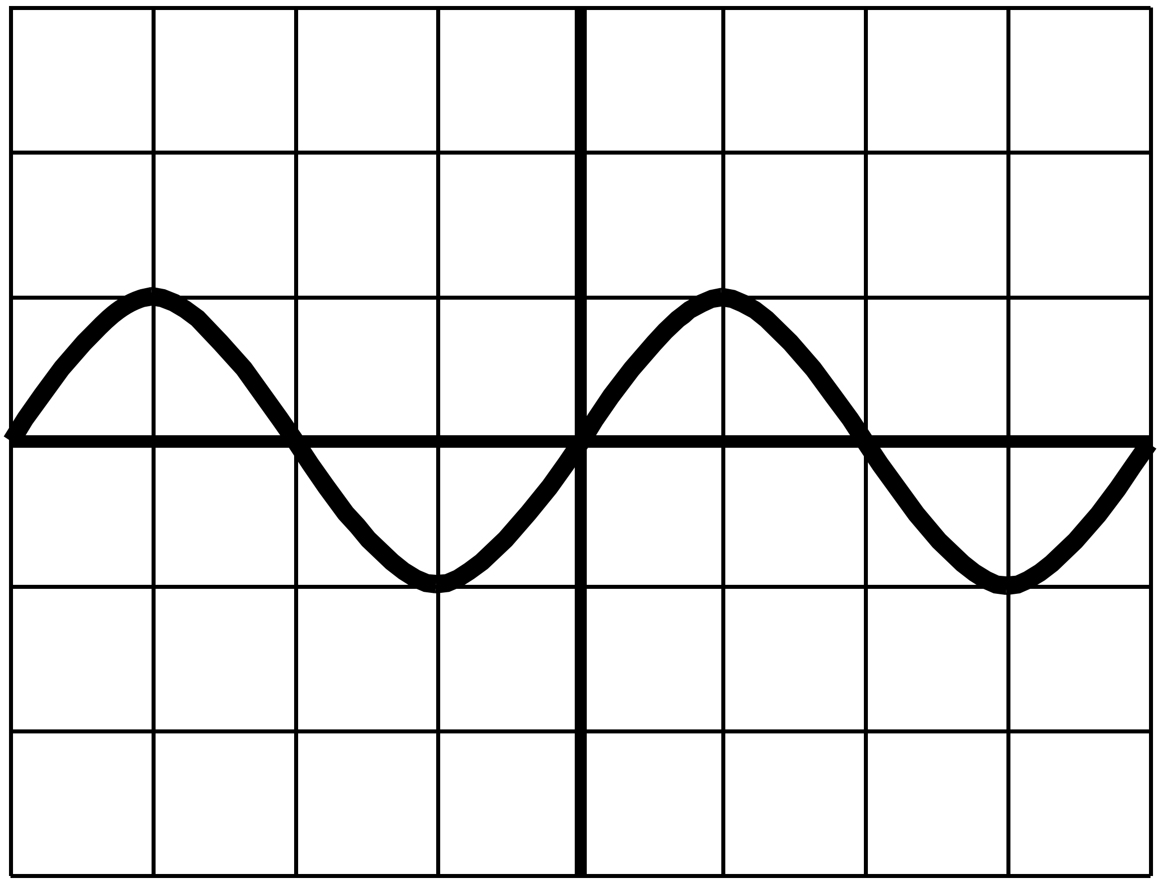
Graph 3



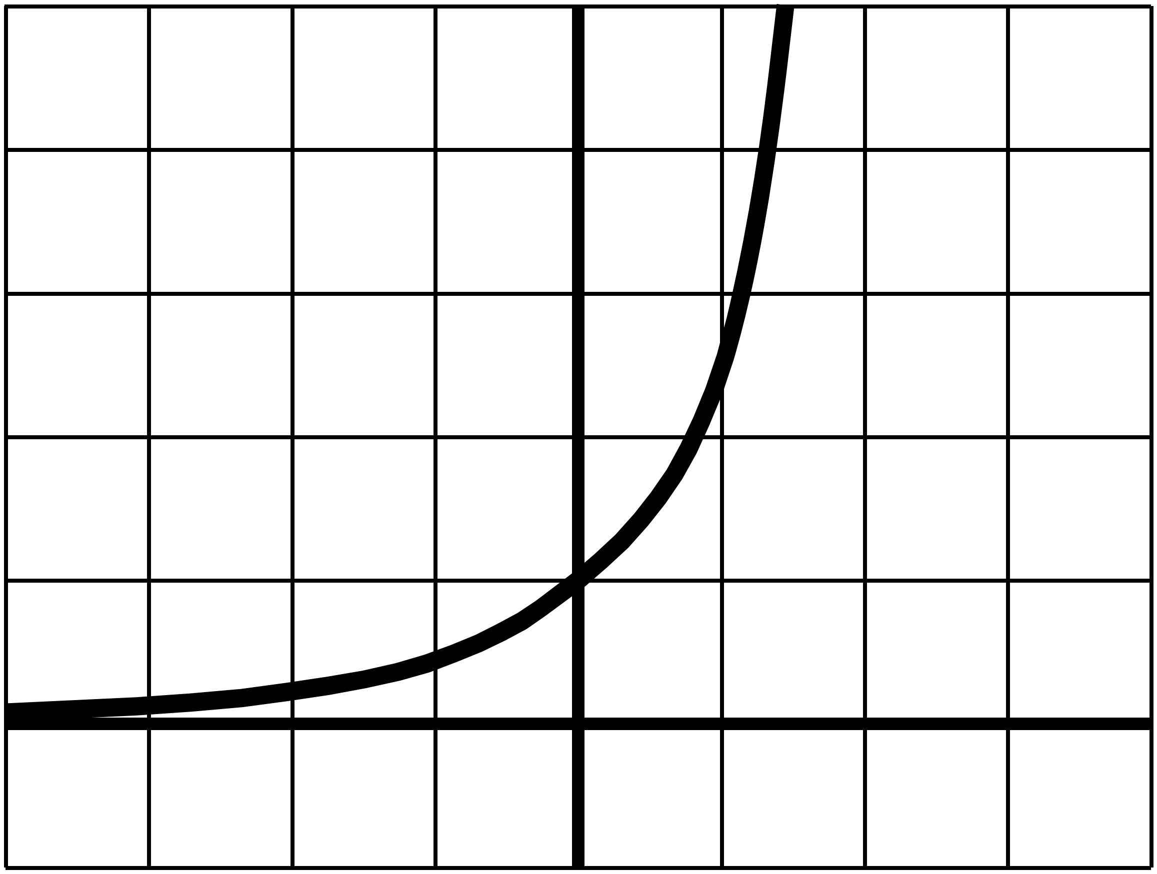
Graph 4



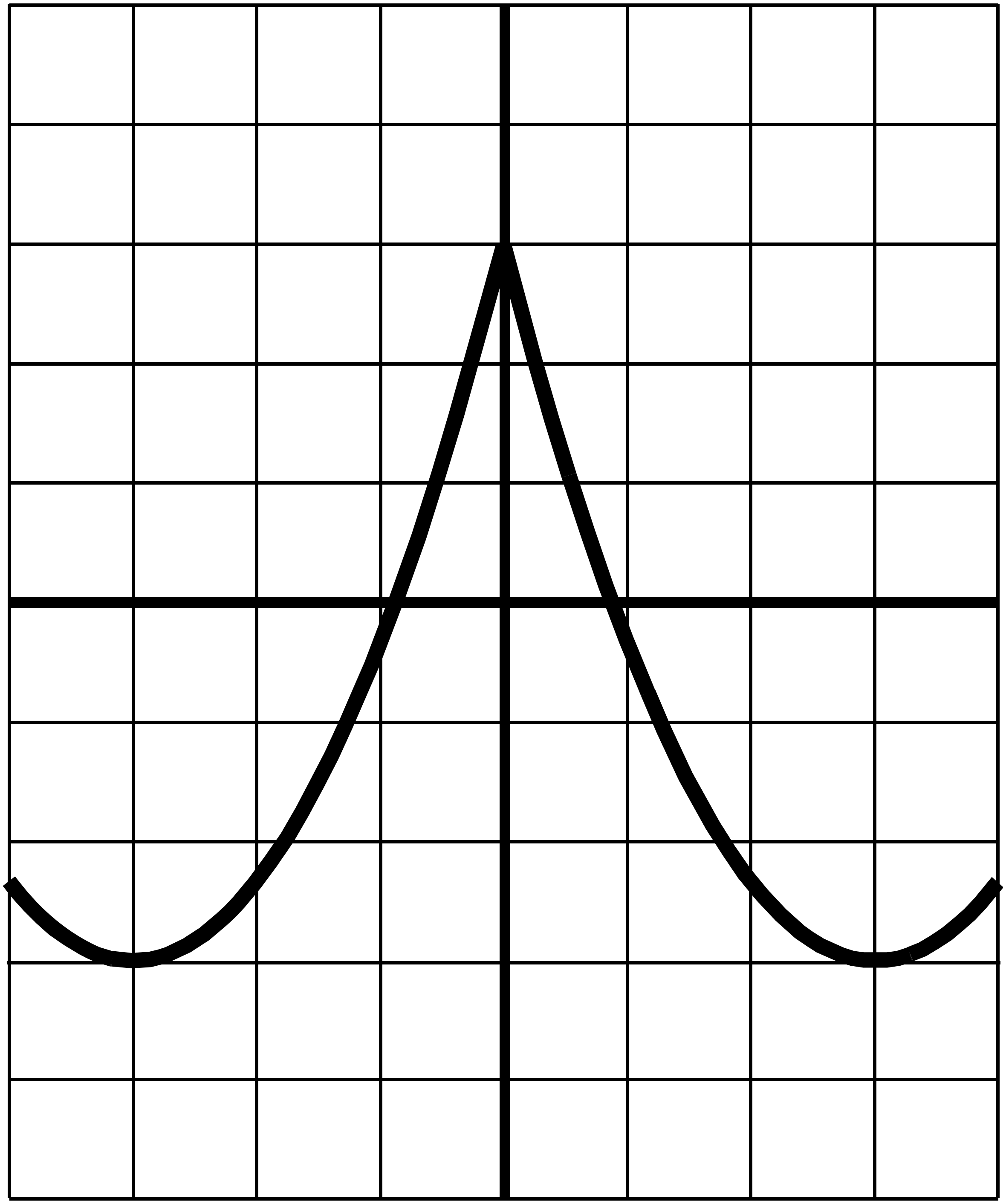
Graph 5



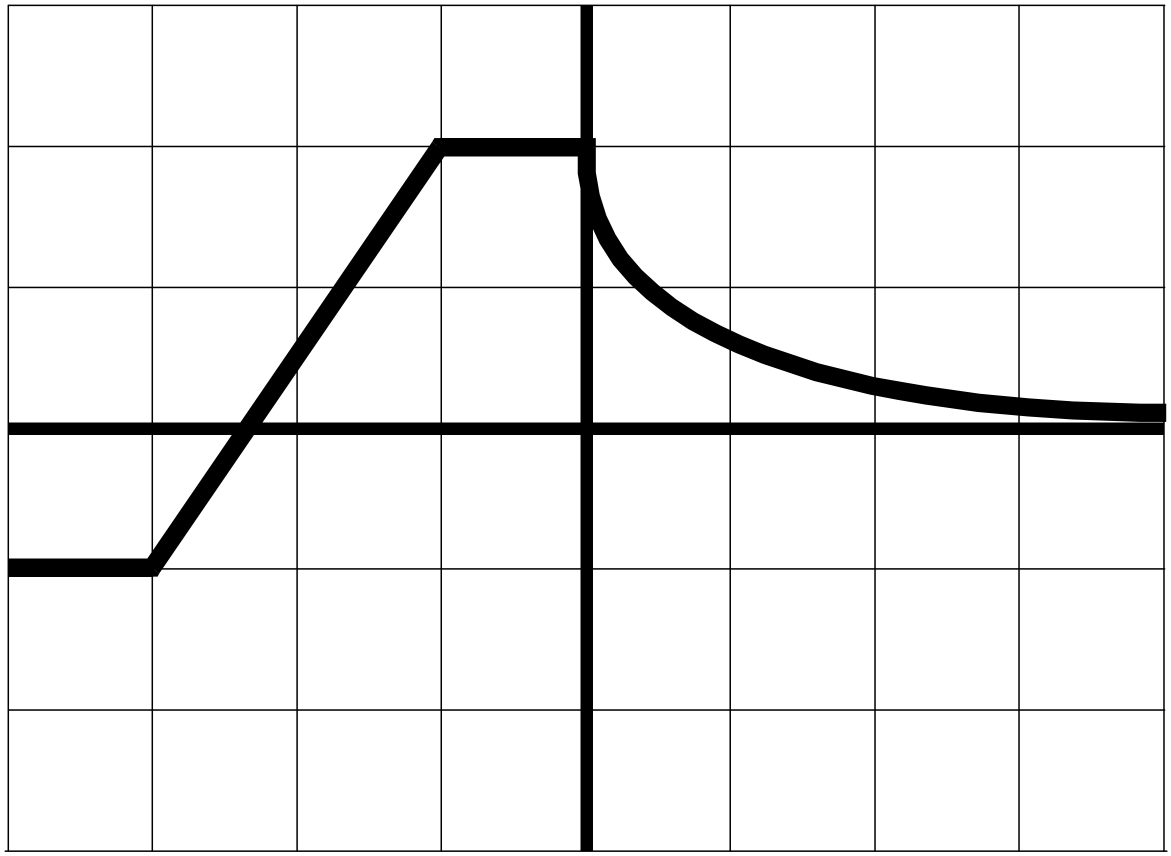
Graph 6



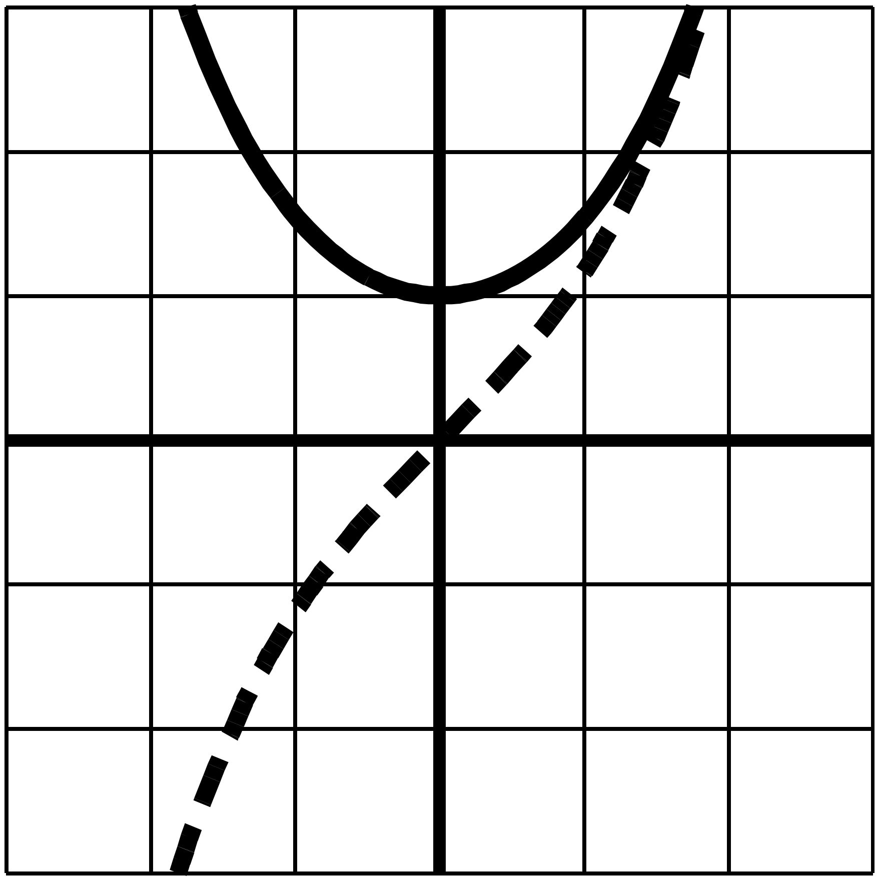
Graph 7



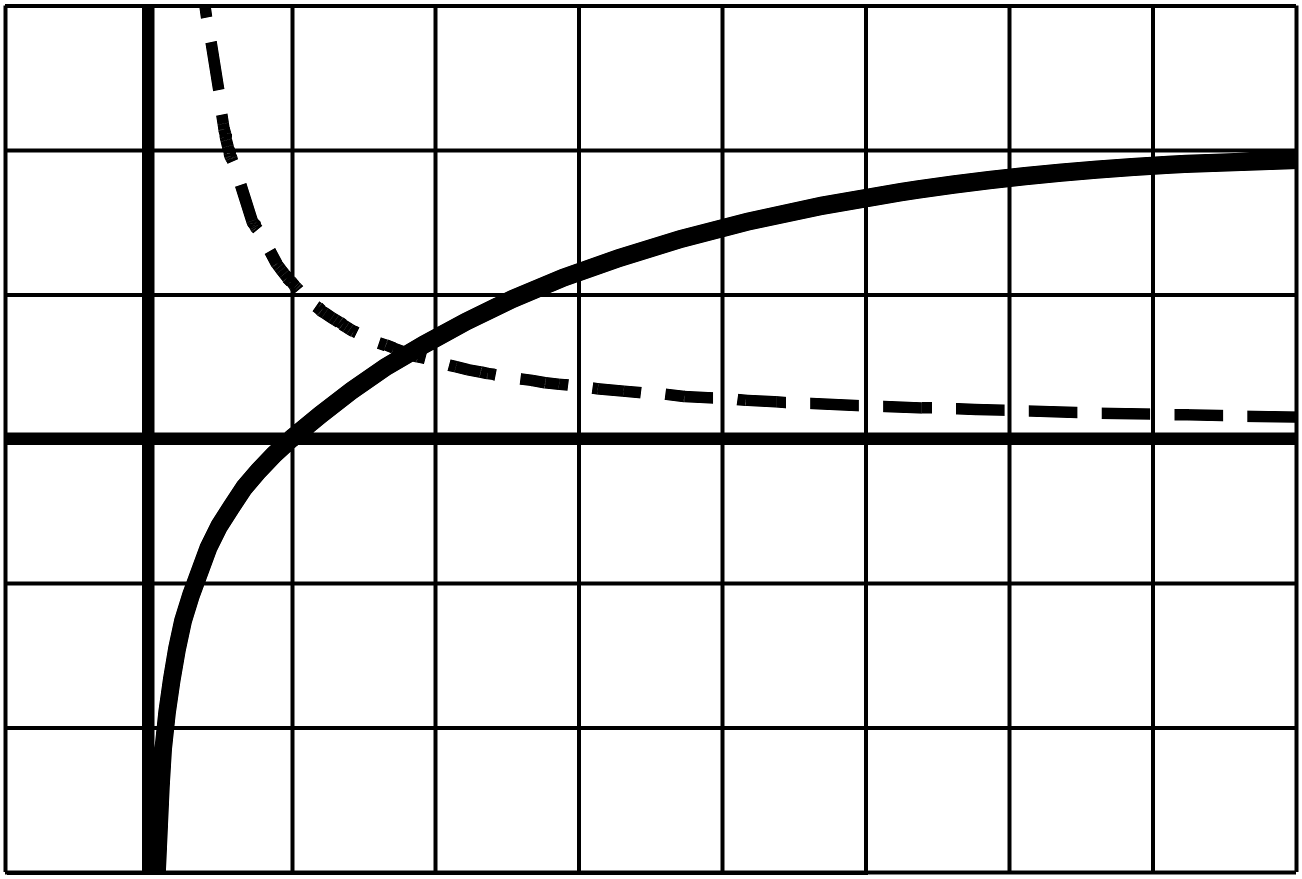
Graph 8



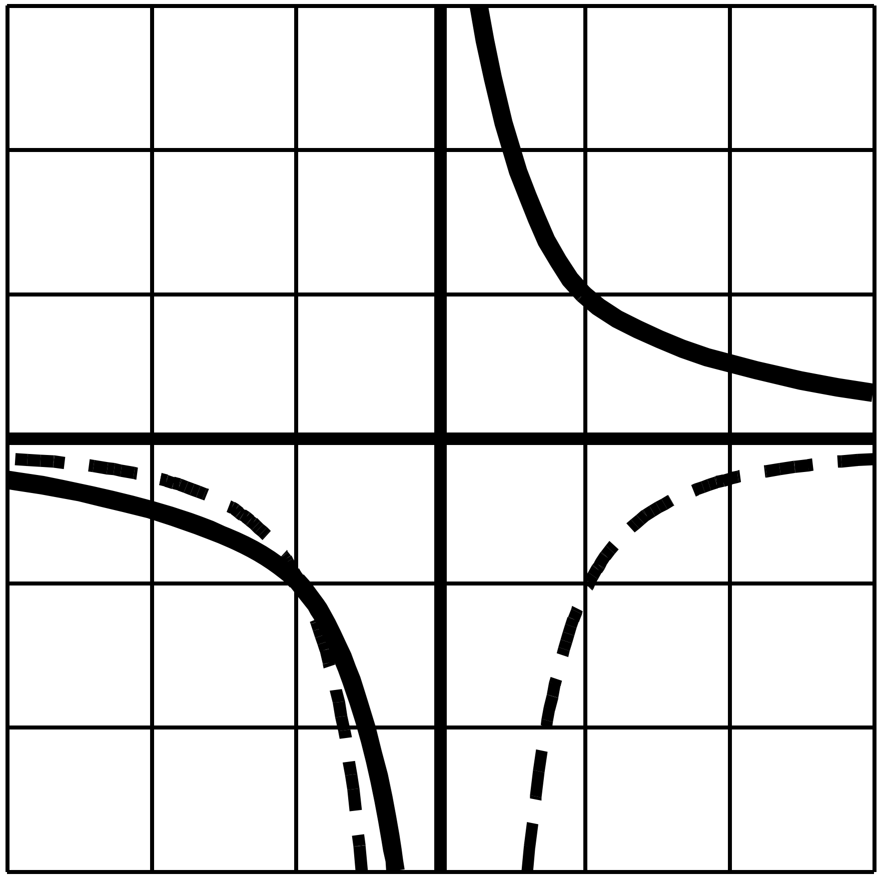
Answer 1



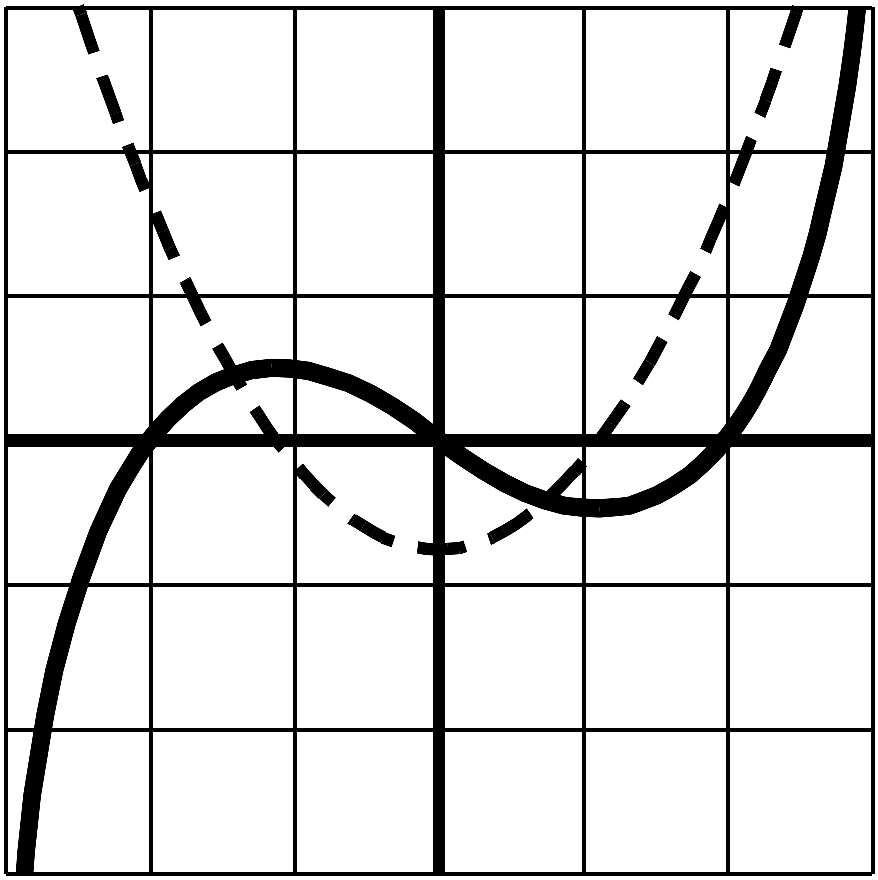
Answer 2



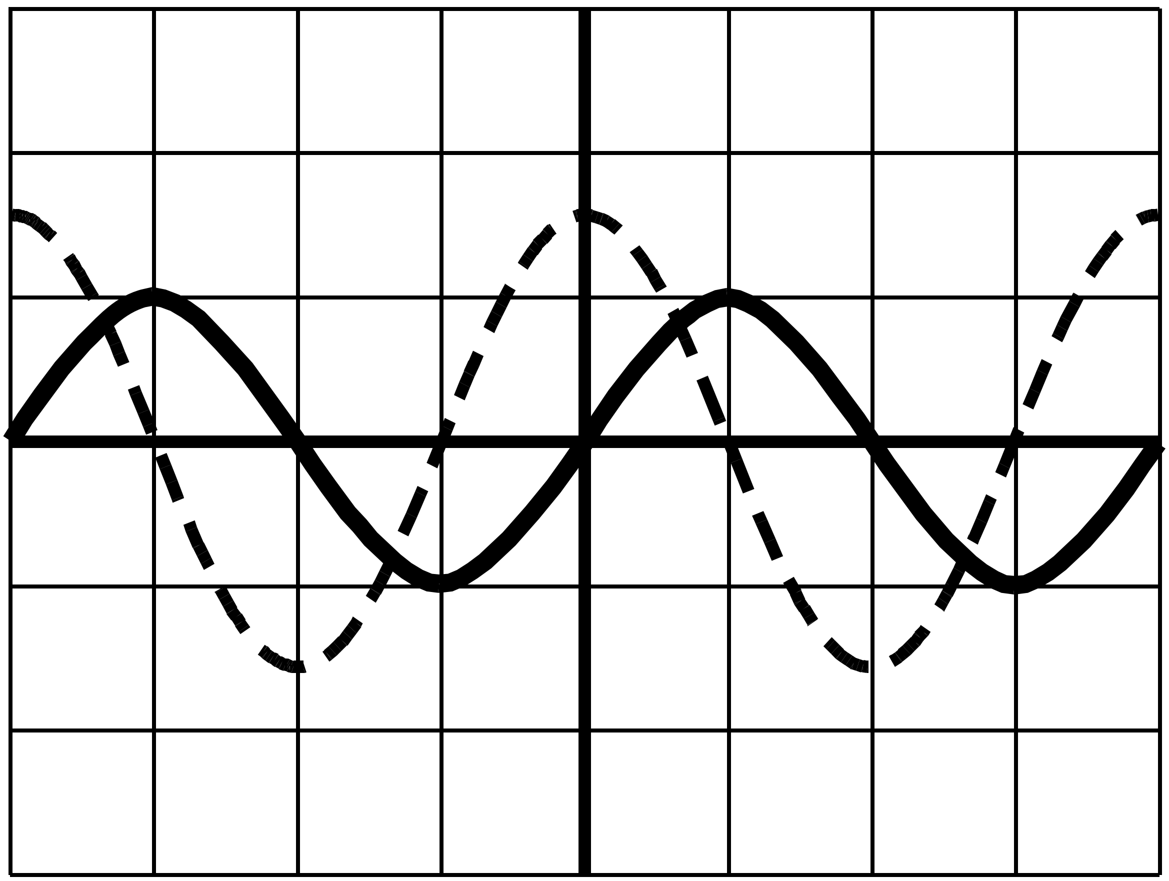
Answer 3



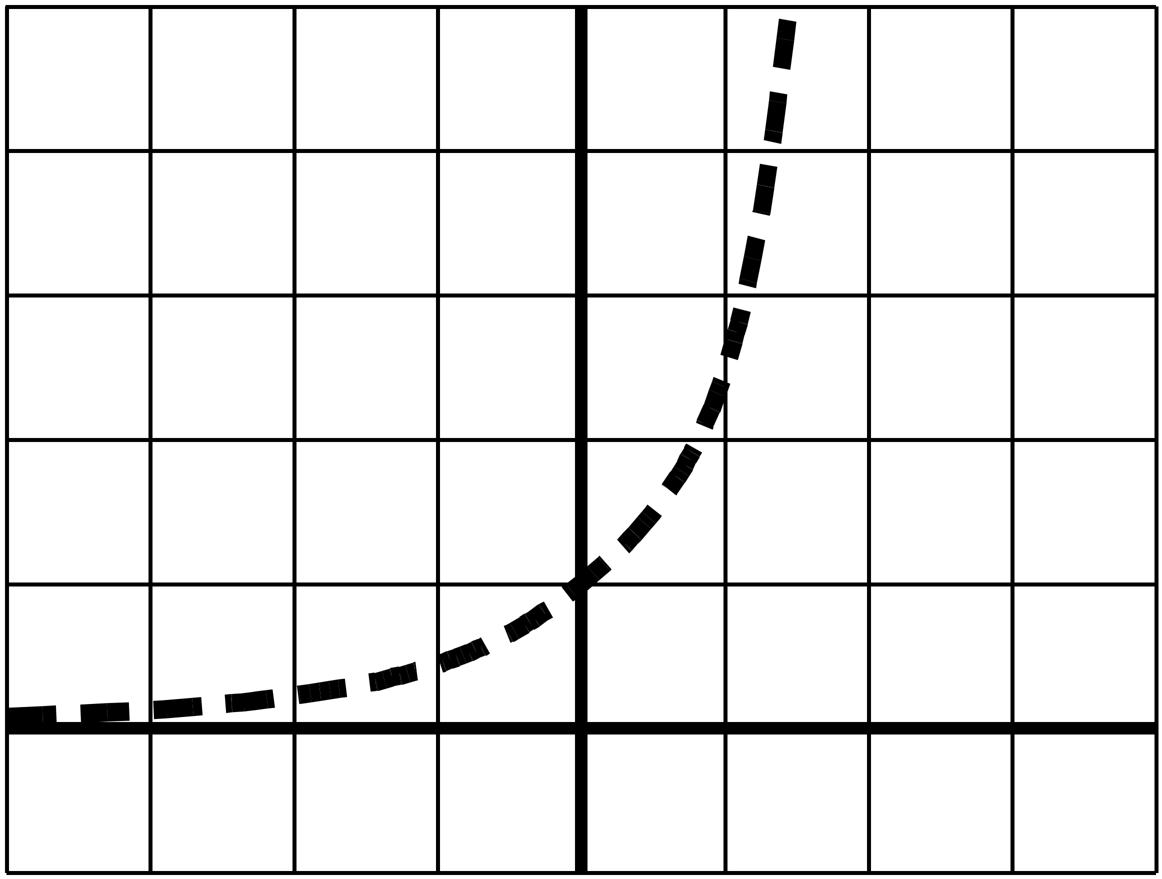
Answer 4



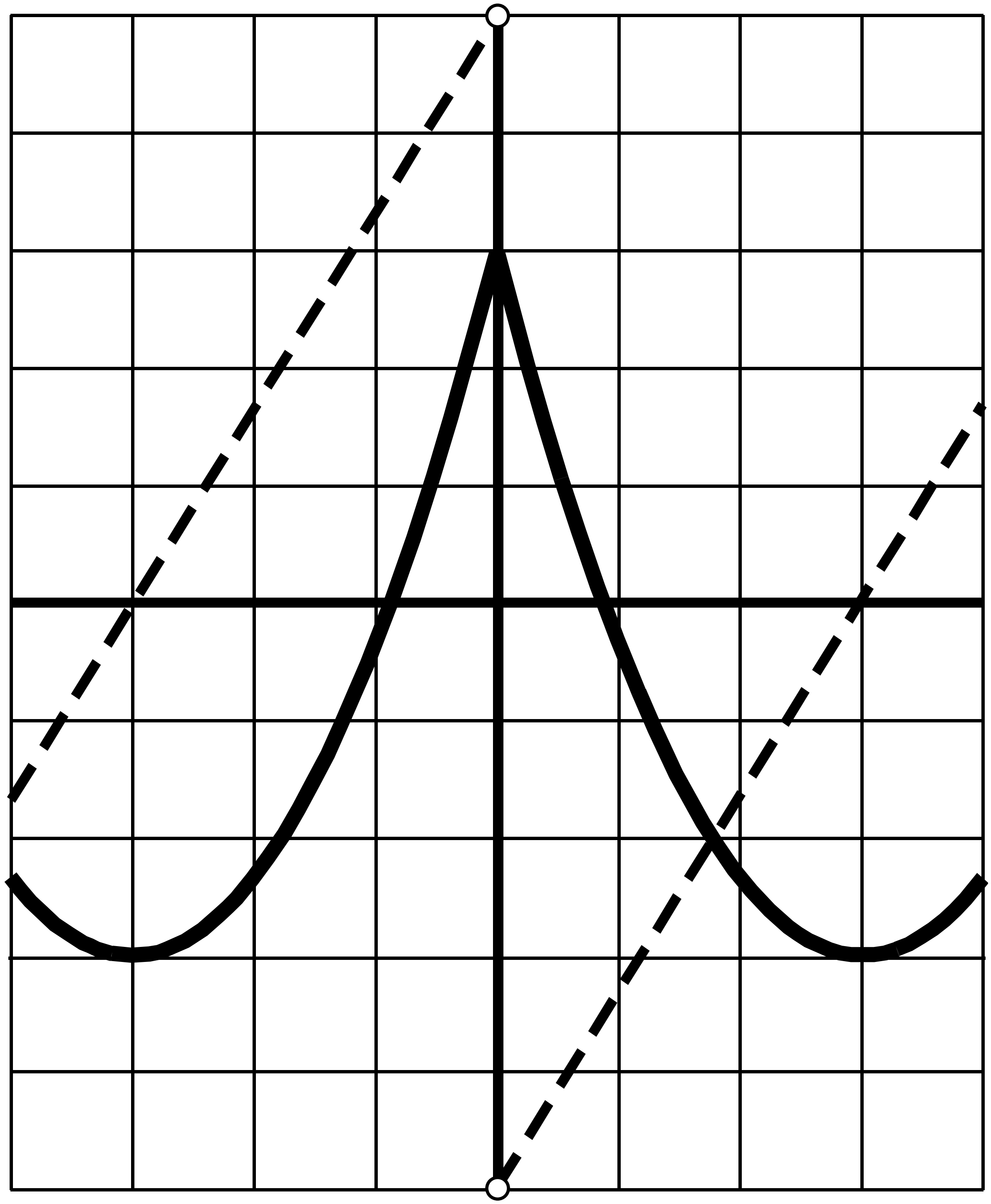
Answer 5



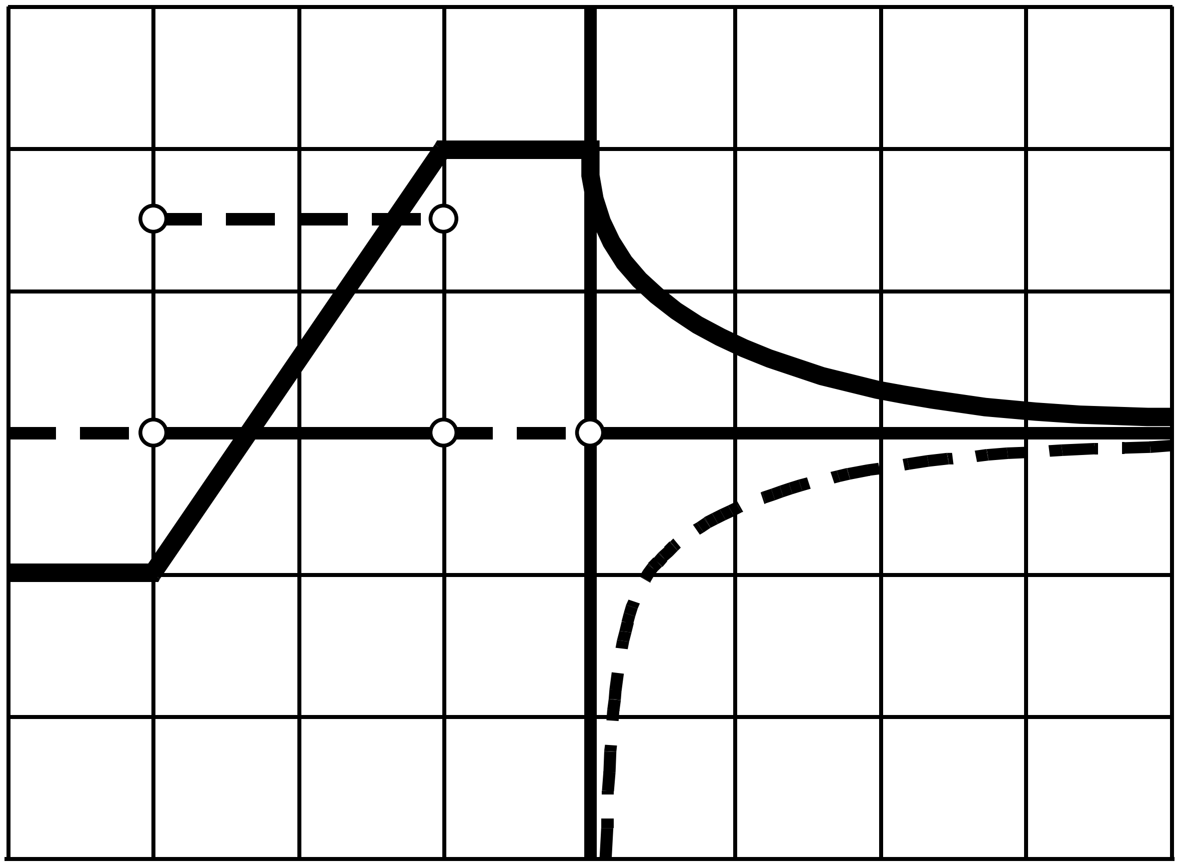
Answer 6



Answer 7



Answer 8



2.3 Differentiation Formulas

SUGGESTED TIME AND EMPHASIS

2–3 classes Essential material

POINTS TO STRESS

1. The Power, Constant Multiple, Sum and Difference Rules, and how they are developed from the limit definition of the derivative.
2. Justification of the Product and Quotient Rules.
3. The computation of derivatives using the above rules.

QUIZ QUESTIONS

- **TEXT QUESTION** Why don't we use the Quotient Rule every time we encounter a quotient?
  ANSWER Sometimes algebraic simplification can make the problem much easier.
- **DRILL QUESTION** Compute the derivative of $\left(x^6 - \frac{1}{8}x^4\right)\left(\sqrt{x} + \pi\right)$.
  ANSWER $\dfrac{x^6 - \frac{1}{8}x^4}{2\sqrt{x}} + \left(6x^5 - \frac{1}{2}x^3\right)\left(\sqrt{x} + \pi\right)$

MATERIALS FOR LECTURE

- As an introductory exercise, draw the function $f(x) = \dfrac{x^3}{3}$. Ask the students to estimate slopes at several points, perhaps using secant lines. Create a table of x versus $f'(x)$ and try to get them to see the pattern. Then review the idea of the derivative function. Similarly, examine the derivatives of $f(x) = 5x + 2$ and $f(x) = 3$.
- Let $f(x) = x^3 + 2x^2 + 3x + 4$. Find a point a, both visually and algebraically, where $f'(a) = 2$. Then ask them to find where the tangent line to the function $f(x) = x^3 - x + 1$ is parallel to the line $y = x$.
- Derive the Product Rule, and show its relationship to the Constant Multiple Rule (For example, one can find $\left[3e^x\right]'$ using *either* rule, but $\left[xe^x\right]'$ requires the Product Rule.)
- State and demonstrate a proof of the Quotient Rule via the Reciprocal Rule:

  Let $fg = 1$. Then by the Product Rule, $f'g + g'f = 0 \quad\Rightarrow\quad f'g = -g'f \quad\Rightarrow\quad f' = -\dfrac{g'f}{g} = -\dfrac{g'}{g^2}$ since $f = \dfrac{1}{g}$. This is the Reciprocal Rule: If $f = \dfrac{1}{g}$, then $f' = -\dfrac{g'}{g^2}$.

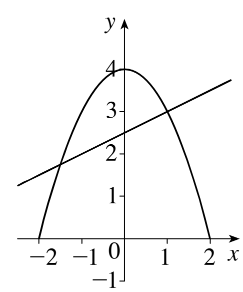
  This result allows us to prove the Quotient Rule:

$$\begin{aligned}\left(\frac{f}{g}\right)' &= \left(f \cdot \frac{1}{g}\right)' = f'\left(\frac{1}{g}\right) + f\left(\frac{1}{g}\right)' &&\text{(by the Product Rule)}\\ &= \frac{f'}{g} + f\left(-\frac{g'}{g^2}\right) &&\text{(by the Reciprocal Rule)}\\ &= \frac{f'g - fg'}{g^2}\end{aligned}$$

- Show that, if $f(x) = x^4 - x^2 + x + 1$, then $f^{(5)}(x) \equiv 0$. Conclude that if $f(x)$ is a polynomial of degree m, then $f^{(m+1)}(x) \equiv 0$.

WORKSHOP/DISCUSSION

- Do a complex-looking differentiation that requires algebraic simplification, such as $f(x) = \dfrac{x^2\sqrt[3]{x} + x\sqrt{x^3} - \left(\sqrt{2x}\right)^2}{\left(x^{2/3}\right)^2}$
- After the students have mastered the basics of the Power Rule, have them differentiate some notationally tricky functions such as x^π, $\sqrt[\pi]{x}$, and $\pi^{\sqrt{2}}$.
- Give some examples in which the automatic use of the Quotient Rule is not the best strategy to follow, for example, $f(x) = \dfrac{x^2 + \sqrt{x} - \sqrt[3]{x}}{x}$, $g(x) = \dfrac{x^3 - 2x}{17}$, or $h(x) = \dfrac{3}{x}$. The idea is to get the students to think and simplify first (if they can) before using any of the rules.
- Do an example like Exercise 53. If you actually use the Witch of Agnesi, the students may be interested to hear the history of the curve: Italian mathematician Maria Agnesi (1718–1799) was a scholar whose first paper was published when she was nine years old. She called a particular curve *versiera*, or "turning curve". John Colson from Cambridge confused the word with *avversiera*, or "wife of the devil," and translated it "witch".
- Graph $f(x) = 4 - x^2$ and compute the equations of the tangent line and the normal line at $x = 1$. Draw those lines and point out that, as predicted, they are perpendicular.



GROUP WORK 1: DOING A LOT WITH A LITTLE

This exercise starts out by showing what can be done with the Power Rule, and ends by foreshadowing the Chain Rule. The first page should be handed out separately, and then the second sheet handed out to groups who finish early. Emphasize that the solution to Problem 5 should resemble that of Problem 4 in form. If a group finishes both sheets far ahead of the others, ask them to figure out a formula for the derivative of $f(x) = (g(x))^n$, and to come up with a few examples to check their formula. (Notice that when we state the Power Rule, we allow n to be any real number.)

ANSWERS (Notation may vary)

1. $f'(x) = 10x^9 + 7x^8 + 4x^7 - 35x^6 - 1.98x^5 + 5\pi x^4 - 4\sqrt{2}x^3$

2. $f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$, $g'(x) = -\dfrac{3}{x^4} + \dfrac{3}{4\sqrt[4]{x^7}}$, $h'(x) = \frac{9}{2}x^{7/2} - x^{-3/2}$

3. $f'(x) = 64x^3$, $g'(x) = 15x^{14}$

4. This follows immediately when the given functions are expanded.

5. $f'(x) = n(kx)^{n-1}k$, $g'(x) = n\left(x^k\right)^{n-1} \cdot kx^{k-1}$

GROUP WORK 2: FIND THE ERROR

This is the first of several exercises where students will try to find mistakes in somebody else's reasoning. When first faced with a task like this, some students will pick a line towards the end, show it is false, and then consider the task completed. It is important to stress you want them to find the reasoning error; what the person who did the work did incorrectly to get that false line.

If a student still doesn't understand the idea, put it this way: "The person who wrote this listens to what you just said, and says, 'What did I do wrong?' Can you give an answer that will help that person avoid making similar mistakes in the future?"

ANSWER The function "$\underbrace{x+x+\cdots+x}_{x \text{ times}}$" is defined only for integer values of x and is thus not a differentiable function.

GROUP WORK 3: BACK AND FORTH

This exercise foreshadows antiderivatives and gives students an opportunity to practice using the derivative rules they've learned so far.

The students pair up, and decide who is A and who is B. Seat the A's on one side of the room and the B's on the other side. All the A's get one sheet, and all the B's get the other sheet. The students compute five derivatives, without simplifying, and write their answers in the space provided. Emphasize that they should write only their unsimplified answers, not the work leading up to them, in the blanks. Then they trade papers with their partner and try to undo what their partner has done, that is, find the antiderivative.

If a pair finishes early, have them repeat the exercise, making up their own functions, and simplifying at will.

When closing this exercise, have the class notice that there was no way to recover the constant terms in Problems 1 and 5. Ask what this implies about the general problem of finding a function whose derivative is equal to a given function.

ANSWERS

FORM A $f'(x)=20x^3+3x$ (the 4 is unrecoverable), $g'(x)=x^{-1/2}-x^{-3/4}$, $h'(x)=\left(x^2+2x+4\right)\left(3x^2-1\right)+(2x+2)\left(x^3-x-3\right)$, $j'(x)=\dfrac{\left(\sqrt{x}+1\right)\left(4x^3-4\right)-\left[\left(x^4-4x+3\right)/\left(2\sqrt{x}\right)\right]}{\left(\sqrt{x}+1\right)^2}$, $k'(x)=-x^{-4/3}$ (the 42 is unrecoverable)

FORM B $f'(x)=-6x^2+8\sqrt{x}$ (the 8 is unrecoverable), $g'(x)=\frac{1}{5}\left[\left(3x^2\right)\left(x^3+x\right)+\left(x^3+1\right)\left(3x^2+1\right)+12x\right]$, $h'(x)=\left(x^3+x^2+2x\right)\left(10x-8x^3+8\right)+\left(5x^2-2x^4+8x\right)\left(3x^2+2x+2\right)$, $j'(x)=1+2x\ (x\neq 0)$, $k'(x)=-\frac{22}{3}x^{-2/3}$

GROUP WORK 4: SPARSE DATA

This exercise allows the students to practice the rules they have learned, with a minimum of algebraic manipulation. The students should work on these problems in groups of three or four, perhaps choosing groups of students with similar algebraic proficiency. Problem 5 uses the General Power Rule, which was illustrated in Group Work 1.

ANSWERS **1.** 0 **2.** -48 **3.** $\frac{43}{25}$ **4.** -18 **5.** $\frac{1}{3}$

HOMEWORK PROBLEMS

CORE EXERCISES 2, 5, 12, 18, 24, 26, 32, 50, 51, 60, 101

SAMPLE ASSIGNMENT 2, 5, 12, 18, 24, 26, 32, 32, 35, 47, 50, 51, 60, 61, 68, 73, 90, 94, 101, 105

EXERCISE	D	A	N	G
2		×		
5		×		
12		×		
18		×		
24		×		
26		×		
32		×		
32		×		
35		×		
47				×
50		×		×
51		×		
60		×		
61	×			
68			×	×
73				×
90		×		
94		×		
101		×		
105		×		

GROUP WORK 1, SECTION 2.3

Doing a Lot with a Little

Section 2.3 introduces the Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$, where n is any real number. The good news is that this rule, combined with the Constant Multiple and Sum Rules, allows us to take the derivative of even the most formidable polynomial with ease! To demonstrate this power, try Problem 1:

1. *A formidable polynomial:*

$$f(x) = x^{10} + \tfrac{7}{9}x^9 + \tfrac{1}{2}x^8 - 5x^7 - 0.33x^6 + \pi x^5 - \sqrt{2}x^4 - 42$$

Its derivative:

$$f'(x) =$$

The ability to differentiate polynomials is only one of the things we've gained by establishing the Power Rule. Using some basic definitions, and a touch of algebra, there are all kinds of functions that can be differentiated using the Power Rule.

2. *All kinds of functions:*

$$f(x) = \sqrt[3]{x} + \sqrt[5]{2} \qquad g(x) = \frac{1}{x^3} - \frac{1}{\sqrt[4]{x^3}} \qquad h(x) = \frac{x^5 - 3\sqrt{x} + 2}{\sqrt{x}}$$

Their derivatives:

$$f'(x) = \qquad g'(x) = \qquad h'(x) =$$

Unfortunately, there are some deceptive functions that look like they should be straightforward applications of the Power and Constant Multiple Rules, but actually require a little thought.

3. *Some deceptive functions:*

$$f(x) = (2x)^4 \qquad g(x) = \left(x^3\right)^5$$

Their derivatives:

$$f'(x) = \qquad g'(x) =$$

The process you used to take the derivative of the functions in Problem 3 can be generalized. In the first case, $f(x) = (2x)^4$, we had a function that was of the form $(kx)^n$, where k and n were constants ($k = 2$ and $n = 4$). In the second case, $g(x) = \left(x^3\right)^5$, we had a function of the form $\left(x^k\right)^n$. Now we are going to find a pattern, similar to the Power Rule, that will allow us to find the derivatives of these functions as well.

4. Show that your answers to Problem 3 can also be written in this form:

$$f'(x) = 4(2x)^3 \cdot 2 \qquad g'(x) = 5\left(x^3\right)^4 \cdot 3x^2$$

And now it is time to generalize the Power Rule. Consider the two general functions, and try to find expressions for the derivatives similar in form to those given in Problem 4. You may assume that n is an integer.

5. *Two general functions*:

$$f(x) = (kx)^n \qquad g(x) = \left(x^k\right)^n$$

Their derivatives:

$$f'(x) =$$
$$g'(x) =$$

GROUP WORK 2, SECTION 2.3

Find the Error

It is a bright Spring morning. You have just finished your Chemistry lab, and have a Physics class starting in a half hour, so you have a little bit of time to sit on a park bench and relax by leafing through your *Calculus* book. Suddenly, you notice a wild-eyed, hungry-looking stranger looking over your shoulder.

“Lies! Lies!” he yells. “That book there is filled with nothing but lies!”

“Why, you are mistaken,” you explain. “My *Calculus* book is chock-a-block with knowledge and useful wisdom.”

“Oh yeah? Well what would your calculus book say about THIS?” he demands, and hands you a piece of paper with the following written on it:

For $x > 0$:

$$\begin{aligned}
x &= \underbrace{1 + 1 + \cdots + 1}_{x \text{ times}} \\
x^2 &= \underbrace{x + x + \cdots + x}_{x \text{ times}} \\
D\left(x^2\right) &= D\left(\underbrace{x + x + \cdots + x}_{x \text{ times}}\right) \\
D\left(x^2\right) &= \underbrace{D(x) + D(x) + \cdots + D(x)}_{x \text{ times}} \\
2x &= \underbrace{1 + 1 + \cdots + 1}_{x \text{ times}} \\
2x &= x \\
2 &= 1
\end{aligned}$$

“Put THAT in your pipe and smoke it!” At that, the gentleman runs off, screaming, “I’ll be back!” into the wind.

Is all of mathematics wrong? Is two really equal to one? Are “two for one” specials really no bargain at all? Is “six of one” really not “half a dozen of the other”? Or is there a mistake in your new friend’s reasoning? If so, what is it?

GROUP WORK 3, SECTION 2.3

Back and Forth (Form A)

Compute the following derivatives. Write your answers at the bottom of this sheet, where indicated. When finished, fold the top of the page backward along the dotted line and hand to your partner.

Do not simplify.

1. $f(x) = 5x^4 + \frac{3}{2}x^2 - 4$

2. $g(x) = 2\sqrt{x} - 4\sqrt[4]{x}$

3. $h(x) = \left(x^2 + 2x + 4\right)\left(x^3 - x - 3\right)$

4. $j(x) = \dfrac{x^4 - 4x + 3}{\sqrt{x} + 1}$

5. $k(x) = \dfrac{3}{\sqrt[3]{x}} + 42$

ANSWERS

$f'(x) =$

$g'(x) =$

$h'(x) =$

$j'(x) =$

$k'(x) =$

GROUP WORK 3, SECTION 2.3

Back and Forth (Form B)

Compute the following derivatives. Write your answers at the bottom of this sheet, where indicated. When finished, fold the top of the page backward along the dotted line and hand to your partner.

Do not simplify.

1. $f(x) = -2x^3 + \frac{\sqrt{8}}{2}x^2 - 8$

2. $g(x) = \dfrac{\left(x^3+1\right)\left(x^3+x\right)+6x^2}{5}$

3. $h(x) = \left(x^3+x^2+2x\right)\left(5x^2-2x^4+8x\right)$

4. $j(x) = \dfrac{x^2+x^3}{x}$

5. $k(x) = \sqrt{11} - 22\sqrt[3]{x}$

- -

ANSWERS

$f'(x) =$

$g'(x) =$

$h'(x) =$

$j'(x) =$

$k'(x) =$

GROUP WORK 4, SECTION 2.3

Sparse Data

Assume that $f(x)$ and $g(x)$ are differentiable functions about which we know very little. In fact, assume that all we know about these functions is the following table of data:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	3	1	-5	8
-1	-9	7	4	1
0	5	9	9	-3
1	3	-3	2	6
2	-5	3	8	?

This isn't a lot of information. For example, we can't compute $f'(3)$ with any degree of accuracy. But we are still able to figure some things out, using the rules of differentiation.

1. Let $h(x) = \left(\sqrt[3]{x}\right)^4 f(x)$. What is $h'(0)$?

2. Let $j(x) = -4f(x)g(x)$. What is $j'(1)$?

3. Let $k(x) = \dfrac{xf(x)}{g(x)}$. What is $k'(-2)$?

4. Let $l(x) = x^3 g(x)$. If $l'(2) = -48$, what is $g'(2)$?

5. Let $m(x) = \dfrac{1}{f(x)}$. What is $m'(1)$?

APPLIED PROJECT **Building a Better Roller Coaster**

This project models a typical hill in a roller coaster ride using two lines as the sides and a parabola for the peak area. It also discusses how to smooth this model to have a continuous second derivative by using cubic connecting functions between the parabola and the two lines. A computer algebra system is needed to solve the resulting equations. In their report, students should address the question, "*Why* do we want the second derivative to be continuous?"

2.4 Derivatives of Trigonometric Functions

SUGGESTED TIME AND EMPHASIS

1 class Essential material

POINTS TO STRESS

Formulas for the derivatives of the standard trigonometric functions.

QUIZ QUESTIONS

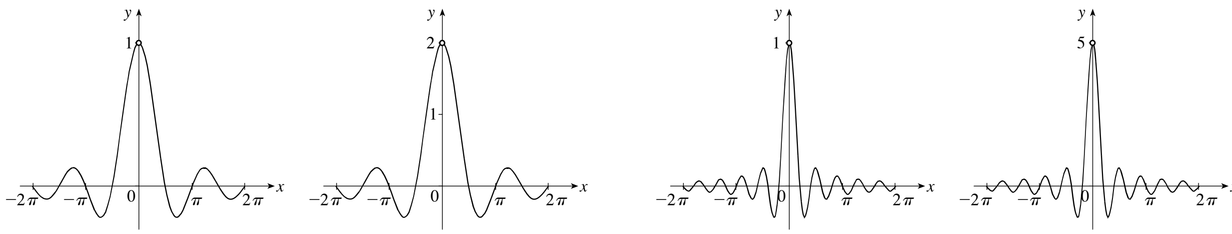
- **TEXT QUESTION** Why does the text bother going through all the fuss of computing $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ and $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta}$?
  ANSWER When deriving the formulas for the derivatives for $\sin\theta$ and $\cos\theta$, these limits arise when taking the limits of the difference quotients. These computations are necessary to finish the derivations.
- **DRILL QUESTION** What is $\lim_{h \to 0} \frac{\tan\left(\frac{\pi}{4} + h\right) - \tan\left(\frac{\pi}{4}\right)}{h}$?
  (A) 2 (B) $-\frac{\sqrt{2}}{2}$ (C) 0 (D) 1 (E) Does not exist
  ANSWER (A)

MATERIALS FOR LECTURE

- Many students may need a review of notation: $\sin^2 x = (\sin x)^2$, $\sin x^2 = \sin\left(x^2\right)$, $(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$, $\sin x^{-1} = \sin\frac{1}{x}$, but $\sin^{-1} x$ represents the inverse sine of x, $\arcsin x$, and not any of the previous functions.
- Demonstrate simple harmonic motion in different ways such as observing the end of a vertical spring, marking the edge of a spinning disk, or swinging an object on a chain.
- Have the students set their calculators to degrees and approximate the derivative of $\cos x$ at $x = \frac{\pi}{2}$ by zooming in on the graph of $\cos x$. Repeat the exercise with their calculators set to radians. Discuss the reason why the answers are different, and why only one is considered correct. Show how the slope of the tangent to the graph of $\sin x$ at $x = 0$ is *not* 1 if the x-axis is calibrated in degrees instead of radians.
  ANSWER The derivation of $(\sin\theta)' = \cos\theta$ involved using the fundamental trigonometric limit, which assumed θ was in radians.

WORKSHOP/DISCUSSION

- Demonstrate that $\lim_{x \to 0} \frac{\sin ax}{ax} = 1$ for any positive a. Then ask students to find $\lim_{x \to 0} \frac{\sin ax}{x}$. Show how this argument can be extended to derive the formulas $\frac{d}{dx} \sin ax = a \cos ax$ and $\frac{d}{dx} \cos ax = -a \sin ax$. Finally, demonstrate that your results make sense by drawing graphs of $\frac{\sin ax}{ax}$ and $\frac{\sin ax}{x}$ for various values of a.



$y = \dfrac{\sin 2x}{2x}$ $\quad y = \dfrac{\sin 2x}{x}$ $\quad y = \dfrac{\sin 5x}{5x}$ $\quad y = \dfrac{\sin 5x}{x}$

- Consider $f(x) = \frac{1}{2}x + \cos x$, $0 \le x \le 2\pi$. Discuss local maxima and minima of $f(x)$. Repeat for $g(x) = \frac{9}{10}x + \cos x$ and $h(x) = x + \cos x$. Discuss why h is qualitatively different from f and g.

GROUP WORK 1: THE MAGNIFICENT SIX

After showing the students that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$, it is possible to use the Quotient Rule to derive the trigonometric derivatives on their own, and the process of deriving these formulas is good practice at using the rules learned so far.

ANSWERS

1. $\cos x$ **2.** $-\sin x$ **3.** $\sec^2 x$

4. $\tan x \sec x$ **5.** $-\csc^2 x$ **6.** $-\cot x \csc x$

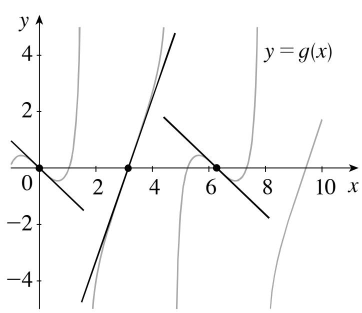
GROUP WORK 2: USING OUR NEW KNOWLEDGE

ANSWERS

1. $-1, 3, -1$

2. $y = -x$, $y = 3x - 3\pi$, $y = -x + 2\pi$

3.



4. There is no tangent line at $y = \frac{\pi}{2}$ because the function has a vertical asymptote there.

GROUP WORK 3: WHEN THE LIGHTS GO DOWN IN THE CITY

This activity will help the students understand the relationship between a trigonometric function in the abstract, and a trigonometric function as a model for real situations.

Creative use of technology can be encouraged here. It is important to stress to the students that Problem 2 assumes that they are looking at only a one-month window. Problem 6 foreshadows the technique of linear approximation covered in Section 2.9.

ANSWERS

1. Maximum: 1, minimum: 20 **2.** It is part of a cosine curve. **3.** December **4.** May
5. 3.095 minutes per day (0.5158 hours per day) **6.** $3.095 \cdot 31 = 95.94$, accurate to within about 1%.

HOMEWORK PROBLEMS

CORE EXERCISES 3, 7, 21, 28, 42, 41, 44

SAMPLE ASSIGNMENT 3, 7, 21, 28, 33, 37, 41, 42, 44

EXERCISE	D	A	N	G
3		×		
7		×		
21		×		
28		×		×
33		×		
37	×	×		
41		×		
42	×	×		
44	×			

GROUP WORK 1, SECTION 2.4

The Magnificent Six

The derivative of $f(x) = \sin x$ was derived for you in class. From this one piece of information, it is possible to figure out formulas for the derivatives of the other five trigonometric functions. Using the trigonometric identities you know, compute the following derivatives. Simplify your answers as much as possible.

1. $(\sin x)' =$

2. $(\cos x)' =$

3. $(\tan x)' =$

4. $(\sec x)' =$

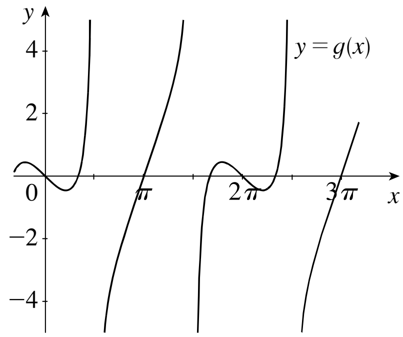
5. $(\cot x)' =$

6. $(\csc x)' =$

GROUP WORK 2, SECTION 2.4

Using Our New Knowledge

The following is a graph of $g(x) = \tan x - 2\sin x$.



There are some things we can say about the graph just by looking at the picture, although our intuition may sometimes mislead us.

1. Compute $g'(0)$, $g'(\pi)$, and $g'(2\pi)$.

2. Find equations of the lines tangent to this curve at $x = 0$, $x = \pi$, and $x = 2\pi$.

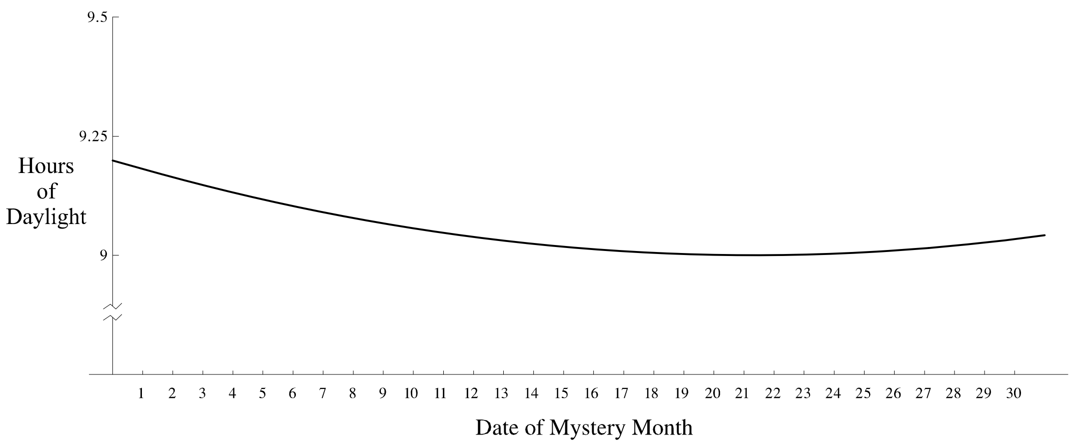
3. Graph the equations you found in Problem 2, and make sure they look as they should.

4. What happens when you try to find the equation of the line tangent to this curve at $x = \frac{\pi}{2}$? Why?

GROUP WORK 3, SECTION 2.4

When the Lights Go Down in the City

The number of hours of daylight in Summitville, Canada varies between 9 hours and 15 hours per day. A model for the number of daylight hours on day t is $D(t) = 12 - 3\cos(0.0172(t + 11))$, $0 < t \leq 365$. ($t = 1$ corresponds to January 1.) The graph for a particular month looks like this:



1. On approximately what day of the month does this graph achieve its minimum? Its maximum?

2. Why does this graph have the shape that it does?

3. What month is this graph likely to represent?

4. For which month would you expect to see a graph shaped like this one, only upside-down?

5. How rapidly are we gaining daylight 90 days after the minimum occurs?

6. A newspaper in Summitville states that during the period of 31 days starting from day 68 after the minimum, we gain 1 hour and 35 minutes of sunlight. Use the rate of change computed in Problem 5 to estimate the change in hours of sunlight over this period. How close is your estimate to the figure reported in the newspaper?

2.5 The Chain Rule

SUGGESTED TIME AND EMPHASIS

$1\frac{1}{2}$–2 classes Essential material

POINTS TO STRESS

1. A justification of the Chain Rule by interpreting derivatives as rates of change.
2. The use of the Chain Rule to compute derivatives.

QUIZ QUESTIONS

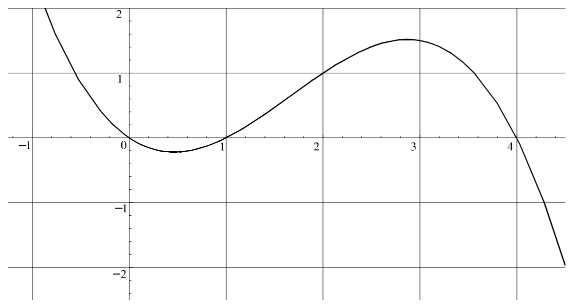
- **TEXT QUESTION** The text presents the two forms of the Chain Rule: $(f(g(x)))' = f'(g(x))\,g'(x)$ and $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. Do these two equations say the same thing? Explain your answer.
  ANSWER They do. Let $y = f(u)$ and $u = g(x)$. Then the statement $f(g(x))' = f'(g(x))\,g'(x)$ becomes $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.
- **DRILL QUESTION** Compute $\frac{d}{dx}\sin x^2$ and $\frac{d}{dx}\sin^2 x$.
  ANSWER $2x\cos x^2$, $2\sin x\cos x$

MATERIALS FOR LECTURE

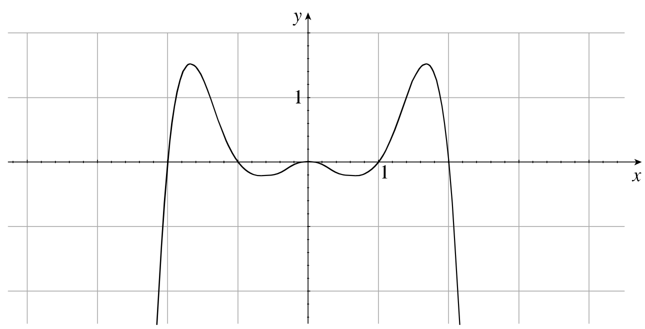
- The following is one way to introduce the Chain Rule:
  Before formally discussing the rule, do two examples of differentiating multi-nested functions. Explain to the students that you aren't going to justify anything yet, but that you just want them to see the pattern before getting into the material. After every step, say something like, "The derivative of $\sin x$ is $\cos x$, so the derivative of the sine of this stuff is the cosine of this stuff, times the derivative of what's left." After the students have seen the pattern with functions like $\left(\sin\left(\cos\left(x^2+4x+5\right)\right)\right)^{33}$, you should justify the Chain Rule and discuss the details.
- Show how to compute derivatives, using the Chain Rule, in one line. Take the derivative of $\sin\left(x^4+1\right)$, first by using the Chain Rule explicitly [$f(u) = \sin u$, $u(x) = x^4+1$], and then by inspection [the derivative of $\sin\left(x^4+1\right)$, which is $\cos\left(x^4+1\right)$ times the derivative of x^4+1, which is $4x^3$.]
- Address the question: "Where do you stop when using the Chain Rule?" For example, why is it false that
  $\frac{d}{dx}\sin\left(x^5+4x^2\right) \stackrel{?}{=} \left[\cos\left(x^5+4x^2\right)\right]\left(5x^4+8x\right)\left(20x^3+8\right)\left(60x^2\right)(120x)(120)$?
  One way to help students decide "when to stop" is to draw their attention to the text's Reference Page 5 (Differentiation Rules). One stops when the derivative is one of the primitive rules such as the ones on that page.
- Justify the Chain Rule using rate of change arguments, such as the following: One factory converts sugar to chocolate ($c = 8s$) and another converts chocolate to candy bars ($b = 16c$). Finding the rate at which sugar is converted to candy bars can be used to help justify the Chain Rule, particularly if the units of the relevant quantities are emphasized.

WORKSHOP/DISCUSSION

- Compute some derivatives, such as those of $\sec\left(x^2+x\right)$, $\left(\dfrac{\cos x}{x^2+1}\right)^3$, $\sqrt[3]{x+\cos x^2}$, and $\cos\left(\sin\left(x^2\right)\right)$.
- Compute the equation of the line tangent to $y=\cos\left(x+\dfrac{\pi}{\sqrt{x+1}}\right)$ at $(0,-1)$.
- Draw the following graph of $f(x)$ (or copy it onto a transparency).



Tell the students that $f'(0)=-1$, $f'(1)=\frac{3}{4}$, $f'(2)=1$, and $f'(4)=-3$. Define $g(x)=f\left(x^2\right)$. First compute $g(-1)$, $g(0)$, $g(1)$, $g\left(\sqrt{2}\right)$, and $g(2)$. Then compute $g'(0)$, $g'(-2)$, and $g'(2)$. Finally, sketch $g(x)$ as below. Use the graph to verify the values for $g(x)$ and $g'(x)$ computed above.



GROUP WORK 1: UNBROKEN CHAIN

This is meant to be a gentle introduction to the mechanics of taking derivatives using the Chain Rule. You may be surprised at the difficulty some groups have with Problem 4 of the activity, but by the end they all should be ready to go home and practice.

Start by "warming the class up" as a large group by having them take the derivatives of functions like $x^{3.24}$, $\sin x$, $\sqrt{x}$, $\tan x$, and so on. This quick review is important, because the activity works best if their mental focus is on the Chain Rule, as opposed to formulas they should already know.

While helping the individual groups, don't volunteer that the answers to most of the questions are supersets of the previous questions. They are supposed to discover this pattern for themselves.

If a group finishes early, give them a function like $\cos\left(x^2+\sqrt{x}\right)\sin(1/x)$ to try.

When they are finished, write the solutions to Problems 4, 6, and 7 on the board. Ask the students if they need you to write the solutions to the earlier ones. After they say "no", try to get them to explain why it isn't necessary. (If they say "yes", refuse and ask them why you are refusing.)

ANSWERS

1. $\cos 3x \cdot 3$ **2.** $3(\sin 3x)^2 \cos 3x \cdot 3$ **3.** $3(\sin 3x)^2 \cos 3x \cdot 3 + 5$

4. $5\left[(\sin 3x)^3 + 5x\right]^4 \left[3(\sin 3x)^2 (\cos 3x) 3 + 5\right]$ (check their parentheses carefully)

5. $1 - x^{-2}$ **6.** $\frac{1}{2}\left[x + (1/x)\right]^{-1/2} (1 - x^{-2})$

7. $\left[(\sin 3x)^3 + 5x\right]^5 \frac{1}{2}\left(x + \frac{1}{x}\right)^{-1/2} (1 - x^{-2}) + 5\left[(\sin 3x)^3 + 5x\right]^4 \left[3(\sin 3x)^2 (\cos 3x) 3 + 5\right] \left(\sqrt{x + \frac{1}{x}}\right)$
(If the students don't write out the answer to Part 7, instead referring to the answers to previous parts, don't penalize them; they have gotten the point.)

GROUP WORK 2: CHAIN RULE WITHOUT FORMULAS

This exercise works best with pairs or groups of three. Before handing it out, write both forms of the Chain Rule on the board. If a group finishes early, ask them where $h' = 0$ and over which intervals h' is constant. (This turns out to be a tricky problem.)

ANSWERS **1.** $f'(3)\,g'(1) \approx 3$ **2.** $f'(0)\,g'(0) \approx -\frac{3}{2}$ **3.** $g'(2)$ does not exist, so $h'(2)$ does not exist.

GROUP WORK 3: EXAMINING A STRANGE GRAPH

Have the students first answer the questions just by looking at the graph, and then go back and verify their intuition using calculus. If the students find this curve interesting, you can point out another interesting property. Consider the line segment going from $(0, -1)$ to $(0, 1)$. The curve gets arbitrarily close to *every* point on this segment, although it never actually touches the segment. If we consider the combined segment and curve we get a mathematical object that is "connected" but not "path connected".
If a group finishes early, perhaps ask them to figure out what the graph of $\tan(1/x)$ will look like, and to verify their guess using their calculators.

ANSWERS

1. $y' = -\dfrac{\cos(1/x)}{x^2}$. As $x \to \infty$, $y' \to 0$. Therefore the function has a horizontal asymptote. Or, one can argue that as $x \to \infty$, $1/x \to 0$, so $\sin(1/x) \to 0$.

2. The function does not approach a specific y-value as $x \to 0$. (One can look at either the function or its derivative as $x \to 0$.)

3. The slope of the curve approaches 0.

4. The slope oscillates, but its peaks and valleys get larger and larger without bound as $x \to 0$.

HOMEWORK PROBLEMS

CORE EXERCISES 4, 7, 10, 12, 24, 44, 53, 67

SAMPLE ASSIGNMENT 4, 7, 10, 12, 19, 24, 44, 53, 63, 65, 67, 68, 73, 80, 84

EXERCISE	D	A	N	G
4		×		
7		×		
10		×		
12		×		
19		×		
24		×		
44		×		
53		×		
63			×	×
65	×		×	
67	×			
68		×		
73	×			
80	×			
84	×	×		

GROUP WORK 1, SECTION 2.5

Unbroken Chain

For each of the following functions of x, write the equation for the derivative function. This will go a lot more smoothly if you remember the Sum, Product, Quotient, and Chain Rules... especially the Chain Rule! Please do us both a favor and don't simplify the answers.

1. $f(x) = \sin 3x$ $\quad f'(x) =$

2. $g(x) = (\sin 3x)^3$ $\quad g'(x) =$

3. $h(x) = (\sin 3x)^3 + 5x$ $\quad h'(x) =$

4. $j(x) = \left[(\sin 3x)^3 + 5x\right]^5$ $\quad j'(x) =$

5. $k(x) = x + \dfrac{1}{x}$ $\quad k'(x) =$

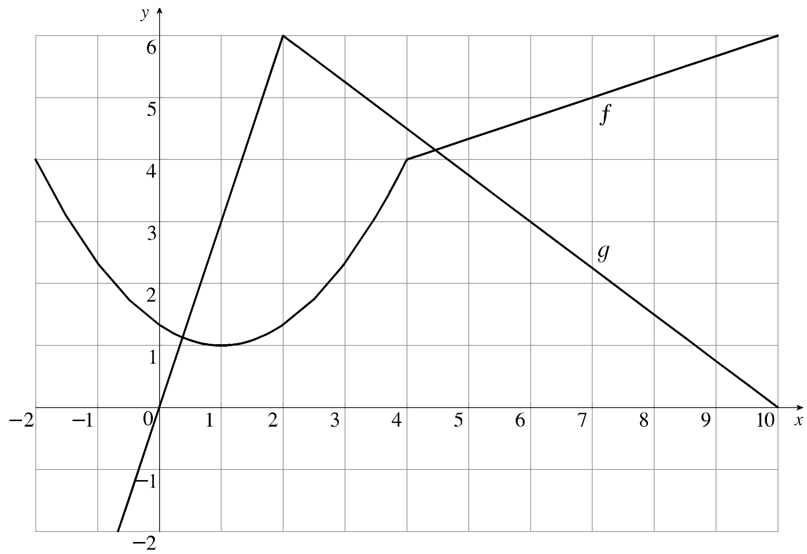
6. $l(x) = \sqrt{x + \dfrac{1}{x}}$ $\quad l'(x) =$

7. $m(x) = \left(\sqrt{x + \dfrac{1}{x}}\right)\left[(\sin 3x)^3 + 5x\right]^5$ $\quad m'(x) =$

GROUP WORK 2, SECTION 2.5

Chain Rule Without Formulas

Consider the functions f and g given by the following graph:



Define $h = f \circ g$.

1. Compute $h'(1)$.

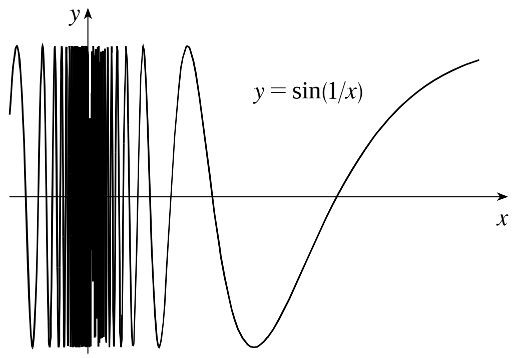
2. Compute $h'(0)$.

3. Does $h'(2)$ exist?

GROUP WORK 3, SECTION 2.5

Examining a Strange Graph

Several times in this course, we have looked at the graph of $y = \sin(1/x)$.



There are some things we can say about the graph just by looking at the picture, although our intuition may deceive us.

1. As we move farther and farther to the right, does the graph oscillate forever, or does it approach some y-value?

2. As we move closer and closer to zero, does the graph oscillate forever, or approach some y-value?

3. What happens to the slope of the curve as we go farther and farther to the right?

4. What happens to the slope of the curve as we approach zero?

Since intuition could fail us, please consider the function $y = \sin(1/x)$ directly, and prove that your answers to the above questions are correct. If it turns out that you were wrong above, then correct your answer and note why your intuition led you astray.

SPECIAL SECTION **Derivative Hangman**

I recommend doing this activity just after covering the Chain Rule, for a class of students who need more practice computing derivatives. It is designed to keep all the students involved and practicing both computing derivatives, and checking their work. Divide the class into teams of 4–6 students each. Put blanks representing the letters of a mystery word or phrase on the board. The game then proceeds as follows:

One representative from each team goes to the blackboard. The teacher then puts up a function either on the blackboard, or using the overhead projector. Everyone in the room tries to compute the derivative. The people at the board cannot speak, but their teammates can work together, speaking quietly.

The first person at the board to compute the derivative slaps the board, blows a whistle, or claps their hands. The teacher calls on him or her to state the solution. Then each other team gets a chance to accept the answer, or challenge.

The team that wins (first to have their representative get it right, or first to challenge successfully) gets to guess a letter of the puzzle. If they guess A, for example, all instances of A in the mystery phrase are filled in:

__ __ A __ __ A __ __ __   __ __ __ __ __ __ A

Whether or not their letter was in the phrase, they then get a chance to guess at the puzzle ("QUADRATIC FORMULA", in this case). If they get it right, the round is over and they win. If not, each team sends up a new representative and the game continues.

If this game is officiated with care and enthusiasm, all the students will be involved and working every time a new problem is put on the board.

APPLIED PROJECT **Where Should a Pilot Start Descent?**

This project can be used as an out-of-class assignment, or as an extended in-class exercise. At this point in the course, some students may be asking about opportunities for extra credit, and an oral report based on this project would be a worthwhile extra-credit activity.

The project includes a computation of the minimum distance from the airport at which an airplane should begin its descent. A nice addition to this project would be the actual figure (or range of figures) used by a local airport, obtained by a few well-placed telephone calls.

2.6 Implicit Differentiation

SUGGESTED TIME AND EMPHASIS

1 class   Essential material

POINTS TO STRESS

1. The concepts of implicit functions and implicit curves.
2. The technique of implicit differentiation.

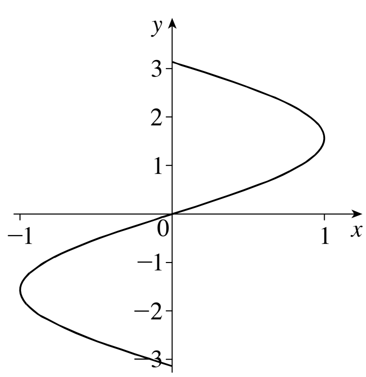
QUIZ QUESTIONS

- **TEXT QUESTION** Describe what is being illustrated by Figure 3. Make sure your answer is as complete as possible.
  ANSWER The implicit curve $x^3 + y^3 = 6xy$ does not define a function. Figure 3 illustrates several functions, each of which is implicitly defined by $x^3 + y^3 = 6xy$.

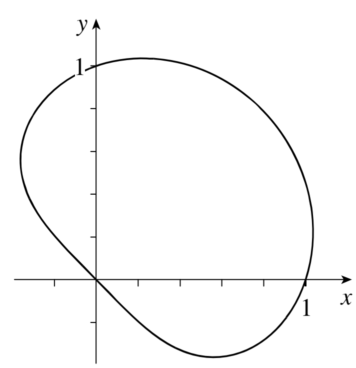
- **DRILL QUESTION** If $x^2 + xy = 10$, find $\dfrac{dy}{dx}$ when $x = 2$.
  ANSWER $-\frac{7}{2}$

MATERIALS FOR LECTURE

- Go over the definition of implicit curves, and the method of implicit differentiation. A good starting example is the curve defined by $x = \sin y$ (which can be easily graphed and visualized). Another example is the curve $x + y = \left(x^2 + y^2\right)^2$, which can be graphed using polar coordinates.
  ANSWER

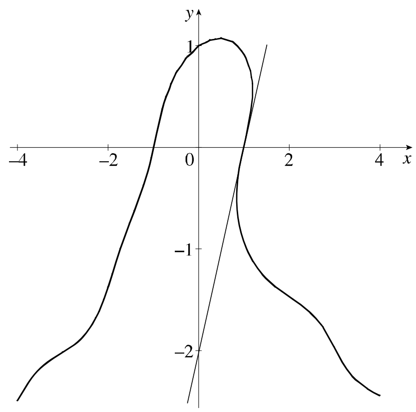


$x = \sin y$

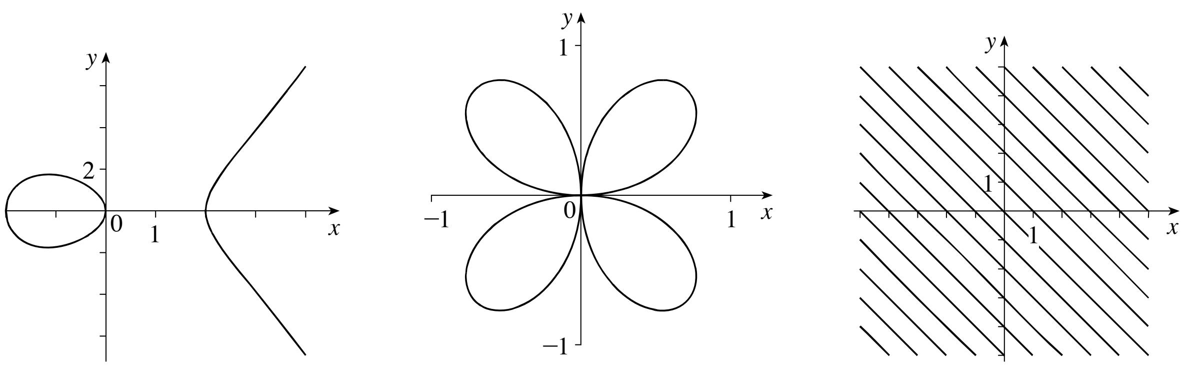


$x + y = \left(x^2 + y^2\right)^2$, $r = \sqrt[3]{\cos\theta + \sin\theta}$

- Derive the equation of the line tangent to the curve $x^2 - \sin(xy) + y^3 = 1$ at the point $(1, 0)$. Sketch the curve as below and draw the tangent line.
  ANSWER The tangent line is $y = 2x - 2$.



- Display some interesting looking implicit curves such as the following:



$y^2 = x^3 - 4x$
an elliptic curve

$x^6 + y^6 = 4x^2y^2 - 3y^2x^4 - 3x^2y^4$
a four-leaved rose

$\sin(\pi(x + y)) = 0$

Have the students figure out a test to see if a given point is on the implicit curve. For example, is $(2, 0)$ on the first graph? Is $(0.6, 0.2)$ on the second? Is $(1.2, 2.8)$ on the third? Have the students determine the slopes of the lines in the third graph, and show that they are parallel.
ANSWER Substituting the coordinates into the equations shows that $(2, 0)$ is on the first graph, $(0.6, 0.2)$ is not on the second, and $(1.2, 2.8)$ is on the third. The lines on the third graph all have slope -1, and are therefore parallel.

WORKSHOP/DISCUSSION

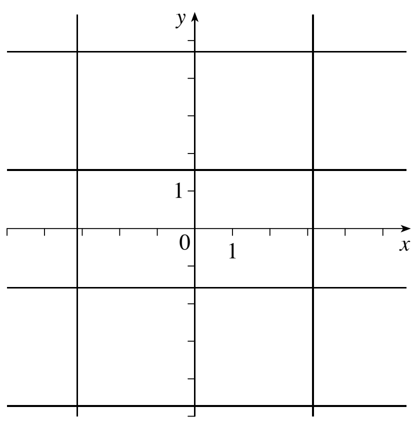
- If the students have access to appropriate graphing technology, have them try to come up with interesting-looking implicit curves. Perhaps have an award for the most aesthetically pleasing one.
- Consider $r^2 + 2s\sqrt{t} = rt$. Show the students how to compute dr/dt when s is held constant, dr/ds and ds/dr when t is held constant, and dt/ds when r is held constant.
- Have the students differentiate $y^2 = x^7 - 6x$ implicitly, and then differentiate $y = \sqrt{x^7 - 6x}$ using the Chain Rule.
- If $f(x)^4 = (x + f(x))^3$ and $f(1) = 2$, find $f'(1)$.

GROUP WORK 1: IMPLICIT CURVES

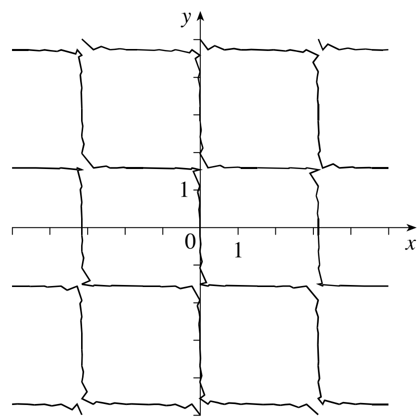
Computer algebra systems are notoriously bad at graphing implicit functions. Even simple functions such as the ones described above in Materials for Lecture point 5 are often poorly graphed by implicit function plotters. This activity describes an implicit curve which many calculators graph inaccurately, but which can be analyzed using a little bit of algebra.

ANSWERS

1. All lines of the form $x = \pi k$, $y = \frac{\pi}{2} + \pi k$, k an integer.



2. Maple gives the graph below.



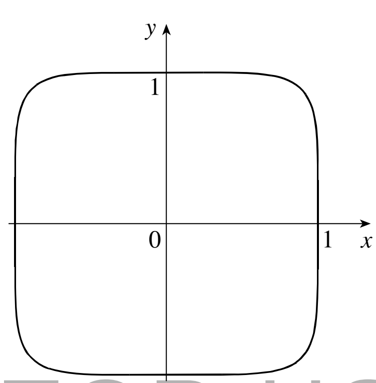
3. $dy/dx = 0$ or is undefined when $x = \pi k$. The derivative must be taken carefully to obtain this result.

GROUP WORK 2: CIRCLES AND ASTROIDS

The basic idea of this activity is for students to visualize flat circles and astroids, and to compute slopes by implicit differentiation. The question about where the slope is 1 or -1 can be addressed first visually and then analytically. As a follow-up question, students can be asked to show that the answers are always the points of intersection with the lines $y = x$ and $y = -x$.

ANSWERS

1. $\dfrac{dy}{dx} = -\left(\dfrac{x}{y}\right)^5$. The slope of the tangent is 1 at $\left(\pm 2^{-1/6}, \mp 2^{-1/6}\right)$ and -1 at $\left(\pm 2^{-1/6}, \pm 2^{-1/6}\right)$.



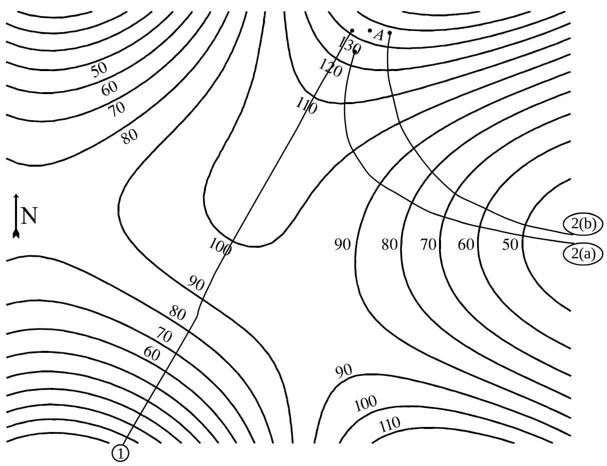
2. If $p/q = \frac{4}{3}$, the slope is 1 at $\left(\pm 2^{-3/4}, \mp 2^{-3/4}\right)$ and -1 at $\left(\pm 2^{-3/4}, \pm 2^{-3/4}\right)$. If $p/q = \frac{2}{5}$, the slope is 1 at $\left(\pm 2^{-5/2}, \mp 2^{-5/2}\right)$ and -1 at $\left(\pm 2^{-5/2}, \pm 2^{-5/2}\right)$.

GROUP WORK 3: A WALK IN THE PARK

Before beginning this activity, discuss the concepts of orthogonal trajectories (discussed in Exercises 49–52) and path of steepest descent. Perhaps do a quick example on the blackboard, and then hand out the activity. Problem 4 requires some deep reasoning.

ANSWERS

1, 2.



3. The steepest descent lines are always perpendicular to the contour lines.

4. Yes, there are. There are precarious balance points between the paths that go to one valley or the other. These are *points of unstable equilibrium*.

HOMEWORK PROBLEMS

CORE EXERCISES 3, 10, 18, 22, 25, 32, 48, 49, 56

SAMPLE ASSIGNMENT 3, 10, 18, 22, 25, 32, 44, 48, 49, 51, 56, 59

EXERCISE	D	A	N	G
3		×		
10		×		
18		×		
22		×		
25		×		
32		×		
44	×			
48	×	×		
49		×		×
51		×		
56	×			
59		×		

GROUP WORK 1, SECTION 2.6

Implicit Curves

Consider the implicit function $\sin x \cos y = 0$.

1. Without using technology, graph this function. You have to think carefully, but you can get it.

2. If you have access to technology that can graph implicit functions, have it graph this function. Do you get a good graph?

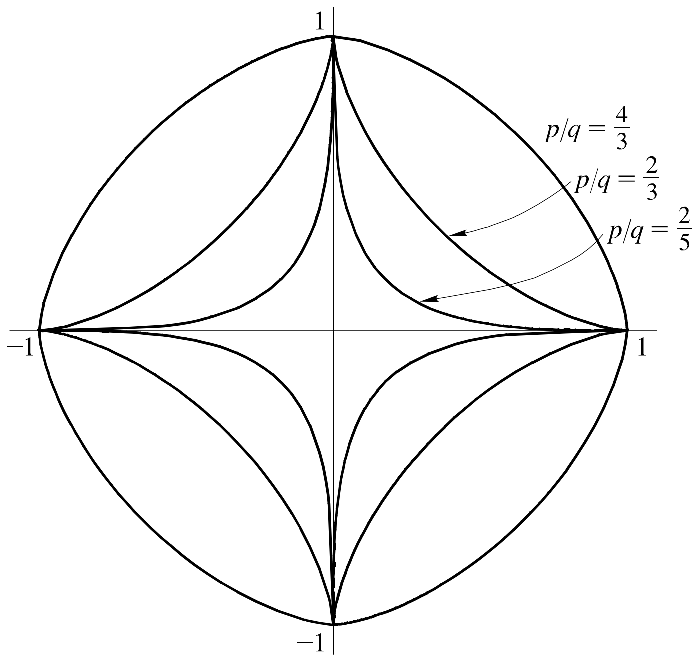
3. Use implicit differentiation to compute $\frac{dy}{dx}$. Does your graph confirm or contradict your answer?

GROUP WORK 2, SECTION 2.6

Circles and Astroids

1. Consider the "flat" circle $x^6 + y^6 = 1$. At what point(s) is the slope of the tangent line equal to 1? Where is it equal to -1?

2. Below are some curves $x^{p/q} + y^{p/q} = 1$, where p is even and q is odd. These curves are sometimes called *astroids* when $p/q < 1$.

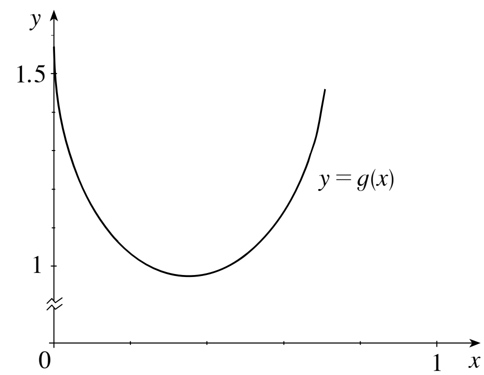


At what point(s) is the slope of the tangent line equal to 1 or -1 if $p/q = \frac{4}{3}$? How about if $p/q = \frac{2}{5}$?

GROUP WORK 3, SECTION 2.6

Looking for the Minimum

The graph of $g(x) = \arcsin\left(x^2 + e^{-x}\right)$ is shown below. Clearly there is a minimum value somewhere between $x = 0.2$ and $x = 0.4$.



1. Find a formula for $g'(x)$.

2. Find an equation of the line tangent to this curve at $x = 0.34$. (Round all numbers to three significant figures.)

3. Does the minimum value of $g(x)$ occur to the left or to the right of $x = 0.34$? How do you know?

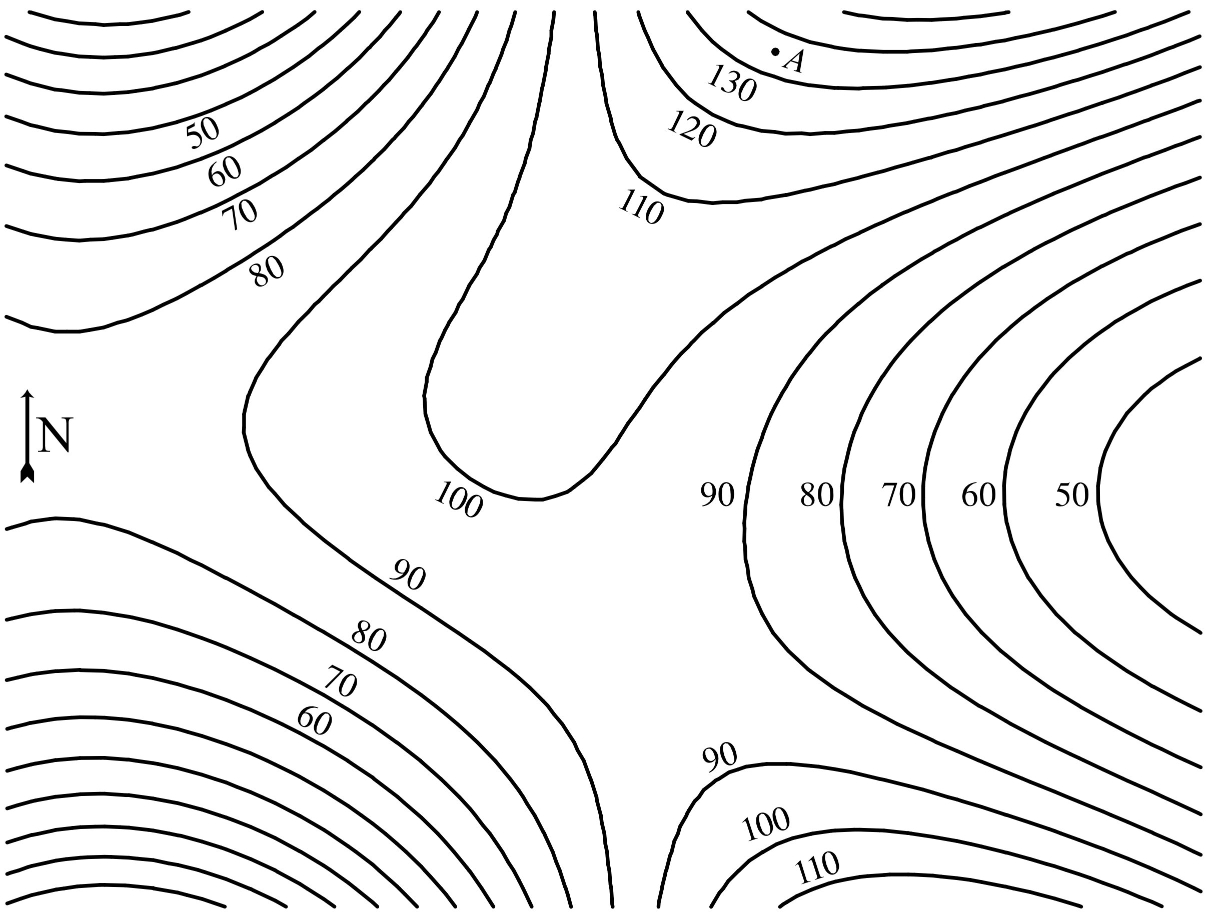
4. Find an equation of the line tangent to the curve at $x = 0.36$. Does the minimum value of $g(x)$ lie to the left or to the right of $x = 0.36$?

5. Estimate the location of the minimum value of $g(x)$. Then use technology to see how close your estimate is to the actual location.

GROUP WORK 4, SECTION 2.6

A Walk in the Park

The following is a contour map of a region in Orange Rock National Park.



1. Suppose you start a little to the west of point A. Draw the path of steepest descent from this point to the edge of the map.

2. (a) Now start a little bit southwest of point A, and trace the path of steepest descent.

(b) Repeat this starting at a point a little east of point A.

3. What assumptions are you making in drawing your paths?

4. Are there any paths starting near point A that do *not* fall into one of the three valleys that are in the park? Explain your reasoning.

LABORATORY PROJECT **Families of Implicit Curves**

This exciting project puts the abilities of a CAS to use quite nicely. Students should be encouraged to take the last part of Problem 1(b) seriously by exploring many values of c, not just the ones explicitly mentioned. With a CAS, this takes only a few keystrokes. In Problem 2, students should be encouraged to play with the equation by putting constants in front of other terms and noting what effect this has on the graph.

2.7 Rates of Change in the Natural and Social Sciences

SUGGESTED TIME AND EMPHASIS

1 class Essential material

POINTS TO STRESS

1. The concepts of average and instantaneous rate of change.
2. Some uses of derivatives in physics and in other disciplines.

QUIZ QUESTIONS

- **TEXT QUESTION** This section discusses many different kinds of examples. What is the main idea underlying them all?

  ANSWER All of them involve expressing quantities as an average rate of change, and then using the idea of the derivative to compute an instantaneous rate of change.

- **DRILL QUESTION** The magnitude F of the force exerted by the Earth on an object is inversely proportional to the square of the distance r from that body to the center of the Earth.

  (a) Write an equation expressing F as a function of r.

  (b) Write an equation expressing dF/dr as a function of r.

  (c) What is the physical meaning of dF/dr?

  ANSWER

  (a) $F = \dfrac{k}{r^2}$ (b) $\dfrac{dF}{dr} = -\dfrac{2k}{r^3}$

  (c) dF/dr tells how fast the force changes as a result of a slight change in the object's distance from the center of the Earth.

MATERIALS FOR LECTURE

- Bring in a taut string, rubber band, violin, or guitar. Illustrate that when the string is plucked, the pitch depends on the length. Discuss Exercise 28, solving it as a class.
- Go over Examples 6 and 7 in detail (or different examples, based on the makeup of the student population).
- Foreshadow Exercise 35 by defining "stable population" and discussing some of the underlying concepts.

WORKSHOP/DISCUSSION

- Discuss some of the issues involved in using a continuous function to model discrete data. For example, ask if taking the derivative of a step function like "cost" is a valid thing to do.
- Do a velocity/distance linear motion problem, such as the one below:

  Let $s(t) = t^4 - 8t^3 + 18t^2$ be the distance function for a particle.

  1. Find the position at $t = 1$, $t = 2$, $t = 3$, and $t = 6$.

  2. Find the velocity at $t = 2$ and $t = 4$.

  3. Determine when the particle is at rest. When is the acceleration zero?

  4. Find the total distance traveled on the intervals [0, 1], [0, 2], [0, 3], and [0, 6].

  5. When is the particle speeding up? Slowing down? This motion can be visualized and analyzed graphically.

GROUP WORK 1: FOLLOW THAT PARTICLE!

Students are asked to analyze the motion of a typical particle.

ANSWERS

1. $0, 3, 22, \approx 1.1$
2. $v(t) = -4t^3 + 15t^2 - 1, -1, 10, 27, \approx -118.6$
3. At rest: at $t \approx 3.7$. Moving forward: $0 \leq t \lesssim 3.7$
4. $\int_1^2 |f(x)|\,dx$ is larger
5. $a(t) = -12t^2 + 30t$
6. Speeding up: $0 < t < 2.5$. Slowing down: $2.5 < t < 5$.

GROUP WORK 2

To help with the homework assignment, put the students into groups, ideally grouping similar majors together, and have each group work on a different problem from the upcoming assignment. After finishing their work, each group should present their solution to the class. Each student will then have a start on several of the problems from the assignment.

HOMEWORK PROBLEMS

CORE EXERCISES 3, 5, 14, 20, 28

SAMPLE ASSIGNMENT 3, 5, 14, 20, 28, 29, 42, 49

EXERCISE	D	A	N	G
3		×		×
5	×		×	
14	×	×		
20	×	×		
28	×	×		
29	×	×		
42	×	×		
49	×	×		

GROUP WORK 1, SECTION 2.7

Follow That Particle!

For 4.95 seconds, a particle moves in a straight line according to the position function

$$f(t) = \left(t^3 + 1\right)(5 - t) - 5$$

where t is measured in seconds and f in feet.

Answer the following questions. You can visualize this motion and verify many of your answers using a graph. First attempt all the problems by hand, and then graph the position function to verify your answers.

1. What is the position of the particle at $t = 0$, $t = 1$, $t = 2$, $t = 4.95$?

2. Find the velocity of the particle at time t. What is the velocity of the particle at $t = 0$, $t = 1$, $t = 2$, $t = 4.95$?

3. When is the particle at rest? When is the particle moving forward?

4. Find the total distance traveled by the particle on the intervals [0, 1] and [1, 2]. Which is larger and why?

5. Find the acceleration of the particle at time t.

6. When was the particle speeding up? Slowing down?

2.8 Related Rates

SUGGESTED TIME AND EMPHASIS

1 class   Recommended material

POINTS TO STRESS

1. The concept of related rates (first two paragraphs of the text).
2. The classic procedure for handling related rates, including the warning to the side of the procedure in the text.
3. The value of careful diagrams and good notation.

QUIZ QUESTIONS

- **TEXT QUESTION** In Example 2 in the text, what is the physical meaning of the negative sign in the expression $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$?
  ANSWER The value of y is getting smaller, because the ladder is moving downward.
- **DRILL QUESTION** If one side of a rectangle, a, is increasing at a rate of 3 inches per minute while the other side, b, is decreasing at a rate of 3 inches per minute, which of the following must be true about the area A of the rectangle?
  (A) A is always increasing
  (B) A is always decreasing
  (C) A is decreasing only when $a < b$
  (D) A is decreasing only when $a > b$
  (E) A is constant.
  ANSWER (D)

MATERIALS FOR LECTURE

- Begin with a quick review of implicit differentiation, particularly when an implicit function in x and y is differentiated with respect to time or some other third variable. Have the students read the first two paragraphs of the section, and try to see why implicit differentiation is going to be useful in solving related rates problems. Then present a sample problem such as Exercise 14, using the strategy outlined in the text. Deliberately start to make the error referred to, to see if the students catch it.
- Bring balloons into class, and show the students (or have them discover for themselves) how the radius naturally grows more slowly as time goes on, assuming air comes in at a constant rate (for example, one breath every 30 seconds).
- Revisit Example 2 in the text. Compute the velocity of the ladder when it is $\frac{1}{1000}$ inch off the ground ($y = 0.001$). Show how that at some point, the tip of the ladder will exceed the speed of light. Have the students discuss what they think the problem is. (This can be done even with a large class; give them a few minutes.) Since the conclusion that "the tip really does exceed the speed of light" is impossible, the only possible conclusion to draw is that the model is faulty. Take a yardstick and actually do the experiment. (The tip of the yardstick does not stay in contact with the wall.) If the room is such that the students cannot all see the result of the experiment, have a few volunteers come up to watch and describe what happens, and encourage the students to try the experiment at home with a ruler or other similar object.

WORKSHOP/DISCUSSION

- Work this problem with the class: You are blowing a bubble with bubble gum and can blow air into the bubble at a rate of 3 in^3/s.
  (a) At what rate is the volume V increasing with respect to the radius when the radius r is 1 inch? When the radius is 3 inches?
  (b) How fast is the radius increasing with respect to time when $r = 1$ inch? When $r = 3$ inches?
  (c) Suppose you increase your effort when $r = 3$ inches and begin to blow in air at a rate of 4 in^3/s. How fast is the radius increasing now?
- Do some challenging related rates problems, such as the ones in the later exercises.
- Many children notice that when they eat a spherical lollipop (as opposed to the disk-shaped kind) it seems like at first they can lick and lick and lick without it seeming to get smaller, and then toward the end it disappears quickly. If they tell an adult, it is usually attributed to imagination or the subjectivity of passing time. Have the students try to come up with a mathematical explanation.
  ANSWER If a student is licking at a constant rate, dV/dt is constant. However, the perceived change in size of the lollipop is based on the *diameter* of the sphere, which decreases more quickly near the end.

GROUP WORK 1: FIND THE ERROR

This activity illustrates a common error that many students make. You may want to project the problem on an overhead, and give the class a few minutes to discuss it. The activity can stand alone, or be handed out as a warm-up.

GROUP WORK 2: NOBODY ESCAPES THE CUBE

This is a good introduction to related rates problems, requiring the students to express the volume of a cube in terms of its surface area.

ANSWERS **1.** 2 in^2/s **2.** $\frac{1}{2}$ in^3/s

GROUP WORK 3: THE SWIMMING POOL

The students shouldn't work on this activity until they've had a chance to see or try some basic related rates problems. Be prepared to give plenty of guidance to the students.

ANSWERS

1. $V = \begin{cases} 500h + \frac{125}{8}h^2 & \text{if } 0 < h < 16 \\ 1500h - 12{,}000 & \text{if } 16 \le h < 20 \end{cases} \quad \Rightarrow \quad \frac{dV}{dt} = \begin{cases} 500 + \frac{125}{4}h & \text{if } 0 < h < 16 \\ 1500\frac{dh}{dt} & \text{if } 16 \le h < 20 \end{cases}$

2. You would need dV/dt, the rate at which the pool is being filled. Note that you would not need h; if you knew dV/dt and the pool was empty at $t = 0$, you could calculate V and then compute h.

HOMEWORK PROBLEMS

CORE EXERCISES 1, 4, 7, 13, 17, 19, 31, 39, 42, 49

SAMPLE ASSIGNMENT 1, 4, 7, 13, 17, 19, 23, 31, 39, 41, 42, 49

EXERCISE	D	A	N	G
1		×		
4		×		
7		×		
13		×		
17		×		
19		×		
23		×		
31	×	×		
39	×			
41				×
42				×
49				×

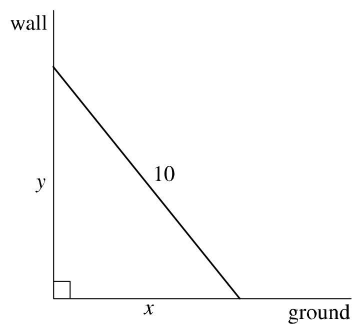
GROUP WORK 1, SECTION 2.8

Find the Error

It is a beautiful Spring evening. You and your wild-eyed, hungry-looking friends are sitting around, reading your Calculus books. You arrive at the following:

> **EXAMPLE 2** A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Your enthused roommates don't read the rest of the example, preferring to do the problem on their own. This is how they proceed:



"We want to find dy/dt. So we set up

$$x^2 + y^2 = 100$$

Now, we want dy/dt when $dx/dt = 1$ and $x = 6$. Substituting $x = 6$ gives us

$$36 + y^2 = 100 \text{ or } y^2 = 64$$

Now we take derivatives:

$$2y\frac{dy}{dt} = 0$$

giving $dy/dt = 0$."

The problem is, of course, that this answer doesn't make any sense.

1. Why does their answer not make any sense?

2. What error did they make? How could they correct it?

GROUP WORK 2, SECTION 2.8

Nobody Escapes the Cube

We are designing a computer graphic in which we zoom in on a cube. The volume V, surface area S, and side length x of the cube are all varying with respect to time. With this information, compute the following quantities, using the steps described in the text:

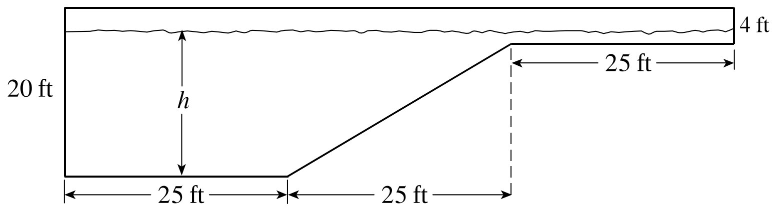
1. dS/dt when $x = 2$ inches and $dV/dt = 1\ \text{in}^3/\text{s}$.

2. dV/dt when $x = 2$ inches and $dS/dt = 1\ \text{in}^2/\text{s}$.

GROUP WORK 3, SECTION 2.8

The Swimming Pool

We wish to find the change in volume of a 20-foot-wide pool as it fills up with water. A cross-section of the pool is shown below.



1. Express dV/dt in terms of h, V, and dh/dt.

2. What additional information would you need to find dh/dt at $t = 10$ minutes?

2.9 Linear Approximations and Differentials

SUGGESTED TIME AND EMPHASIS

1 class   Essential material (linear approximation) and optional material (differentials)

POINTS TO STRESS

1. The general equation of a line tangent to the graph of a function, and its use in approximating that function near a point.
2. The differential as the difference between the linearization of a function and the function itself.

QUIZ QUESTIONS

- **TEXT QUESTION** What is the difference between the function $L(x)$ defined in the text and the equation of the tangent line $y = f(a) + f'(a)(x - a)$?
  ANSWER None
- **DRILL QUESTION** Write the equation of the straight line that best approximates the graph of $y = x + \cos x$ at the point $(0, 1)$.
  ANSWER $y = x + 1$

MATERIALS FOR LECTURE

- Discuss the motivation for studying linear approximations. Ask, "Why use an approximation to a function when a computer can find the answer precisely?"
  ANSWERS
  1. A common modeling technique is to assume a function is locally linear, and then use the linear equation in calculations, since it is easier to manipulate.
  2. It is often easier physically to measure the derivative of a function than the function itself. Then the derivative measurements can be used to obtain an approximation of the function.
  3. When measuring a real phenomenon, there is often no easy-to-understand function that can be written in a line or two, and the best that can be obtained is a set of sample data points. The "underlying" function must be approximated.
  4. In the real world, the input to functions can be noisy or wiggly. It is easier to handle small input fluctuations if we assume that the output varies linearly.
  5. When a function is called thousands of times by a computer program, as occurs in computer graphics applications, the small time savings from using a linear function can result in savings of hours or even days.
- Discuss the meaning of the phrase "approximating along the tangent line" and its connections to linear approximation. Then present examples of linear approximation, such as $\sin x \approx x$ for x near 1 and $x + \cos x \approx x + 1$ for x near 0.
- Raise the question, "What if we want a more accurate model of a function?" Foreshadow the quadratic approximation (Taylor polynomial of order two) as an extension of the linear approximation. (The linear approximation matches the function in the first derivative, so how can you make a function match the second derivative as well?)

- Graph $y = \sin x$ with its approximations at $x = 0$ and $x = \frac{\pi}{4}$. Discuss which is "better".
- To illustrate how controversial differentials once were, cite the quotation from Bishop Berkeley (1734) on differentials: "And what are these evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?"
- Bring in a carpenter's level. Show how, when the level is held perfectly straight, it can be used to measure acceleration. (The bubble moves when the level is accelerated, and returns to center at constant velocity). This can be done on an overhead projector, if the floor is flat. This is the principle used to make a simple accelerometer. Then discuss how, given acceleration measurements, it is possible to approximate velocity using the technique of linear approximation.

WORKSHOP/DISCUSSION

- Let $f(x) = x^{1.857}$. Find the linear approximation of $f(x)$ at $a = 1$ and use it to approximate f at $x = 1.1$, $x = 1.01$, and $x = 1.001$. Compare the approximations to the actual values the calculator gives for f at these points.
- In Example 1, discuss why we base our linear approximation at $x = 1$ rather than at $x = 0.99$ or 1.01.
- Practice using linear approximations with $y = \dfrac{1}{\sqrt{x}}$ at $x = 4$, and use differentials to approximate Δy for $\Delta x = -1$ and $\Delta x = 1$.
- Have the students try to find a linear approximation for $|x|$ near $x = 0$, and explain why it is impossible.

GROUP WORK 1: FOUR VARIATIONS ON A THEME

This activity explores four different functions that have identical linear approximations near $x = 0$.

ANSWERS

1. $y = x$ in all cases.

2.

Function	Function Value at $x = 0.1$	Approximation at $x = 0.1$
f	0.09545	0.1
g	0.101	0.1
h	0.11007	0.1
j	0.09983	0.1

3. If the students need to, they can check the approximations for $x = 0.2$ or $x = 0.3$. The best approximation is the one to $j(x)$, and the worst is the one to $h(x)$. This is immediate from looking at the graphs. Notice that j and g have inflection points at $x = 0$.

GROUP WORK 2: LINEAR APPROXIMATION

Some students may try to find approximations of the derivative functions. They should be reminded that we are approximating f, using the graph of f' as an aid.

ANSWERS

1. $f(x) \approx 1.75(x-2)+4$, so $f(1.98) \approx 3.965$ and $f(2.02) \approx 4.035$.
2. The graph of f lies below its tangent line, so the approximations are overestimates.
3. The estimates are both 7, because the function is horizontal when $x = 3$.

HOMEWORK PROBLEMS

CORE EXERCISES 3, 13, 17, 25, 33, 39

SAMPLE ASSIGNMENT 3, 5, 10, 13, 17, 25, 33, 38, 39, 41

EXERCISE	D	A	N	G
3		×		
5		×		×
10		×		
13		×		
17		×		
25		×		
33	×	×		
38	×	×		
39		×		
41		×		×

GROUP WORK 1, SECTION 2.9

Four Variations on a Theme

Consider the following four functions:

$$f(x) = -1 + \sqrt{2x+1} \qquad g(x) = x^3 + x \qquad h(x) = \tan^2 x + x \qquad j(x) = \sin x$$

1. Find the linearizations of f, g, h, and j at $a = 0$.

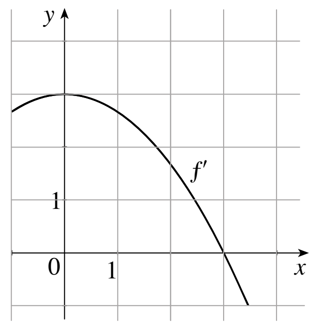
2. Compute the values of each of these functions at $x = 0.1$ and the values of their linearizations.

3. For which function is the approximation best? For which is it worst? Why?

GROUP WORK 2, SECTION 2.9

Linear Approximation

Consider this graph of $f'(x)$, the *derivative* of $f(x)$.



1. Suppose that $f(2) = 4$. Approximate $f(1.98)$ and $f(2.02)$ as best you can. Don't just guess. Show your work.

2. Determine whether your approximations were overestimates or underestimates.

3. Suppose you also know that $f(3) = 7$. Can you approximate $f(2.98)$ and $f(3.02)$? Explain your answer.

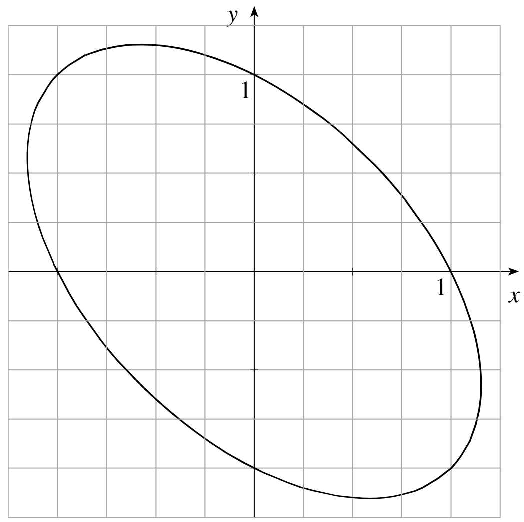
LABORATORY PROJECT **Taylor Polynomials**

This project provides a solid early introduction to Taylor polynomials as extensions of the tangent line approximation concept. A few examples involving $\cos x$ and $\sqrt{x+3}$ are explored in more detail. Students may be asked to explore their own function, and see what happens. Have them go beyond just working through the six questions, and try to demonstrate that they understand the pretty concept introduced in this project.

2 SAMPLE EXAM

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. Consider the graph of $x^2 + xy + y^2 = 1$.



(a) Find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

(b) Find all points where the tangent line is horizontal.

(c) Find all points where the tangent line is parallel to the line $y = -x$.

2. Let $f(x) = 7\sin(x + \pi) + \cos 2x$.

(a) Compute $f'(x)$, $f''(x)$, $f^3(x)$, and $f^4(x)$.

(b) Compute $f^{13}(0)$.

3. Assume that $f(x)$ and $g(x)$ are differential functions that we know very little about. In fact, assume that all we know of these function is the following table of data:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-2	3	1	-5	8
-1	-9	7	4	1
0	5	9	9	-3
1	3	-3	2	6
2	-5	3	8	0

(a) Let $h(x) = g(x)\sin x$. What is $h'(0)$?

(b) Let $j(x) = \left[f(x) + x^2\right]^3$. What is $j'(1)$?

(c) Let $l(x) = \dfrac{\tan \pi x}{g(x)}$. What is $l'(-1)$?

4. Let $u(x)$ be an always positive function such that $u'(x) < 0$ for all real numbers.

(a) Let $f(x) = [u(x)]^2$. For what values of x will $f(x)$ be increasing?

(b) Let $g(x) = u(u(x))$. For what values of x will $g(x)$ be increasing?

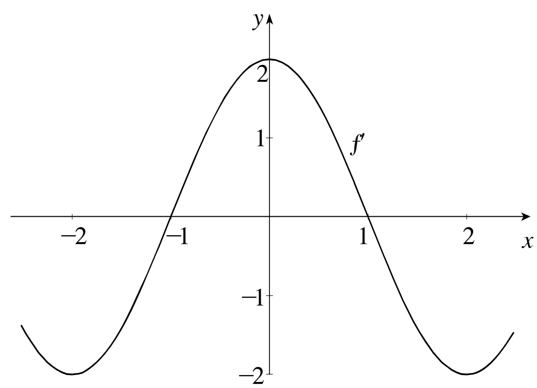
5. Let $f(x) = -x^3 - 2x^2 + x + 1$ and $g(x) = \sin x + 1$.

(a) Find the equation of the line tangent to $f(x)$ at $x = 0$.

(b) Show that $g(x)$ has the same tangent line as $f(x)$ at $x = 0$.

(c) Does this tangent line give a better approximation of $f(x)$ or $g(x)$ at $x = 1$? Give reasons for your answer.

6. The following is a graph of f', the derivative of some function f.



(a) Where is f increasing?

(b) Where does f have a local minimum? Where does f have a local maximum?

(c) Where is f concave up?

(d) Assuming that $f(0) = -1$, sketch a possible graph of f.

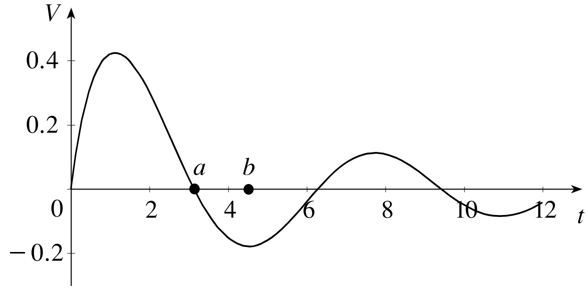
7. As a spherical raindrop evaporates, its volume changes at a rate proportional to its surface area A.

(a) If the constant of proportionality is K, find the rate of change of the radius r when $r = 4$.

(b) Show that the rate of change of the radius is always constant.

(c) Does part (b) mean that the rate of change of the volume is always constant? Why or why not?

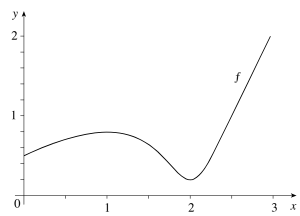
8. The voltage across a resistor R is given by $V(t) = \dfrac{1}{1+t}\sin t$. A graph of $V(t)$ is shown below.



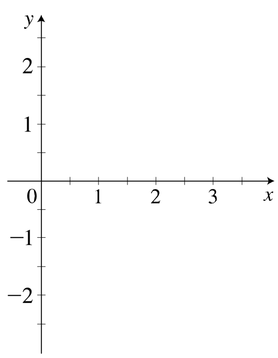
(a) How fast is the voltage changing after 2 seconds?

(b) Would you be better off using the linear approximation at $x = a$ to estimate $V(b)$, or using the linear approximation at $x = b$ to estimate $V(a)$? Justify your answer.

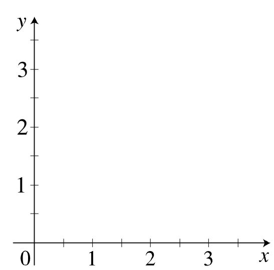
9. Let f be the function whose graph is given below.



(a) Sketch a plausible graph of f'.



(b) Sketch a plausible graph of a function F such that $F' = f$ and $F(0) = 1$.



10. Suppose that the line tangent to the graph of $y = f(x)$ at $x = 3$ passes through the points $(-2, 3)$ and $(4, -1)$.

(a) Find $f'(3)$.

(b) Find $f(3)$.

(c) What is the equation of the line tangent to f at 3?

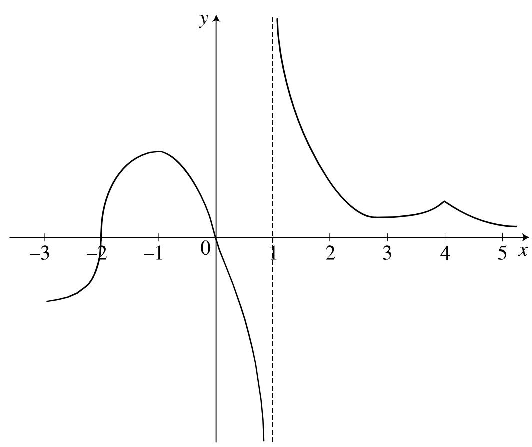
11. Each of the following limits represent the derivative of a function f at some point a. State a formula for f and the value of the point a.

(a) $\lim_{h\to 0} \frac{(3+h)^2 - 9}{h}$

(b) $\lim_{x\to 3} \frac{(x+1)^{3/2} - 8}{x-3}$

(c) $\lim_{h\to 0} \frac{\sin(\pi(2+h)) - 0}{h}$

12. The graph of $f(x)$ is given below. For which value(s) of x is $f(x)$ not differentiable? Justify your answer(s).



2 SAMPLE EXAM SOLUTIONS

1. (a) $2x + x\dfrac{dy}{dx} + y + 2y\dfrac{dy}{dx} = 0$; $\dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$

(b) Set $y + 2x = 0$ and $y = -2x$. Then $x^2 - 2x^2 + 4x^2 = 1 \Leftrightarrow 3x^2 = 1 \Leftrightarrow x = \pm\frac{1}{\sqrt{3}} \Leftrightarrow y = \mp\frac{2}{\sqrt{3}}$, so the points are $\left(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.

(c) Set $-\dfrac{2x+y}{x+2y} = -1$ to get $y = x$. Then $x^2 + x^2 + x^2 = 3x^2 = 1 \Leftrightarrow x = \pm\frac{1}{\sqrt{3}} \Leftrightarrow y = \pm\frac{1}{\sqrt{3}}$, so the points are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

2. (a) $f'(x) = 7\cos(x+\pi) - 2\sin 2x$; $f''(x) = -7\sin(x+\pi) - 4\cos 2x$; $f^{(3)}(x) = -7\cos(x+\pi) + 8\sin 2x$; $f^{(4)}(x) = 7\sin(x+\pi) + 16\cos 2x$

(b) $f^{(13)}(x) = 7\cos(x+\pi) - 2^{13}\sin 2x$; $f^{(13)}(0) = 7\cos\pi = -7$

3. (a) $h'(x) = g'(x)\sin x + g(x)\cos x$; $h'(0) = g(0) = 9$

(b) $j'(x) = 3\left(f(x) + x^2\right)^2\left(f'(x) + 2x\right)$; $j'(1) = 3\left(f(1)+1\right)^2\left(f'(1)+2\right) = 3\cdot 4^2\cdot 4 = 192$

(c) $l'(x) = \dfrac{g(x)\cdot\pi\sec^2\pi x - g'(x)\tan\pi x}{g(x)^2}$; $l'(-1) = \dfrac{7\pi - 0}{7^2} = \dfrac{\pi}{7}$

4. (a) $f'(x) = 2u(x)u'(x) < 0$ for all x, since $u(x) > 0$ and $u'(x) < 0$. Never increasing.

(b) $g'(x) = u'(u(x))\cdot u'(x) > 0$, since $u'(u(x))$ and $u'(x) < 0$. Always increasing.

5. (a) $f'(x) = -3x^2 - 4x + 1$; $f'(0) = 1$, $f(0) = 1$. Tangent line is $y = 1 + 1\cdot x = 1 + x$

(b) $g'(x) = \cos x$; $g'(0) = 1$, $g(0) = 1$. Tangent line is $y = 1 + x$

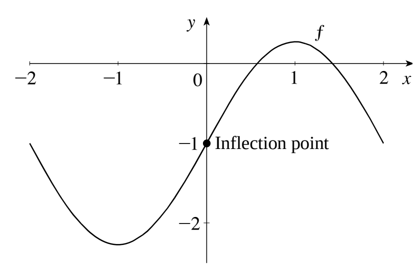
(c) At $x = 1$, $f(1) = -1$, $g(1) = \sin 1 + 1 \approx 1.841$. The tangent line approximation is $y = 1 + 1 = 2$. This is better for $g(x)$ at $x = 1$.

6. (a) f is increasing on $(-1, 1)$.

(b) Local minimum at $x = -1$; local maximum at $x = 1$

(c) f is concave up where $f'(x)$ is increasing, that is, on $(-2, 0)$.

(d)



7. (a) $\dfrac{dV}{dt} = KA$. $V = \frac{4}{3}\pi r^3$, so $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. Since $A = 4\pi r^2$, we have $K4\pi r^2 = 4\pi r^2\dfrac{dr}{dt}$. Thus, $\dfrac{dr}{dt} = K$.

(b) By part (a), $\dfrac{dr}{dt} = K$ is constant. $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt} = 4K\pi r^2$. So $\dfrac{dV}{dt}$ depends on r^2 and is not constant.

8. (a) $V'(t) = \dfrac{1}{1+t}\cos t - \dfrac{1}{(1+t^2)}\sin t$; $V'(2) = \frac{1}{3}\cos t - \frac{1}{9}\sin 2 \approx -0.240$

(b) The tangent line at $x = b$ is horizontal. So the estimate for $V(a)$ using this linear approximation is $V(b)$, which is not very good. Thus, it is better to use the linear approximation at $x = a$ to estimate $V(b)$.

9. (a) Answers will vary. Look for:

(i) zeros at 1 and 2

(ii) f' positive for $x \in (0, 1)$ and $(2, 4)$

(iii) f' negative for $x \in (1, 2)$

(iv) f' flattens out for $x > 2.5$

(b) Answers will vary. Look for

(i) $F(0) = 1$

(ii) F always increasing

(ii) F is never perfectly flat

(iv) F is closest to being flat at $x = 2$

(v) F is concave up for $x \in (0, 1)$ and $x \in (2, 4)$

(vi) F is concave down for $x \in (1, 2)$

10. (a) $\frac{3-(-1)}{-2-4} = -\frac{2}{3}$

(b) The equation of the tangent line is $y - 3 = -\frac{2}{3}(x+2)$, so $f(3) = -\frac{2}{3}(3+2) + 3 = -\frac{1}{3}$.

(c) The equation of the tangent line is $y - 3 = -\frac{2}{3}(x+2)$.

11. (a) $f(x) = x^2$, $a = 3$ (b) $f(x) = (x+1)^{3/2}$, $a = 3$ (c) $f(x) = \sin(\pi x)$, $a = 2$

12. f isn't differentiable at $x = 1$, because it is not continuous there; at $x = -2$, because it has a vertical tangent there; and at $x = 4$, because it has a cusp there.